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A note on weak convergence of singular integrals in metric spaces

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We prove that in any metric space (X, d) the singular integral operators

$$T^k_{\mu,\varepsilon}(f)(x) = \int_{X\setminus B(x,\varepsilon)} k(x,y) f(y)\, d\mu(y)$$

converge weakly in some dense subspaces of $L^2(\mu)$ under minimal regularity assumptions for the measures and the kernels.

1. Introduction

A Radon measure on a metric space (X, d) has s-growth if there exists some constant c_μ such that $\mu(B(x, r)) \le c_\mu r^s$ for all $x \in X$, $r > 0$.

We say that $k(\cdot,\cdot) : X \times X \setminus \{(x, y) \in X \times X : x = y\} \to \mathbb{R}$ is an s-dimensional kernel if there exists a constant $c > 0$ such that for all $x, y \in X$, $x \neq y$:

$$|k(x, y)| \le c\, d(x, y)^{-s}.$$

The kernel k is antisymmetric if $k(x, y) = -k(y, x)$ for all distinct $x, y \in X$.

Given a positive Radon measure ν on X and an s-dimensional kernel k, we define

$$T^k\nu(x) := \int k(x, y)\, d\nu(y), \quad x \in X \setminus \operatorname{spt} \nu.$$

This integral may not converge when $x \in \operatorname{spt} \nu$. For this reason, we consider the following ε-truncated operators T^k_ε, $\varepsilon > 0$:

$$T^k_\varepsilon \nu(x) := \int_{d(x,y)>\varepsilon} k(x, y)\, d\nu(y), \quad x \in X.$$

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Given a fixed positive Radon measure μ on X and $f \in L^1_{\mathrm{loc}}(\mu)$, we write

$$T^k_\mu f(x) := T^k(f\,\mu)(x), \quad x \in X \setminus \operatorname{spt}(f\,\mu)$$

and

$$T^k_{\mu,\varepsilon} f(x) := T^k_\varepsilon(f\,\mu)(x).$$

Concerning the limit properties of the operators $T^k_{\mu,\varepsilon}$ one can ask if the limit, the so-called principal value of T,

$$\lim_{\varepsilon \to 0} T^k_{\mu,\varepsilon}(f)(x),$$

exists μ almost everywhere. When μ is the Lebesgue measure in $\mathbb{R}^d$, and k is a standard Calderón–Zygmund kernel, due to cancellations and the denseness of smooth functions in L^1, the principal values exist almost everywhere for L^1-functions. For more general measures, the question is more complicated. Let n be an integer, $0 < n < d$, and consider the coordinate Riesz kernels

$$R^n_i(x) = \frac{x_i}{|x|^{n+1}}, \qquad \text{for } i = 1, \dots, d.$$

Tolsa proved in [17] that if $E \subset \mathbb{R}^d$ has finite n-dimensional Hausdorff measure $\mathcal{H}^n$ the principal values

$$\lim_{\varepsilon \to 0} \int_{E \setminus B(x,\varepsilon)} \frac{x_i - y_i}{|x-y|^{m+1}} d\mathcal{H}^n(y)$$

exist $\mathcal{H}^n$ almost everywhere in E if and only if the set E is n-rectifiable i.e., if there exist n-dimensional Lipschitz surfaces M_i, $i \in \mathbb{N}$, such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i=1}^{\infty} M_i\right) = 0.$$

Mattila and Preiss had obtained the same result earlier, in [12] under some stronger assumptions for the set E. It becomes obvious that the existence of principal values is deeply related to the geometry of the set E.

Assuming $L^2(\mu)$-boundedness for the operators T^k_μ one could have expected that more could be deduced about the structure of μ and the existence of principal values, but this is a hard and, in a large extent, open problem. Dating from 1991 the David–Semmes conjecture, see [5], asks if the $L^2(\mu)$-boundedness of the operators associated with the n-dimensional Riesz kernels suffices to imply n-uniform rectifiabilty, which can be thought as a quantitative version of rectifiability. In the very recent deep work [14], Nazarov et al. resolved the conjecture in the codimension 1 case, that is for $n = d - 1$. Mattila et al. in [11], using a special symmetrization property of the Cauchy kernel, had earlier proved the conjecture in the case of one-dimensional Riesz kernels. For all other dimensions and for other kernels few things are known. In fact, there are several examples of kernels whose boundedness does not imply rectifiability, see [1, 4, 9]. For some recent positive results involving other kernels see [2].

