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Rigidity of time-flat surfaces in the Minkowski spacetime

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A time-flat condition on spacelike 2-surfaces in spacetime is considered here. This condition is analogous to the constant torsion condition for curves in a three-dimensional space and has been studied in [2, 5, 6, 13, 14]. In particular, any 2-surface in a static slice of a static spacetime is time-flat. In this paper, we address the question in the title and prove several local and global rigidity theorems for such surfaces in the Minkowski and Schwarzschild spacetimes. Higher-dimensional generalizations are also considered.

1. Introduction

The geometry of spacelike 2-surfaces in spacetime plays a crucial role in general relativity. Penrose's singularity theorem predicts future singularity formation from the existence of a trapped 2-surface. A black hole is quasi-locally described by a marginally outer trapped 2-surface. These conditions can be expressed in terms of the mean curvature vector field H of the 2-surface. H is the unique normal vector field determined by the variation of the area functional and is ultimately connected to the warping of spacetime in the vicinity of the 2-surface. It is thus not surprising that several definitions of quasi-local mass in general relativity are closely related to the mean curvature vector field. In particular, both the Hawking mass [8] and the Brown–York–Liu–Yau mass [4, 10] involve the norm of the mean curvature vector field $|H|$. In the new definition of quasi-local mass in [13, 14], in addition to $|H|$, the direction of the mean curvature vector field is also utilized. When the mean curvature vector field is spacelike everywhere on Σ (thus $|H| > 0$), the direction of H defines a connection one-form α_H of the normal bundle (see Definition 2 for the precise definition of α_H). The quasi-local mass in [13, 14] is defined in terms of the induced metric σ on the surface, $|H|$, and α_H. In particular, the condition

$$\operatorname{div}_\sigma(\alpha_H) = 0 \tag{1.1}$$

implies that the isometric embedding of Σ into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ is an optimal isometric embedding in the sense of [13, 14]. Recently, Bray and Jauregui [2] discovered a very interesting monotonicity property of the Hawking mass along surfaces that satisfy the condition (1.1). Such surfaces are said to be "time-flat" in [2] and include all 2-surfaces in a time-symmetric initial data set.

For curves in $\mathbb{R}^3$, the direction of the mean curvature vector corresponds to the normal of the curve [7, pages 16–17]. The connection 1-form α_H is nothing but the torsion of the curve (see Definition 2) and condition (1.1) simply says the torsion is constant. Weiner [16, 17] constructed simple closed curves with constant torsion in $\mathbb{R}^3$ that do not lie in a totally geodesic $\mathbb{R}^2$. On the other hand, in [3], Bray and Jauregui proved that a curve in $\mathbb{R}^3$ with constant torsion must lie in a totally geodesic $\mathbb{R}^2$ if it can be written as a graph over a simple closed curve in $\mathbb{R}^2$.

A natural rigidity question raised by Bray [1] is "Must a time-flat surface in the Minkowski spacetime lie in a totally geodesic $\mathbb{R}^3$?" From the above analogy between curves and surfaces, one expects some global condition is needed in order for the rigidity property of time-flat surfaces to hold. In this paper, we prove several global and local rigidity theorems for time-flat surfaces in the Minkowski and Schwarzschild spacetimes under various conditions.

We first prove a local rigidity theorem that holds in the Minkowksi spacetime in all dimensions:

Theorem 4. *Let $n \geq 3$. Suppose Σ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n+1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then Σ is locally rigid as a time-flat $n-1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of Σ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.*

We also prove three global rigidity theorems:

Theorem 6. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that $\alpha_H = 0$ and Σ is a topological sphere, then Σ lies in a totally geodesic hyperplane.*

Theorem 7. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ and Σ is a topological sphere. If Σ is invariant under a rotational Killing vector field, then Σ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$.*

The last global rigidity theorem holds for the $(n+1)$-dimensional Schwarzschild spacetime with metric

$$-\left(1-\frac{2m}{r^{n-2}}\right)dt^2+\frac{1}{1-\frac{2m}{r^{n-2}}}\,dr^2+r^2g_{S^{n-1}},$$

where $m \geq 0$ is the mass and $g_{S^{n-1}}$ is the standard metric on a unit sphere S^{n-1}.

