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Non-trapping surfaces of revolution with long-living resonances

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We study resonances of surfaces of revolution obtained by removing a disk from a cone and attaching a hyperbolic cusp in its place. These surfaces include ones with non-trapping geodesic flow (every maximally extended non-reflected geodesic is unbounded) and yet infinitely many long-living resonances (resonances with uniformly bounded imaginary part, i.e., decay rate).

1. Introduction

Let $a < 0 < b$, and let (X, g) be the surface of revolution

$$X = \mathbb{R} \times S^1, \quad g = dr^2 + f(r)^2 d\theta^2, \quad f(r) = \begin{cases} 1 + ar & \text{if } r \leq 0, \\ e^{-br} & \text{if } r > 0. \end{cases}$$

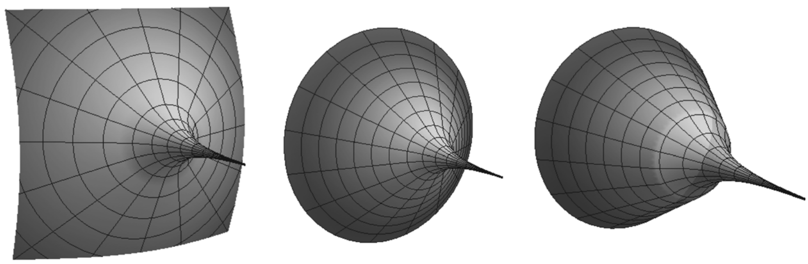
Let Δ_g be the non-negative Laplacian on (X, g). The resolvent $(\Delta_g - \lambda^2)^{-1}$ is holomorphic $L^2(X) \to L^2(X)$ for $\operatorname{Im} \lambda > 0$. Poles of the continuation of its integral kernel from $\{\operatorname{Im} \lambda > 0\}$ to $\{\operatorname{Im} \lambda \leq 0, \operatorname{Re} \lambda > b/2\}$ are called *resonances* (see [Me, DyZw]).

Theorem. *The surface of revolution (X, g) has a sequence of resonances $(\lambda_k)_{k \geq k_0}$ satisfying*

$$\begin{aligned} \operatorname{Re} \lambda_k &= \pi b \frac{k}{\log k} \left(1 + O\left(\frac{\log \log k}{\log k}\right)\right), \\ \operatorname{Im} \lambda_k &= -\frac{bj}{2} + O\left(\frac{1}{\log k}\right), \end{aligned} \tag{1.1}$$

where $j = 1$ if $a + b \neq 0$, and $j = 2$ if $a + b = 0$.

The most interesting case is $a + b = 0$, as illustrated in Figure 1b. Then $f \in C^{1,1}$, so the geodesic flow on (X, g) is well defined and *non-trapping* (see Proposition 2.1) and yet there exists a sequence of *long living* resonances, i.e., a sequence λ_k with $|\operatorname{Im} \lambda_k|$ bounded. This seems to be a new phenomenon (Figure 2).



(a) $a + b < 0$ (b) $a + b = 0$ (c) $a + b > 0$

Figure 1: The surface (X, g) is obtained by attaching a conic end to a hyperbolic cusp. Three geometric pictures arise depending on the relationship between the widths of the cone and cusp. Case (b) is the most interesting, as it exhibits both non-trapping geodesic flow and long-living resonances.

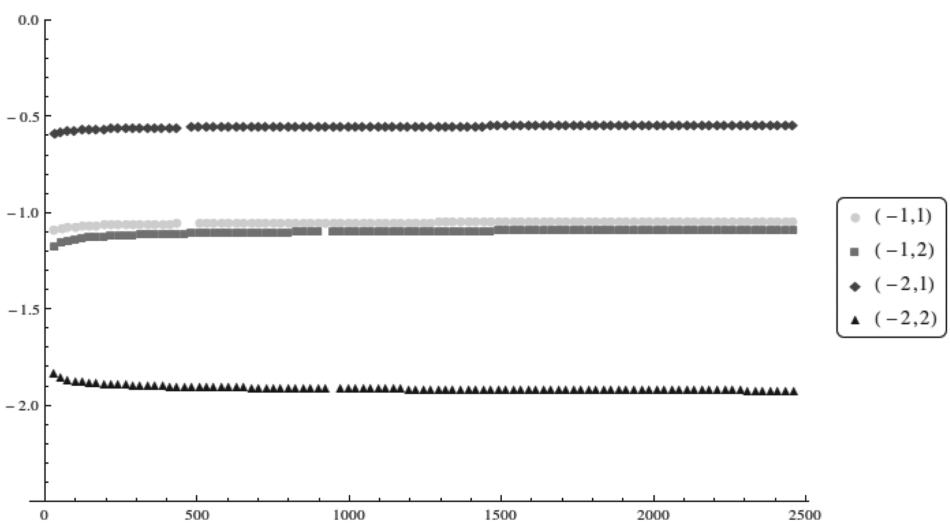


Figure 2: Resonances λ_k are plotted here in the complex plane, with k ranging from 10 to 1000 in steps of 10, for $(a, b) = (-1, 1), (-2, 1), (-1, 2)$, and $(-2, 2)$. They were computed by solving equation (2.7) numerically in Mathematica using FindRoot, initialized with the leading term of (1.1).

For many scattering problems it is known that sequences of long-living resonances are impossible when the geodesic or bicharacteristic flow is non-trapping: such results go back to Lax and Phillips [LaPh], and Vainberg [Va1] for asymptotically Euclidean scattering, and have been recently extended to asymptotically hyperbolic scattering by Vasy [Va2], Melrose, Sá Barreto, and Vasy [MSV], and Wang [Wa]. The result closest to the setting of the theorem is that of [Da], where it is shown that *smooth* non-trapping manifolds with cusp and funnel have no sequences of long-living resonances (many more references can be found in that paper).