Let μ be a finite Radon measure and let k be an antisymmetric kernel in a complete metric space (X, d) where the Vitali covering theorem holds for μ and the family of closed balls defined by d. Mattila and Verdera in [13] showed that in this case the $L^2(\mu)$-boundedness of the operators $T^k_{\mu,\varepsilon}$ forces them to converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator $T^k_\mu : L^2(\mu) \to L^2(\mu)$ such that for all $f, g \in L^2(\mu)$,

$$\lim_{\varepsilon \to 0} \int T^k_{\mu,\varepsilon}(f)(x) g(x)\, d\mu(x) = \int T^k_\mu(f)(x) g(x)\, d\mu(x).$$

Furthermore notions of weak convergence have been recently used by Nazarov et al. [14].

Motivated by these developments it is natural to ask whether limits of this type might exist if we remove the very strong L^2-boundedness assumption. We prove that the operators $T^k_{\mu,\varepsilon}$ converge weakly in dense subspaces of $L^2(\mu)$ under minimal assumptions for the measures and the kernels in general metric spaces.

The following standard definition is from [7, 2.8.16].

Definition 1.1. Let μ be a measure on a metric space (X, d). Let $\mathcal{V}$ be a family of Borel sets in X and for $x \in X$ and $\mathcal{F} \subset \mathcal{V}$ denote $\mathcal{F}_x = \{S \in \mathcal{F} : x \in S\}$. We say that $\mathcal{V}$ is a μ-*Vitali relation* if for any $\mathcal{F} \subset \mathcal{V}$ and any $A \subset X$ such that for any $x \in A$,

$$\inf\{\operatorname{diam}(S) : S \in \mathcal{F}_x\} = 0,$$

then the family $\{S \in \mathcal{F}_x : x \in A\}$ has a countable disjointed subfamily $\mathbb{C}$ such that

$$\mu(A \setminus \cup_{S \in \mathbb{C}} S) = 0.$$

When $X = \mathbb{R}^d$ the closed balls defined by various metrics (including the standard d_p metrics for $1 \le p \le \infty$) are Vitali relations for any Radon measure μ as a consequence of Besicovitch's covering theorem, see [10, Theorem 2.8].

Furthermore, the same is true in any doubling metric measure space, that is when there exists some constant C such that for all closed balls B, $\mu(2B) \leq C\mu(B)$, see [8, Theorem 1.6]. These facts are frequently referred to as *Vitali Covering Theorems.* We would also like to point out that there exist metric measure spaces where the closed balls are not Vitali relations. In [6], Davies constructed an example of a compact metric space equipped with a probability measure such that the family of closed balls is not a Vitali relation. Moreover, Preiss [15] provided an example of a Gaussian measure on a separable Hilbert space such that the family of closed balls is not a Vitali relation.

Denote by $\mathcal{X}_B$ the space of finite linear combinations of characteristic functions of balls in (X,d),

$$\mathcal{X}_B = \left\{ \sum_{i=1}^{n} a_i \chi_{B(z_i,r_i)} : n \in \mathbb{N},\, a_i \in \mathbb{R},\, z_i \in X,\, r_i > 0 \right\}.$$

We remark that whenever the family of closed balls is a μ-Vitali relation the space $\mathcal{X}_B$ is dense in $L^2(\mu)$.

Theorem 1.2. *Let μ be a finite Radon measure with s-growth and k an antisymmetric s-dimensional kernel on a metric space (X,d). If the family of closed balls in (X,d) is a μ-Vitali relation then there exists subsets $\mathcal{X}'_B \subset \mathcal{X}_B$ which are dense in $L^2(\mu)$ and the weak limits*

$$\lim_{\varepsilon \to 0} \int T^k_{\mu,\varepsilon} f(x)\, g(x)\, d\mu(x)$$

exist for all $f, g \in \mathcal{X}'_B$.

