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Kurosh rank of intersections of subgroups of free products of right-orderable groups

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We prove that the reduced Kurosh rank of the intersection of two subgroups H and K of a free product of right-orderable groups is bounded above by the product of the reduced Kurosh ranks of H and K.

In particular, taking the fundamental group of a graph of groups with trivial vertex and edge groups, and its Bass–Serre tree, our theorem becomes the desired inequality of the usual strengthened Hanna Neumann conjecture for free groups.

1. Introduction

Let H and K be subgroups of a free group F, and $\overline{r}(H) = \max\{0, \mathrm{rank}(H) - 1\}$. It was an open problem dating back to the 1950s to find an optimal bound of the rank of $H \cap K$ in terms of the ranks of H and K.

In [14], Hanna Neumann proved the following:

$$\overline{r}(H \cap K) \leq 2 \cdot \overline{r}(H) \cdot \overline{r}(K). \tag{1}$$

The *Hanna Neumann conjecture* says that (1) holds replacing the 2 with a 1.

Later, in [15], Walter Neumann improved (1) to

$$\sum_{g \in K\backslash F/H} \overline{r}(H^g \cap K) \leq 2 \cdot \overline{r}(H) \cdot \overline{r}(K), \tag{2}$$

where $H^g = g^{-1}Hg$. The *strengthened Hanna Neumann conjecture*, introduced by Walter Neumann, says that (2) holds replacing the 2 with a 1.

These two conjectures have received a lot of attention, and recently, Igor Mineyev [13] proved that both conjectures are true (independently and at the same time, Joel Friedman also proved these conjectures (see [5, 8])). The fact that F is right-orderable plays a crucial role in Mineyev's proof.

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Definition 1.1. A group G is *right-orderable* if it admits a total order which is invariant under the right G-multiplication action.

In particular, right-orderability is inherited by subgroups and implies torsion freeness. The class of right-orderable groups is also closed by free products (see reference below) and extensions. The integers with the natural order is a right-ordered group. It follows that the class of right-orderable groups contains all poly-free groups. This last class contains, for example, all fundamental groups of surfaces (excluding the projective plane), finitely generated right-angled Artin groups, pure braid groups and poly-{infinite cyclic} groups.

We recall the well-known Kurosh Subgroup Theorem [11]. A proof can be found in [6].

Theorem 1.2 (Kurosh Subgroup Theorem). *Let $G = *_{i\in I} A_i$ be a free product and H a subgroup of G. Then*

$$H = *(H \cap A_i^g) * F, \tag{3}$$

where the g ranges over a set of double coset representatives in $A_i\backslash G/H$ for each $i \in I$ and F is a free group.

In view of the theorem, one would like to define the Kurosh rank of H with respect to the free product $*_{i\in I} A_i$ as the number of non-trivial factors $(H \cap A_i^g)$ in (3) plus the rank of F. One has to prove that this number is independent of the double coset representatives. This is done in [10, Lemma 3.4].

However, we prefer to give a different definition based on groups acting on trees. In our approach, the Kurosh rank of $H \leqslant G$ will depend on the action on a G-tree T rather in a free product decomposition of G. We will denote the Kurosh rank by $\kappa_T(H)$ and the reduced Kurosh rank by $\overline{\kappa}_T(H) = \max\{0, \kappa_T(H) - 1\}$. We will give the formal definition of κ_T in Section 2.

It is natural to consider to what extent Mineyev's Theorem can be generalized to intersections of subgroups of free products. It turns out that the Kurosh rank is the appropriate concept to consider in this context. A proof of the Howson property for free products with respect to the Kurosh rank can be found in [17, Theorem 2.13(1)].

Let G be a group, T a G-tree with trivial edge stabilizers and $H, K \leqslant G$. Implicitly in [16, Theorem 3], Soma proved that

$$\overline{\kappa}_T(H \cap K) \leq 18\overline{\kappa}_T(H)\overline{\kappa}_T(K).$$

To our knowledge, Burns *et al.* were the first studying explicitly the Kurosh rank of the intersection of subgroups in [2]. Ivanov [9] found that the inequality (1) holds for factor free subgroups of free products of right-orderable groups. Later, Dicks and Ivanov [7] extended this result to factor free subgroups of free products of torsion-free groups. In [10], Ivanov using results obtained in [7] proved that if G is torsion free then

$$\overline{\kappa}_T(H \cap K) \leq 2\overline{\kappa}_T(H)\overline{\kappa}_T(K).$$