Sequences of long-living resonances have been found for a variety of scattering problems, going back to work of Selberg [Se] on finite-volume hyperbolic quotients and Ikawa [Ik] on Euclidean obstacle scattering, but in each case the geodesic or bicharacteristic flow has been trapping; see, e.g., Dyatlov [Dy] for some recent results and many references.

It is interesting to compare the asymptotic formula (1.1) to analogous asymptotics for one-dimensional potential scattering. By work of Regge [Re], Zworski [Zw1] (see also Stepin and Tarasov [StTa]), if $V : \mathbb{R} \to \mathbb{R}$ is supported in $[-L/2, L/2]$, is smooth away from $\pm L/2$ and vanishes to order $j \geq 0$ at $\pm L/2$, then the resonances of $-\partial_r^2 + V(r)$ satisfy

$$\lambda_k = \frac{\pi}{L} k - i \frac{j+2}{L} \log k + O(1) \tag{1.2}$$

in the right-half-plane. Note that, as in our result, the decay rates (values of $|\operatorname{Im} \lambda_k|$) are related to the regularity of the potential, with more regularity giving faster decay.

Sequences of resonances asymptotic to logarithmic curves (as in (1.2)) have been found in other situations where the coefficients of the differential operator are not C^∞: see work of Zworski [Zw2] for scattering by radial potentials and Burq [Bu] for scattering by two convex obstacles, one of which has a corner (and see also similar results by Galkowski on scattering by potentials supported on hypersurfaces in [Ga] and in the references therein). In Section 3, we prove that if we take $b < 0 < a$, and

$$X_1 = (-1/a, \infty) \times S^1, \quad g_1 = dr^2 + f_1(r)^2 d\theta^2, \quad f_1(r) = \begin{cases} 1 + ar & \text{if } r \leq 0, \\ e^{-br} & \text{if } r > 0, \end{cases}$$

then for a fixed Fourier mode the resonances closest to the real axis obey an asymptotic more similar to (1.2), namely

$$\lambda_k = \pi a k - \frac{ija}{2} \log k + O(1).$$

This suggests that the reflected geodesics (which, unlike the transmitted ones, may be trapped) are not by themselves enough to produce sequences of long-living resonances.

In [BaWu], Baskin and Wunsch study another scattering problem with non-smooth coefficients (and in particular trapping of non-transmitted geodesics): they show that there are no long-living resonances for non-trapping Euclidean scattering by a metric perturbation with cone points. For this, they use the result of Melrose and Wunsch [MeWu] that at cone points diffracted singularities for the wave equation are weaker than transmitted ones, as well as the method of Vainberg [Va1] and Tang and Zworski [TaZw] which relates resolvent continuation and propagation of singularities. For a general metric $g \in C^{1,\alpha}$, $\alpha > 0$, with only conormal singularities (as in the case of a jump such as we have) de Hoop, Uhlmann, and Vasy [DUV] show that reflected singularities are weaker than transmitted ones: these results lead one to conjecture that there are no long-living resonances for non-trapping Euclidean scattering by a metric perturbation with $C^{1,\alpha}$ jump singularities. When the discontinuities are more severe, as in the case of the transmission obstacle problem (where g itself is discontinuous), long-living resonances and even resonances with $|\operatorname{Im}\lambda_k| \to 0$ have been observed (see Cardoso, Popov, and Vodev [CPV] and references therein): note, however, that in that case, unlike in our $C^{1,1}$ case, the geodesic equation does not have unique solutions.

To prove the theorem we use the fact that, on each Fourier mode in the angular variable, Δ_g is an ordinary differential operator and the integral kernel of $(\Delta_g - \lambda^2)^{-1}$ can be written in terms of Bessel functions. Resonances occur at values of λ satisfying a transcendental equation (see (2.7) below), and we use Bessel function asymptotics to analyze these solutions for a fixed Fourier mode m.

It would be interesting to find out whether increasing the regularity of (X, g) further leads to further increases in $|\operatorname{Im}\lambda_k|$. The results of [Da], that smooth non-trapping manifolds with cusp and funnel have no sequences of long-living resonances, suggest that the answer is yes. To study this, one could modify f on $(-R, 0)$ for some $R > 0$ to make more derivatives of f continuous at 0. Then one could replace the Hankel function asymptotics used in (2.14) by WKB asymptotics (as in e.g., [Ol, Chapter 10, Section 3.1]), and compute further terms of the asymptotic expansion in Lemma 2.4.

Another interesting problem is to find *all* the long-living resonances of (X, g). This would require asymptotics uniform in the Fourier mode m. Such asymptotics for Bessel functions in terms of Airy functions are known (see, e.g., [AbSt, Section 9.3.37]) but the resulting transcendental equation seems difficult to solve.

2. Proof of theorem

Lemma 2.1. *When* $a+b=0$ *the manifold* (X,g) *has a non-trapping geodesic flow.*

Proof. The geodesic equations of motion for this flow are

$$\ddot{r} - f'f(\dot{\theta})^2 = 0, \tag{2.1}$$

$$\ddot{\theta} = 0. \tag{2.2}$$

If $f \in C^{1,1}$, there is a unique solution. Additionally, if $\dot{\theta} = 0$, then $\dot{r} = 0$ is disallowed because that would describe a stationary solution. In this case (2.1) reduces to $\ddot{r} = 0$ supplemented with the condition $\dot{r} \neq 0$, and this forces $r \to \pm\infty$ linearly.

