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Exceptional collections on 2-adically uniformized fake projective planes

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We show that there exist exceptional collections of length 3 consisting of line bundles on the three fake projective planes that have a 2-adic uniformization with torsion-free covering group. We also compute the Hochschild cohomology of the right orthogonal of the subcategory of the bounded derived category of coherent sheaves generated by these exceptional collections.

1. Introduction

Mumford showed in the article [12] that discrete cocompact torsion-free subgroups $\Gamma \subset \mathrm{PGL}_3(\mathbb{Q}_2)$ which act transitively on the vertices of the Bruhat–Tits building of $\mathrm{PGL}_{3,\mathbb{Q}_2}$ give rise to so-called fake projective planes — surfaces of general type with the same Betti numbers as $\mathbb{P}^2$ — and he gave one example of such a subgroup. All such subgroups were classified by Cartwright et al. [4]; there are two others and they give rise to two other fake projective planes [8].

More recently, all fake projective planes over $\mathbb{C}$ were classified by Prasad and Yeung [14] and Cartwright and Steger [5]. However, from the algebro-geometric point of view these surfaces are still not well understood; for example, it is still not known whether Bloch's conjecture on zero cycles on surfaces with $p_g = 0$ holds for any of these surfaces. Another question about surfaces of general type with $p_g = 0$ that has arisen very recently is the existence of exceptional collections of maximal possible length in their bounded derived categories of coherent sheaves. The first such example was found by Böhning et al. [3] and subsequently several other examples have been found. In the article [7] of Galkin et al. conjecture that such exceptional collections exist on all fake projective planes admitting a cube root of the canonical bundle. The following is a consequence of the main result of this paper, Theorem 4.1.

Theorem. *Let M be a fake projective plane having a 2-adic uniformization with a torsion-free covering group. There is an exceptional collection of length 3 in $\mathcal{D}^b(M)$ consisting of line bundles.*

We note that 3 is the smallest possible length of an exceptional collection in $\mathcal{D}^b(X)$, for X any smooth projective variety, so that the right orthogonal $\mathcal{A}$ to the exceptional collection is a quasi-phantom subcategory of $\mathcal{D}^b(X)$, i.e., such that the Hochschild homology $HH_\bullet(\mathcal{A}) = 0$. Using a spectral sequence recently constructed by Kuznetsov [11] which has a very simple form in our setting, we are also able to compute the Hochschild cohomology of $\mathcal{A}$. While we do not prove Bloch's conjecture for these surfaces, we formulate a general conjecture, Conjecture 4.2, on the K_0 of quasi-phantom subcategories, which we hope might lead to a proof.

Our method of proof depends crucially on the fact that the fake projective planes with 2-adic uniformizations have natural regular proper models over $\mathrm{Spec}(\mathbb{Z}_2)$. The special fibre in all cases is an explicit irreducible rational surface whose normalization is the blowup of $\mathbb{P}^2_{\mathbb{F}_2}$ along its rational points. The main technical result is the computation of all the cohomology groups of a natural class of line bundles on these fake projective planes, Proposition 3.1. The particular case of this relevant to the construction of exceptional collections is proved by using a Galois theoretic argument and specialization, eventually reducing this to an explicit computation on $\mathbb{P}^2_{\mathbb{F}_2}$.

After the first version of this paper was put on arXiv, we learned from Katzarkov that the authors of [7] had proved their conjecture for six fake projective planes over $\mathbb{C}$, distinct from the ones we have considered, and by different methods. This is included in v2 of [7].

2. Line bundles on fake projective planes

2.1.

We denote by $\mathscr{X}$ the formal scheme over $\mathrm{Spec}\,(\mathbb{Z}_2)$ corresponding to $\mathrm{PGL}_3(\mathbb{Q}_2)$ constructed by Mustafin [13] and Kurihara [9]; the reader may also consult [12] for an exposition in the case we use. The irreducible components of the special fibre of $\mathscr{X}$ are in bijection with the vertices of the Bruhat–Tits building of $\mathrm{PGL}_{3,\mathbb{Q}_2}$ and each of these components is isomorphic to the surface B obtained by blowing up $\mathbb{P}^2_{\mathbb{F}_2}$ at all its $\mathbb{F}_2$-rational points. There is a faithful action of $\mathrm{PGL}_3(\mathbb{Q}_2)$ on $\mathscr{X}$ which is transitive on the irreducible components of the special fibre and the stabilizer of each component is isomorphic to $\mathrm{PGL}_3(\mathbb{Z}_2)$. This action restricts to a faithful action on $\widehat{\Omega}^2$, the two-dimensional Drinfeld upper half space over $\mathbb{Q}_2$, which is the generic fibre (as a rigid analytic space over $\mathbb{Q}_2$) of $\mathscr{X}$. Moreover, $\widehat{\Omega}^2$ is an admissible open subset (in the sense of rigid analytic geometry) of $\mathbb{P}^2_{\mathbb{Q}_2}$ and the $\mathrm{PGL}_3(\mathbb{Q}_2)$ action on it is compatible with this inclusion and the natural action of $\mathrm{PGL}_3(\mathbb{Q}_2)$ on $\mathbb{P}^2_{\mathbb{Q}_2}$.

