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Finite-time extinction of the Kähler–Ricci flow

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We investigate the limiting behavior of the unnormalized Kähler–Ricci flow on a Kähler manifold with a polarized initial Kähler metric. We prove that the Kähler–Ricci flow becomes extinct in finite time if and only if the manifold has positive first Chern class and the initial Kähler class is proportional to the first Chern class of the manifold. This proves a conjecture of Tian for the smooth solutions of the Kähler–Ricci flow.

1. Introduction

The limiting behavior of the Kähler–Ricci flow is deeply related to the existence of canonical metrics and the minimal model program in algebraic geometry. Let X be an n-dimensional compact Kähler manifold. We consider the following unnormalized Kähler–Ricci flow starting with a Kähler metric ω_0.

$$\begin{cases} \dfrac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \\ \omega|_{t=0} = \omega_0. \end{cases} \tag{1.1}$$

If $c_1(X) = 0$, then the unnormalized Kähler–Ricci flow (1.1) converges to the unique Ricci-flat metric in $[\omega_0]$ [Cao, Y1]. If $c_1(X) < 0$, then the normalized Kähler–Ricci flow

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) - \omega \tag{1.2}$$

converges to the unique hyperbolic Kähler–Einstein metric [Cao, Y1, A].

If $c_1(X)$ is nonpositive and not strictly negative, then there does not exist a smooth Kähler–Einstein metric on X. However, the normalized Kähler–Ricci flow (1.2) has long-time existence. If X is a minimal model of general type, then the Kähler–Ricci flow (1.2) converges to the unique singular Kähler–Einstein metric as time tends to infinity [Ts, TZha]. In general, if the canonical bundle K_X is semi-ample, the normalized Kähler–Ricci flow (1.2) can possibly collapse but it always converges to the unique generalized Kähler–Einstein metric on the canonical model of X [SoT1, SoT2].

If $c_1(X) > 0$, X is called a Fano manifold in algebraic geometry. It is conjectured by Yau [Y2] that the existence of a Kähler–Einstein metric is equivalent to suitable stability in the sense of geometric invariant theory. The condition of K-stability is later proposed by Tian [T1] and has been refined by Donaldson [D]. The Kähler–Ricci flow provides an approach to the Yau–Tian–Donaldson conjecture and it has attracted considerable current interest. We refer the reader to an incomplete list of literatures [PS1, PS2, PSSW1, PSSW2, Sz, To, CW, CS] for some recent development. If one assumes the existence of a Kähler–Einstein metric, then according to the unpublished work of Perelman [P2] (see [TZhu]), the normalized Kähler–Ricci flow $\frac{\partial}{\partial t} g = -\mathrm{Ric}(g) + g$ with the initial Kähler metric in $c_1(X)$ converges to a Kähler–Einstein metric (this is due to [H, Ch] in the case of one complex dimension).

Let us first define finite-time extinction for the Ricci flow.

Definition 1.1. Let M be a closed Riemannian manifold and $g(t)$ be a smooth solution of the Ricci flow for $t \in [0, T)$. The Ricci flow is said to become extinct at $t = T$ if $(M, g(t))$ converges to a point in the sense of Gromov–Hausdorff, or equivalently, the diameter of $g(t)$ tends to 0 as $t \to T$.

It is proved in [IK] that the diameter of $g(t)$ is uniformly bounded away from 0 if there exists an element of infinite order in $H_1(M, \mathbf{Z})$. However, many projective manifolds do not admit elements of infinite order in $H_1(M, \mathbf{Z})$. In particular, Fano manifolds are all simply connected. Indeed, it is proved by Perelman [P1, P2] (see [SeT]) that the Kähler–Ricci flow (1.1) on Fano manifolds becomes extinct in finite time if the initial Kähler class is proportional to the first Chern class of the manifold. Thus it is of our interest to completely understand the extinction conditions for the Kähler–Ricci flow (1.1). In general, the Kähler–Ricci flow can be weakly defined on singular projective varieties with mild singularities and surgeries might be performed, as investigated in [SoT3]. Therefore the notion of extinction can be generalized to the Kähler–Ricci flow with surgery. The following conjecture is proposed by Tian in [T2].

