

Sharp constants for composition with a bi-Lipschitz measure-preserving map

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In this note, we aim to describe sharp constants for the composition operator with a bi-Lipschitz measure-preserving map in several functional spaces (BMO, Hardy space, Carleson measures, etc.). It is interesting to see how the measure preserving property allows us to improve these constants. Moreover, we will prove the optimality of our results for the BMO space and describe improved estimates for solutions of transport PDEs.

Denote by $\mathcal{L} := \mathcal{L}(\mathbb{R}^d)$ the group of all bi-Lipschitz homeomorphisms of \mathbb{R}^d (equipped with the composition law \circ) and for every $\phi \in \mathcal{L}$ let

$$K(\phi) = K_\phi := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} + \frac{|x - y|}{|\phi(x) - \phi(y)|}.$$

An easy computation yields that $K(\phi) \geq 2$, with equality if and only if ϕ is an isometry of \mathbb{R}^d , and moreover this quantity is sub-multiplicative: for $\phi, \psi \in \mathcal{L}$ then

$$K(\phi \circ \psi) \leq K(\phi)K(\psi).$$

Consequently, $\phi \mapsto \log(K(\phi))$ is a pseudo-norm on $\mathcal{L}/E(d)$, where $E(d)$ is the set of isometries on \mathbb{R}^d to \mathbb{R}^d , which enables us to consider the group $\mathcal{L}/E(d)$ as a topological object.

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A natural question then arises: for which functional Banach space X (that is, a space of functions from \mathbb{R}^d to \mathbb{R}), is the bilinear map

$$(f, \phi) \rightarrow f \circ \phi$$

bounded from $X \times (\mathcal{L}/E(d))$ to X ? By boundedness, we mean that we have the estimate

$$\|f \circ \phi\|_X \lesssim \log(K(\phi)) \|f\|_X.$$

We shall answer this question when $\phi \in \mathcal{L}$ is measure preserving and X is the space of functions of bounded mean oscillation BMO (Section 2.1 provides the precise definition of this space). We also consider the space of Carleson measures (Section 2.3) and their dual spaces.

For a measure preserving $\phi \in \mathcal{L}$, the bi-Lipschitz property implies that for a ball B , $\phi(B)$ can be contained in a ball of radius $K_\phi r_B$, where r_B is the radius of B . As a consequence, $\phi(B)$ can be covered by a collection of K_ϕ^d balls, with same radius as B . This observation easily yields that for a BMO_p function (the BMO space with L^p oscillation) f on \mathbb{R}^d

$$\|f \circ \phi\|_{BMO_p} \leq K_\phi^{1+d/p} \|f\|_{BMO_p}$$

and since, by the John–Nirenberg property, BMO_p coincides with BMO, by varying $p \in (1, \infty)$, it follows that for all $\epsilon > 0$ we have

$$(1) \quad \|f \circ \phi\|_{BMO} \lesssim_\epsilon K_\phi^{1+\epsilon} \|f\|_{BMO}.$$

Let us point out that without any assumption on the map ϕ , this inequality is optimal up to the loss $\epsilon > 0$. Indeed, on \mathbb{R} choose the BMO-function $f := x \mapsto \log(|x|)$ and a map ϕ such that $\phi(x) = x^n$ for $x \in [0, 1]$ with $K_\phi \lesssim n$. Then $f \circ \phi(x) = n \log(|x|)$ for every $x \in (0, 1]$ hence,

$$K_\phi \lesssim n \lesssim \|f \circ \phi\|_{BMO},$$

which proves the reverse inequality and so the optimality of (1) (up to the loss ϵ).

The interesting point of this work is to prove that the dependence on the constant K_ϕ can be improved, under the measure preservation of ϕ . In fact, the optimal dependence on K_ϕ is logarithmic.