If Γ is a discrete torsion-free cocompact subgroup of $\mathrm{PGL}_3(\mathbb{Q}_2)$, then one may form the quotient formal scheme $\mathscr{X}/\Gamma$. The dualizing sheaf $\omega_{\mathscr{X}}$ descends to a line bundle on $\mathscr{X}/\Gamma$ which is ample on the special fibre, hence by Grothendieck's existence theorem, $\mathscr{X}/\Gamma$ is the formal completion of a unique regular projective scheme over $\mathrm{Spec}\,(\mathbb{Z}_2)$. If Γ acts transitively on the irreducible components of $\mathscr{X}$, then Mumford shows that the generic fibre M of $\mathscr{M}$, the projective scheme over $\mathrm{Spec}(\mathbb{Z}_2)$ corresponding to $\mathscr{X}/\Gamma$, is a fake projective plane. The special fibre M_0 of $\mathscr{M}$ is an irreducible surface over $\mathbb{F}_2$ whose normalisation is isomorphic to B.

Lemma 2.1. *Let F be any finite unramified extension of $\mathbb{Q}_2$. For all line bundles L on M_F we have $9 \mid c_1(L)^2$. In particular, ω_M does not have a cube root defined over F.*

Proof. Let A_F be the ring of integers of F. Since $\mathscr{M}$ is regular and F is unramified, it follows that $\mathscr{M}_F := \mathscr{M} \otimes_{\mathbb{Z}_2} A_F$ is also regular. Since M_0 is geometrically irreducible, it follows that the restriction map $\mathrm{Pic}(\mathscr{M}_F) \to \mathrm{Pic}(M_F)$ is an isomorphism. In particular, any line bundle L on M_F extends uniquely to a line bundle $\mathscr{L}$ on $\mathscr{M}_F$. Since M is a fake projective plane, $\mathrm{Pic}(M_F)$ modulo its (finite) torsion subgroup is isomorphic to $\mathbb{Z}$. Since $c_1(\omega_M)^2 = 9 \neq 0$, it follows that there exists a positive integers m, n so that $\mathscr{L}^{\otimes m}$ is isomorphic to $\omega_{\mathscr{M}}^{\otimes n}$.

From the computations of [12, p. 238], it follows that the degree of ω_{M_0}, which is the restriction of $\omega_{\mathscr{M}}$ to M_0, on the image of any exceptional divisor in B is 1. Since the degree of $\mathscr{L}$ on the curve must also be an integer, it follows from $\mathscr{L}^{\otimes m} \cong \omega_{\mathscr{M}}^{\otimes n}$ that $m \mid n$, so $9 = c_1(\omega_M)^2 \mid c_1(L)^2$. $\square$

2.2.

We now show that ω_M does have cube roots defined over certain cubic extensions of $\mathbb{Q}_2$. The existence of cube roots over some extension also follows from the classification results of Prasad and Yeung [14], and the basic principle of our proof is the same. However, the argument below is elementary and is essentially immediate from the construction of the groups Γ. More importantly, it also gives precise information about the field of definition of the cube roots.

Let $\overline{\mathbb{Q}}_2$ be an algebraic closure of $\mathbb{Q}_2$. There is a natural surjection $q : \mathrm{SL}_3(\overline{\mathbb{Q}}_2) \to \mathrm{PGL}_3(\overline{\mathbb{Q}}_2)$; we let G denote the group $q^{-1}(\mathrm{PGL}_3(\mathbb{Q}_2))$ so that there is a short exact sequence

$$1 \to \mu_3(\overline{\mathbb{Q}}_2) \to G \to \mathrm{PGL}_3(\mathbb{Q}_2) \to 1.$$

Suppose Γ' is a subgroup of G mapping isomorphically onto Γ by q and so that all elements of Γ' are defined over a finite extension k of $\mathbb{Q}_2$. Then Γ' acts on $\widehat{\Omega}^2 \otimes_{\mathbb{Q}_2} k$ via q and the base change of the action of Γ on $\widehat{\Omega}^2$. We denote by $O(-1)$ the inverse of the standard generator of $\mathrm{Pic}(\mathbb{P}^2_{\mathbb{Q}_2})$ as well as its restriction to $\widehat{\Omega}^2$ and $\widehat{\Omega}^2 \otimes_{\mathbb{Q}_2} k$. Since the line bundle $O(-1)$ on $\mathbb{P}^2_k$ has a natural $\mathrm{SL}_{3,k}$ linearization, the inclusion of Γ' in $\mathrm{SL}_3(k)$ gives rise to a linearization of $O(-1)$ on $\widehat{\Omega}^2 \otimes_{\mathbb{Q}_2} k$, so it descends to a line bundle L on $M_k = (\widehat{\Omega}^2 \otimes_{\mathbb{Q}_2} k)/\Gamma$. Since ω_M is the line bundle on M corresponding to $O(-3)$ with its induced linearization, it follows that $L^{\otimes 3} \cong \omega_{M_k}$.

2.3.

