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Log-concavity and symplectic flows

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The Duistermaat-Heckman measure of a Hamiltonian torus action on a symplectic manifold (M, ω) is the push forward of the Liouville measure on M by the momentum map of the action. In this paper we prove the logarithmic concavity of the Duistermaat-Heckman measure of a complexity two Hamiltonian torus action, for which there exists an effective commuting symplectic action of a 2-torus with symplectic orbits. Using this, we show that given a complexity two symplectic torus action satisfying the additional 2-torus action condition, if the fixed point set is non-empty, then it has to be Hamiltonian. This implies a classical result of McDuff: a symplectic S^1-action on a compact connected symplectic 4-manifold is Hamiltonian if and only if it has fixed points.

1. Introduction

One leading problem in the symplectic theory of group actions is to understand the relationship between: (i) a symplectic action being Hamiltonian; and (ii) the existence of fixed points. Frankel [9] proved that on compact connected Kähler manifolds, a 2π-periodic vector field whose flow preserves the Kähler form (equivalently, a Kähler S^1-action) is Hamiltonian if and only if it has a fixed point. This implies that (i) and (ii) are equivalent for any torus T in the Kähler setting.

In the general symplectic setting, consider the symplectic action of a k-dimensional torus on a $2n$-dimensional symplectic manifold. The complexity i of the action is defined to be $n - k$, where $k \leqslant n$. We note that when i increases, it becomes more difficult to deal with a symplectic torus action with non-empty fixed point set. When $i = 0$, a symplectic torus action with a fixed point is Hamiltonian (this is the famous "toric" case) and it is classified by its momentum map image, cf. Delzant [5]. When $i = 1$ and $n = 2$, McDuff [24] proved that if the fixed point set is non-empty, then the action is Hamiltonian. Later, Kim [18] proved that McDuff's result still holds for general complexity one symplectic torus actions. However, McDuff [24] also constructed a counterexample of a non-Hamiltonian symplectic S^1-action on a closed symplectic 6-manifold with non-empty fixed point set. Therefore, when $i = 2$, one must have additional conditions to guarantee that a symplectic torus action is indeed Hamiltonian.

Consider the Hamiltonian action of a torus T with moment map $\mu \colon M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra of T. The *Duistermaat-Heckman measure* on $\mathfrak{t}^*$ is the push-forward of the Liouville measure on M by μ. The Duistermaat-Heckman measure is absolutely continuous with respect to the Lebesgue measure, and its density function [1] is called the *Duistermaat--Heckman function* (see Section 2). In a different direction, Lin [22] proved that given a Hamiltonian torus action with complexity two, if the b_+ of a reduced space equals 1, then its Duistermaat-Heckman function has to be log-concave. In [4], this result was used to show that if a complexity two symplectic torus action has non-empty fixed points set, and if the b_+ of a reduced space is one, then the action has to be Hamiltonian. In the current paper, we introduce another condition which guarantees that a complexity two symplectic torus action with fixed points is Hamiltonian. More precisely, we prove the following theorem.

Theorem 1. *Let T be an $(n-2)$-dimensional torus which acts effectively with non-empty fixed point set on a compact, connected, symplectic $2n$-dimensional manifold (M, ω). Suppose that there is an effective commuting symplectic action of a 2-torus on M whose orbits are symplectic. Then the action of T on (M, ω) is Hamiltonian.*

The existence of a commuting 2-torus action with symplectic orbits is critical for this paper (such actions are described in Section 7). This assumption is nevertheless satisfied in a number of interesting situations. For instance, consider the symplectic action of an $(n-1)$-dimensional torus T^{n-1} on a compact connected symplectic manifold X of dimension $2n$. Let $M = X \times T^2$ be the cartesian product of X and the symplectic two torus, and let T^{n-1} act trivially on the second factor of M. Then the action of T^{n-1} on X extends to M, and there is a commuting action of T^2 with symplectic orbits. Indeed, T^2 acts on the first factor trivially and on the second factor by rotations. Therefore, the category of complexity two symplectic torus actions considered in Theorem 1 can be regarded as a natural generalization of complexity one symplectic torus actions.