Similarly, it follows that for a Carleson measure μ on \mathbb{R}^d , the pull-back measure

$$\mu^{\sharp\phi} = \mu(Id \otimes \phi^{-1})$$

is a Carleson measure with

$$(2) \quad \|\mu^{\sharp\phi}\|_C \lesssim K_\phi^d \|\mu\|_C.$$

This can also be improved to a logarithmic dependence on K_ϕ in certain circumstances.

Our main results are summarized in the following theorem.

Theorem 1. *Let us assume that ϕ bi-Lipschitz function preserves the Lebesgue measure on \mathbb{R}^d , then*

- [BMO FUNCTIONS] *there exists an implicit constant (independent of K_ϕ) such that for every BMO function f*

$$\|f \circ \phi\|_{\text{BMO}} \lesssim \log(K_\phi) \|f\|_{\text{BMO}}.$$

- [CARLESON MEASURES] *there exist a class \mathcal{SC} of Carleson measures and an implicit constant (independent of K_ϕ) such that for every Carleson measure $\mu \in \mathcal{SC}$, $\mu^{\sharp\phi}$ belongs to \mathcal{SC} and*

$$\|\mu^{\sharp\phi}\|_C \lesssim \log(K_\phi) \|\mu\|_C.$$

Moreover, the logarithmic growth is optimal for the estimates involving the BMO norms.

Remark 1. In the light of the John–Nirenberg inequality, the result is valid for the BMO_p -norm, $p \in [1, \infty)$.

As a corollary and using H^1 – BMO duality, we have

Corollary 1. *Let us assume that ϕ is a bi-Lipschitz function preserving the measure on \mathbb{R}^d , then there exists an implicit constant (independent of K_ϕ) such that for every function belonging to the Hardy space $f \in H^1$*

$$\|f \circ \phi\|_{H^1} \lesssim \log(K_\phi) \|f\|_{H^1}.$$

That means in some sense that the image of an atom by the composition $a \circ \phi$ can be split into the sum of $\log(K_\phi)$ atoms.

One of the main motivations of this work is the study of transport PDEs, associated with a free-divergence vector field. Indeed, such a vector field gives rise to a bi-Lipschitz measure preserving flow, which plays a crucial role for solving the transport equation. We also describe some consequences (we obtain an improved growth of the solution) in the last section for such PDEs. We point out that such study have already been done for Besov spaces, see [4, Theorem 4.2] where Vishik obtained a logarithmic growth (as our result for BMO space) for the Besov space $B_{\infty,1}^0$ with applications to Euler equation. More recently, the authors have used similar ideas in [1] to get well-posedness results for Euler equation, with a vorticity belonging to a space strictly imbricated between L^∞ and BMO. In these two results, the spaces are of completely different nature but the same idea is to understand and to have sharp inequalities for the composition (by a measure-preserving map) in these spaces.

1. A geometric lemma

Before proving Theorem 1, we would like to point out the key argument: a geometric lemma, which describes how a ball is modified by a measure-preserving map.

Lemma 1. *For every ball $B = B(x_0, r)$ in \mathbb{R}^d , there exists a collection $(O_k)_k$ of balls such that:*

- *The collection $(2O_k)_k$ is a bounded covering of $\phi(B)$.*
- *The collection $(O_k)_k$ is disjoint.*
- *By writing r_k the radius of O_k we have*

$$(3) \quad \frac{1}{|B|} \sum_k |O_k| \log(r/r_k) \lesssim \log(K_\phi),$$

with an implicit constant dependent only on the dimension n .

Proof. Let us consider a Whitney covering of the open set $\phi(B)$: that is a collection of open balls $(O_k)_k$ such that:

- the collection of double balls is a bounded covering:

$$\phi(B) \subset \cup_k 2O_k;$$

- the collection is disjoint and for all k , $O_k \subset \phi(B)$;

- the Whitney property is satisfied:

$$(4) \quad r_k \simeq d(O_k, \phi(B)^c).$$

So it remains for us to check (3). Indeed, this is a combinatorial argument.