We now check that subgroups Γ' as above exist for all the three fake projective planes and also determine the extensions k corresponding to these subgroups. This requires explicit knowledge of the groups Γ, so we have to consider Mumford's example and the CMSZ examples separately. However, in both cases, Γ is contained in a larger lattice Γ_1 which can be lifted to a lattice Γ_1' in G in a very simple way.

2.3.1. The Mumford lattice. Mumford's lattice Γ is a sublattice of index 21 of the subgroup Γ_1 of $\mathrm{PGL}_3(\mathbb{Q}_2)$ generated by the images of the matrices

$$\sigma = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1+\lambda \\ 0 & 1 & \lambda \end{bmatrix}, \quad \text{and} \quad \rho = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & -\lambda^3/2 \\ 0 & 0 & \lambda^2/2 \end{bmatrix},$$

where λ is a certain element of $\mathbb{Q}_2$ of the form $2u$ with u a unit [12, Section 2]. One sees immediately that $\sigma, \tau \in \mathrm{SL}_3(\mathbb{Q}_2)$ while $\det(\rho) = \lambda^2/2$ has valuation 1. Let μ be a cube root of $\det(\rho)$, $k = \mathbb{Q}_2(\mu)$, and $\rho' = \mu^{-1}\rho$. Clearly $\rho' \in \mathrm{SL}_3(k)$ and the image in $\mathrm{PGL}_3(k)$ of the subgroup Γ_1' of $\mathrm{SL}_3(k)$ generated by σ, τ, and ρ' is equal to Γ_1. Moreover, since k does not contain a primitive cube root of 1, the only scalar matrix in Γ_1 is the identity. It follows that Γ_1' maps isomorphically onto Γ_1. We then let Γ' be the inverse image of Γ in Γ_1'. The three choices for μ give rise to three such subgroups Γ_1'. The three subgroups Γ' which they give rise to are also distinct since, by Lemma 2.1, none of the Γ' can be subgroups of $\mathrm{SL}_3(\mathbb{Q}_2)$.

2.3.2. The CMSZ lattices. The lattices constructed by Cartwright et al. [4, p. 181] are both sublattices of Γ_1, the image of the subgroup of $\mathrm{GL}_3(\mathbb{Q}_2)$ generated by the elements

$$a_3 = \begin{bmatrix} 0 & 0 & -(S-1)/4 \\ 1 & 0 & 1 \\ 0 & 1 & (S-1)/4 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 0 & -1 & -(S-1)/4 \\ 1 & -1 & -(S-5)/4 \\ 0 & 0 & 1 \end{bmatrix},$$

where $S \in \mathbb{Z}_2$ is the square root of -15 which is congruent to 1 modulo 4 [4, p. 182]. Clearly, $s \in \mathrm{SL}_3(\mathbb{Q}_2)$ while $\det(a_3) = (S-1)/4$. Since $(S-1)(S+1) = -16$, it follows that in fact the valuation of $S-1$ is 3 and so $(S-1)/4$ is a uniformizer. By letting k be the extension of $\mathbb{Q}_2$ obtained by adjoining a cube root as above and then modifying a_3, it follows as in the previous case that we get three distinct lifts of Γ (for both choices of Γ).

2.3.3. In each of the cases discussed above, it can be seen that Hom (Γ, μ_3) has order three, so the three lifts that we have constructed are in fact all.

3. Cohomology of line bundles

Henceforth, M denotes any one of the fake projective planes considered earlier. We let K be the Galois closure of any of the cubic extensions k of $\mathbb{Q}_2$ of the previous section, so it is a Galois extension of $\mathbb{Q}_2$ with Galois group S_3, containing the unramified quadratic extension $F = \mathbb{Q}_2(\zeta)$, with $\zeta^2 + \zeta + 1 = 0$; the extension K/F is totally ramified. We will compute the dimensions of the cohomology groups of all line bundles on M_K contained in the subgroup P of $\mathrm{Pic}(M_K)$ generated by the cube roots of ω_{M_K} constructed above and the line bundles of order two coming from characters of Γ in $\mathbb{Q}_2^\times$. Using the explicit description of the groups Γ given in [4, 12] or the figures at the end of [8], one can see that this group is isomorphic to $\mathbb{Z} \times \mathbb{Z}/3 \times (\mathbb{Z}/2)^2$ for the Mumford surface and one of the CMSZ surfaces and to $\mathbb{Z} \times \mathbb{Z}/3$ for the other CMSZ surface. We do not use these computations in the sequel so we do not give the details and the arguments that follow do not depend on a case by case analysis of the surfaces.

We define the degree of L, $\deg(L)$, to be the positive square root of $c_1(L)^2$ if L is ample and its negative otherwise. Since ω_M is ample, in order to compute the cohomology of all line bundles it suffices, by Serre duality, to consider only line bundles L which are ample.

Proposition 3.1. *For any $L \in P$ let $h^i(L)$ denote the dimension of $H^i(M_K, L)$. Let $L \in P$ be ample and let $d = \deg(L)$.*

(1) *If $d \in \{1, 2\}$, then $h^i(L) = 0$ for all i.*

(2) *If $d = 3$, then $h^i(L) = 0$ for $i = 0, 1$ and $h^2(L) = 1$ if $L \cong \omega_M$ else $h^0(L) = 1$ and $h^i(L) = 0$ for $i = 1, 2$.*

(3) *If $d > 3$, then $h^0(L) = (d-1)(d-2)/2$ and $h^i(L) = 0$ for $i = 1, 2$.*

Furthermore, $h^1(L) = 0$ for any $L \in P$.