The log concavity of the Duistermaat-Heckman function was shown for torus actions on compact Kähler manifolds by W. Graham [12], and for

[1] which is well defined once the normalization of the Lebesgue measure is declared

compact connected group actions on projective varieties, possibly singular, by A. Okounkov [26–28]. However, in the general symplectic category, Karshon [17] constructed a counterexample of a Hamiltonian circle action on a compact six manifold for which the Duistermaat- Heckman function is non-log-concave. Inspired by Karshon's work, recently Lin [22] found a general construction of complexity two Hamiltonian torus actions with non-log concave Duistermaat-Heckman function, in the case when the b_+ of reduced spaces are greater than one.

It is interesting to note that for the complexity two Hamiltonian torus actions considered in Theorem 1, the extra 2-torus condition implies that there is a symplectic 2-torus action on any regular symplectic quotient, which is a 4-dimensional symplectic orbit. Therefore, by the classification of symplectic 2-torus actions on symplectic four manifolds, cf. Pelayo [29] and Duistermaat-Pelayo [7], the smooth parts of these symplectic quotients are Kähler. As a result, we are able to refine the methods used by Graham in the Kähler setting to establish the log-concavity of the Duistermaat-Heckman function. The following result is important in our proof of Theorem 1.

Theorem 2. *Let T be an $(n-2)$-dimensional torus which acts effectively and Hamiltonianly on a connected symplectic $2n$-dimensional manifold (M, ω) with a proper momentum map. Suppose that the list of stabilizer subgroups of the T action on M is finite, and that there is an effective commuting symplectic action of a 2-torus S on M whose orbits are symplectic. Then the Duistermaat-Heckman function DH_T of the Hamiltonian T-action is log-concave, i.e. its logarithm is a concave function.*

We note that Theorem 1 yields a quick alternative proof of the well known results due to McDuff and Kim (Corollary 23, Theorem 22). Moreover, it applies to a more general class of symplectic torus actions, as we explain by examples in Section 6 that the category of complexity two symplectic torus actions considered in Theorem 1 is larger than the category of complexity one torus actions. The literature on the topic discussed in the current paper is extensive. We refer to [8, 11, 20–23, 31, 36] and the references therein for related results.

Structure of the paper. In Section 2 we review the basic elements of symplectic manifolds and torus actions that we need in the remaining of the paper. We prove Theorem 2 in Section 4, and we prove Theorem 1 in Section 5. In Section 6 we explain how our theorems can be used to derive proofs of some well-known results.

2. Duistermaat-Heckman theory preliminaries

Symplectic group actions. Let (M, ω) be a symplectic manifold, i.e. the pair consisting of a smooth manifold M and a symplectic form ω on M (a non-degenerate closed 2-form on M). Let T be a torus, i.e. a compact, connected, commutative Lie group with Lie algebra $\mathfrak{t}$ (T is isomorphic, as a Lie group, to a finite product of circles S^1). Suppose that T acts on (M, ω) symplectically (i.e., by diffeomorphisms which preserve the symplectic form). We denote by $(t, m) \mapsto t \cdot m$ the action $T \times M \to M$ of T on M. Any element $X \in \mathfrak{t}$ generates a vector field X_M on M, called the *infinitesimal generator*, given by $X_M(m) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(tX) \cdot m$, where $\exp\colon \mathfrak{t} \to T$ is the exponential map of Lie theory and $m \in M$. As usual, we write $\iota_{X_M}\omega := \omega(X_M, \cdot) \in \Omega^1(M)$ for the contraction 1-form. The T-action on (M, ω) is said to be *Hamiltonian* if there exists a smooth invariant map $\mu\colon M \to \mathfrak{t}^*$, called the *momentum map*, such that for all $X \in \mathfrak{t}$ we have that

$$(1) \qquad \iota_{X_M}\omega = \mathrm{d}\langle \mu, X \rangle,$$

where $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$ is the duality pairing.