Conjecture 1.1. *The Kähler–Ricci flow (1.1) with surgery becomes extinct in finite time if and only if the initial projective manifold is birationally equivalent to a Fano manifold.*

It is shown in [SW] that the conjecture holds in the special case of Hirzebruch surfaces when the initial Kähler metric satisfies the Calabi symmetry.

We prove Conjecture 1.1 in the case of the usual Kähler–Ricci flow with smooth solutions.

Theorem 1.1. *Let X be an n-dimensional Kähler manifold. Then the Kähler–Ricci flow (1.1) with an initial Kähler metric in the class of $H^2(X, \mathbf{Z})$ becomes extinct in finite time if and only if X is Fano and the initial Kähler class is proportional to $c_1(X)$.*

More generally, we have the following general diameter estimates.

Theorem 1.2. *Let X be an n-dimensional Kähler manifold. Let ω_0 be a Kähler metric on X such that $[\omega_0] \in H^2(X, \mathbf{Z})$. Then the diameter is uniformly bounded away from 0 along the Kähler–Ricci flow (1.1) if one of the following conditions holds.*

1) *K_X is semi-ample.*
2) *K_X is not nef and K_X^{-1} is not ample.*
3) *K_X^{-1} is ample and $[\omega_0]$ is not proportional to $[K_X^{-1}]$.*

In particular, if K_X is semi-ample and the Kodaira dimension of X is positive, the diameter tends to infinity of order $\sqrt{t}$ as $t \to \infty$ along the Kähler–Ricci flow (1.1).

It is known that the abundance conjecture holds for dimension three and so K_X is semi-ample whenever it is nef. The following corollary is then an immediate consequence of Theorem 1.2.

Corollary 1.1. *Let X be a Kähler manifold of $\dim X \leq 3$. Then the Kähler–Ricci flow (1.1) on X with an initial Kähler metric in $H^2(X, \mathbf{Z})$ becomes extinct if and only if X is Fano and the initial Kähler class is proportional to $c_1(X)$.*

The above statement holds for $\dim X \geq 4$ if we assume the abundance conjecture.

We further propose the following conjecture as a natural attempt to generalize Theorem 1.1 and Corollary 1.1 by removing the condition for the initial Kähler class and the assumption of the abundance conjecture.

Conjecture 1.2. *Let X be an n-dimensional Kähler manifold. Then the Kähler–Ricci flow on X with an initial Kähler metric becomes extinct if and only if X is Fano and the initial Kähler class is proportional to $c_1(X)$.*

Remark 1.1 Theorem 1.1 also holds if the initial Kähler metric is in a multiple of a class in $H^2(X, \mathbf{Z})$. This can be easily proved by suitable scaling of the unnormalized flow (1.1). It is pointed out by V. Tosatti to the author that Theorem 1.1 should also be true if the initial Kähler metric sits in the real Néron–Severi group $(H^2(X, \mathbf{Z}) \cap H^{1,1}(X, \mathbf{C})) \otimes \mathbf{R}$ as the base-point-free theorem holds for $\mathbf{R}$-divisors due to Shokurov.

2. Base point freeness

Let X be an n-dimensional projective manifold and $L \to X$ a holomorphic line bundle over X. Let $N(L)$ be the semi-group defined by

$$N(L) = \{m \in \mathbf{N} \mid H^0(X, L^m) \neq 0\}.$$

Given any $m \in N(L)$, the linear system $|L^m| = \mathbf{P}H^0(X, L^m)$ induces a rational map

$$\Phi_m \ : \ X \dashrightarrow \mathbf{CP}^{d_m}$$

by any basis $\{\sigma_{m,0}, \ \sigma_{m,1}, \ \dots \ , \sigma_{m,d_m}\}$ of $H^0(X, L^m)$ as

$$\Phi_m(z) = [\sigma_{m,0}, \ \sigma_{m,1}, \ \dots, \sigma_{m,d_m}]\,(z),$$

where $d_m + 1 = \dim H^0(X, L^m)$. Let $Y_m = \overline{\Phi_m(X)} \subset \mathbf{CP}^{d_m}$ be the closure of the image of X by Φ_m.