Theorem 10. *Let $n \geq 3$ and Σ^{n-1} be a connected spacelike codimension-2 submanifold in the (n+1)-dimensional Schwarzschild spacetime with mass m. Suppose $\alpha_H = 0$ and Σ is star-shaped with respect to Q (see Definition 9) where $Q = rdr \wedge dt$. Then, when $m > 0$, Σ lies in a time-slice; when $m = 0$, Σ lies in a totally geodesic hyperplane.*

Note that Q is a conformal Killing–Yano 2-form on the Schwarzschild spacetime, see [9]. Theorem 4 follows from applying the Reilly formula to the linearized equation (3.7). Theorem 6 is proved using the Codazzi equation and a theorem of Yau from [18]. The proof of Theorem 7 is reduced to Theorem 6 using the structure of axially symmetric surfaces in $\mathbb{R}^{3,1}$. Theorem 10 is proved using a Minkowski-type integral formula involving the conformal Killing–Yano 2-form on the Schwarzschild spacetime.

We review the geometry of spacetime surfaces in Section 2 and define the time-flat condition and the higher dimensional generalization. Theorem 4 is proved in Section 3, Theorem 6 is proved in Section 4, Theorem 7 is proved in Section 5, and Theorem 10 is proved in Section 6.

2. Geometry of spacelike 2-surface in spacetime

Let N be a time-oriented spacetime. Denote the Lorentzian metric on N by $\langle\cdot,\cdot\rangle$ and covariant derivative by ∇^N. Let Σ be a closed space-like two-surface embedded in N. Denote the induced Riemannian metric on Σ by σ and the gradient and Laplace operator of σ by ∇ and Δ, respectively.

Given any two tangent vector X and Y of Σ, the second fundamental form of Σ in N is given by $\mathrm{II}(X,Y) = (\nabla^N_X Y)^\perp$ where $(\cdot)^\perp$ denotes the projection onto the normal bundle of Σ. The mean curvature vector is the trace of the second fundamental form, or $H = \mathrm{tr}_\Sigma \mathrm{II} = \sum_{a=1}^2 \mathrm{II}(e_a, e_a)$, where $\{e_1, e_2\}$ is an orthonormal basis of the tangent bundle of Σ.

The normal bundle is of rank two with structure group $SO(1,1)$ and the induced metric on the normal bundle is of signature $(-,+)$. Since the Lie algebra of $SO(1,1)$ is isomorphic to $\mathbb{R}$, the connection form of the normal bundle is a genuine 1-form that depends on the choice of the normal frames. The curvature of the normal bundle is then given by an exact 2-form which reflects the fact that any $SO(1,1)$ bundle is topologically trivial. Connections of different choices of normal frames differ by an exact form. We define (see [14]):

Definition 1. Let e_3 be a space-like unit normal along Σ, the connection one-form determined by e_3 is defined to be

$$\alpha_{e_3} = \langle \nabla^N_{(\cdot)} e_3, e_4 \rangle, \tag{2.1}$$

where e_4 is the future-directed time-like unit normal that is orthogonal to e_3.

Definition 2. Suppose the mean curvature vector field H of Σ in N is a spacelike vector field. The connection one-form in mean curvature gauge is

$$\alpha_H = \langle \nabla^N_{(\cdot)} e_3, e_4 \rangle,$$

where $e_3 = -\frac{H}{|H|}$ and e_4 is the future-directed timelike unit normal that is orthogonal to e_3.

Definition 3. We say Σ is time-flat if $\mathrm{div}_\sigma(\alpha_H) = 0$.