We say that the T-action on M *has fixed points* if

$$(2) \qquad M^T := \{m \in M \mid t \cdot m = m, \text{ for all } t \in T\} \neq \varnothing.$$

The T-action is *effective* if the intersection of all stabilizer subgroups $T_m := \{t \in T \,|\, t \cdot m = m\}$, $m \in M$, is the trivial group. The T-action is *free* if T_m is the trivial group for all points $m \in N$. Finally, the action of T is called *quasifree* if the stabilizer subgroup of every point is connected.

Atiyah-Guillemin-Sternberg convexity. We will need the following refinement of the Atiyah-Guillemin-Sternberg convexity theorem due to Lerman, Meinrenken, Tolman, and Woodward [19], which holds for Hamiltonian torus action with a proper momentum map.

Theorem 3 ([2, 15, 19]). *Consider the Hamiltonian action of a torus T with Lie algebra $\mathfrak{t}$ on a connected $2n$-dimensional symplectic manifold (M, ω). Suppose that $\mu : M \to \mathfrak{t}^*$ is a momentum map for the T action and that μ is proper as a map into a convex open set $V \subset \mathfrak{t}^*$. Then*

a) *the image $\mu(M)$ is convex;*

b) *each fiber of μ is connected;*

c) *if the list of stabilizer subgroups for the T-action is finite, then $\mu(M)$ is a rational polyhedral set.*

Reduced symplectic forms and Duistermaat-Heckman formula. Consider the Hamiltonian action of a torus T on a symplectic manifold (M, ω). Let $a \in \mathfrak{t}^*$ be a regular value of the momentum map $\mu : M \to \mathfrak{t}^*$ of the action. When the action of T on M is not quasi-free, the quotient $M_a := \mu^{-1}(a)/T$ taken at a regular value of the momentum map is not a smooth manifold in general. However, M_a admits a smooth orbifold structure in the sense of Satake [34]. Even though smooth orbifolds are not necessarily smooth manifolds, they carry differential structures such as differential forms, fiber bundles, etc. In fact, the usual definition of symplectic structures extends to the orbifold case. In particular, the restriction of the symplectic form ω to the fiber $\mu^{-1}(a)$ descends to a symplectic form ω_a on the quotient space M_a, cf. Weinstein [37]. We refer to [1, 25] and to [29, Appendix] background on orbifolds. We are ready to state an equivariant version of the celebrated Duistermaat-Heckman theorem.

Theorem 4 (Duistermaat-Heckman [6]). *Consider an effective Hamiltonian action of a k-dimensional torus T^k on a connected, $2n$-dimensional symplectic manifold (M, ω) with a proper momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that there is another symplectic action of 2-torus T^2 on M that commutes with the action of T^k. Then the following hold:*

(a) *at a regular value $a \in \mathfrak{t}^*$ of μ, the Duistermaat-Heckman function is given by*

$$\mathrm{DH}_{T^k}(a) = \int_{M_a} \frac{\omega_a^{n-k}}{(n-k)!},$$

where $M_a := \mu^{-1}(a)/T^k$ is the symplectic quotient, ω_a is the corresponding reduced symplectic form, and M_a has been given the orientation of ω_a^{n-k};

(b) *if $a, a_0 \in \mathfrak{t}$ lie in the same connected component of the set of regular values of the momentum map μ, then there is a T^2-equivariant diffeomorphism $F \colon M_a \longrightarrow M_{a_0}$, where $M_{a_0} := \mu^{-1}(a_0)/T^k$ and $M_a := \mu^{-1}(a)/T^k$. Furthermore, using F to identify M_a with M_{a_0}, the reduced symplectic form on M_a may be identified with $\omega_a = \omega_{a_0} + \langle c, a - a_0 \rangle$, where $c \in \Omega^2(M, \mathfrak{t}^*)$ is a closed $\mathfrak{t}^*$–valued two-form representing the Chern class of the principal torus bundle $\mu^{-1}(a_0) \to M_{a_0}$.*

3. Primitive decomposition of differential forms

Here we assume that (X, ω) is a $2n$-dimensional symplectic orbifold. Let $\Omega^k(X)$ be the space of differential forms of degree k on X. Let $\Omega(X)$ be the space corresponding to the direct sum of all $\Omega^k(X)$, for varying k's.