First, since ϕ is measure preserving, it follows that $|O_k| \leq |B|$ and so $r_k \leq r$ for all k . For a nonnegative integer $j \geq 1$, we write

$$u_j := \sum_{k, 2^{-j}r \leq r_k < 2^{-j+1}r} |O_k|.$$

Since $(2O_k)_k$ is a bounded covering of $\phi(B)$ (and that the balls (O_k) are disjoint), we have

$$(5) \quad \sum_l u_j \simeq |\phi^{-1}(B)| = |B|.$$

Moreover, we see that

$$(*) := \frac{1}{|B|} \sum_k |O_k| \log(r/r_k) \leq \frac{1}{|B|} \sum_{j \geq 0} j u_j.$$

If we denote by $C \geq 1$ the implicit constant appearing in (4), then

$$u_j \leq |\{y \in \phi(B) : d(y, \phi(B)^c) \leq C2^{-j}r\}|.$$

The preservation of Lebesgue measure by ϕ yields

$$|\{y \in \phi(B) : d(y, \phi(B)^c) \leq C2^{-j}r\}| = |\{x \in B : d(\phi(x), \phi(B)^c) \leq C2^{-j}r\}|,$$

Since $\phi(B)^c = \phi(B^c)$ then

$$u_j \leq |\{x \in B : d(\phi(x), \phi(B^c)) \leq C2^{-j}r\}|.$$

We set

$$D_j = \{x \in B : d(\phi(x), \phi(B^c)) \leq C2^{-j}r\}.$$

Since $\phi(\partial B) = \partial[\phi(B)]$ and ϕ^{-1} is Lipschitz

$$D_j \subset \{x \in B : \exists y \in \partial B : |x - y| \leq C\|\phi^{-1}\|_{Lip} 2^{-j}r \leq CK_\phi 2^{-j}r\}.$$

Thus,

$$(6) \quad u_j \leq |D_j| \lesssim K_\phi 2^{-j}r^d.$$

Coming back to (*). Let N be a large integer to be chosen later. We split the sum in the right-hand side into two parts

$$(*) \lesssim \sum_{k \leq N} (...) + \sum_{k > N} (.....) := I + II.$$

Using (5) we estimate

$$(7) \quad I \leq N.$$

On the other hand, using (6),

$$II \lesssim \sum_{k > N} K_\phi j 2^{-j} \lesssim K_\phi N 2^{-N}.$$

This yields

$$(*) \lesssim N + K_\phi N 2^{-N}.$$

Taking $N = [\log(K_\phi)] + 1$ gives

$$(*) \lesssim \log(K_\phi),$$

as claimed. □

2. The behavior of BMO functions and Carleson measures

2.1. BMO functions

For $p \in [1, \infty)$ and a locally integrable function f set

$$\|f\|_{BMO_p} = \sup_B \left(\frac{1}{|B|} \int_B |f(y) - \text{Avg}_B(f)|^p dy \right)^{1/p}.$$

The John–Nirenberg property shows us that all the spaces BMO_p ($1 \leq p < \infty$) coincide, so we write $BMO_1 = BMO$.

The following lemma is classical. For sake of completeness, we will provide the proof.

Lemma 2. *Let $f \in BMO$ and B be a ball in \mathbb{R}^d , which contains $x \in \mathbb{R}^d$, then for all $\lambda > 1$*

$$|\text{Avg}_B(f) - \text{Avg}_{\lambda B}(f)| \lesssim \log(\lambda) \|f\|_{BMO}.$$

Proof of Lemma 2. First we remark that, using the doubling property of the Euclidean measure,

$$\begin{aligned} |\text{Avg}_B(f) - \text{Avg}_{2B}(f)| &\leq \text{Avg}_B|f - \text{Avg}_{2B}(f)| \lesssim \text{Avg}_{2B}|f - \text{Avg}_{2B}(f)| \\ &\lesssim \text{Avg}_{2B}|f - \text{Avg}_{2B}(f)|. \end{aligned}$$

So it follows that

$$|\text{Avg}_B(f) - \text{Avg}_{2B}(f)| \lesssim \|f\|_{\text{BMO}},$$

which corresponds to the desired result for $\lambda = 2$.