Remark 1. For spacelike codimension-2 submanifolds in a time-oriented $(n+1)$-dimensional spacetime N, the second fundamental form II and the mean curvature vector H can be defined in the same manner. Let $e_n = -\frac{H}{|H|}$ and e_{n+1} be the future timelike normal orthogonal to e_n. The connection one-form with respect to mean curvature gauge is defined to be

$$\alpha_H = \langle \nabla^N_{(\cdot)} e_n, e_{n+1} \rangle.$$

3. Local rigidity of mean convex hypersurfaces in $\mathbb{R}^n \subset \mathbb{R}^{n,1}$

The local rigidity problem can be formulated as follows. Suppose Σ is time-flat and is given by an embedding X. Suppose V is a smooth vector field along Σ such that the image of $X(s) = X + sV$ is infinitesimally time-flat in the sense the derivatives of $\mathrm{div}_\sigma(\alpha_H)$ along the image with respect to s is zero when $s = 0$. Do all such V correspond to trivial deformations? It is easy to see that submanifolds lying in a totally geodesic slice is time-flat. We assume $\partial\Omega = \Sigma \subset \{t = 0\} = \mathbb{R}^n$. It is clear that any deformation in the $\mathbb{R}^n$ direction preserves the time-flat condition. On the other hand, a Lorentz transformation preserves the geometry of Σ and thus preserves the time-flat condition.

Let ∇, Δ denote the covariant derivative and Laplacian of the induced metric σ. Let h_{ab}, h be the second fundamental form and mean curvature of $\Sigma \subset \mathbb{R}^n$ with respect to the outward unit normal ν.

Theorem 4. *Let $n \geq 3$. Suppose Σ is a mean convex hypersurface which lies in a totally geodesic $\mathbb{R}^n$ in the $n+1$ dimensional Minkowski spacetime $\mathbb{R}^{n,1}$, then Σ is locally rigid as a time-flat $n-1$ dimensional submanifold in $\mathbb{R}^{n,1}$. In other words, any infinitesimal deformation of Σ that preserves the time-flat condition must be a deformation in the $\mathbb{R}^n$ direction, a deformation that is induced by a Lorentz transformation of $\mathbb{R}^{n,1}$, or a combination of these two types of deformations.*

Proof. In this proof, we denote α_H by α. Since $\delta(\mathrm{div}_\sigma \alpha)$ depends linearly on infinitesimal deformation and any deformation in $\mathbb{R}^n$ corresponds to trivial deformations, it suffices to consider deformations in the time direction. Let $V = f\frac{\partial}{\partial t}$ for a smooth function f defined on Σ be such an infinitesimal deformation and $X(s) = (\tau(s), X^1(s), \ldots, X^n(s))$ be the corresponding deformation. Since we only vary the surface in the time direction, $X^i(s) = X^i(0)$ for $i = 1, \ldots, n$, and $\delta\tau = f$. Here δ stands for $\frac{\partial}{\partial s}\big|_{s=0}$. We start by computing the variation of $\mathrm{div}_\sigma \alpha$. The induced metrics satisfy

$$\sigma(s)_{ab} = \sigma_{ab} - \frac{\partial\tau(s)}{\partial u^a}\frac{\partial\tau(s)}{\partial u^b}. \tag{3.1}$$

Since $\tau(0) = 0$, $\delta\sigma = 0$. Let Δ_s be the Laplacian of the induced metric on $X(s)$. We have

$$H = (\Delta_s\tau(s), \Delta_s X^1, \ldots, \Delta_s X^n). \tag{3.2}$$

$\delta\sigma = 0$ implies the infinitesimal variation of Laplacian is zero. Therefore, we have

$$\delta H = (\Delta f)\frac{\partial}{\partial t}, \tag{3.3}$$

$$\delta |H|^2 = 2\langle \delta H, -h e_n\rangle = 0, \tag{3.4}$$

$$\delta e_n = -\frac{\delta H}{h} + \frac{\delta |H|}{h^2} H = -\frac{\Delta f}{h}\frac{\partial}{\partial t}. \tag{3.5}$$

Since $0 = \delta\langle e_{n+1}, \frac{\partial}{\partial u^a}\rangle = \delta\langle e_{n+1}, e_n\rangle$, we have

$$0 = \left\langle \delta e_{n+1}, \frac{\partial}{\partial u^a}\right\rangle + \left\langle e_{n+1}, \frac{\partial f}{\partial u^a}\frac{\partial}{\partial t}\right\rangle$$

and

$$0 = \langle \delta e_{n+1}, e_n\rangle + \left\langle e_{n+1}, -\frac{\Delta f}{h}\frac{\partial}{\partial t}\right\rangle.$$