We note that, in analogy with the case of symplectic manifolds, there are three natural operators on $\Omega(X)$ given as follows:

$$\begin{aligned} &\mathrm{L} : \Omega^k(X) \to \Omega^{k+2}(X), \ \ \alpha \mapsto \omega \wedge \alpha, \\ &\Lambda : \Omega^k(X) \to \Omega^{k-2}(X), \ \ \alpha \mapsto \iota_\pi \alpha, \\ &\mathrm{H} : \Omega^k(X) \to \Omega^k(X), \ \ \alpha \mapsto (n-k)\alpha, \end{aligned}$$

where $\pi = \omega^{-1}$ is the Poisson bi-vector induced by the symplectic form ω. These operators satisfy the following bracket relations.

$$[\Lambda, \mathrm{L}] = \mathrm{H}, \quad [\mathrm{H}, \Lambda] = 2\Lambda, \quad [\mathrm{H}, \mathrm{L}] = -2\mathrm{L}.$$

Therefore they define a representation of the Lie algebra sl(2) on $\Omega(X)$. A primitive differential form in $\Omega(X)$ is by definition a highest weight vector in the sl_2-module $\Omega(X)$. Equivalently, for any integer $0 \leqslant k \leqslant n$, we say that a differential k-form α on X is primitive if and only if $\omega^{n-k+1} \wedge \alpha = 0$. Although the sl_2-module $\Omega(X)$ is infinite dimensional, there are only finitely many eigenvalues of H. The sl_2-modules of this type is studied in great details in [39]. The following result is an immediate consequence of [39, Corollary 2.6].

Lemma 5. *A differential form $\alpha \in \Omega^k(X)$ admits a unique primitive decomposition*

$$\alpha = \sum_{r \geqslant \max(k-n,0)} \frac{\mathrm{L}^r}{r!} \beta_{k-2r}, \tag{3}$$

where β_{k-2r} is a primitive form of degree $k-2r$.

4. Proof of Theorem 2

Toric fibers and reduction ingredients.

Lemma 6. *Let a torus T with Lie algebra $\mathfrak{t}$ act on a connected symplectic manifold (M, ω) in a Hamiltonian fashion with momentum map $\mu : M \to \mathfrak{t}^*$.*

Suppose that the fixed point set of the T action is non-empty, and that there is another action of a torus S on M which is symplectic and commutes with the action of T. Then for any $a \in \mathfrak{t}^*$, the action of S preserves the level set $\mu^{-1}(a)$ of the momentum map.

Proof. Let X and Y be two vectors in the Lie algebra of S and T respectively, and let X_M and Y_M be the vector fields on M induced by the infinitesimal action of X and Y respectively. Let $\mathcal{L}_{X_M}\mu^Y$ denote the Lie derivative of $\mu^Y := \langle \mu, Y \rangle \colon M \to \mathbb{R}$ with respect to X_M. Let $[\cdot, \cdot]$ denote the Lie bracket on vector fields. It suffices to show that $\mathcal{L}_{X_M}\mu^Y = 0$.

Since the action of T and S commute, we have that $[X_M, Y_M] = 0$. By the Cartan identities, we have

$$\begin{aligned} 0 &= \iota_{[X_M,Y_M]}\omega = \mathcal{L}_{X_M}\iota_{Y_M}\omega - \iota_{Y_M}\mathcal{L}_{X_M}\omega \\ &= \mathcal{L}_{X_M}\iota_{Y_M}\omega \quad \text{(because the action of } S \text{ is symplectic.)} \\ &= \mathcal{L}_{X_M}(\mathrm{d}\mu^Y) = \mathrm{d}\mathcal{L}_{X_M}\mu^Y. \end{aligned}$$