We iterate this argument k_0 times, where k_0 is such that $2^{k_0} \leq \lambda < 2^{k_0+1}$, and obtain

$$\begin{aligned} |\text{Avg}_B(f) - \text{Avg}_{2^{k_0}B}(f)| &\leq \sum_{k=0}^{k_0-1} |\text{Avg}_{2^k B}(f) - \text{Avg}_{2^{k+1}B}(f)| \\ &\lesssim \sum_{k=0}^{k_0-1} \|f\|_{\text{BMO}} \lesssim \log(2^{k_0}) \|f\|_{\text{BMO}}. \end{aligned}$$

To conclude, it remains for us to estimate the following term:

$$\begin{aligned} |\text{Avg}_{2^{k_0}B}(f) - \text{Avg}_{\lambda B}(f)| &\lesssim \text{Avg}_{2^{k_0}B} |f - \text{Avg}_{\lambda B}(f)| \\ &\lesssim \text{Avg}_{\lambda B} |f - \text{Avg}_{\lambda B}(f)| \\ &\lesssim \|f\|_{\text{BMO}} \lesssim \log(\lambda) \|f\|_{\text{BMO}}, \end{aligned}$$

where we have one more time used the doubling property. \square

We can now prove the first part of Theorem 1.

Fix $x \in \mathbb{R}^d$ and let B be a ball in \mathbb{R}^d containing x . We wish to estimate

$$I := \text{Avg}_B |f \circ \phi - \text{Avg}_B(f \circ \phi)|$$

and with this in mind, using the measure-preserving property of ϕ , we see that

$$I = \frac{1}{|B|} \int_{\phi(B)} |f(y) - \text{Avg}_{\phi(B)}(f)| dy.$$

Now we would like to compare $\phi(B)$ with the ball

$$\tilde{B} := B(\phi(x), r).$$

We claim that we have the following inequality

$$(8) \quad II := \text{Avg}_{\phi(B)} |f - \text{Avg}_{\tilde{B}}(f)| \lesssim \log(K_\phi) \|f\|_{\text{BMO}}.$$

Let us first deduce the first part of Theorem 1 from (8).

We have

$$\left| \text{Avg}_{\phi(B)}(f) - \text{Avg}_{\tilde{B}}(f) \right| \lesssim \log(K_\phi) \|f\|_{\text{BMO}},$$

hence

$$I \lesssim \log(K_\phi) \|f\|_{\text{BMO}},$$

where we have used (8) a second time and so we deduce the desired result.

It remains for us to prove (8). To achieve this we use the collection $(O_k)_k$ given by Lemma 1. This yields

$$II \leq \frac{1}{|B|} \sum_k \int_{2O_k} |f(y) - \text{Avg}_{\tilde{B}}(f)| dy.$$

Let us first remark that since ϕ is measure preserving then $|O_k| \leq |\phi(B)| = |B|$ so $r_k \leq r$. As a consequence, we see that $2O_k \subset 2K_\phi \tilde{B}$ and so

$$\begin{aligned} |\text{Avg}_{2O_k}(f) - \text{Avg}_{\tilde{B}}(f)| &\leq \left| \text{Avg}_{2O_k}(f) - \text{Avg}_{2K_\phi \tilde{B}}(f) \right| \\ &\quad + \left| \text{Avg}_{2K_\phi \tilde{B}}(f) - \text{Avg}_{\tilde{B}}(f) \right| \\ &\lesssim \left| \text{Avg}_{2O_k}(f) - \text{Avg}_{2K_\phi \tilde{B}}(f) \right| + \log(2K_\phi) \|f\|_{\text{BMO}} \\ &\lesssim \left| \text{Avg}_{2O_k} \left(f - \text{Avg}_{2K_\phi \tilde{B}}(f) \right) \right| \\ &\quad + \log(2K_\phi) \|f\|_{\text{BMO}} \\ &\lesssim \left| \text{Avg}_{B(x_{O_k}, 4K_\phi r)} \left(f - \text{Avg}_{2O_k}(f) \right) \right| \\ &\quad + \log(2K_\phi) \|f\|_{\text{BMO}} \\ &\lesssim \left| \text{Avg}_{B(x_{O_k}, 4K_\phi r)} f - \text{Avg}_{2O_k}(f) \right| \\ &\quad + \log(2K_\phi) \|f\|_{\text{BMO}} \\ &\lesssim [\log(2K_\phi r/r_k) + \log(2K_\phi)] \|f\|_{\text{BMO}} \\ &\lesssim [\log(r/r_k) + \log(K_\phi)] \|f\|_{\text{BMO}}, \end{aligned}$$