Until now Theorem 1.2 was only known for measures with $(d-1)$-growth in $\mathbb{R}^d$ under some smoothness assumptions for the kernels, see [3]. We thus extend the result from [3] to measures with s-growth for arbitrary s in metric spaces where the family of closed balls is a Vitali relation without requiring any smoothness for the kernels. Our proof follows a completely different strategy using an "exponential growth" lemma for probability measures on intervals and is self-contained (unlike the proof from [3] which depends on several $L^2(\nu)$ to $L^2(\mu)$ boundedness results for separated measures ν and μ).

Recall that if k is the $(d-1)$-dimensional Riesz kernel in $\mathbb{R}^d$ and μ has $(d-1)$-growth and is $(d-1)$ purely unrectifiable, that is $\mu(E) = 0$ for all $(d-1)$-rectifiable sets E, the principal values diverge μ almost everywhere and the weak convergence in $L^2(\mu)$ fails. On the other hand, it is of interest that weak convergence in the sense of Theorem 1.2 holds as it holds for any s-dimensional antisymmetric kernel and any finite measure with s-growth.

2. Proof of Theorem 1.2

We first prove the following lemma about exponential growth of probability measures on compact intervals. It is motivated by a similar result proved in [16]. Here Leb stands for the Lebesgue measure on the real line and $|I|$ denotes the length of an interval $I \subset \mathbb{R}$.

Lemma 2.1. *For every integer $\lambda > 2$ the following holds. Let ν be a finite Borel measure on a compact interval $\Delta \subset \mathbb{R}$. Then for every interval $I \subset \Delta$ there exists a subset $I'(\lambda) \subset I$ such that $\mathrm{Leb}(I'(\lambda)) > |I|(1 - 3(\lambda^{-1} + \lambda^{-2} + \cdots))$ and for every $t \in I'(\lambda)$,*

$$\nu([t - \lambda^{-3n}|I|, t + \lambda^{-3n}|I|]) < \lambda^{-n}\nu(I)$$

for all integers $n \geq 1$.

Proof. Without loss of generality we can assume that $\nu(\Delta) \leq 1$. Let us partition the interval I into λ^2 subintervals J of length $|I|\lambda^{-2}$. Let B_1 be the family of all intervals J from this partition for which $\nu(J) < \lambda^{-1}$. Obviously, there are at most λ intervals in B_1^c. Thus

$$\#B_1 > \lambda^2 - \lambda = \lambda^2 \left(1 - \frac{\lambda}{\lambda^2}\right)$$

and

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_1\}\right) \geq |I| \left(1 - \frac{\lambda}{\lambda^2}\right) = |I| \left(1 - \frac{1}{\lambda}\right).$$

Next, each interval in B_1 is divided into λ^2 subintervals with disjoint interiors and of length $|I|\lambda^{-4}$, and we remove those subintervals for which $\nu(J) \geq \lambda^{-2}$. Denoting by B_2 the family of remaining intervals, we see that

$$\#B_2 \geq (\lambda^2)^2 \left(1 - \frac{\lambda}{\lambda^2}\right) - \lambda^2 = (\lambda^2)^2 \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right)$$

and

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_2\}\right) \geq |I| \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2}\right).$$

Proceeding inductively, we partition the interval I into disjoint intervals of length $|I|\lambda^{-2n}$. Next, we define in the same way the family B_n. It is formed by the intervals J of this partition of nth generation, which are contained in some interval of the family B_{n-1} and for which $\nu(J) < \lambda^{-n}$. Then

$$\mathrm{Leb}\left(\bigcup\{J : J \in B_n\}\right) \geq \left(1 - \frac{1}{\lambda} - \frac{1}{\lambda^2} - \cdots - \frac{1}{\lambda^n}\right)|I|.$$