This proves that $\mathcal{L}_{X_M}\mu^Y$ must be a constant. On the other hand, we have

$$\mathcal{L}_{X_M}\mu^Y = \iota_{X_M}\mathrm{d}\mu^Y = \iota_{X_M}\iota_{Y_M}\omega = \omega(Y_M, X_M).$$

Since the fixed point set of the action of T is non-empty, there must exist a point $m \in M$ such that Y_M vanishes at m. Thus $(\mathcal{L}_{X_M}\mu^Y)(m) = 0$, and therefore $\mathcal{L}_{X_M}\mu^Y$ is identically zero on M. □

Remark 7. It follows from the above argument that if the action of T is symplectic with a generalized momentum map, cf. Definition 16, and if the fixed point set is non-empty, the assertion of Lemma 6 still holds.

Lemma 8. *Let an $(n-2)$-dimensional torus T^{n-2} with Lie algebra $\mathfrak{t}$ act effectively on a connected symplectic manifold (M, ω) in a Hamiltonian fashion with a proper momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that there is an effective commuting T^2-action on (M, ω) which has a symplectic orbit. Then for any regular value $a \in \mathfrak{t}^*$, the symplectic quotient $M_a := \mu^{-1}(a)/T^{n-2}$ inherits an effective symplectic T^2-action with symplectic orbits.*

Proof. Let a be a regular value of the momentum map $\mu \colon M \to \mathfrak{t}^*$. Then the symplectic quotient M_a is a four dimensional symplectic orbifold. By assumption, the T^2-action on M has a symplectic orbit. Hence all the orbits of the T^2-action on M are symplectic [29, Corollary 2.2.4]. By Lemma 6, the T^2-action preserves the level set $\mu^{-1}(a)$, and thus it descends to an action on the reduced space M_a. It follows that the orbits of the induced T^2-action on M_a are symplectic two dimensional orbifolds. Thus, by dimensional considerations, any stabilizer subgroup of the T^2-action on M_a is zero-dimensional. Since these stabilizer subgroups must be closed subgroups of T^2, they must be finite subgroups of T^2. Now let H be the intersection of the all stabilizer subgroups of T^2. Then H is a finite subgroup of T^2 itself. Note that the quotient group T^2/H is a two-dimensional compact, connected commutative Lie group. Hence T^2/H is isomorphic to T^2. Therefore M_a admits an effective action of $T^2 \cong T^2/H$ with symplectic orbits. □

Symplectic T-model of (M,ω). Let (X,σ) be a connected symplectic 4-manifold equipped with an effective symplectic action of a 2-torus T^2 for which the T^2-orbits are symplectic. We give a concise overview of a model of (X,σ), cf. Section 7. Consider the quotient map $\pi : X \to X/T^2$. Choose a base point $x_0 \in X$, and let $p_0 := \pi(x_0)$. For any homotopy class $[\gamma] \in \widetilde{X/T^2}$, i.e., the homotopy class of a loop γ in X/T^2 with base point p_0, denote by $\lambda_\gamma : [0,1] \to X$ the unique horizontal lift of γ, with respect to the flat connection Ω of symplectic orthogonal complements to the tangent spaces to the T-orbits, such that $\lambda_\gamma(0) = x_0$, cf. [29, Lemma 3.4.2]. Then the map

$$\Phi : \widetilde{X/T^2} \times T^2 \to X, \quad ([\gamma], t) \mapsto t \cdot \lambda_\gamma(1)$$

is a smooth covering map and it induces a T^2-equivariant symplectomorphism

$$\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2 \to X.$$

We refer to see Section 7, and in particular Theorem 27 and the remarks below it, for a more thorough description.

Kähler ingredients.

Lemma 9. *Consider an effective symplectic action of a 2-torus T^2 on a connected [2] symplectic 4-manifold (X,σ) with a symplectic orbit. Then X admits a T^2-invariant Kähler structure. It consists of a T^2-invariant complex structure, and a Kähler form equal to σ.*

Proof. If X is compact, the lemma is a particular case of Duistermaat-Pelayo [7, Theorem 1.1]. The proof therein is given by constructing the complex

[2] not necessarily compact

structure on the model $\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2$ of M. In fact, the same proof therein applies even if X is not compact. We review it for completeness.