Our main result is the following

Theorem A. *Let G be a right-orderable group and T a G-tree with trivial edge stabilizers. Let H, K be subgroups of G. Then*

$$\sum_{g \in K \backslash G / H} \overline{\kappa}_T(H^g \cap K) \leq \overline{\kappa}_T(H)\overline{\kappa}_T(K).$$

Remark 1.3. We note that if one considers a free group acting freely on a tree, then the Kurosh rank of a subgroup and its genuine rank agree. Therefore the strengthened Hanna Neumann conjecture is a corollary of Theorem A.

Remark 1.4. The main interest is when all these quantities are finite, but it is also true when they are infinite, in which case we adopt the convention that $0 \cdot \infty = 0$.

Mineyev's brilliant Hilbert-module proof [13] yields to a very general result. Our proof of Theorem A is basically the same as the simplified version of Mineyev's proof due to Warren Dicks [4], which only applies to the special case of the strengthened Hanna Neumman conjecture. A proof based on [4, 13] that uses neither Hilbert-module theory nor Bass–Serre theory was given in [12].

We note that the main tool for passing from the free group case and the free product case is the proof of Theorem 2.4 and its application in Theorem 3.8, which says that the reduced Kurosh rank is equal to the number of orbits of edges in the Dicks' tree.

To obtain examples of groups and trees for Theorem A, we can consider G to be a graph of groups with right-orderable vertex groups and trivial edge groups, and T the corresponding Bass–Serre tree. In this case, the group is a free product of right-orderable groups. One can prove that such group is right orderable using the Kurosh subgroup theorem and [3, Theorem 2].

Another fairly simple proof of the fact that right orderability is preserved under free products is given in [1, Corollary 36].

2. Kurosh rank

Our notation and basic reference for groups acting on trees is [6]. The groups are acting on the right.

Definition 2.1 (Kurosh rank). Let G be a group, T a G-tree with trivial edge stabilizers and H a subgroup of G.

Let $c_T(H) \in \mathbb{N} \cup \{\infty\}$ be the number of vertices $vH \in VT/H$ such that v has a non-trivial H-stabilizer. This is well defined since it is independent of the choice of the representative of vH.

The *Kurosh rank* of H with respect to T is defined to be

$$\kappa_T(H) := c_T(H) + \operatorname{rank}(T/H),$$

where the $\operatorname{rank}(T/H)$ is the number of edges outside of a maximal subtree of T/H (equivalently, $\operatorname{rank}(T/H)$ is the rank of fundamental group of the graph T/H, which is a free group).

The *reduced Kurosh rank* of H with respect to T is defined to be

$$\overline{\kappa}_T(H) := \max\{0, \kappa_T(H) - 1\}.$$

Remark 2.2. We note that the Kurosh rank of a subgroup depends on the free product decomposition and not just the isomorphism type of the subgroup. Therefore, suppose we took the free product of two surface groups, S_1, S_2. Then a free subgroup of infinite rank inside S_1 would have Kurosh rank 1, whereas a free subgroup of infinite rank meeting no conjugate of either S_1 or S_2 would have infinite Kurosh rank.

Proposition 2.3. *Let H be a subgroup of G. Let T' be an H-subtree of T. Then $\kappa_T(H) = \kappa_{T'}(H)$.*

In particular, T_H/H is finite (as a graph) if and only if $\kappa_T(H)$ is finite.

Proof. This is an exercise in Bass–Serre theory and the proof is left to the reader.

The proposition also follows as a special case of a more general result of Sykiotis ([17, Proposition 2.2.] in view of [17, Lemma 2.5]), since the Sykiotis' "complexity" of a subgroup of a free product is equal to its Kurosh rank. □

Let T be an H-tree and F an H-subforest of T. The *tree obtained from T by collapsing F* is the tree $\mathcal{T}$, whose edges set is exactly $ET - EF$ and the vertex set is the set of connected components of $T - E\mathcal{T}$. Note that since F is an H-subforest, $E\mathcal{T} = ET - EF$ is and H-set. It follows that the connected components of $T - E\mathcal{T}$ is also an H-set. There is a natural equivariant quotient graph map $\phi\colon T \to \mathcal{T}$, that restricted to $ET - EF$ is the identity, and collapses each connected component of $T - (ET - EF)$ to its corresponding vertex in $\mathcal{T}$.