We take a negative and b positive, so f' is always negative and thereby from the first equation of motion (2.1) we find $\ddot{r} \leq 0$. Thus as t increases, r must tend toward either a constant or negative infinity. In the former case, we must also have that $\ddot{r} \to 0$. However, (2.1) would require that $f'f(\dot{\theta})^2 \to 0$ which is impossible if $\dot{\theta}$ does not vanish, so r cannot tend to a constant. Hence r tends to negative infinity and describes a non-trapping geodesic. ☐

In (r,θ) coordinates the Laplacian on (X,g) is given by

$$\Delta_g = -f^{-1}\partial_r f \partial_r - f^{-1}\partial_\theta f^{-1}\partial_\theta = -\partial_r^2 - f'f^{-1}\partial_r - f^{-2}\partial_\theta^2.$$

Let

$$P(m) = -\partial_r^2 - f^{-1}f'\partial_r + f^{-2}m^2.$$

On functions of the form $u(r)\,e^{im\theta}$ the Laplacian acts as follows:

$$\Delta_g u(r)\,e^{im\theta} = P(m)u(r)\,e^{im\theta}.$$

Lemma 2.2. *The outgoing resolvent, defined by*

$$(P(m)-\lambda^2)^{-1}\colon L^2(\mathbb{R}, f(r)dr) \to L^2(\mathbb{R}, f(r)dr), \quad \operatorname{Im}\lambda > 0,$$

is holomorphic on the half-plane $\operatorname{Im}\lambda > 0$*. Its integral kernel continues meromorphically to a covering space of* $\{\lambda \in \mathbb{C}\colon \lambda \neq 0,\ \lambda \neq \pm b/2\}$*.*

Proof. For $\lambda \in \mathbb{C}\setminus\{0\}$ and $m>0$, the general solution to the Helmholtz equation

$$(P(m)-\lambda^2)u = 0 \tag{2.3}$$

is given by

$$u(r) = \begin{cases} c_1 H^{(1)}_{m/a}(\lambda(r+1/a)) + c_2 H^{(2)}_{m/a}(\lambda(r+1/a)) & \text{if } r \le 0,\\ c_3 e^{br/2} I_\nu(\frac{m}{b}e^{br}) + c_4 e^{br/2} K_\nu(\frac{m}{b}e^{br}) & \text{if } r > 0, \end{cases}$$

where $H^{(1)}$ and $H^{(2)}$ are the Hankel functions (as in [AbSt, Section 9.1]), I and K are the modified Bessel functions (as in [AbSt, Section 9.6]), ν is given by

$$\nu := \sqrt{\frac{1}{4} - \frac{\lambda^2}{b^2}} = -i\frac{\lambda}{b}\left(1 + O(\lambda^{-2})\right), \tag{2.4}$$

and $c_1,\ c_2,\ c_3,\ c_4 \in \mathbb{C}$ are taken such that u and u' are continuous at 0. Indeed, for $r \neq 0$ this follows from the differential equations solved by $H^{(1)}$, $H^{(2)}$, I, and K ([AbSt, Sections 9.1.1, 9.6.1]), and the condition at 0 follows from the fact that f has, at worst, a jump singularity there.

By the method of variation of parameters, the inhomogeneous Helmholtz equation

$$(P(m)-\lambda^2)u = v \tag{2.5}$$

is solved by

$$\begin{aligned} u(r) &= \int_{-\infty}^{\infty} R(r,r')v(r')dr',\\ R(r,r') &:= -\psi_1(\max\{r,r'\})\psi_2(\min\{r,r'\})/W(r'), \end{aligned} \tag{2.6}$$

for ψ_1 and ψ_2 linearly independent solutions of (2.3), and $W = \psi_1\psi_2' - \psi_2\psi_1'$ their Wronskian.

The resolvent operator $v \mapsto u$ is bounded on $L^2(\mathbb{R}, f(r)dr)$ for $\operatorname{Im}\lambda > 0$ if and only if

$$\psi_1(r) = e^{br/2} K_\nu \left(\frac{m}{b} e^{br}\right), \qquad r > 0,$$
$$\psi_2(r) = H^{(2)}_{m/a}(\lambda(r + 1/a)), \qquad r < 0,$$

up to an overall constant factor. The condition on ψ_1 is justified by the asymptotics [AbSt, Sections 9.2.3, 9.2.4] ($H^{(1)}$ grows exponentially and $H^{(2)}$ decays exponentially as $r \to -\infty$). The condition on ψ_2 is justified by the asymptotics [AbSt, Sections 9.7.1, 9.7.2] (I grows double exponentially and K decays double exponentially as $r \to \infty$).