The orbifold universal covering $\widetilde{X/T^2}$ of the orbisurface X/T^2 may be identified with the Riemann sphere, the Euclidean plane, or the hyperbolic plane, on which the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/T^2)$ of X/T^2 acts by means of orientation preserving isometries, see Thurston [35, Section 5.5]. Provide $\widetilde{X/T^2}$ with the standard complex structure. Let σ^{T^2} be the unique T^2-invariant symplectic form on T^2 defined by $\sigma^{\mathfrak{t}}$ at the beginning of Section 7: the restriction of σ to any T-orbit. Equip T^2 with a T^2-invariant complex structure such that σ^T is equal to a Kähler form. In this way we obtain a T^2-invariant complex structure on $\widetilde{X/T^2} \times_{\pi_1^{\mathrm{orb}}(X/T^2)} T^2$ with symplectic form σ equal to the Kähler form. □

Let an $(n-2)$-dimensional torus T^{n-2} with Lie algebra $\mathfrak{t}$ act effectively on a connected symplectic manifold (M, ω) in a Hamiltonian fashion with a proper momentum map $\mu : M \to \mathfrak{t}^*$. Suppose that a is a regular value of $\mu : M \to \mathfrak{t}^*$. Let M_a^{reg} be the set of smooth points in M_a. Then it follows from Lemma 9 that M_a^{reg} is a smooth manifold that admits a T^2 invariant Kähler structure, which consists of a T^2-invariant complex structure and a T^2-invariant Kähler form ω_a.

Duistermaat-Heckman theorem in a complex setting. The following lemma is a key step in our approach.

Lemma 10. *Consider an effective Hamiltonian action of an $(n-2)$-torus T^{n-2} with Lie algebra $\mathfrak{t}$ on a connected, $2n$-dimensional symplectic manifold (M, ω) with a proper momentum map $\mu\colon M \to \mathfrak{t}^*$. Suppose that there is an effective commuting symplectic T^2-action on M which has a symplectic orbit, that a and b are two regular values of μ which lie in the same connected component of the regular values of the momentum map, and that*

$$M_a := \mu^{-1}(a)/T^{n-2} \ \text{ and } \ M_b := \mu^{-1}(b)/T^{n-2}$$

are symplectic quotients with reduced symplectic structure ω_a and ω_b respectively. Let M_a^{reg} and M_b^{reg} be the complement of the sets of orbifold singularities in M_a and M_b, respectively. Then both M_a^{reg} and M_b^{reg} are T^2-invariant Kähler manifolds as in Lemma 9, whose Kähler forms are ω_a and ω_b respectively. Moreover, there is a diffeomorphism $F : M_a^{\mathrm{reg}} \to M_b^{\mathrm{reg}}$ such that $F^*\omega_b$ is a $(1,1)$-form on the Kähler manifold M_a^{reg}.

Proof. By Lemma 8, both (M_a, ω_a) and (M_b, ω_b) are symplectic four orbifolds that admit an effective symplectic T^2 action with symplectic orbits. The smooth parts M_a^{reg} and M_b^{reg} are symplectic manifolds that admit an effective symplectic action of T^2 with symplectic orbits. It follows from Lemma 9 that $(M_a^{\mathrm{reg}}, \omega_a)$ and $(M_b^{\mathrm{reg}}, \omega_b)$ admit a T^2-invariant complex structure, and that the corresponding Kähler form is equal to ω_a, ω_b, respectively.

By Theorem 4, there exists a T^2-equivariant diffeomorphism from M_a to M_b. Let F be the restriction to M_a^{reg} of this T^2-equivariant diffeomorphism. We will show that $F^*\omega_b$ is a $(1,1)$-form on M_a^{reg} by writing down a local expression of F using holomorphic coordinate charts on M_a^{reg} and M_b^{reg} respectively.