Definition 2.1. The Iitaka dimension of L is defined to be

$$\kappa(X, L) = \max_{m \in N(L)} \{\dim Y_m\}$$

if $N(L) \neq \phi$, and $\kappa(X, L) = -\infty$ if $N(L) = \phi$.

Definition 2.2. Let X be an projective manifold and K_X the canonical line bundle over X. Then the Kodaira dimension $\mathrm{kod}(X)$ of X is defined to be

$$\mathrm{kod}(X) = \kappa(X, K_X).$$

The Kodaira dimension is a birational invariant of projective varieties and the Kodaira dimension of a singular variety is equal to that of its smooth model.

Definition 2.3. Let $L \to X$ be a holomorphic line bundle over a projective manifold X. L is called semi-ample if the linear system $|L^m|$ is base point free for some $m > 0$.

For any $m \in \mathbf{N}$ such that $|L^m|$ is base point free, the linear system $|L^m|$ induces a holomorphic map Φ_m

$$\Phi_m \ : \ X \to \mathbf{CP}^{d_m}$$

by any basis of $H^0(X, L^m)$. Let $Y_m = \Phi_m(X)$ and so

$$\Phi_m \ : \ X \to Y_m \in \mathbf{CP}^{d_m}.$$

The following theorem is well known (see [L, U]).

Theorem 2.1. *Let $L \to X$ be a semi-ample line bundle over an algebraic manifold X. Then there is a projective fibration*

$$\pi : X \to Y$$

such that for any sufficiently large integer m with L^m being globally generated,

$$Y_m = Y \quad \textit{and} \quad \Phi_m = \pi,$$

where Y is a normal projective variety.

Furthermore, there exists an ample line bundle A on Y such that $L^m = \pi^* A$.

If L is semi-ample, the graded ring $R(X, L) = \oplus_{m \geq 0} H^0(X, L^m)$ is finitely generated and so $R(X, L) = \oplus_{m \geq 0} H^0(X, L^m)$ is the coordinate ring of Y.

Definition 2.4. Let $L \to X$ be a semi-ample line bundle over a projective manifold X. Then the algebraic fibration $\pi : X \to Y$ as in Theorem 2.1 is called the Iitaka fibration associated with L. It is completely determined by the linear system $|L^m|$ for sufficiently large m.

The following theorems are known as the rationality theorem and base-point-free theorem in the minimal model program (see [KMM, KM]).

Theorem 2.2. *Let X be a projective manifold such that K_X is not nef. Let H be an ample divisor and let*

$$\lambda = \max\{t \in \mathbf{R} \mid H + tK_X \text{ is nef }\}. \tag{2.1}$$

Then $\lambda \in \mathbf{Q}$.

Theorem 2.3. *Let X be a projective manifold. Let D be a nef divisor such that $aD - K_X$ is nef and big for some $a > 0$. Then D is semi-ample.*

We now will apply the base-point-free theorem to the Kähler–Ricci flow at the finite blow-up time. We consider the unnormalized Kähler–Ricci flow (1.1) with the initial Kähler class $H = [\omega_0] \in H^2(X, \mathbf{Z}) \cap H^{1,1}(X, \mathbf{C})$. The evolution of the Kähler class satisfies the following ordinary differential equation

$$\frac{\partial}{\partial t}[\omega] = [K_X], \quad [\omega]|_{t=0} = [H]. \tag{2.2}$$

Therefore, $[\omega] = [H + tK_X]$ for $t \geq 0$.