Meromorphic continuation now follows from the fact that the Hankel function terms are entire in $\log \lambda$ and the modified Bessel function terms are entire in ν. Poles of $R(r, r')$ occur at the values of λ for which the Wronskian W vanishes, or equivalently for which ψ_1 is a multiple of ψ_2, i.e., for which there is $c \in \mathbb{C}$ such that

$$u(r) = \begin{cases} H^{(2)}_{m/a}\left(\lambda\left(r + \frac{1}{a}\right)\right) & \text{if } r \le 0, \\ c e^{br/2} K_\nu\left(\frac{m}{b} e^{br}\right) & \text{if } r > 0 \end{cases}$$

is continuous along with its first derivative at 0. That is,

$$H^{(2)}_{m/a}\left(\frac{\lambda}{a}\right) = c K_\nu\left(\frac{m}{b}\right),$$
$$\lambda H^{(2)\prime}_{m/a}\left(\frac{\lambda}{a}\right) = c m K_\nu'\left(\frac{m}{b}\right) + \frac{cb}{2} K_\nu\left(\frac{m}{b}\right).$$

Dividing the equations to eliminate c gives

$$\frac{K_\nu'(\frac{m}{b})}{K_\nu(\frac{m}{b})} = \frac{\lambda H^{(2)\prime}_{m/a}(\frac{\lambda}{a})}{m H^{(2)}_{m/a}(\frac{\lambda}{a})} - \frac{b}{2m}. \tag{2.7}$$

It is important to keep track of the branch of $H^{(2)}$: as the range of λ extends from $\{\operatorname{Im}\lambda > 0\}$ to $\{\operatorname{Im}\lambda > 0\} \cup \{|\operatorname{Re}\lambda| > b/2\}$, the range of $\arg(\lambda/a)$ extends from $(-\pi, 0)$ to $(-3\pi/2, \pi/2)$.

Note that no poles (i.e., solutions to (2.7)) can occur when $\operatorname{Im}\lambda > 0$. This follows from the self-adjointness of Δ_g on a suitable domain, but can also be checked directly from the fact that at such a pole we would have $\langle P(m)u, u\rangle_{L^2} = \lambda^2 \|u\|^2_{L^2}$ for the corresponding u, and the left-hand side of this equation is non-negative $\langle P(m)u, u\rangle_{L^2} \geq 0$ via integration by parts, but this is a contradiction since $\lambda^2 \notin [0,\infty)$. This also shows that (2.6) is the unique L^2 solution to (2.5) when $v \in L^2$, $\operatorname{Im}\lambda > 0$.

For $m = 0$, the analysis is similar and slightly simpler: the only change is that $I_\nu(\frac{m}{b}e^{br})$ is replaced by $e^{b\nu r}$ and $K_\nu(\frac{m}{b}e^{br})$ is replaced by $e^{-b\nu r}$. $\square$

Poles of the meromorphic continuation of the integral kernel $R(r,r')$ (i.e., solutions to (2.7)) are called *resonances* of $P(m)$.

Proposition 2.3. *For any fixed $a < 0 < b$, $m > 0$, $\varepsilon > 0$, there are $\lambda_0 > 0, k_0 \in \mathbb{N}$ such that the resonances of $P(m)$ in the region $\{|\lambda| > \lambda_0, |\arg\lambda| < \pi/2 - \varepsilon\}$ form a sequence $(\lambda_k)_{k\geq k_0}$ satisfying*

$$\begin{aligned} \operatorname{Re}\lambda_k &= \pi b \frac{k}{\log k}\left(1 + O\left(\frac{\log\log k}{\log k}\right)\right), \\ \operatorname{Im}\lambda_k &= -\frac{bj}{2} + O\left(\frac{1}{\log k}\right), \end{aligned} \tag{2.8}$$

where $j = 1$ if $a + b \neq 0$, and $j = 2$ if $a + b = 0$.

We begin by using Bessel function asymptotics to simplify (2.7).

Lemma 2.4. *For fixed $a < 0 < b$ and $m > 0$, $\varepsilon > 0$, there are $\lambda_0 > 0$ and a function r_0 such that $\lambda \in \{|\lambda| > \lambda_0, |\arg\lambda| < \pi/2 - \varepsilon\}$ solves (2.7) if and only if*

$$\left(\frac{m}{2b}\right)^{-2\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)} = \frac{c_0}{\nu^j}(1 + r_0), \tag{2.9}$$

where ν is given by (2.4), $j = 1$ if $a + b \neq 0$, and $j = 2$ if $a + b = 0$, and $c_0 \in \mathbb{R}\setminus\{0\}$. Moreover r_0 is holmorphic for $\lambda \in \{|\lambda| > \lambda_0, |\arg\lambda| < \pi/2 - \varepsilon\}$ and is $O(\nu^{-1})$ there.

In the proofs below it will be convenient to write

$$z = m/b, \quad g(\nu) = (z/2)^{-2\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}.$$

Proof of Lemma 2.4. We first write the left-hand side of (2.7) in terms of Gamma functions and remainders which are small for $|\nu|$ large. We recall the definitions [AbSt, Sections 9.6.10, 9.6.2]:

$$I_\nu(z) = \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)}, \quad K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\left[I_{-\nu}(z) - I_\nu(z)\right].$$

Define the remainders R_1, R_2, R_3, and R_4 by the equations

(2.10)

$$I_\nu(z) = \frac{z^\nu}{2^\nu\Gamma(1+\nu)}(1+R_1), \quad I_{-\nu}(z) = \frac{z^{-\nu}}{2^{-\nu}\Gamma(1-\nu)}(1+R_2),$$