It seems reasonable to expect that a similar result holds for all line bundles on all fake projective planes.

Proof. Since $c_1(\omega_M)^2 = 9$, we have $c_1(L) \cdot c_1(\omega_M) = 3d$. Moreover, $\chi(O_M) = 1$, so by the Riemann–Roch theorem for surfaces we have $\chi(L) = (d-1)(d-2)/2$.

If $d > 3$ then $L \otimes \omega^{-1}$ is ample so the result in this case follows from the Kodaira vanishing theorem.

We now consider the case $d = 2$ and assume that $h^0(L) > 0$. By the definition of P, L is isomorphic to $L_1^{\otimes 2} \otimes T$, where L_1 is one of the cube roots of ω_{M_K} constructed in Section 2 and T is either a trivial line bundle or has order 2. The action of $\mathrm{Gal}(K/\mathbb{Q}_2)$ on P preserves all the bundles of order 2 and permutes the three cube roots of ω_{M_K}, so it follows that L has two other Galois conjugates, say L' and L'', and $L \otimes L' \otimes L'' \cong \omega_{M_K}^{\otimes 2} \otimes T$.

Let $k/\mathbb{Q}_2$ be the cubic extension over which L is defined and let D be any divisor (defined over k) in the linear system corresponding to L. It follows that D has two Galois conjugates, D' and D'', such that $O(D') \cong L'$ and $O(D'') \cong L''$. Then $C_K := D + D' + D''$ is a Galois invariant divisor in the linear system corresponding to $\omega_{M_K}^{\otimes 2} \otimes T$, so it is the base change of a divisor C on M. Let C_0 be the specialization of C in M_0. Since T corresponds to a character of Γ of order at most 2, the specialization of T is trivial, hence C_0 is a Cartier divisor in the linear system corresponding to $\omega_{M_0}^{\otimes 2}$.

We claim that the Weil divisor associated with C_0 is divisible by 3 in the group of Weil divisors on M_0. It suffices to prove this over $M_{0,\mathbb{F}_4}$, and since $F/\mathbb{Q}_2$ is unramified (so specialization commutes with base change) it is enough to consider the specialization of C_F as a divisor on $M_F \subset \mathscr{M}_R$, where R is the ring of integers in F.

Observe that no prime divisor in the support of D can be preserved by $\mathrm{Gal}(K/F) \cong \mathbb{Z}/3$, since any such divisor would descend to a divisor of degree 1 or 2 defined over F, which is not possible by Lemma 2.1 (since $F/\mathbb{Q}_2$ is unramified). It follows that each prime divisor Z in the support of C_F splits into a sum of three prime divisors over K. We show that the specialization of any such Z has multiplicity three along each component of its support.

Let $\mathscr{Z}$ be the Zariski closure of Z in $\mathscr{M}_R$ and $\widetilde{\mathscr{Z}}$ its normalization. Since Z splits into three components over K, the function field of Z must contain K. Thus, since $\widetilde{\mathscr{Z}}$ is normal, the morphism $\widetilde{\mathscr{Z}} \to \mathrm{Spec}(R)$ factors though $\mathrm{Spec}(S)$, where S is the ring of integers of K. The specialization of $\widetilde{\mathscr{Z}}$ is given by the valuations of a uniformizer of R with respect to the discrete valuation corresponding to the generic point of each irreducible component of the closed fibre. Since K/F is ramified and of degree 3, so a uniformizer in R is (up to a unit) the cube of a uniformizer in S, it follows that each irreducible component of $\widetilde{\mathscr{Z}} \times_R \mathbb{F}_4$ has multiplicity divisible by 3. Since specialization commutes with proper pushforward [6, Proposition 20.3], the same holds for the irreducible components of $\mathscr{Z} \times_R \mathbb{F}_4$, thereby proving the claim.

From the claim, we see that all the irreducible components of C_0 have multiplicity divisible by 3 in the corresponding Weil divisor. By Lemma 3.2, below no such divisor exists. Thus, if $d = 2$, we must have $h^0(L) = 0$.

If $d = 1$ and $h^0(L) \neq 0$, then $h^0(L^{\otimes 2}) \neq 0$. It follows from the $d = 2$ case already considered that this is not possible.

If $d \in \{1, 2\}$, then $L^{\otimes -1} \otimes \omega_{M_K}$ has degree $3 - d \in \{1, 2\}$, so it follows from the above and Serre duality that $h^2(L) = 0$. Since $\chi(L) = 0$, it follows that we also have $h^1(L) = 0$.