Theorem 2.4. *Let T be an H-tree and let $\mathcal{T}$ be the H-tree obtained from T by collapsing an H-subforest of T. Then*

$$\kappa_T(H) = \sum_{v \in V\mathcal{T}/H} \kappa_T(H_v) + \operatorname{rank}(\mathcal{T}/H). \tag{4}$$

In particular, the left-hand side is infinite if and only if the right-hand side is infinite.

Proof. Let $\phi\colon T \to \mathcal{T}$ be the natural H-equivariant quotient graph map. We let $\overline{\phi}$ be the induced graph map $T/H \to \mathcal{T}/H$.

As ϕ consists of the collapse of various subtrees to points, we have that $\phi^{-1}(v) = T_v$ is connected for any vertex v of $\mathcal{T}$. Hence T_v is an H_v-subtree of T, where H_v is the stabilizer of $v \in V\mathcal{T}$.

Note that if for some $h \in H$ and some $v \in V\mathcal{T}$, $T_v h \cap T_v \neq \emptyset$, then $vh = v$ and hence $h \in H_v$. In particular, the graph map $T_v/H_v \to T/H$ is injective.

For every $vH \in V\mathcal{T}/H$ we let Γ_v be the subgraph T_v/H_v of T/H and $(\Gamma_v, H_v(-))$ be the associated graph of groups, which has fundamental group H_v by Bass–Serre Theory. Since every vertex of T is in the pre-image of some vertex of $\mathcal{T}$, we have that

$$VT/H = \bigsqcup_{vH \in V\mathcal{T}/H} V\Gamma_v.$$

We view the edge set $E\mathcal{T}$ as a subset of ET, and the remaining edges of T are precisely those which map to a vertex in $\mathcal{T}$. Hence it is clear that

$$ET/H = \bigsqcup_{vH \in V\mathcal{T}/H} E\Gamma_v \sqcup \bigsqcup_{eH \in E\mathcal{T}/H} eH. \tag{5}$$

We now consider the rank of the graph, T/H, which is simply the number of edges outside of any maximal subtree. We construct a maximal subtree first by taking the union of maximal subtrees of each Γ_v, and then enlarging the resulting forest to a maximal subtree, Y, of T/H. We note that the map $\overline{\phi}$ simply consists of collapsing each Γ_v to a point. Since each $Y \cap \Gamma_v$ is connected, $\overline{\phi}(Y)$ must be a tree, and contains every vertex of $\mathcal{T}/H$ as $\overline{\phi}$ is surjective. Hence $\overline{\phi}(Y)$ is a maximal subtree of $\mathcal{T}/H$. As before, we think of edges of $\mathcal{T}/H$ as being a subset of the edges of T/H and, hence the edges of $\overline{\phi}(Y)$ as being a subset of the edges of Y.

Therefore, counting edges outside of Y and bearing in mind the disjoint union given by (5) immediately gives that

$$\operatorname{rank}(T/H) = \sum_{vH \in V\mathcal{T}/H} \operatorname{rank}(\Gamma_v) + \operatorname{rank}(\mathcal{T}/H). \tag{6}$$

In particular, the left-hand side is infinite if and only if (any term of) the right-hand side is infinite.

By Proposition 2.3, $\kappa_T(H_v) = \kappa_{T_v}(H_v)$. Since $T_v/H_v = \Gamma_v$, we have that $\kappa_T(H_v) = \operatorname{rank}(\Gamma_v) + c_{T_v}(H_v)$. Moreover, since the $\cup_{v \in V\mathcal{T}/H} V\Gamma_v = VT/H$ and $(\Gamma_v, H(-))$ is the restriction of the graph of groups $(T/H, H(-))$ to the subgraph Γ_v we have that $c_T(H) = \sum_{v \in V\mathcal{T}/H} c_{T_v}(H_v)$. Thus

$$c_T(H) + \sum_{vH \in V\mathcal{T}/H} \operatorname{rank}(\Gamma_v) = \sum_{vH \in V\mathcal{T}/H} \kappa_T(H_v). \tag{7}$$

Again, the left-hand side is infinite if and only if the right-hand side is infinite.