(2.11)

$$I'_\nu(z) = \frac{\nu}{z}\frac{z^\nu}{2^\nu\Gamma(1+\nu)}(1+R_3), \quad I'_{-\nu}(z) = -\frac{\nu}{z}\frac{z^{-\nu}}{2^{-\nu}\Gamma(1-\nu)}(1+R_4),$$

and observe that R_1, R_2, R_3, and R_4 are all $O(\nu^{-1})$ for fixed z. For later use, we record the fact that by the recurrence relation [AbSt, Section 9.6.26], we have

$$\begin{aligned}\frac{z^\nu(R_1-R_3)}{2^\nu\Gamma(1+\nu)} &= I_\nu(z) - \frac{z}{\nu}I'_\nu(z) = -\frac{z}{\nu}I_{\nu+1}(z)\\ &= -\frac{z^{\nu+2}}{2^{\nu+1}\nu\Gamma(2+\nu)}\left(1+O\left(\frac{1}{\nu}\right)\right),\end{aligned}$$

so that

$$R_1 - R_3 = -\frac{z^2}{2\nu^2} + O\left(\frac{1}{\nu^3}\right). \tag{2.12}$$

We simplify the resulting formula for K using the Gamma reflection formula [AbSt, Section 6.1.17]:

$$\begin{aligned}K_\nu(z) &= \frac{\pi}{2\sin\pi\nu}\left[\left(\frac{z}{2}\right)^{-\nu}\frac{1+R_2}{\Gamma(1-\nu)} - \left(\frac{z}{2}\right)^{\nu}\frac{1+R_1}{\Gamma(1+\nu)}\right]\\ &= \frac{\Gamma(1-\nu)\Gamma(\nu)}{2}\left[\left(\frac{z}{2}\right)^{-\nu}\frac{1+R_2}{\Gamma(1-\nu)} - \left(\frac{z}{2}\right)^{\nu}\frac{1+R_1}{\nu\,\Gamma(\nu)}\right]\\ &= \frac{1}{2}\left[\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)(1+R_2) + \left(\frac{z}{2}\right)^{\nu}\Gamma(-\nu)(1+R_1)\right].\end{aligned}$$

Similarly

$$K'_\nu(z) = \frac{\nu}{2z}\left[\left(\frac{z}{2}\right)^{\nu}\Gamma(-\nu)(1+R_3) - \left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)(1+R_4)\right].$$

The quotient is

$$\frac{K'_\nu(z)}{K_\nu(z)} = \frac{\nu}{z}\left[\frac{1+R_3-g(\nu)(1+R_4)}{1+R_1+g(\nu)(1+R_2)}\right]. \tag{2.13}$$

On the other hand, by Hankel's asymptotics, the right-hand side of (2.7) has an expansion in powers of ν. Indeed, applying (A.1) with $n = m/a$ and $x = \lambda/a$, and using

$$-i\lambda/b = \nu\left(1 - \frac{1}{8\nu^2} + O(\nu^{-4})\right)$$

(see, (2.4)) we obtain

(2.14)

$$\begin{aligned}\frac{\lambda H^{(2)'}_{m/a}(\frac{\lambda}{a})}{m H^{(2)}_{m/a}(\frac{\lambda}{a})} - \frac{b}{2m} &= \frac{-i\lambda}{m}\left(1 + \frac{a+b}{2i\lambda} + \frac{a^2-4m^2}{8\lambda^2} + O(\lambda^{-3})\right)\\ &= \frac{\nu}{z}\left(1 - \frac{a+b}{2b\nu} + \frac{4m^2-b^2-a^2}{8b^2\nu^2} + O(\nu^{-3})\right).\end{aligned}$$

Plugging (2.13) and (2.14) into (2.7) gives

$$\begin{aligned}&\frac{1+R_3-g(\nu)(1+R_4)}{1+R_1+g(\nu)(1+R_2)} = 1+R_5,\\ &R_5 := -\frac{a+b}{2b\nu} + \frac{4m^2-b^2-a^2}{8b^2\nu^2} + O(\nu^{-3}).\end{aligned}$$

Solving for $g(\nu)$ we find, using the fact that R_2, R_4, and R_5 are each $O(\nu^{-1})$, that

$$g(\nu) = \frac{1+R_3-(1+R_1)(1+R_5)}{(1+R_2)(1+R_5)+1+R_4} = \frac{R_3-R_1-R_5+R_1R_5}{2+O(\nu^{-1})}.$$

Using (2.12) and the formula for R_5, we obtain

$$g(\nu) = \begin{cases} -(a+b+O(\nu^{-1}))/(4b\nu), & a+b \neq 0, \\ -(1+O(\nu^{-1}))/(8\nu^2), & a+b=0. \end{cases}$$

□

Proof of Proposition 2.3. By Stirling's approximation (see, e.g., [AbSt, Section 6.1.37]):

$$\begin{aligned} g(\nu) &= \left(\frac{z}{2}\right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} = \left(\frac{z}{2}\right)^{-2\nu} \frac{\sqrt{2\pi}\nu^{\nu+1/2}e^{-\nu}\left(1+O\left(\frac{1}{\nu}\right)\right)}{\sqrt{2\pi}(-\nu)^{-\nu+1/2}e^{\nu}\left(1+O\left(\frac{1}{\nu}\right)\right)} \\ &= \left(\frac{ez}{2}\right)^{-2\nu} \nu^{\nu+1/2}(-\nu)^{\nu-1/2}\left(1+O\left(\frac{1}{\nu}\right)\right). \end{aligned}$$

Using log to denote the principal branch of the logarithm, we have

$$\log g(\nu) = \left(\nu+\frac{1}{2}\right)\log\nu + \left(\nu-\frac{1}{2}\right)\log(-\nu) - 2\nu\log\frac{ez}{2} + O\left(\frac{1}{\nu}\right).$$