Lemma 2.1. *Let*

$$T = \sup\{t \geq 0 \mid H + tK_X \text{ is nef}\}. \tag{2.3}$$

Then $T \in \mathbf{Q}$ if K_X is not nef and $T = \infty$ if K_X is nef.

Proof. The lemma is an immediate corollary of the rationality Theorem 2.2. □

The following theorem is proved in [TZha] for the maximal existence of the Kähler–Ricci flow.

Proposition 2.1. *The Kähler–Ricci flow (1.1) exists for $t \in [0, T)$.*

Lemma 2.2. *Let $L = H + TK_X$. Then L is semi-ample.*

Proof. Notice that $L - \epsilon K_X = H + (T - \epsilon)K_X$ is ample for sufficiently small $\epsilon > 0$. The lemma follows from the base-point-free theorem 2.3. □

3. The case when K_X is not nef and $\kappa(L) = 0$

If K_X is not nef, $T < \infty$ and the unnormalized Kähler–Ricci flow (1.1) does not have long-time existence and it must develop singularities at $t = T$. In particular, it is shown in [Z] that the scalar curvature must blow up at $t = T$. Let $L = H + TK_X$. Then either $\kappa(L) = 0$ or $\kappa(L) > 0$ as L is semi-ample.

Proposition 3.1. *If $\kappa(L) = 0$, X is Fano and $[H]$ is proportional to $c_1(X)$.*

Proof. Since L semi-ample and $\kappa(L) = 0$, L^m is trivial for some m and $[L] = 0$. Hence

$$T\ c_1(X) = -T\ [K_X] = [H] > 0.$$

□

The following theorem is proved by Perelman [P2] (see [SeT]).

Theorem 3.1. *Let X be a Fano manifold of complex dimension $n \geq 2$. The unnormalized Kähler–Ricci flow becomes extinct in finite time if the initial Kähler class is proportional to the first Chern class $c_1(X)$.*

More precisely, Perelman shows that if X is Fano and the initial Kähler metric lies in $c_1(X)$, the diameter of the evolving metrics is uniformly bounded above along the normalized Kähler–Ricci flow $\frac{\partial}{\partial t} g = -\mathrm{Ric}(g) + g$. The above theorem follows immediately after scaling the normalized flow back to the unnormalized flow.

4. The case when K_X is not nef and $\kappa(L) > 0$

Now we assume $k = \kappa(L) > 0$. Since L is semi-ample, $H^0(X, L^m)$ induces a holomorphic map for sufficiently large m

$$\pi : X \to Y \subset \mathbf{CP}^{d_m},$$

where Y is a normal variety of $\dim Y = k$ and $d_m + 1 = \dim H^0(X, L^m)$.

Since $L = H + TK_X$ is semi-ample, there exists a smooth $(1,1)$-form $\chi \in [K_X]$ such that

$$\omega_T = \omega_0 + T\chi \geq 0$$

is proportional to the pullback of the Fubini–Study metric ω_{FS} on $\mathbf{CP}^{d_m}$ by π. There also exists a smooth volume form Ω on X such that

$$\sqrt{-1}\partial\overline{\partial} \log \Omega = \chi.$$

Let $\omega_t = \omega_0 + t\chi$. Then

$$\omega_t = \frac{t}{T}\omega_T + \frac{T-t}{T}\omega_0 \geq \frac{T-t}{T}\omega_0 > 0$$

and the following lemma holds immediately.