Now summing (7) and (6) we get that

$$\begin{aligned}\kappa_T(H) + \sum_{vH \in V\mathcal{T}/H} \operatorname{rank}(\Gamma_v) = &\sum_{vH \in V\mathcal{T}/H} \kappa_T(H_v) + \operatorname{rank}(\mathcal{T}/H) \\ &+ \sum_{vH \in V\mathcal{T}/H} \operatorname{rank}(\Gamma_v).\end{aligned}$$

Hence we are done if $\sum_{vH \in \mathcal{T}/H} \operatorname{rank}(\Gamma_v)$ is finite. However, in the case that $\sum_{vH \in \mathcal{T}/H} \operatorname{rank}(\Gamma_v)$ is infinite, we deduce from (6) that the left-hand side of (4) is infinite, and from (7) that the right-hand side of (4) is infinite. □

3. Main argument

Throughout this section G will be a right-orderable group and T a G-tree with trivial edge stabilizers. G will act on T on the right.

An element $g \in G$ is called *elliptic* if it fixes a point in T and is called *hyperbolic* if it does not.

Given a hyperbolic $g \in G$, the *axis* of g, denoted A_g, consists of the subtree of points displaced by the minimal amount by g (with respect to the path metric). This is always non-empty and homeomorphic to the real line.

Associated to any non-trivial subgroup (not necessarily finitely generated), $H \leqslant G$ there is a minimal H-invariant subtree T_H of T. In general, this will be the union of the axes of hyperbolic elements of H except when every element of H is elliptic. In this case, a result of Serre says that any finite set of elements of H have a common fixed point and therefore, as edge stabilizers are trivial, there will be a unique point for the whole of H, in which case T_H will be the fixed vertex for H.

Remark 3.1. If, in Theorem A, either $\kappa_T(H)$ or $\kappa_T(K)$ is equal to infinity, then the theorem holds. So, without loss of generality, we can assume that $\kappa_T(H)$ and $\kappa_T(K)$ are finite. Hence $\kappa_T(\langle H \cup K\rangle)$ is also finite, and in view of Proposition 2.3, we can change G to $\langle H \cup K\rangle$, T to $T_{\langle H\cup K\rangle}$ and hence we can assume that T/G is finite.

Throughout the rest of the section T/G will be a finite graph.

We fix an order $<$ of G, and we use it to construct an order on the edges of T. We first put any total order on ET/G, which is a finite set, and we denote it again by $<$. Then, we order $(ET/G) \times G$ lexicographically, and use the natural bijection of this set with ET to order it. That is $(eG, g) \leq (fG, h)$ if and only if $eG < fG$ or $eG = fG$ and $g \leq h$. This ordering on the edges of T is invariant under the action of G. We henceforth fix this ordering.

Definition 3.2 (Dicks' trees). Let T' be a subtree of T.

An edge e of T' is a *T'-bridge* if there is a reduced bi-infinite path in T', containing e and in which e is the $<$-largest edge.

For any subgroup H of G, we call an edge an *H-bridge* if it is a T_H-bridge. Note that H acts freely on the set of H-bridges.

Note that if $T_0 \subseteq T_1$ are subtrees of T then any T_0-bridge is a T_1-bridge. Hence, if $H \leqslant K$ are subgroups of G, then any H-bridge is also a K-bridge.

An *H-island* is a component of T_H after all the H-bridges have been removed. Note that H acts on the set of H-islands.

The Dicks' H-tree, $\mathcal{T}_H$, is the H-tree whose vertices are the H-islands and whose edges are the H-bridges. In other words, $\mathcal{T}_H$ is the tree obtained from T_H by collapsing the H-forest consisting in the union of all H-islands. Note that all the edge stabilizers in $\mathcal{T}_H$ are trivial.

Remark 3.3. The purely combinatoric concept of a T-bridge is introduced by Dicks in [4] and corresponds to an order-essential edges in Mineyev's terminology [13, Definition 2]. The concept of islands corresponds to relative components to the set of order-essential edges. The relationship between bridges and order-essential edges is not at all obvious from the definitions and that is why we use Dicks terminology.