Plugging this into (2.9), we obtain

$$\begin{aligned} &\left(\nu+\frac{1}{2}\right)\log\nu + \left(\nu-\frac{1}{2}\right)\log(-\nu) - 2\nu\log\frac{ez}{2} \\ &\quad = \tilde{c}_0 - j\log\nu - 2\pi i k + O\left(\frac{1}{\nu}\right), \end{aligned}$$

where $\exp(\tilde{c}_0) = c_0$, and $k \in \mathbb{Z}$. With $\tilde{\nu} = i(2\nu + j)/ez$ this becomes

$$\frac{2\pi k}{ez} = \tilde{\nu}\log\tilde{\nu} + r(\tilde{\nu}), \tag{2.15}$$

where $r(\tilde{\nu})$ is holomorphic, independent of k, and bounded as $\operatorname{Re}\tilde{\nu} \to +\infty$ (since $\operatorname{Re}\lambda \to +\infty$ implies $\operatorname{Im}\nu \to -\infty$ and hence $\operatorname{Re}\tilde{\nu} \to +\infty$). Indeed, since $\operatorname{Im}\nu < 0$ (because $\operatorname{Re}\lambda > b/2$), we have $\log(-\nu) = \log\nu + \pi i$ and hence

$$\begin{aligned}
\tilde{c}_0 - 2\pi i k &= 2\nu \log \nu + j \log \nu + \left(\nu - \frac{1}{2}\right) i\pi - 2\nu \log \frac{ez}{2} + O\left(\frac{1}{\nu}\right) \\
&= (2\nu + j)\left(\log \nu + \frac{\pi i}{2} - \log \frac{ez}{2}\right) + j \log \frac{ez}{2} - i\pi \frac{j+1}{2} + O\left(\frac{1}{\nu}\right) \\
&= (2\nu + j)\left(\log(2\nu + j) - \frac{1}{2\nu + j} + \frac{\pi i}{2} - \log ez\right) - i\pi \frac{j+1}{2} \\
&\quad + j \log \frac{ez}{2} + O\left(\frac{1}{\nu}\right) \\
&= (2\nu + j)\left(\log(i(2\nu + j)) - \log ez\right) - i\pi \frac{j+1}{2} + \log \frac{ez}{2} - 1 \\
&\quad + O\left(\frac{1}{\nu}\right) \\
&= (2\nu + j)\left(\log \frac{i(2\nu + j)}{ez}\right) - i\pi \frac{j+1}{2} + \log \frac{ez}{2} - 1 + O\left(\frac{1}{\nu}\right),
\end{aligned}$$

giving (2.15). We will show that if λ_0 and k are large enough, then (2.15) has a unique solution $\tilde{\nu}_k$ corresponding to a λ_k in $\{\lambda \in \mathbb{C}\colon \operatorname{Re} \lambda \geq \lambda_0\}$, and we will compute its asymptotics. Changing variables back to λ will give (2.8).

For this, we use the Lambert W function. Recall that for $\operatorname{Re} \zeta > 0$, the equation

$$\tilde{\nu} \log \tilde{\nu} = \zeta \tag{2.16}$$

has a unique solution with $\operatorname{Re} \log \tilde{\nu} > 1$ and $|\operatorname{Im} \log \tilde{\nu}| < \pi/2$ (and hence a unique solution with $\operatorname{Re} \tilde{\nu} > 1$), and it is given by the principal branch of the Lambert W function (see [CGHJK, Figures 4 and 5]). As $\operatorname{Re} \zeta \to \infty$, it obeys (see [CGHJK, (4.20)])

$$\tilde{\nu}(\zeta) = \frac{\zeta}{\log \zeta}\left(1 + O\left(\frac{\log \log \zeta}{\log \zeta}\right)\right). \tag{2.17}$$

For k sufficiently large, this reduces solving (2.15) with $\operatorname{Re} \tilde{\nu} > 1$ to solving

$$\frac{2\pi k}{ez} = \zeta + r(\tilde{\nu}(\zeta)),$$

for ζ with $\operatorname{Re} \zeta > 0$. But, e.g., [Ol, Chapter 1, Theorem 5.1] guarantees that such a solution exists and is unique, provided k is sufficiently large. We denote the solution by ζ_k, and the corresponding $\tilde{\nu}$ by $\tilde{\nu}_k$.

Then (2.17) becomes

$$\log \tilde{\nu}_k = \log\left(\frac{2\pi k}{ez} + O(1)\right) - \log\log\left(\frac{2\pi k}{ez} + O(1)\right) + O\left(\frac{\log\log\left(\frac{2\pi k}{ez} + O(1)\right)}{\log\left(\frac{2\pi k}{ez} + O(1)\right)}\right),$$

that is to say

$$\tilde{\nu}_k = \frac{2\pi}{ez}\frac{k}{\log k}\left(1 + O\left(\frac{\log\log k}{\log k}\right)\right). \tag{2.18}$$

To obtain more precise asymptotics for $\operatorname{Im}\tilde{\nu}$, we take the imaginary part of (2.15):

$$\operatorname{Re}\tilde{\nu}_k \arg\tilde{\nu}_k + \operatorname{Im}\tilde{\nu}_k \log|\tilde{\nu}_k| = O(1).$$

Note that, since $\operatorname{Re}\tilde{\nu}_k > 0$, both terms on the left-hand side have the same sign. Consequently $\operatorname{Im}\tilde{\nu}_k \log|\tilde{\nu}_k| = O(1)$ and so

$$\operatorname{Im}\tilde{\nu}_k = O\left(\frac{1}{\log k}\right).$$

Recalling that in terms of λ we have

$$\lambda_k = \frac{b\tilde{\nu}_k ez - ibj}{2}\left(1 + O\left(\frac{1}{\tilde{\nu}_k^2}\right)\right),$$

we conclude, using $z = m/b$, that

$$\operatorname{Re}\lambda_k = \frac{bez}{2}\operatorname{Re}\tilde{\nu}_k\left(1 + O\left(\frac{1}{\tilde{\nu}_k^2}\right)\right) = \pi b\frac{k}{\log k}\left(1 + O\left(\frac{\log\log k}{\log k}\right)\right),$$
$$\operatorname{Im}\lambda_k = \left(\frac{bez}{2}\operatorname{Im}\tilde{\nu}_k - \frac{bj}{2}\right)\left(1 + O\left(\frac{1}{\tilde{\nu}_k^2}\right)\right) = -\frac{bj}{2} + O\left(\frac{1}{\log k}\right). \qquad \square$$