If $d = 3$ and $L \cong \omega_M$, then the statements follow since $p_g(M) = q(M) = 0$. Otherwise, $L_0 = L \otimes \omega_M^{-1}$ is a non-trivial torsion line bundle, so by Serre duality $h^2(L) = 0$. Since $\chi(L) = 1$, it follows that $h^0(L) > 0$. Since $L \in P$, the line bundle L_0 corresponds to a non-trivial homomorphism f from Γ into $K^\times$. Let $\pi : M' \to M_K$ be the cover of M_K corresponding to $\mathrm{Ker}(f)$. Then $\pi_*(O_{M'})$ is isomorphic to $\oplus_{i=0}^{e-1}(L_0)^{\otimes i}$, where e is the order of L_0 in $\mathrm{Pic}(M_K)$. From the projection formula, $\pi_*(\omega_{M'})$ is isomorphic to $\oplus_{i=0}^{e-1}\omega_M \otimes (L_0)^{\otimes i}$. We have $\chi(\omega_{M'}) = e\chi(\omega_M)$ and $q(M') = 0$ by Mumford [12, p. 238], so $p_g(M') = e - 1$. It follows that $\sum_{i=0}^{e-1} h^0(\omega_M \otimes (L_0)^{\otimes i}) = e - 1$. Since all summands except for $i = 0$ must be at least 1, it follows that they are all equal to 1. In particular, $h^0(\omega_M \otimes L_0) = h^0(L) = 1$. Since $\chi(L)$ is also 1, it follows that $h^1(L) = 0$.

The last statement follows from the previous claims if L is ample. If L is not ample, then $\omega_{M_K} \otimes L^{-1}$ is ample so the claim follows by Serre duality. □

Lemma 3.2. *There is no (Cartier) divisor C_0 in the linear system on M_0 associated with $\omega_{M_0}^{\otimes 2}$ such that the associated Weil divisor has multiplicity divisible by 3 along each geometric component of its support.*

Proof. Suppose such a divisor C_0 exists. Let $\nu : B \to M_0$ be the normalization morphism and $\pi : B \to \mathbb{P}^2_{\mathbb{F}_2}$ the morphism blowing up all $\mathbb{F}_2$-rational points of $\mathbb{P}^2_{\mathbb{F}_2}$. The morphism ν identifies each exceptional curve E_i with the strict transform of a $\mathbb{F}_2$-rational line F_i in $\mathbb{P}^2_{\mathbb{F}_2}$ in a way that depends on the particular fake projective plane under consideration. According to [12, p. 238], $\nu^*(\omega_{M_0})$ is equal to $O_B(\pi^* 4H - \sum_{i=1}^7 E_i)$, where H is the class of a line in $\mathbb{P}^2_{\mathbb{F}_2}$. It follows that $\widetilde{C} := \nu^*(C_0) = \pi^*(C_1) - \sum_{i=1}^7 2E_i$ (as Cartier divisors) where C_1 is a curve of degree 8 in $\mathbb{P}^2_{\mathbb{F}_2}$ passing through all the $\mathbb{F}_2$-rational points and having multiplicity at least 2 at each such point.

If $\nu(E_i) \subset M_0$ is contained in the support of C_0 for some i, then both E_i and F_i must be contained in the support of $\widetilde{C}$. Moreover, since the multiplicity of this component is divisible by 3, the sum of the multiplicities of E_i and F_i in $\widetilde{C}$ must be divisible by 3.

First suppose that the support of C_0 is not contained in the double point locus of M_0. Then C_1 has a component which is not a rational line and the multiplicity of each such component must be be divisible by 3; let C_1' be the union of all such components (with multiplicity). Since $\deg(C_1) = 8$, it follows that $\deg(C_1')$ is 3 or 6. If $\deg(C_1') = 3$, then C_1' must be a triple (rational) line which is not possible by assumption. If $\deg(C_1') = 6$ then the corresponding reduced curve must be a conic and $C_1'' := C_1 - C_1'$ is either a double line or a union of two distinct lines.

Suppose C_1'' is a double line. Since C_1 must contain all rational points it follows that C_1' must contain at least four rational points. Since a smooth conic over $\mathbb{F}_2$ has only three rational points and a singular (irreducible) conic has only one rational point, it follows that C_1' must be a union of two rational lines which is a contradiction.

Suppose C_1'' is a union of two distinct rational lines, say F_i and F_j. It then follows that both E_i and E_j must have multiplicity at least 2 in $\widetilde{C}$, so the corresponding points p_i and p_j in $\mathbb{P}^2_{\mathbb{F}_2}$ have multiplicity at least 4 in C_1. Since the union of F_i and F_j contains five rational points, C_1' must be a smooth conic. Since a smooth conic contains three rational points, it follows that $(F_i \cup F_j) \cap C_1'$ contains at most one rational point. Since F_i and F_j have multiplicity 1 in C_1, it follows that there is at most one point of multiplicity at least 4 on C_1, a contradiction.

It remains to consider the case that C_0 is contained in the singular locus of M_0, so C_1 is a union of rational lines (with multiplicity). Since every rational point must lie on C_1, C_1 must have at least three irreducible components.