Proposition 3.4. *Let $1 \neq H$ be a subgroup of G with $\kappa_T(H) < \infty$. Suppose that for every H-island I in T_H, the H-stabilizer H_I has $\kappa_T(H_I) = 1$. Then $\overline{\kappa}_T(H)$ is equal to $|E\mathcal{T}_H/H|$, the number of orbits of edges in the Dicks' H-tree.*

Proof. As $\kappa_T(H) < \infty$, T_H/H is finite, and hence so is $\mathcal{T}_H/H$.

Since $\kappa_T(H_I) = 1$ for all H-island I, we have that

$$\sum_{I \in V\mathcal{T}_H/H} \kappa_T(H_I) = |V\mathcal{T}_H/H|.$$

By Theorem 2.4 we have that

$$\kappa_T(H) = |E\mathcal{T}_H/H| - |V\mathcal{T}_H/H| + 1 + \sum_{I \in V\mathcal{T}_H/H} \kappa_T(H_I) = |E\mathcal{T}_H/H| + 1.$$

Since $H \neq 1$, $\overline{\kappa}_T(H) = \kappa_T(H) - 1 = |E\mathcal{T}_H/H|$. □

Therefore the goal will be to show that for the tree $\mathcal{T}_H$, the stabilizer of any vertex has Kurosh rank 1 (to obtain our main result Theorem 3.8). Equivalently, we need to show that stabilizers of H-islands have Kurosh rank less than 2 (Proposition 3.5) and are non-trivial (Proposition 3.7).

Proposition 3.5. *Suppose that $H \leqslant G$ and that $\kappa_T(H) \geq 2$. Then there is an H-bridge in T_H.*

Proof. If H fixes a vertex, T_H will simply be this fixed vertex and $\kappa(H) \leq 1$, contradicting the hypothesis.

Now suppose that T_H is a single line. If some $1 \neq h \in H$ fixes a vertex, then h^2 fixes T_H and hence $h^2 = 1$. This is impossible since H is right-orderable, and hence torsion free. Thus H acts freely on T_H, i.e., $H \cong \mathbb{Z}$, so has Kurosh rank 1, again contradicting the hypothesis.

Hence there exist two hyperbolic elements $g, h \in H$ with distinct axes, $A_g \neq A_h$. These two might intersect non-trivially, but they can only intersect in finitely many edges. If they did intersect in an infinite ray, then the commutator $ghg^{-1}h^{-1}$ would fix an infinite subray and hence an edge. However, the action is free on the edge set, and hence the commutator would be the trivial element, implying that g and h commute and therefore have the same axis, contradicting our choice of g and h.

Now choose a vertex $v \in A_g$ and let p denote a path from vg^{-1} to vg with $<$-largest edge e. By replacing g with g^{-1} if necessary, we may assume that $e > eg$. It follows that $e > eg > eg^2 > \cdots > eg^n \ldots$. Therefore e is the largest edge in the infinite ray starting with e and continuing in the positive direction of the axis. (Note that the action of an hyperbolic element g on its axis A_g induces an orientation of A_g with respect to which g translates in the positive direction.) Likewise eg^n is the largest edge in the infinite ray starting with eg^n.

It follows that every infinite ray p_∞ in A_g starting at any vertex of A_g and going in the positive direction has a $<$-largest edge. Similarly, every infinite ray r_∞ in A_h starting at any vertex of A_h and going in the positive direction has a $<$-largest edge.

Thus there will exist a reduced bi-infinite path of the form $p_\infty^{-1} \cdot q \cdot r_\infty$ where q is a finite path from A_g to A_h; in the case where A_g and A_h are disjoint, q is the path from one axis to the other, and in the case where they intersect, q is a subpath of the intersection, possibly a single vertex. In either case, $p_\infty^{-1} \cdot q \cdot r_\infty$ has a $<$-largest edge which is then an H-bridge in T_H. □

Lemma 3.6. *Suppose that $H \leqslant G$ and that $\kappa_T(H) = \infty$. Then there are infinitely many H-orbits of H-bridges in $\mathcal{T}_H$.*

Proof. If $|E\mathcal{T}_H/H| < \infty$, then $|V\mathcal{T}_H/H| < \infty$ and also $\mathrm{rank}(\mathcal{T}/H) < \infty$. Hence, by Theorem 2.4, $\sum_{vH \in V\mathcal{T}/H} \kappa_T(H_v) = \infty$. Therefore there exists a $vH \in V\mathcal{T}/H$ such that $\kappa_T(H_v) = \infty$. So by Proposition 3.5, T_{H_v} contains an H_v-bridge, and hence an H-bridge. This contradicts the fact that H_v is the stabilizer of an island. □

Proposition 3.7. *Let H be a non-trivial subgroup of G with $\kappa_T(H) < \infty$ and let I be an H-island in T_H with stabilizer H_I. Then H_I is non-trivial.*

Proof. If there is no H-bridge, then $H_I = H$ and, since H is non-trivial, H_I is non-trivial.