This completes the proof of the theorem. To interpret (2.8) as a Weyl asymptotic, put

$$N(\lambda) = \#\left\{k : \lambda_0 \le \operatorname{Re}\lambda_k \le \lambda\right\}.$$

Then, if $W = -\log k$,

$$\lambda_k = \pi b\frac{k}{\log k}\left(1 + r(k)\right) = -\frac{\pi b}{We^W}\left(1 + r\right).$$

We obtain from this that $We^W = -\pi b\,(1+r)\,/\lambda_k$, and thus, by (2.17),

$$\begin{aligned}W &= \log\left(\frac{\pi b}{\lambda_k}(1+r)\right) - \log\left(-\log\left(\frac{\pi b}{\lambda_k}(1+r)\right)\right)\\ &\quad + O\left(\frac{\log\left(-\log\left(\frac{\pi b}{\lambda_k}(1+r)\right)\right)}{\log\left(\frac{\pi b}{\lambda_k}(1+r)\right)}\right)\\ &= -\log\lambda_k + \log\pi b + O(r) - \log\log\lambda_k + O\left(\frac{1}{\log\lambda_k}\right) + O\left(\frac{\log\log\lambda_k}{\log\lambda_k}\right).\end{aligned}$$

From this, we get the result

$$k = e^{-W} = \frac{\lambda_k\log\lambda_k}{\pi b}\left(1 + O\left(\frac{\log\log\lambda_k}{\log\lambda_k}\right)\right),$$

and hence

$$N(\lambda) = \frac{\lambda\log\lambda}{\pi b}\left(1 + O\left(\frac{\log\log\lambda}{\log\lambda}\right)\right).$$

On the other hand,

$$\frac{1}{2\pi}\operatorname{Vol}\{\rho\in\mathbb{R}, r\ge 0\colon \rho^2 + m^2e^{2br}\le\lambda^2\} = \frac{\lambda\log\lambda}{\pi b} + O(\lambda),$$

allowing us to interpret our result as a Weyl asymptotic for $N(\lambda)$: note this is the same as the asymptotic obeyed by the eigenvalues of $-\partial_r^2 + m^2e^{2br}$ on $(0,\infty)$: see, e.g., [Ti, (7.4.2)].

3. Non-trapping surfaces with funnel

If instead $b < 0 < a$ and (X_1, g_1) is given by

$$X_1 = (-1/a,\infty)\times S^1, \quad g_1 = dr^2 + f_1(r)^2d\theta^2, \quad f_1(r) = \begin{cases}1 + ar & \text{if } r\le 0,\\ e^{-br} & \text{if } r > 0,\end{cases}$$

then (X_1, g_1) is a surface of revolution with a cone point and a funnel. In the case $a = 1$ the cone point disappears and we have a surface of revolution with a disk and a funnel. Let

$$P_1(m) = -\partial_r^2 - f_1^{-1}f_1'\partial_r + f_1^{-2}m^2.$$

On functions of the form $u(r)\,e^{im\theta}$ the Laplacian acts as

$$\Delta_{g_1}u(r)\,e^{im\theta} = P_1(m)u(r)\,e^{im\theta}.$$

Proposition 3.1. *For any $m > 0$ there are $\lambda_0 > 0, k_0 \in \mathbb{N}$ such that the resonances of $P_1(m)$ in the region $\{\operatorname{Re}\lambda \geq \lambda_0\}$ form a sequence $(\lambda_k)_{k\geq k_0}$ satisfying*

$$\lambda_k = \pi a k - \frac{ija}{2}\log k + O(1)$$

where $j = 1$ if $a + b \neq 0$, and $j = 2$ if $a + b = 0$.

Proof. Arguing as in the proof of Proposition 2.2, with the difference that the asymptotics [AbSt, Sections 9.2.3, 9.2.4] for the Hankel functions must now be replaced by [AbSt, Sections 9.1.7, 9.1.8, 9.1.9], we find that resonances are given by solutions to

$$\frac{I'_\nu(\frac{m}{b})}{I_\nu(\frac{m}{b})} = \frac{\lambda J'_{m/a}(\frac{\lambda}{a})}{m J_{m/a}(\frac{\lambda}{a})} - \frac{b}{2m}, \tag{3.1}$$

where now (2.4) is replaced by

$$\nu = \sqrt{\frac{1}{4} - \frac{\lambda^2}{b^2}} = i\frac{\lambda}{b}\left(1 - \frac{b^2}{8\lambda^2} + O\left(\frac{1}{\lambda^4}\right)\right),$$

so that again $\operatorname{Re}\nu > 0$ when $\operatorname{Im}\lambda > 0$. (We are using here the Friedrichs extension of the Laplacian at the cone point — see, e.g., [MeWu, (3.3)].)