Hence

$$\delta e_{n+1} = \nabla f - \frac{\Delta f}{h} e_n. \tag{3.6}$$

We are ready to compute the variation of α.

$$\begin{aligned}(\delta\alpha)_a &= \delta\langle D_a e_n, e_{n+1}\rangle \\ &= \langle (\delta D)_a e_n, e_{n+1}\rangle + \left\langle D_{\frac{\partial(\delta X)}{\partial u^a}} e_n, e_{n+1}\right\rangle \\ &\quad + \langle D_a(\delta e_n), e_{n+1}\rangle + \langle D_a e_n, \delta e_{n+1}\rangle.\end{aligned}$$

Since $\delta\sigma = 0$, $\delta D = 0$. By (3.5) and (3.6), we get

$$\begin{aligned}(\delta\alpha)_a = &\left\langle D_{\frac{\partial f}{\partial u^a}\frac{\partial}{\partial t}} e_n, e_{n+1}\right\rangle \\ &+ \left\langle D_a\left(-\frac{\Delta f}{h}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right\rangle + \left\langle D_a e_n, \nabla f - \frac{\Delta f}{h} e_n\right\rangle \\ = &\nabla_a\left(\frac{\Delta f}{h}\right) + h_{ab}\nabla^b f\end{aligned}$$

and

$$\delta(\mathrm{div}_\sigma \alpha) = \Delta\left(\frac{\Delta f}{h}\right) + \nabla^a\left(h_{ab}\nabla^b f\right). \tag{3.7}$$

We remark that the linearization of this operator was also derived in [5, 11].

To prove the theorem, it suffices to show that f is the restriction of linear coordinate functions on Σ. Let u solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma. \end{cases}$$

On $\mathbb{R}^n$, the Reilly formula [12] reads

$$\int_\Omega |D^2 u|^2 = -\int_\Sigma \left(h^{ab} \nabla_a f \nabla_b f + 2(\Delta f) e_n(u) + h(e_n(u))^2 \right). \tag{3.8}$$

This was used in [11] to derive minimizing property of the Wang–Yau quasi-local energy. On the other hand, multiplying $\delta(\mathrm{div}_\sigma \alpha) = 0$ by f and integrating over Σ yield

$$\int_\Sigma \left(\frac{(\Delta f)^2}{h} - h^{ab} \nabla_a f \nabla_b f \right) = 0. \tag{3.9}$$

Adding (3.8) and (3.9) together and completing square, we obtain

$$\int_\Omega |D^2 u|^2 + \int_\Sigma \left(\frac{\Delta f}{\sqrt{h}} + \sqrt{h} e_n(u) \right)^2 = 0.$$

Hence, u is a linear function up to a constant. □

4. Global rigidity for surfaces with $\alpha_H = 0$

We first recall a general sufficient condition of Yau that implies a submanifold lies in another submanifold which is totally geodesic.

Theorem 5 ([18, Theorem 1]). *Let M be a submanifold in a p-dimensional pseudo-Riemannian manifold P with constant curvature. Let N_1 be a subbundle of the normal bundle of M with fiber dimension k. Suppose the second fundamental form of M with respect to any direction in N_1 vanishes and N_1 is parallel in the normal bundle. Then M lies in a $(p-k)$-dimensional totally geodesic submanifold.*

Although in [18], the theorem is proved only for Riemannian manifold, the same argument works in the pseudo-Riemannian case. We apply Theorem 5 to prove the following global rigidity theorem.

Theorem 6. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ such that $\alpha_H = 0$ and Σ is a topological sphere, then Σ lies in a totally geodesic $\mathbb{R}^3$.*

Proof. Denote by $e_3 = -\frac{H}{|H|}$ and e_4 to be the unit future timelike normal that is orthogonal to e_3. The second fundamental form of Σ can be written as $h^3_{ab}e_3 - h^4_{ab}e_4$ and h^4_{ab} is trace-free. The Codazzi equation for h^4_{ab} reads

$$\nabla^a h^4_{ab} - \nabla_b \mathrm{tr}_\sigma h^4 + (\alpha_H)^a h^3_{ab} - \mathrm{tr}_\sigma h^3 (\alpha_H)_b = 0.$$