Lemma 4.1. *For any* $(t,z) \in [0,T) \times X$,

$$\frac{\omega_t^n}{\omega_0^n} \geq (\frac{T-t}{T})^n. \tag{4.1}$$

We consider the following Monge–Ampère flow induced by the unnormalized Kähler–Ricci flow.

$$\begin{cases} \frac{\partial}{\partial t}\varphi = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}, \\ \varphi|_{t=0} = 0. \end{cases} \tag{4.2}$$

Lemma 4.2. *There exists* $C > 0$ *such that for* $(t,z) \in [0,T) \times X$,

$$\varphi \leq C. \tag{4.3}$$

Proof. Let $\varphi_{\max}(t) = \max_{z\in X} \varphi(t,z)$. Then

$$\frac{\partial}{\partial t}\varphi_{\max} \leq \log \frac{\omega_t^n}{\Omega} \leq \log \frac{(\omega_0 + \omega_T)^n}{\Omega} \leq C.$$

Hence

$$\varphi(t,z) \leq CT.$$

□

Proposition 4.1. *There exists* $C > 0$ *such that for all* $(t,z) \in [0,T) \times X$,

$$\mathrm{tr}_\omega(\omega_T) \leq C. \tag{4.4}$$

Proof. This is a parabolic Schwarz lemma similar to the one given in [SoT1]. Suppose $\omega_T = c\pi^*\omega_{\mathrm{FS}}$ and $\omega_{\mathrm{FS}} = h_{\alpha\overline{\beta}} dx^\alpha \wedge d\overline{x}^\beta$.

Choose normal coordinate systems for $g = \omega(t,\cdot)$ on X and h on $\mathbf{CP}^{d_m}$, respectively. Let $u = \mathrm{tr}_g(h) = g^{i\overline{j}}\pi_i^\alpha \pi_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}$ and we will calculate the evolution of u. u is nonnegative as π is holomorphic. Standard calculation shows that

$$\begin{aligned}\Delta u &= g^{k\bar{l}}\partial_k\partial_{\bar{l}}\left(g^{i\bar{j}}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}h_{\alpha\overline{\beta}}\right)\\ &= g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}h_{\alpha\overline{\beta}} + g^{i\bar{j}}g^{k\bar{l}}\pi_{i,k}^\alpha\pi_{\bar{j},\bar{l}}^{\overline{\beta}}h_{\alpha\overline{\beta}} - g^{i\bar{j}}g^{k\bar{l}}S_{\alpha\overline{\beta}\gamma\overline{\delta}}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}\pi_k^\gamma\pi_{\bar{l}}^{\overline{\delta}},\end{aligned}$$

where $S_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is the curvature tensor of $h_{\alpha\overline{\beta}}$. By the definition of u we have

$$\Delta u \geq g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}h_{\alpha\overline{\beta}} - Ku^2 + g^{i\bar{j}}g^{k\bar{l}}\pi_{i,k}^\alpha\pi_{\bar{j},\bar{l}}^{\overline{\beta}}h_{\alpha\overline{\beta}}$$

for some fixed constant $K > 0$. Now

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g^{i\bar{l}}g^{k\bar{j}}\frac{\partial g_{k\bar{l}}}{\partial t}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}h_{\alpha\overline{\beta}}\\ &= g^{i\bar{l}}g^{k\bar{j}}R_{k\bar{l}}\pi_i^\alpha\pi_{\bar{j}}^{\overline{\beta}}h_{\alpha\overline{\beta}},\end{aligned}$$

therefore

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\log u &= \frac{1}{u}\left(\frac{\partial}{\partial t} - \Delta\right)u - \frac{|\nabla u|^2}{u^2}\\ &\leq Ku - \frac{1}{u^2}(ug^{i\bar{j}}g^{k\bar{l}}\pi_{i,k}^\alpha\pi_{\bar{j},\bar{l}}^{\overline{\beta}}h_{\alpha\overline{\beta}} - |\nabla u|^2)\\ &\leq Ku,\end{aligned}\tag{4.5}$$

where the last inequality follows from $ug^{i\bar{j}}g^{k\bar{l}}\pi_{i,k}^\alpha\pi_{\bar{j},\bar{l}}^{\overline{\beta}}h_{\alpha\overline{\beta}} - |\nabla u|^2 \geq 0$ (see p. 627 [SoT1]).