where we have used Lemma 2 once and then a second time with the fact that $B(x_{O_k}, 4K_\phi r)$ is a dilation of $2O_k$ by a factor $2K_\phi r/r_k$. Consequently (using that $(2O_k)_k$ is a bounded covering of $\phi(B)$ and that the measure is

preserved), we deduce that

$$\begin{aligned}
 II &\lesssim \left(\frac{1}{|B|} \sum_k \left[\int_{2O_k} |f - \text{Avg}_{2O_k}(f)| + |O_k| ([\log(r/r_k) + \log(K_\phi)] \|f\|_{\text{BMO}}) \right] \right) \\
 &\lesssim \left(\frac{1}{|B|} \sum_k |O_k| (\log(r/r_k) + \log(K_\phi)) \right) \|f\|_{\text{BMO}} \\
 &\lesssim \log(K_\phi) \|f\|_{\text{BMO}} + \left(|B|^{-1} \left[\sum_k |O_k| \log(r/r_k) \right] \right) \|f\|_{\text{BMO}},
 \end{aligned}$$

where we have used that $\sum_k |O_k| \simeq |B|$. The proof is also achieved by invoking property (3) to compute the remaining sum and estimate it by the expected quantity $\log(K_\phi) \|f\|_{\text{BMO}}$.

2.2. Optimality of the logarithmic growth

We begin by observing that if ϕ is a measure preserving bi-Lipschitz function then the Lipschitz norms of ϕ and ϕ^{-1} are logarithmically equivalent.

Lemma 3. *There exists $C = C(d)$ such that for every smooth measure preserving $\phi \in \mathcal{L}$*

$$\frac{1}{C} \|\phi\|_{\text{Lip}}^{\frac{1}{d-1}} \leq \|\phi^{-1}\|_{\text{Lip}} \leq C \|\phi\|_{\text{Lip}}^{d-1}.$$

In particular, if $\|\phi\|_{\text{Lip}}$ is large enough,

$$\log(K_\phi) \lesssim \log(\|\phi\|_{\text{Lip}}).$$

Proof. By symmetry it is enough to prove the second inequality only. Since ϕ is a measure preserving map then

$$|\det(D\phi(x))| = 1, \quad \forall x \in \mathbb{R}^d.$$

Take $x \in \mathbb{R}^d$ and denote $y = \phi^{-1}(x)$. Since $D(\phi^{-1})(x) = (D\phi(y))^{-1}$ then

$$\|D(\phi^{-1})(x)\| \lesssim \max_{\lambda \in \Sigma(y)} |\lambda|^{-1} \leq \left[\max_{\lambda \in \Sigma(y)} |\lambda| \right]^{d-1} \lesssim \|D\phi(y)\|^{d-1},$$

where $\Sigma(y)$ is the spectrum of $\sqrt{D\phi(y)^* D\phi(y)}$. The second inequality is due to the fact that $\prod_{\lambda \in \Sigma(y)} \lambda = 1$.