Since $\alpha_H = 0$ and h^4_{ab} is trace-free, this reduces to

$$\nabla^a h^4_{ab} = 0.$$

A divergence-free symmetric trace-free 2-tensor corresponds to a holomorphic quadratic differential, which must vanish since Σ is a topological sphere. Let N_1 be the subbundle of the normal bundle spanned by e_4. Since $\alpha_H = 0$, N_1 is parallel in the normal bundle. Hence by Theorem 5 above, Σ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$. □

5. Global rigidity for axially symmetric and time-flat surfaces in $\mathbb{R}^{3,1}$

In this section, we prove the following global rigidity theorem for time-flat axially symmetric surfaces in $\mathbb{R}^{3,1}$.

Theorem 7. *Suppose Σ is a time-flat 2-surface in $\mathbb{R}^{3,1}$ and Σ is a topological sphere. If Σ is invariant under a rotational Killing vector field, then Σ lies in a totally geodesic hyperplane in $\mathbb{R}^{3,1}$.*

Proof. Without loss of generality, we assume that the rotational Killing vector field is $\frac{\partial}{\partial\phi}$ where (t, r, θ, ϕ) is the standard spherical coordinate in $\mathbb{R}^{n,1}$, and in terms of this coordinate system Σ is locally given by the embedding

$$F : (\theta, \phi) \to (t(\theta), r(\theta), \theta, \phi).$$

A basis of the tangent space of Σ consists of

$$t'\frac{\partial}{\partial t} + r'\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta} \text{ and } \frac{\partial}{\partial\phi},$$

where $()'$ stands for differentiation with respect to θ. On the other hand, a basis of the normal bundle is given by

$$Y_3 = r^2\frac{\partial}{\partial r} - r'\frac{\partial}{\partial\theta} \text{ and } Y_4 = r'\frac{\partial}{\partial t} + t'\frac{\partial}{\partial r}.$$

By axisymmetry, both normal vectors $e_3 = -\frac{H}{|H|}$ and its orthogonal complement e_4 can be written as linear combinations of Y_3 and Y_4 with coefficients depending only on θ. We compute

$$\alpha_H\left(\frac{\partial}{\partial\phi}\right) = \langle D_{\frac{\partial}{\partial\phi}} e_3, e_4\rangle = \left\langle a(\theta)\bar{\Gamma}^{\phi}_{\phi r}\frac{\partial}{\partial\phi} + b(\theta)\bar{\Gamma}^{\phi}_{\phi\theta}\frac{\partial}{\partial\phi}, e_4\right\rangle = 0.$$

Hence $\alpha_H = \varphi(\theta)d\theta$ and $d\alpha_H = 0$. Since Σ is a topological 2-sphere, $d\alpha_H = \mathrm{div}\alpha_H = 0$ imply $\alpha_H = 0$. By Theorem 6, Σ lies in a totally geodesic hyperplane. □

6. Global rigidity of codimension-2 submanifolds with $\alpha_H = 0$ in the Schwarzschild spacetime

We generalize Theorem 6 to "star-shaped" (see Definition 9) codimension-2 submanifolds in the $(n+1)$ dimensional Schwarzschild spacetime with mass $m \geq 0$. In Schwarzschild coordinates, the metric $\bar{g}$ takes the form

$$\bar{g} = -(1 - \frac{2m}{r^{n-2}})\, dt^2 + \frac{1}{1-\frac{2m}{r^{n-2}}}dr^2 + r^2 g_{S^{n-1}}.$$

We include the case $m = 0$ which corresponds to the Minkowski spacetime.

Let $Q = rdr \wedge dt$ be the conformal Killing–Yano 2-form and Q^2 be the symmetric 2-tensor given by

$$(Q^2)_{\alpha\beta} = Q_\alpha{}^\gamma Q_{\gamma\beta}.$$

We need the following lemma from [15, Lemma B.1] relating the curvature tensor and Q.