Consider $H = \log u - 2A\varphi$. If A is chosen to be sufficiently large,

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right)H\\ &\leq -\mathrm{tr}_\omega(2A\omega_t - K\omega_T) - 2A\log\frac{\omega^n}{\Omega} + 2An\\ &\leq -\mathrm{tr}_\omega(A\omega_t) + 2A\log\frac{\omega_t^n}{\omega^n} + 2A\log\frac{\Omega}{\omega_t^n} + 2An\\ &\leq -CAn\log(T-t) + CA.\end{aligned}$$

$H|_{t=0} = \log\mathrm{tr}_{\omega_0}(\omega_T)$ is bounded from above. By the maximum principle,

$$H \leq -C\int_0^T \log(T-t)dt + C \leq C'.$$

The proposition is proved as φ is uniformly bounded from above. □

Corollary 4.1. *There exists $C > 0$ such that for $(t, z) \in [0, T) \times X$*

$$\omega_t \geq C\pi^* \omega_{\mathrm{FS}}. \tag{4.6}$$

Theorem 4.1. *Let $g(t)$ be the Kähler metric associated with $\omega(t, \cdot)$. Then there exists $C > 0$, such that for $t \in [0, T)$,*

$$\operatorname{diam}(X, g(t)) \geq C. \tag{4.7}$$

Proof. Since Y is normal, the singular set of Y is an analytic subvariety of Y of codimension greater than one. Let x_0 be a point in the nonsingular part Y_{reg} of Y. Then there exists a geodesic ball $B_{2r}(x_0, g_{\mathrm{FS}}) \subset Y_{\mathrm{reg}}$ of radius $2r > 0$ centered at x_0 with respect to the Fubini–Study metric g_{FS} restricted on Y. There exist x_1 and $x_2 \in B_r(x_0, g_{\mathrm{FS}})$ such

$$d_{Y, g_{\mathrm{FS}}}(x_1, x_2) \geq r,$$

where $d_{Y, g_{\mathrm{FS}}}(x_1, x_2)$ is the distance between x_1 and x_2 on Y with respect to g_{FS}.

Choose $z_1 \in \pi^{-1}(x_1)$ and $z_2 \in \pi^{-1}(x_2)$. Then for $i = 1$, 2, let

$$d_{\pi^{-1}(B_{2r}(x_0, g_{\mathrm{FS}})), g(t)}(z_i, \pi^{-1}\partial(B_{2r}(x_0, g_{\mathrm{FS}})))$$

be the distance from z_i to $\pi^{-1}\partial(B_{2r}(x_0, g_{\mathrm{FS}}))$ in $\pi^{-1}(B_{2r}(x_0, g_{\mathrm{FS}}))$ with respect to $g(t)$ and $d_{Y, g_{\mathrm{FS}}}(x_i, \partial B_{2r}(x_0, g_{\mathrm{FS}}))$ be the distance from x_i to $\partial B_{2r}(x_0, g_{\mathrm{FS}})$ on Y with respect to g_{FS}. Then

$$\begin{aligned} &d_{\pi^{-1}(B_{2r}(x_0, g_{\mathrm{FS}})), g(t)}(z_i, \pi^{-1}\partial(B_{2r}(x_0, g_{\mathrm{FS}}))) \\ &\quad \geq C d_{Y, g_{\mathrm{FS}}}(x_i, \partial B_{2r}(x_0, g_{\mathrm{FS}})) \geq Cr, \end{aligned}$$

and so

$$d_{X, g(t)}(z_1, z_2) \geq C d_{Y, g_{\mathrm{FS}}}(x_1, x_2) \geq Cr.$$

□

5. The case when K_X is nef

When K_X is nef, it follows from Proposition 2.1 that the unnormalized Kähler–Ricci flow (1.1) has long-time existence.