Finally, we obtain

$$\|D\phi^{-1}\|_{L^\infty} \lesssim \|D\phi\|_{L^\infty}^{d-1},$$

as claimed. \square

Our argument for the optimality of the logarithmic growth relies on properties of quasi-conformal mappings. Given that we can restrict our attention to bi-Lipschitz functions, we can take as our definition of $\phi \in \mathcal{L}$ being K -quasi-conformal that it satisfies the inequality

$$\|D\phi(x)\|_{L^\infty}^d \leq K \det(D\phi)(x)$$

for all $x \in \mathbb{R}^d$. If ϕ is also measure preserving, then this inequality reduces to

$$(9) \quad \|D\phi\|_{L^\infty}^d \leq K,$$

so in this context, K -quasiconformality is simply that the size of the Lipschitz constant of ϕ is bounded by $K^{1/d}$. We will use the following theorem by H.M. Riemann [3, Theorem 3].

Theorem 2. *Assume that $\phi \in \mathcal{L}$ is orientation preserving. If the induced map $f \mapsto f \circ \phi$ is a bijective isomorphism of BMO and satisfies*

$$(10) \quad \|f \circ \phi\|_{BMO} \leq \kappa \|f\|_{BMO}$$

for all $f \in BMO$ then ϕ is a K -quasiconformal mapping with $K = e^{(d-1)(C\kappa-1)}$, for a fixed $C > 0$ depending only on the dimension d .

If in the context of this theorem one also assumes that ϕ satisfies (9) (which is the case if ϕ preserves the Lebesgue measure) then

$$C_d \log(\|D\phi\|_{L^\infty}) \leq \kappa.$$

Combined with Lemma 3 this proves the optimality of the logarithmic growth in Theorem 1.

2.3. The behavior of some Carleson measures

Let μ be a Carleson measure on $\mathbb{R}^+ \times \mathbb{R}^d$. We recall that this means that

$$\|\mu\|_C := \sup_{\text{ball } B \subset \mathbb{R}^d} |B|^{-1} \mu(T(B)) < \infty$$

where $T(B)$ is the Carleson box over the ball B defined by

$$T(B) := \{(x, t), x \in B, 0 < t \leq r\} = B \times (0, r].$$

Definition 1. Let μ be a Carleson measure and ϕ a bi-Lipschitz measure-preserving map on \mathbb{R}^d . We denote $\mu^{\sharp\phi}$ the pull-back measure, defined by

$$\mu^{\sharp\phi}(I \times A) = \mu(I \times \phi^{-1}(A)),$$

for every time interval I and measurable set $A \subset \mathbb{R}^d$.

Definition 2. Let β be a measurable map from $\mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ and we define the measure

$$\mu := \mu_\beta = |\beta(t, x)|^2 \frac{dtdx}{t}.$$

Consequently, we infer

$$d\mu^{\sharp\phi}(t, x) = |\beta(t, \phi(x))|^2 \frac{dtdx}{t}$$

so $\mu^{\sharp\phi} = \mu_{\beta\phi}$ with $\beta\phi(t, x) = \beta(t, \phi(x))$.

Theorem 3. Let ϕ a bi-Lipschitz measure-preserving map on \mathbb{R}^d and $\mu = \mu_\beta$ be a Carleson measure associated to some $\beta \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. Then there exists an implicit constant (only dependent on n) such that $\mu^{\sharp\phi}$ is a Carleson measure with

$$\|\mu^{\sharp\phi}\|_C \lesssim \|\mu\|_C + \log(K_\phi) \|\beta\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)}^2.$$

Proof. Let consider $B = B(x_0, r)$ a ball of \mathbb{R}^d and its Carleson box $T(B)$. We have to estimate

$$\begin{aligned} \mu^{\sharp\phi}(T(B)) &= \int_{[0, r] \times B} |\beta(t, \phi(x))|^2 \frac{dtdx}{t} \\ &= \int_{[0, r] \times \phi(B)} |\beta(t, x)|^2 \frac{dtdx}{t}. \end{aligned}$$