Lemma 8. *The curvature tensor $\bar{R}_{\alpha\beta\gamma\delta}$ of the Schwarzschild metric $\bar{g}$ can be expressed as*

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\delta} = {}& \frac{2m}{r^n}\left(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right) - \frac{n(n-1)m}{r^{n+2}} \\ & \times\left(\frac{2}{3}Q_{\alpha\beta}Q_{\gamma\delta} - \frac{1}{3}Q_{\alpha\gamma}Q_{\delta\beta} - \frac{1}{3}Q_{\alpha\delta}Q_{\beta\gamma}\right) \\ & - \frac{nm}{r^{n+2}}\left(\bar{g}\circ Q^2\right)_{\alpha\beta\gamma\delta}\end{aligned} \tag{6.1}$$

where $(\bar{g}\circ Q^2)_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\gamma}(Q^2)_{\beta\delta} - \bar{g}_{\alpha\delta}(Q^2)_{\beta\gamma} + \bar{g}_{\beta\delta}(Q^2)_{\alpha\gamma} - \bar{g}_{\beta\gamma}(Q^2)_{\alpha\delta}.$

Let Σ be a spacelike codimension-2 submanifold in the Schwarzschild spacetime of $(n+1)$-dimension. Let e_n, e_{n+1} and α_H be as in Remark 1. We consider a natural condition that generalizes the star-shaped condition for hypersurfaces in the Euclidean space.

Definition 9. Σ is said to be star-shaped with respect to Q if $Q(e_n, e_{n+1}) > 0$.

We are now ready to prove the main theorem in this section:

Theorem 10. *Let $n \geq 3$ and Σ^{n-1} be a connected spacelike codimension-2 submanifold in the (n+1)-dimensional Schwarzschild spacetime with mass $m \geq 0$. Suppose $\alpha_H = 0$ and Σ is star-shaped with respect to Q where $Q = rdr \wedge dt$. Then, when $m > 0$, Σ lies in a time-slice; when $m = 0$, Σ lies in a totally geodesic hyperplane.*

Proof. Let σ denote the induced metric on Σ. Let h_n and h_{n+1} be the second fundamental forms with respect to e_n and e_{n+1}, respectively. We pick a tangential basis $\partial_a, a = 1, \ldots, n-1$ and express the components of a tensor in terms of this basis together with e_n and e_{n+1}. For example, $Q_{nb} = Q(e_n, \partial_b)$. In the following computations, we lower and raise indices with respect to the induced metric $\sigma_{ab} = \sigma(\partial_a, \partial_b)$ and its inverse σ^{ab}. Consider the divergence quantity on Σ:

$$\nabla_a \left[\left(\mathrm{tr}_\sigma (h_{n+1}) \sigma^{ab} - {h_{n+1}}^{ab} \right) Q_{nb} \right]. \tag{6.2}$$

By the assumption $\alpha_H = 0$ and the Codazzi equation (see, for example, [15, Theorem 2.1]), we obtain

$$\nabla_a \left(\mathrm{tr}_\sigma (h_{n+1}) \sigma^{ab} - {h_{n+1}}^{ab} \right) = -\bar{R}^{ab}{}_{a,n+1}.$$

On the other hand, Q satisfies the conformal Killing–Yano equation [9, Definition 1]

$$D_\alpha Q_{\beta\gamma} + D_\beta Q_{\alpha\gamma} = \frac{2}{n} \left(\bar{g}_{\alpha\beta} \xi_\gamma - \frac{1}{2} \bar{g}_{\alpha\gamma} \xi_\beta - \frac{1}{2} \bar{g}_{\beta\gamma} \xi_\alpha \right)$$

where $\xi^\beta = D_\alpha Q^{\alpha\beta}$. In our case $Q = rdr \wedge dt$ and $\xi = -n\frac{\partial}{\partial t}$. Therefore,

$$\begin{aligned}\frac{1}{2} \Big(\nabla_a (Q(e_n, \partial_b)) + \nabla_b (Q(e_n, \partial_a)) \Big) = \sigma_{ab} \left\langle \frac{\partial}{\partial t}, e_n \right\rangle - h_{n+1,ab} Q_{n+1,n} \\ - \frac{1}{2} (Q_{bc} {h_{na}}^c + Q_{ac} h_{nb}^c).\end{aligned}$$