Consider the set $\{e \ : \ e$ is a bridge whose initial point is in $I\}$. Note that T_H/H is a finite graph, therefore if the set above is infinite, it must contain edges e and eh for some $1 \neq h \in H$. Clearly, by looking at initial points, h is a non-trivial element preserving I and we would have that H_I would be non-trivial.

Therefore, we may assume that the set above is finite and list the elements in order, $e_1 < e_2 < \cdots < e_s$.

Then e_1 is the largest edge in a reduced bi-infinite path in T_H. The tail of this path is a ray whose initial vertex is in I and hence the entire ray must remain in I due to the minimality of e_1. Therefore, again as T_H/H is finite, there are two distinct edges of I in the same H-orbit, and therefore H_I is non-trivial (and note that this is really a contradiction, since it implies that the set above is infinite unless $\mathcal{T}_H$ is a single vertex). □

We now summarise the three propositions in one theorem.

Theorem 3.8. *Let G be a right-orderable group and T an G-tree with trivial edge stabilizers and finitely many orbits of edges. Let H be a subgroup of G. Then $\overline{\kappa}_T(H)$ is equal to $|E\mathcal{T}_H/H|$, the number of orbits of edges in Dicks' H-tree.*

Proof. If H is the trivial group, then the theorem holds. So we assume H is non-trivial. If $\kappa_T(H) = \infty$, then by Lemma 3.6, $\mathcal{T}_H/H$ is also infinite, and the theorem holds. So we assume $\kappa_T(H) < \infty$. For every H-island I, by Proposition 3.7, its H-stabilizer H_I is non-trivial. By Proposition 3.5, if $\kappa(H_I) \geq 2$ then there would be a T_{H_I}-bridge, which would imply the existence of an H-bridge in I, contradicting the definition of I. Then $\kappa(H_I) = 1$. The theorem now follows from Proposition 3.4. □

Proof of Theorem A. If either H or K is trivial, the theorem trivially holds. Hence, we assume that H and K are non-trivial.

By Remark 3.1, the theorem also holds when the Kurosh rank of H or K is infinite. Therefore, again by Remark 3.1, we can assume that T/G, T_H/H and T_K/K are finite.

The inclusions $T_{(H^g\cap K)} \subseteq T_{H^g} = (T_H)g$ and $T_{(H^g\cap K)} \subseteq T_K$ induce a (diagonal) graph map, sending $e(H^g \cap K)$ to $(eg^{-1}H, eK)$. Therefore we get a map

$$\bigcup_{g\in K\backslash G/H} T_{(H^g\cap K)}/(H^g\cap K) \to T_H/H \times T_K/K$$

which is injective on edges as edge stabilizers are trivial and the union is over distinct double coset representatives.

In turn, this induces a map on the following edge sets:

$$\bigcup_{g\in K\backslash G/H} E\mathcal{T}_{(H^g\cap K)}/(H^g\cap K) \to E\mathcal{T}_H/H \times E\mathcal{T}_K/K, \tag{8}$$

which is also injective.

By Theorem 3.8 we know that the number of edges in $\mathcal{T}_H/H$ is equal to $\overline{\kappa}_T(H)$. And similarly for K and $H^g \cap K$ whenever they are non-trivial.

Therefore, the injectivity of the map (8) on edges, gives us the result. □

Remark 3.9. We note that in general, there is no obvious way to extend the map (8) to a graph map. That is because, while $(H\cap K)$-bridges map to H-bridges, it does not follow that non-bridges map to non-bridges. In general, an $(H\cap K)$-island will consist of multiple H islands as well as some H-bridges.

For instance, if H is free of rank 2 and K is any cyclic subgroup. Then $T_{H\cap K} = T_K$ is a line and a K-island. However, this line will contain H-bridges unless the generator of K acts elliptically on $\mathcal{T}_H$, which is not always the case.

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