For any $t \in I$ let $J_n = J_n(t)$ be the interval of the nth partition such that $t \in J_n$. Thus, for every $t \in \bigcap_{n=1}^{\infty} \bigcup_{J\in B_n} J$, we have that $J_n(t) \in B_n$. Consequently, for all $t \in \bigcap_{n=1}^{\infty} \bigcup_{J\in B_n} J$, it holds that $\nu(J_n(t)) < \lambda^{-n}$ for all $n \geq 1$. Let now

$$C_n = \{t \in I : [t - |I|\lambda^{-3n}, t + |I|\lambda^{-3n}] \subset J_n(t)\}.$$

It is easy to see that $\mathrm{Leb}(C_n^c) < 2|I|\lambda^{-n}$, and, therefore,

$$\mathrm{Leb}\left(\bigcap_{n=1}^{\infty} C_n\right) > |I|\left(1 - 2\left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots\right)\right).$$

Finally, setting

$$I' := \left(\bigcap_{n=1}^{\infty} C_n\right) \cap \left(\bigcap_{i=1}^{\infty} \bigcup_{J\in B_i} J\right)$$

completes the proof. □

Proof of Theorem 1.2. We can assume that $\mu(X) \leq 1$. We define finite Borel measures on the unit interval for all $z \in \operatorname{spt}\mu$ by

$$\mu_z(F) = \mu\{x \in X : d(x,z) \in F\},\ F \subset [0,1].$$

Let $G_z = \cup_{\lambda>2} I'_z(\lambda)$ where $I'_z(\lambda)$ are the sets we obtain after we apply Lemma 2.1 to the measures μ_z. Then Lemma 2.1 implies that $\mathrm{Leb}(G_z) = \mathrm{Leb}([0,1])$. Let

$$\mathcal{X}'_B := \left\{\sum_{i=1}^{n} a_i \chi_{B(z_i,r_i)} : n \in \mathbb{N},\ a_i \in \mathbb{R},\ z_i \in \operatorname{spt}\mu,\ r_i \in G_{z_i}\right\}.$$

Since G_z is dense in $[0,1]$ we have that $\mathcal{X}'_B$ is dense $\mathcal{X}_B$ and hence it is also dense in $L^2(\mu)$.

Let $f, g \in \mathcal{X}'_B$ such that

$$f = \sum_i^n a_i \chi_{B_i} \quad \text{and} \quad g = \sum_j^m b_j \chi_{S_j},$$

where $a_i, b_j \in \mathbb{R}$ and B_i, S_j are closed balls. Then for $0 < \delta < \varepsilon$,

$$\begin{aligned}\int T^k_{\mu,\varepsilon} f(x) g(x)\, d\mu(x) - &\int T^k_{\mu,\delta} f(x) g(x)\, d\mu(x) \\ &= \sum_{j=1}^m \sum_{i=1}^n a_i b_j \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j} \int_{B_i}} k(x,y)\, d\mu(y)\, d\mu(x).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left| \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j} \int_{B_i}} k(x,y)\, d\mu(y)\, d\mu(x) \right| \\ &\leq \left| \underset{\delta < d(x,y) < \varepsilon}{\int_{B_i \cap S_j} \int_{B_i \cap S_j}} k(x,y)\, d\mu(y)\, d\mu(x) \right| + \left| \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j \setminus B_i} \int_{B_i \cap S_j}} k(x,y)\, d\mu(y)\, d\mu(x) \right| \\ &\quad + \left| \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j \setminus B_i} \int_{B_i \setminus S_j}} k(x,y)\, d\mu(y)\, d\mu(x) \right| + \left| \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j \cap B_i} \int_{B_i \setminus S_j}} k(x,y)\, d\mu(y)\, d\mu(x) \right| \\ &\leq \underset{\delta < d(x,y) < \varepsilon}{\int_{B_i} \int_{B_i^c}} |k(x,y)|\, d\mu(y)\, d\mu(x) + 2 \underset{\delta < d(x,y) < \varepsilon}{\int_{S_j} \int_{S_j^c}} |k(x,y)|\, d\mu(y)\, d\mu(x).\end{aligned}$$