The assumption $\alpha_H = 0$ and Ricci equation (see, for example, [15, Theorem 2.1]) together imply

$$Q_{bc}h_{na}{}^{c}h_{n+1}{}^{ab} = \frac{1}{2}\bar{R}^{ab}{}_{n+1,n}Q_{ba}.$$

Therefore, (6.2) becomes

$$-|h_{n+1}|^2 Q_{n,n+1} - \bar{R}^{ab}{}_{a,n+1}Q_{nb} + \frac{1}{2}\bar{R}^{ab}{}_{n+1,n}Q_{ba}.$$

Here we use the fact $\mathrm{tr}_\sigma(h_{n+1}) = 0$. In the following, we apply the curvature formula (6.1) and an algebraic relation of components of Q to simplify the last two terms.

By (6.1),

$$\begin{aligned}
\bar{R}^{ab}{}_{a,n+1} &= -\frac{n(n-1)m}{r^{n+2}}Q^{ab}Q_{a,n+1} - \frac{n(n-2)m}{r^{n+2}} \\
&\quad \times \left(-Q^{ab}Q_{a,n+1} + Q^{b}{}_{n}Q_{n,n+1}\right), \\
\bar{R}^{ab}{}_{n+1,n} &= -\frac{n(n-1)m}{r^{n+2}}\left(\frac{2}{3}Q^{ab}Q_{n+1,n} - \frac{1}{3}Q^{a}{}_{n+1}Q_{n}{}^{b} - \frac{1}{3}Q^{a}{}_{n}Q^{b}{}_{n+1}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
&- \bar{R}^{ab}{}_{a,n+1}Q_{nb} + \frac{1}{2}\bar{R}^{ab}{}_{n+1,n}Q_{ba} \\
=&\frac{nm}{r^{n+2}}\Big[(n-1)Q^{ab}Q_{a,n+1}Q_{nb} - (n-2)Q^{ab}Q_{a,n+1}Q_{nb} \\
&+ (n-2)Q^{bn}Q_{nb}Q_{n,n+1} \\
&- \frac{n-1}{3}(Q^{ab}Q_{ba}Q_{n+1,n} - Q^{a}{}_{n+1}Q_{n}{}^{b}Q_{ba})\Big].
\end{aligned}$$

From [15, Lemma B.3], we have

$$Q^{ab}Q_{a,n+1}Q_{bn} = -\frac{1}{2}Q^{ab}Q_{ab}Q_{n,n+1}.$$

Therefore,

$$\begin{aligned}
&\bar{R}^{ab}{}_{a,n+1}Q_{nb} - \frac{1}{2}\bar{R}^{ab}{}_{n+1,n}Q_{ba} \\
&\quad = \frac{n(n-2)m}{r^{n+2}}\left(\frac{1}{2}Q^{ab}Q_{ab}Q_{n,n+1} + Q^{b}{}_{n}Q_{bn}Q_{n,n+1}\right),
\end{aligned}$$

and we obtain

$$
\begin{aligned}
0 &= \int_\Sigma \nabla_a \left[\left(\mathrm{tr}_\sigma (h_{n+1}) \sigma^{ab} - {h_{n+1}}^{ab} \right) Q_{nb} \right] d\mu \\
&= - \int_\Sigma \left(|h_{n+1}|^2 + \frac{n(n-2)m}{r^{n+2}} \left(\frac{1}{2} Q^{ab} Q_{ab} + Q^b{}_n Q_{bn} \right) \right) Q_{n,n+1} d\mu.
\end{aligned}
$$

If $m > 0$, we obtain $Q_{ab} = Q_{bn} = 0$. As $Q = rdr \wedge dt$, it is not hard to see that $\frac{\partial}{\partial t}$ is orthogonal to Σ. Hence Σ lies on a time-slice. If $m = 0$, we can only deduce $h_{n+1} = 0$. However, this is the case of the Minkowski spacetime on which Theorem 5 is applicable. We conclude that Σ lies in a totally geodesic hyperplane of the Minkowski spacetime. □

Remark 2. Theorem 10 holds on a class of spherically symmetric spacetimes that satisfy null convergence condition. See [15, Theorem 5.11].

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