If there are exactly three components, then there must be a (unique) point p contained in all of them since this is the only way that the union of three lines in $\mathbb{P}^2_{\mathbb{F}_2}$ can contain all rational points. Since $\deg(C_1) = 8$, C_1 has multiplicity 8 at p, so the exceptional divisor E_p has multiplicity $6 > 0$, in C. It follows that the strict transform of the corresponding line F_p must also be in the support of $\widetilde{C}$, so F_p must be one of the lines in the support of C_1 and its multiplicity is divisible by 3. If the multiplicity is 3, then since $\deg(C_1) = 8$ one of the other lines must also have multiplicity ≥ 3. But, then the five points on the union of these lines will have multiplicity > 2 in C_1, so the corresponding exceptional divisors must have multiplicity > 0 in C. But this implies that C_1 must contain the five rational lines corresponding to these exceptional divisors which is a contradiction. If the multiplicity of F_p in C_1 is 6 then both the other lines must have multiplicity 1, but then there would exist rational points on C_1 of multiplicity 1 which is not possible. Thus C_1 cannot have three components.

Suppose C_1 has exactly four components. Since all rational points must lie on C_1, one sees that there is only one such configuration (up to automorphisms of $\mathbb{P}^2_{\mathbb{F}_2}$) consisting of the union of all three lines passing through a distinguished point together with one other line F. It follows that exactly one rational point lies on three lines, three rational points lie on two lines each and the remaining three rational points on a single line each. These three lines must have multiplicity at least 2, since the multiplicity of any rational point on C_1 must be at least 2. Furthermore, these three lines intersect F in distinct points, so the rational points p_1, p_2, and p_3 on this line have multiplicity > 2 on C_1. It follows that the exceptional divisors E_{p_1}, E_{p_2}, and E_{p_3} corresponding to these points must be contained in the support of $\widetilde{C}$, so the corresponding lines F_{p_1}, F_{p_2}, and F_{p_3} must be contained in the support of C_1. If F has multiplicity 2, then the multiplicity of C_1 at all p_i, $i = 1, 2, 3$ is exactly 4. But then the multiplicity of each E_{p_i} in C is 2 so the multiplicity of each F_{p_i} in C_1 must be congruent to 1 modulo 3. Since $2 \not\equiv 1 \mod 3$ this is a contradiction.

It follows that F must have multiplicity 1 and so one of the other lines must have multiplicity 3. But then there are at least five points on C_1 with multiplicity at least 3 which implies that C_1 has at least five lines in its support, also a contradiction.

Suppose that there are five lines in the support of C_1 so there are exactly five rational points on C_1 with multiplicity > 2. The union of any three lines contains at least six rational points so at most two of the lines are multiple and the multiplicities must be one line of multiplicity 2 and another of multiplicity 3 or a single line of multiplicity 4.

There is a unique configuration of five lines up to automorphism. In such a configuration, there is one point lying on a single line, four on two lines each and the remaining two lie on three lines. The line F containing the point p which lies on only one line must be multiple, since the multiplicity of each point must be at least two. If the multiplicity is 4, then there is no other multiple component, so there are four points of multiplicity 2 which is not possible.

If the multiplicity of F is 2, then there is another component of multiplicity 3. Then there are still three rational points of multiplicity 2 which is not possible.

If the multiplicity of F is 3, then there is another component of multiplicity 2. Then the multiplicities of the points are $2, 2, 3, 3, 3, 5, 6$ so the multiplicities of the exceptional divisors are $0, 0, 1, 1, 1, 3, 4$. Since the sum of the multiplicity of any exceptional divisor and the line corresponding to it is divisible by 3, it follows that there must be four multiple lines which is a contradiction.

Suppose there are six lines in the support of C_1. Then there are three collinear points which lie on two lines each and the remaining four points lie on three lines each. Also, there must be exactly six rational points with multiplicity at least 3 on C_1. Thus, there are exactly two points in the three element set of points lying on only two lines and a line containing each point of multiplicity 2. The intersection point of these two lines will then have multiplicity 5 and the two other points on them have multiplicity 4. Thus, the multiplicities of the rational points on C_1 must be $2, 3, 3, 3, 4, 4, 5$ which, using the congruence modulo 3 as above, implies that there must be at least three lines of multiplicity > 1 which is a contradiction.

Finally, we consider the case that the support of C_1 is the union of all seven rational lines. Since each rational point lies on three lines and exactly one of the lines, call it F, must be double, the four rational points not on F have multiplicity 3 on C_1. By the congruence argument as before, this implies that there must be four multiple lines, a contradiction. $\square$

4. Exceptional collections

Let X be a smooth projective variety over a field K. A sequence of objects $E_1, E_2, \ldots, E_n$ of $\mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on X, is called an exceptional collection if $\mathrm{Hom}(E_j, E_i[k])$ is non-zero for $j \geq i$ and $k \in \mathbb{Z}$ iff $i = j$ and $k = 0$, in which case it is one dimensional. Galkin et al. have conjectured [7, Conjecture 3.1] that if X is an n-dimensional fake projective space over $\mathbb{C}$ such that the canonical bundle ω_X has an $(n+1)$th root $O_X(-1)$, then the line bundles $O_X, O_X(-1), \ldots, O_X(-n)$ form an exceptional collection. If X is a surface, they observe that to prove this it suffices to show that $H^0(X, O_X(2)) = 0$. This conjecture appears to be difficult to prove in general, but the computations of the previous section lead to the following:

Theorem 4.1. (a) *Let M be a 2-adically uniformized fake projective plane over $\mathbb{Q}_2$ and let $L_1, L_2 \in P$ be line bundles of degree -1 and -2. Then the sequence of line bundles O_{M_K}, L_1, L_2 is an exceptional collection.*

(b) *Let $\mathcal{B} = \langle O_{M_K}, L_1, L_2 \rangle$, the subcategory of $\mathcal{D}^b(X)$ generated by O_{M_K}, L_1, and L_2 and let $\mathcal{A} = \mathcal{B}^{\perp}$. Then $HH_\bullet(\mathcal{A}) = 0$, i.e., $\mathcal{A}$ is a quasiphantom category. Moreover, the dimensions of the vector spaces $HH^t(\mathcal{A})$, $t \geq 0$, are given by the sequence $1, 0, 0, 28, 54, 27, 0, 0, \ldots$. In particular, the product of any two elements of $HH^\bullet(\mathcal{A})$ of positive degree is 0.*

Proof. For ease of notation, we denote O_{M_K} by L_0.

By definition $\mathrm{Hom}(L_j, L_i[k]) = H^k(M_K, L_i \otimes L_j^{\otimes -1})$. If $j > i$, then the degree of $L_i \otimes L_j^{\otimes -1}$ is 1 or 2 so the cohomology groups vanish for all k by Proposition 3.1. If $i = j$, then $L_i \otimes L_j^{\otimes -1} \cong O_{M_K}$, and since $p_g(M) = q(M) = 0$, (a) follows.

The first part of (b) follows from the fact that the Betti numbers $b_i(M)$ are equal to 1 for $i = 1, 2, 4$ and 0 for all other i together with Kuznetsov's additivity theorem for Hochschild homology [10, Theorem 7.3].

To compute the Hochschild cohomology of $\mathcal{A}$ we first compute that of $\mathcal{D}^b(X)$. Recall, for example, from [10], that this is given by

$$HH^t(\mathcal{D}^b(X)) = \bigoplus_{p=0}^{n} H^{t-p}(M, \wedge^p T_M).$$

The only non-zero cohomology group of O_M is in degree 0 where the dimension is 1. We have $H^0(M,T_M)=0$ since M is of general type and $H^1(M,T_M)=0$ since M is (infinitesimally) rigid. By the Hirzebruch–Riemann–Roch theorem, we then have

$$\begin{aligned} h^2(M,T_M) &= \chi(T_M) = \int_M ch(T_M)\cdot td(T_M) \\ &= \int_M (2+c_1(T_M)+(c_1(T_M)^2-2c_2(T_M))/2)\cdot(1+c_1(T_M)/2 \\ &\quad + (c_1(T_M)^2+c_2(T_M))/12) \\ &= \int_M (c_1(T_M)^2+c_2(T_M))/6 + c_1(T_M)^2/2 + (c_1(T_M)^2 \\ &\quad - 2c_2(T_M))/2 = 8, \end{aligned}$$

where we use that $c_1(T_M)^2=9$ and $c_2(T_M)=3$.

The only non-zero cohomology group of $\wedge^2 T_M = \omega_M^{-1}$ is in degree 2. Moreover, $h^2(M,\omega_M^{-1}) = h^0(M,\omega_M^2) = 10$ by Proposition 3.1. Thus, the dimensions of the vector spaces $HH^t(\mathcal{D}^b(X))$, for $t\geq 0$, are given by the sequence $1,0,0,8,10,0,0,\ldots$.

In the article [11], Kuznetsov defines the normal Hochschild cohomology of $\mathcal{B}$ in $\mathcal{D}^b(X)$, denoted by $NHH^\bullet(\mathcal{B},\mathcal{D}^b(X))$, for $\mathcal{B}$ any admissible subcategory of $\mathcal{D}^b(X)$ with X a smooth projective variety, which sits in a distinguished triangle

$$NHH^\bullet(\mathcal{B},\mathcal{D}^b(X)) \to HH^\bullet(\mathcal{D}^b(X)) \to HH^\bullet(\mathcal{A}) \xrightarrow{+1}, \tag{4.1}$$

where $\mathcal{A}=\mathcal{B}^\perp$. Moreover, if $\mathcal{B}$ is generated by an exceptional collection $E_1,E_2,\ldots,E_n$ then he constructs a spectral sequence converging to $NHH^\bullet(\mathcal{B},\mathcal{D}^b(X))$ whose $\mathbf{E}_1^{-p,q}$ term is given by

$$\begin{aligned} \bigoplus_{\substack{1\leq a_0<a_1\cdots<a_p\leq n,\\ k_0+\cdots+k_p=q}} &\operatorname{Ext}^{k_0}(E_{a_0},E_{a_1}) \\ &\otimes\cdots\otimes \operatorname{Ext}^{k_{p-1}}(E_{a_{p-1}},E_{a_p}) \otimes \operatorname{Ext}^{k_p}(E_{a_p},S^{-1}E_{a_0}) \end{aligned}$$

where S^{-1} denotes the inverse of the Serre functor on $\mathcal{D}^b(X)$. The differentials in this spectral sequence are given by (higher) multiplication maps, but in our situation it degenerates for trivial reasons.