Aiming that, we use the collection (O_k) given by Lemma 1 to cover $\phi(B)$ (with $p = 1$). Then (we remember that as previously we have $r_k \leq r$), it follows:

$$\begin{aligned}
 \mu^{\sharp\phi}(T(B)) &= \sum_k \int_{[0,r] \times 2O_k} |\beta(t,x)|^2 \frac{dt dx}{t} \\
 &\leq \sum_k \int_{[0,r_k] \times 2O_k} |\beta(t,x)|^2 \frac{dt dx}{t} \\
 &\quad + \sum_k \int_{[r_k,r] \times 2O_k} |\beta(t,x)|^2 \frac{dt dx}{t} \\
 &\leq \sum_k \mu(T(2O_k)) + \|\beta\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)}^2 \sum_k |O_k| \log\left(\frac{r}{r_k}\right) \\
 &\lesssim \|\mu\|_{\mathcal{C}} \left(\sum_k |O_k| \right) + \|\beta\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)}^2 \log(K_\phi) |B| \\
 &\lesssim \left(\|\mu\|_{\mathcal{C}} + \log(K_\phi) \|\beta\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)}^2 \right) |B|,
 \end{aligned}$$

where we used the doubling property of the Euclidean measure, the disjointness of the balls $(O_k)_k$ and the property (3). The proof is also concluded. \square

Corollary 2. *Let define \mathcal{SC} the class of Carleson measure $d\mu = |\beta(t,x)|^2 \frac{dt dx}{t}$ satisfying $\|\beta\|_{L^\infty} \lesssim \|\mu\|_{\mathcal{C}}$, equipped with the norm $\|\mu\|_{\mathcal{SC}} := \|\mu\|_{\mathcal{C}}$. Then we have*

$$\|\mu^{\sharp\phi}\|_{\mathcal{SC}} \lesssim \log(K_\phi) \|\mu\|_{\mathcal{SC}}.$$

Example. We know that for some standard “approximations of unity” kernels (K_t) , for a L^1_{loc} function g we can build the measure

$$d\mu_g(t,x) = \left| \int K_t(x,y) g(y) dy \right|^2 \frac{dt dx}{t}.$$

Then it is well known that $d\mu_g$ is a Carleson measure if and only if $g \in BMO$ (see [2]). Moreover, it is easy to check that such measures belong to \mathcal{SC} .

3. Applications to some PDEs

3.1. The Transport equation by a free-divergence vector field

Let $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a divergence free¹ Lipschitz vector field and consider the transport equation:

$$(11) \quad \begin{cases} \partial_t u - v \cdot \nabla(u) = 0, \\ u|_{t=0} = u_0, \end{cases}$$

with an initial data u_0 . Then it is well known that a smooth solution is constant along the characteristics given by the vector field. Indeed, consider the flow $\phi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, solution of

$$\begin{cases} \partial_t \phi = v(\phi), \\ \phi(0, x) = x, \end{cases}$$

then the divergence free assumption on v yields that $\phi(t, \cdot)$ is a Lebesgue measure preserving diffeomorphism, for every $t \in \mathbb{R}$. Moreover, any smooth solution u of the transport equation is unique and is given by

$$u(t, x) = u_0(\phi_t^{-1}(x)).$$

It is well known using Gronwall lemma that

$$K(\phi(t, \cdot)) \lesssim e^{\|v\|_{\text{Lip}} t},$$

where $\|v\|_{\text{Lip}}$ is the Lipschitz constant of the vector field.

As a consequence, the previous Theorem and Corollary imply the following:

Theorem 4. *Let u be the unique solution of (11).*

1) *If $u_0 \in BMO$, then $u \in L_{loc}^\infty(BMO)$ and*

$$\|u(t)\|_{BMO} \lesssim [1 + \|v\|_{\text{Lip}} t] \|u_0\|_{BMO}, \quad \forall t \geq 0.$$

2) *If $u_0 \in Lip_p(a)$ (for some $a \in (0, 1]$ and $p \in (1, \infty)$), then $u \in L_{loc}^\infty(Lip_p(a))$ and*

$$\|u(t)\|_{Lip_p(a)} \lesssim e^{a\|v\|_{\text{Lip}} t} \|u_0\|_{Lip_p(a)}, \quad \forall t \geq 0.$$

¹That means $\nabla \cdot v = 0$.