Using (2.10)–(2.12), the left-hand side of (3.1) becomes

$$\frac{I'_\nu(z)}{I_\nu(z)} = \frac{\nu}{z}\frac{(1+R_3)}{(1+R_1)} = \frac{\nu}{z} + \frac{z}{2\nu} + O\left(\frac{1}{\nu^2}\right),$$

giving

$$\frac{I'_\nu(\frac{m}{b})}{I_\nu(\frac{m}{b})} = \frac{i\lambda}{m} - \frac{i(b^2+4m^2)}{8m\lambda} + O\left(\frac{1}{\lambda^2}\right),$$

so (3.1) becomes

$$\frac{J'_{m/a}(\frac{\lambda}{a})}{J_{m/a}(\frac{\lambda}{a})} = i + \frac{b}{2\lambda} - \frac{i(b^2+4m^2)}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right) =: i + j_0.$$

Plugging this j_0 into (A.2), with $x = \lambda/a$ and $n = m/a$, and simplifying gives

$$e^{-2i\chi} = \begin{cases} (i(a+b) + O(\lambda^{-1}))/(4\lambda), & a+b \neq 0, \\ (a^2 + O(\lambda^{-1}))/(8\lambda^2), & a+b = 0, \end{cases}$$

where $\chi = \frac{\lambda}{a} - \frac{\pi m}{2a} - \frac{\pi}{4}$. As in the previous section, we write

$$e^{-2i\chi} = c_0 \lambda^{-j}(1 + O(\lambda^{-1})).$$

We solve for λ by taking the log of both sides:

$$\begin{aligned} -2i\chi - 2\pi ki &= \log c_0 - j\log\lambda + O(\lambda^{-1}), \\ -2i\left(\frac{\lambda}{a} - \frac{\pi m}{2a} - \frac{\pi}{4}\right) - 2\pi ki &= \log c_0 - j\log\lambda + O(\lambda^{-1}), \\ \frac{2i}{a}\lambda - j\log\lambda &= -2\pi ki + O(1). \end{aligned}$$

Substituting $\tilde{\lambda} = \frac{-2i}{ja}\lambda$, we get

$$\begin{aligned} \tilde{\lambda} + \log\frac{\tilde{\lambda} jai}{2} &= \frac{2\pi ki}{j} + O(1), \\ \tilde{\lambda} + \log\tilde{\lambda} &= \frac{2\pi ik}{j} + O(1). \end{aligned}$$

Using the Lambert W function to solve this (as we solved (2.15) above), we see that

$$\tilde{\lambda}_k = -\frac{2\pi ik}{j} - \log k + O(1), \quad \lambda_k = \pi ak - \frac{ija}{2}\log k + O(1). \qquad \square$$

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Appendix. Hankel's asymptotics for Bessel functions

Here we collect some consequences of Hankel's asymptotics for $H^{(2)}$ and J. Let $n \in \mathbb{C}$ be fixed. For x in the universal cover of $\mathbb{C} \setminus \{0\}$, as $|x| \to \infty$

with $\arg x$ varying in a compact subset of $(-2\pi,\pi)$, by Hankel's asymptotics [AbSt, Section 9.2.8–16] we have

$$\frac{H_n^{(2)\prime}(x)}{H_n^{(2)}(x)} = \frac{-iR(n,x) - S(n,x)}{P(n,x) - iQ(n,x)},$$

where

$$P(n,x) = 1 - \frac{(4n^2-1)(4n^2-9)}{128x^2} + O(x^{-4}),\quad Q(n,x) = \frac{4n^2-1}{8x} + O(x^{-3}),$$
$$R(n,x) = 1 - \frac{(4n^2-1)(4n^2+15)}{128x^2} + O(x^{-4}),\quad S(n,x) = \frac{4n^2+3}{8x} + O(x^{-3}).$$

Consequently,

$$\text{(A.1)}\qquad \begin{aligned}\frac{H_n^{(2)\prime}(x)}{H_n^{(2)}(x)} &= \frac{-i - \frac{4n^2+3}{8x} + i\frac{(4n^2-1)(4n^2+15)}{128x^2} + O(x^{-3})}{1 - i\frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)}{128x^2} + O(x^{-3})} \\ &= -i - \frac{1}{2x} + i\frac{4n^2-1}{8x^2} + O(x^{-3}).\end{aligned}$$

On the other hand, for $x \in \mathbb{C}$, as $|x| \to \infty$ with $\arg x$ varying in a compact subset of $(-\pi,\pi)$, by Hankel's asymptotics [AbSt, Sections 9.2.5, 9.2.11] we have

$$\frac{J_n'(x)}{J_n(x)} = \frac{-R\sin\chi - S\cos\chi}{P\cos\chi - Q\sin\chi},$$

with P, Q, R, S as above and $\chi = x - (2n+1)\pi/4$. Rearranging terms we find that

$$e^{-2i\chi} = \frac{\frac{J_n'(x)}{J_n(x)}(Q - Pi) - R - Si}{\frac{J_n'(x)}{J_n(x)}(Q + Pi) - R + Si}.$$

Now if $J_n'(x)/J_n(x) = i + j_0$, with $j_0 = O(x^{-1})$, then

$$\frac{J_n'(x)}{J_n(x)}(Q + Pi) - R + Si = -2 + O(x^{-1})$$

and

$$\begin{aligned}\frac{J_n'(x)}{J_n(x)}(Q - Pi) - R - Si = \frac{-i}{2x} &+ \frac{3(4n^2-1)}{16x^2} \\ &+ j_0\left(-i + \frac{4n^2-1}{8x}\right) + O(x^{-3}),\end{aligned}$$

giving

(A.2)
$$e^{-2i\chi} = \left(\frac{i}{4x} - \frac{3(4n^2-1)}{32x^2} + j_0\left(\frac{i}{2} - \frac{4n^2-1}{16x}\right) + O(x^{-3})\right)(1 + O(x^{-1})).$$

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