We apply the above with $X=M$, so $n=3$ and $(E_1,E_2,E_3)=(L_0,L_1,L_2)$. The only possibilities for p are $0,1,2$ and we see from Proposition 3.1 that to get a non-zero summand in the spectral sequence we must have $k_i = 2$ for $i < p$ and $k_p = 4$ (since $S^{-1}E_{a_0} = E_{a_0} \otimes \omega_M^{-1}[-2]$). We compute each term in the spectral sequence as follows:

- $p = 0$. We have $\mathrm{Ext}^4(E_i, E_i \otimes \omega_M^{-1}[-2]) = H^2(M, \omega_M^{-1})$ which has dimension 10 as we have seen above and all other groups are 0. Thus, we must have $q = 4$ and $\dim(\mathbf{E}_1^{0,4}) = 3 \times 10 = 30$.
- $p = 1$. Then $\dim(\mathrm{Ext}^2(E_i, E_j)) =$ is 3 if $(i, j) = (1, 2)$ or $(2, 3)$ and 6 if $(1, j) = (1, 3)$. On the other hand, $\dim(Ext^4(E_j, S^{-1}E_i)) = 6$ if $(i, j) = (1, 2)$ or $(2, 3)$ and 3 if $(i, j) = (1, 3)$ and all other groups are 0. Thus, we must have $q = 2 + 4 = 6$ and each of the three possibilities for (i, j) contributes a summand of dimension $3 \cdot 6 = 6 \cdot 3 = 18$. It follows that $\dim(\mathbf{E}_1^{-1,6}) = 3 \cdot 18 = 54$.
- $p = 2$. We must have $(a_0, a_1, a_2) = (1, 2, 3)$, so from computations in the previous case we must have $q = 2 + 2 + 4 = 8$ and $\dim(\mathbf{E}_1^{-2,8}) = 3 \cdot 3 \cdot 3 = 27$.

Since $\mathbf{E}_1^{0,4}$, $\mathbf{E}_1^{-1,6}$ and $\mathbf{E}_1^{-2,8}$ are the only non-zero terms in the spectral sequence it follows that it degenerates at $\mathbf{E}_1$. Consequently, the dimension of $NHH^t(\mathcal{B}, \mathcal{D}^b(M))$, for $t \geq 0$, are given by the sequence $0, 0, 0, 0, 30, 54, 27, 0, 0, \ldots$.

From the computations of $HH^\bullet(\mathcal{D}^b(M))$ and $NHH^\bullet(\mathcal{B}, \mathcal{D}^b(M))$ above, it follows from (4.1) that to compute the dimensions of all $HH^t(\mathcal{A})$ it suffices to compute the rank of the map from $HH^4(\mathcal{B}, \mathcal{D}^b(M))$ to $HH^4(\mathcal{D}^b(M))$. Since $E_1 = O_M$, it follows from the case $\mathcal{B} = \langle O_X \rangle$, where X is any smooth projective variety, considered by Kuznetsov [10, Theorem 8.5], and the functoriality of restriction maps on Hochschild cohomology for admissible subcategories, that this map is surjective. □

We note that the dimensions of the Ext groups in our exceptional collection satisfy the same duality with respect to those for the exceptional collection $O_{\mathbb{P}^2}(-2)$, $O_{\mathbb{P}^2}(-1)$, $O_{\mathbb{P}^2}$ on $\mathbb{P}^2$ as discussed by Alexeev and Orlov [1, p. 757] in the case of Burniat surfaces.

As mentioned in Section 1, Bloch's conjecture on zero cycles is still not known for any fake projective plane. Based on standard motivic conjectures, we make the following:

Conjecture 4.2. *Let X be a smooth projective variety over a field k of characteristic zero. If $\mathcal{A} \subset \mathcal{D}^b(X)$ is an admissible subcategory with $HH_\bullet(\mathcal{A}) = 0$ then $K_0(\mathcal{A})$ is a torsion group.*

Although the usual motivic conjectures are notoriously intractable, we hope that the extra structure here might make this more accessible. If true, together with Theorem 4.1 it would clearly imply Bloch's conjecture for the fake projective planes we have considered.

Remark 4.3. Because of the existence of 2-torsion in P for two out of the three 2-adically uniformized fake projective planes, we get more exceptional collections in these cases than was conjectured in [7]. Moreover, this suggests that for general fake projective planes the condition on the existence of cube roots of the canonical bundle might be unnecessary.

Remark 4.4. Recently, Allcock and Kato [2] have found a cocompact lattice Γ in $\mathrm{PGL}_3(\mathbb{Q}_2)$ containing non-trivial torsion such that $\widehat{\Omega}^2/\Gamma$ is still a fake projective plane and have suggested that there might be other examples as well. Our methods do not immediately apply to their example, but the results of Section 2.3 do, so one might expect that a more detailed knowledge of the special fibre could lead to the proof of the conjecture of Galkin et al., in this case as well. Of course, for fake projective planes without any p-adic uniformization completely new methods would be needed.

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