3.2. The perturbed transport equation

Consider the following transport equation with a linear Riesz-type second member term:

$$(12) \quad \begin{cases} \partial_t \omega + (u \cdot \nabla) \omega = \mathcal{R} \omega, \\ \omega|_{t=0} = \omega_0, \end{cases}$$

where \mathcal{R} is a Riesz operator. This type of equation naturally arises when one considers for example the perturbed two-dimensional (2D) Euler equations which is obtained by adding a zero order term to the incompressible 2D Euler system. Let us for example consider the following system

$$(13) \quad \begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + Au, \\ \operatorname{div}(u) = 0, \\ u|_{t=0} = u_0, \end{cases}$$

with $Au = (u^1, 0)$. Then the vorticity $\omega := \operatorname{curl}(u) = \partial_1 u^2 - \partial_2 u^1$ satisfies the following equation

$$\partial_t \omega + u \cdot \nabla \omega = \partial_{22} \Delta^{-1} \omega.$$

The continuity of Riesz operator on L^p for every $1 < p < \infty$, the divergence free condition and Gronwall inequality imply together

$$(14) \quad \|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} e^{C_p t}, \quad \forall t \geq 0.$$

Here $C_p = \|\mathcal{R}\|_{\mathcal{L}(L^p, L^p)} \simeq \frac{p^2}{p-1}$. However, it is not clear at all how one can obtain an L^∞ estimates since the Riesz operator $\partial_{22} \Delta^{-1}$ is not continuous on that space. A natural idea is to replace the space L^∞ by another space with similar "scaling" but stable for \mathcal{R} (such that BMO for example). However, in this case a problem of composition arises: an extra term depending of the Lipschitz norm of u appears and the estimate is no longer closable.

Theorem 2 shows that we cannot avoid the constants generated by the composition with the flow (nor even to improve them). A bound for the BMO -norm similar to (14) cannot be obtained directly: indeed with the best constants, we already have a quadratic estimate which is not *Gronwallisable*. However, Theorem 1 can be applied in order to get sharper a priori estimates. In fact, consider u be a smooth solution of (12) and the corresponding flow

$\phi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, solution of

$$\begin{cases} \partial_t \phi = u(t, \phi), \\ \phi(0, x) = x. \end{cases}$$

Hence by Gronwall Lemma and using the BMO -boundedness of the Riesz transform, we get

$$\|\omega(t, \phi(t, \cdot))\|_{BMO} \lesssim \|\omega_0\|_{BMO} \exp(ct),$$

for some numerical constant $c > 0$, and so by Theorem 1

$$\|\omega(t, \cdot)\|_{BMO} \lesssim \|\omega_0\|_{BMO} (1 + \|u\|_{L_t^1 \text{Lip}}) \exp(ct).$$

If the vector-field satisfies

$$\|u\|_{L_t^1 \text{Lip}} \leq \exp(\alpha t),$$

for some $\alpha > 0$, then one has a similar estimate than all the L^p norm. It is worthy of noticing that a rough estimate (involving $K(\phi)$ instead of $\log K(\phi)$) gives

$$\|\omega(t, \cdot)\|_{BMO} \lesssim \|\omega_0\|_{BMO} \exp(\|u\|_{L_t^1 \text{Lip}}) \exp(ct).$$

The merit of the next result is only the improvement of the estimate of the growth of the BMO norm of the solution.

Proposition 1. *Let u be a divergence free vector fields and ω a smooth solution of (12). If $\omega_0 \in BMO$, then*

$$\|\omega(t)\|_{BMO} \lesssim \|\omega_0\|_{BMO} (1 + \|u\|_{L_t^1 \text{Lip}}) \exp(ct),$$

for all $t \geq 0$.

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