Proposition 5.1. *Let X be n-dimensional projective manifold of* $\mathrm{kod}(X) = 0$. *If K_X is semi-ample, then there exists $C > 0$ depending on g_0 such that along the Kähler–Ricci flow (1.1)*

$$\mathrm{diam}(X, g(t)) \geq C. \tag{5.1}$$

Proof. If $\mathrm{kod}(X) = 0$ and K_X is semi-ample, $c_1(X) = 0$. The proposition is a result of Cao [Cao, Y1]. □

Proposition 5.2. *Let X be n-dimensional projective manifold of* $\mathrm{kod}(X) > 0$. *If K_X is semi-ample, then there exists $C > 0$ depending on g_0 such that along the Kähler–Ricci flow (1.1)*

$$\mathrm{diam}(X, g(t)) \geq C\sqrt{t}. \tag{5.2}$$

Proof. We consider the following normalized Kähler–Ricci flow when K_X is nef.

$$\begin{cases} \dfrac{\partial}{\partial s}\tilde{g} = -\mathrm{Ric}(\tilde{g}) - \tilde{g}, \\ \tilde{g}|_{s=0} = g_0. \end{cases} \tag{5.3}$$

The relation between the solution $g(t)$ of the unnormalized Kähler–Ricci flow (1.1) and the solution $\tilde{g}(s)$ of the normalized flow (5.3) is given by

$$t = e^s - 1,$$
$$g(t) = e^s \tilde{g}(s) = (t+1)\tilde{g}(\log(t+1)).$$

If X is of general type, there exists a birational holomorphic map $f : X \to X_{\mathrm{can}}$ from X to its canonical model X_{can} as $R(X, K_X)$ is finitely generated. Let E be the set of all the points on X such that f is not isomorphic. Then E is a subvariety of X and it is proved in [Ts, TZha] that $g(t)$ converges in $C^\infty(X \setminus E)$ as $t \to \infty$ along the normalized flow (5.3). Since the limiting singular Kähler–Einstein metric is smooth and nondegenerate on $X \setminus E$, $\mathrm{diam}(X, \tilde{g}(t))$ is then uniformly bounded below from 0.

If $0 < k = \mathrm{kod}(X) < n$, then there exists a unique holomorphic map $f : X \to X_{\mathrm{can}} \in \mathbf{CP}^N$, where X_{can} is the canonical model of X of

$\dim X_{\rm can} = k$. Let

$$X^{\circ}_{\rm can} = \{x \in X_{\rm can} \mid x \text{ is nonsingular and } f^{-1}(x) \text{ is nonsingular}\}.$$

Let h be the pullback of the Fubini–Study metric on $X_{\rm can}$. Then it is proved in [SoT1, SoT2] by the parabolic Schwarz lemma, that for any $K \subset\subset X^{\circ}_{\rm can}$, there exists $C_K > 0$ such that on $[0, \infty) \times K$,

$$\tilde{g}(t) \geq C_K f^* h \tag{5.4}$$

along the normalized Kähler–Ricci flow (5.3). Therefore, $\mathrm{diam}(X, \tilde{g}(t))$ is uniformly bounded from below from 0.

Combining the above estimates, for any initial metric g_0, there exists $C > 0$ such that along the normalized Kähler–Ricci flow (5.3),

$$\mathrm{diam}(X, \tilde{g}(s)) \geq C. \tag{5.5}$$

The proposition is proved as $g(t) = (t + 1)\tilde{g}(\log(t + 1))$. □

Now we are able to prove the main theorems.

Proof of Theorem 1.2. If K_X is not nef and K_X^{-1} is not ample, $L = H + TK_X$ has positive Iitaka dimension at the singular time $t = T$. If K_X^{-1} is ample and $[\omega_0]$ is not proportional to K_X^{-1}, K_X is not nef and $L = H + TK_X$ has positive Iitaka dimension at the singular time $t = T$ by Proposition 3.1. Then the proposition follows by combining Theorem 4.1, Propositions 5.1 and 5.2. □

Proof of Theorem 1.1. The Kähler–Ricci flow (1.1) has long-time existence if K_X is nef. Theorem 1.1 follows easily from Theorems 1.2 and 3.1. □

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