The last inequality follows because by antisymmetry and Fubini's theorem

$$\underset{\delta < d(x,y) < \varepsilon}{\int_{B_i \cap S_j} \int_{B_i \cap S_j}} k(x,y)\, d\mu(y)\, d\mu(x) = 0.$$

Therefore it is enough to show that for any "good" ball $B = B(z,r)$ with $z \in \operatorname{spt} \mu$ and $r \in G_z$

$$\lim_{\substack{0 < \delta < \varepsilon \\ \varepsilon \to 0}} \underset{\delta < d(x,y) < \varepsilon}{\int_B \int_{B^c}} |k(x,y)| d\mu(y)\, d\mu(x) = 0,$$

which will follow by the monotone convergence theorem if we show that

$$\int_B \int_{B^c} |k(x,y)| d\mu(y)\, d\mu(x) < \infty. \tag{2.1}$$

Since $B = B(z,r)$ and $r \in G_z$ Lemma 2.1 implies that $\mu(\partial B) = 0$ hence it is enough to show that

$$\int_{B^o} \int_{B^c} |k(x,y)| d\mu(y)\, d\mu(x) < \infty,$$

where B^o stands for the interior of B. For any $x \in B^o$ let $n(x) > 0$ such that

$$2^{n(x)} d(x, \partial B) = 3$$

and $N(x) =$ integer part of $n(x)+1$. Therefore, since $\operatorname{diam}(B) \leq 2$,

$$B(x,2) \setminus B \subset \bigcup_{i=1}^{N(x)} B(x, 2^i d(x,\partial B)) \setminus B(x, 2^{i-1} d(x, \partial B)).$$

Hence for all $x \in B^o$

$$\begin{aligned}
\int_{B(x,2)\setminus B} |k(x,y)| d\mu(y) &\leq \int_{B(x,2)\setminus B} d(x,y)^{-s} d\mu(y) \\
&= \sum_{i=1}^{N(x)} \int_{B(x,2^i d(x,\partial B))\setminus B(x,2^{i-1} d(x,\partial B))} d(x,y)^{-s} d\mu(y) \\
&\leq \sum_{i=1}^{N(x)} \mu(B(x,2^i d(x,\partial B))(2^{i-1} d(x,\partial B))^{-s} d\mu(y) \\
&\lesssim N(x) \lesssim |\log d(x, \partial B)|
\end{aligned}$$

and

$$\begin{aligned}
\int_{B^c} |k(x,y)| d\mu(y) &\lesssim \int_{B(x,2)^c} d(x,y)^{-s} d\mu(y) + |\log d(x,\partial B)| \\
&\lesssim 1 + |\log d(x, \partial B)|.
\end{aligned}$$

Since $r \in G_z$ there exists some $\lambda \in \mathbb{N}$ such that $r \in I'_z(\lambda)$. We write

$$\int_{B(z,r)^o} |\log d(x, \partial B)| d\mu(x) = \int_{B(z,r-\lambda^{-3})^o} |\log d(x, \partial B)| d\mu(x) + \sum_{n=1}^{\infty} \int_{\{x : r-\lambda^{-3n} \leq d(z,x) < r-\lambda^{-3(n+1)}\}} |\log d(x, \partial B)| d\mu(x).$$

Notice that by Lemma 2.1

$$\begin{aligned} \mu(\{x : r - \lambda^{-3n} \leq d(z,x) < r - \lambda^{-3(n+1)}\}) &= \mu_z([r - \lambda^{-3n}, r - \lambda^{-3(n+1)})) \\ &\leq \mu_z([r - \lambda^{-3n}, r + \lambda^{-3n})) \leq \lambda^{-n}. \end{aligned}$$

Therefore,

$$\int_{B(z,r)^o} |\log d(x, \partial B)| d\mu(x) \lesssim 3 \log(\lambda)(r - \lambda^{-3})^s + \sum_{i=1}^{n} \lambda^{-n} |\log(\lambda^{-3(n+1)})| < \infty,$$

and this completes the proof of Theorem 1.2. □

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