Let $\Sigma_a := M_a^{\mathrm{reg}}/T^2$, $\Sigma_b := M_b^{\mathrm{reg}}/T^2$. Let $\pi_a : M_a^{\mathrm{reg}} \to \Sigma_a$ and $\pi_b : M_b^{\mathrm{reg}} \to \Sigma_b$ be the quotient maps. Choose a base point $x_0 \in M_a^{\mathrm{reg}}$ and a base point $p_0 = \pi_a(p_0) \in \Sigma_a$. Then choose a base point $F(x_0) \in M_b^{\mathrm{reg}}$ and $q_0 = \pi_b(F(x_0)) \in \Sigma_b$. Let $\widetilde{\Sigma_a}$ and $\widetilde{\Sigma_b}$ be the orbifold universal covers of Σ_a and Σ_b, respectively, with the base points chosen above.

Choose Ω_a to be the flat connections given by the ω_a-orthogonal complements to the tangent spaces to the T^2-orbits on M_a^{reg}. Similarly choose Ω_b. As before, for any loop γ in Σ_a based at p_0, we would denote by λ_γ the unique horizontal lifting of γ with respect to Ω_a such that $\lambda_\gamma(0) = x_0$. Then we have two smooth covering maps $\Phi_a : \widetilde{\Sigma_a} \times T^2 \to M_a^{\mathrm{reg}}$ and $\Phi_b : \widetilde{\Sigma_b} \times T^2 \to M_b^{\mathrm{reg}}$. For an arbitrary point $t_0\lambda_{\gamma_0}(1) \in M_a^{\mathrm{reg}}$, where $t_0 \in T^2$, choose an open neighborhood U_a of $[\gamma_0]$ in $\widetilde{\Sigma_a}$, and an open neighborhood V_a of t_0 in T^2, such that the restriction of Φ_a to $U_a \times V_a$ is a diffeomorphism.

Since F is T^2-equivariant, we have that for any $[\gamma] \in U$ and $t \in V$, $F(t\lambda_\gamma(1)) = t \cdot F(\lambda_\gamma(1))$. On the other hand, F induces a diffeomorphism between two 2-dimensional orbifolds

$$\varphi_{ab} : \Sigma_a \to \Sigma_b. \tag{4}$$

These orbifolds can be shown to be good orbifolds [29, Lemma 3.4.1], and hence their orbifold universal covers are smooth manifolds. Then we have an induced diffeomorphism between smooth manifolds $\widetilde{\varphi}_{ab} : \widetilde{\Sigma_a} \to \widetilde{\Sigma_b}$. Note that for any loop γ in Σ_a based at p_0, $\widetilde{\varphi}_{ab}([\gamma]) = [\varphi_{ab}(\gamma)]$, where $\varphi_{ab}(\gamma)$ is a loop based at q_0. Denote by $\lambda_{\widetilde{\varphi}_{ab}(\gamma)}$ its horizontal lift with respect to Ω_b. Then we have $F(\lambda_\gamma(1)) = \tau_\gamma \cdot \lambda_{\widetilde{\varphi}_{ab}(\gamma)}(1)$, for some $\tau_\gamma \in T^2$. In particular, $F(t_0\lambda_{\gamma_0}(1)) = (\tau_{\gamma_0} t_0) \cdot \lambda_{\gamma_0}(1)$.

Now choose an open neighborhood U_b of $[\varphi(\gamma_0)]$ in $\widetilde{\Sigma_b}$, and open neighborhood V_b of $\tau_{\gamma_0} t_0$ in T^2, such that the restriction of Φ_b to $U_b \times V_b$ is a diffeomorphism. Without the loss of generality, we may assume that $F \circ \Phi_a(U_a \times$

$V_a) \subset \Phi_b(U_b \times V_b)$. Then, the map $\Phi_b^{-1} \circ F \circ \Phi_a : U_a \times V_a \to U_b \times V_b$ has the following expression.

$$([\gamma], t) \mapsto ([\widetilde{\varphi}_{ab}(\gamma)],\, \tau_\gamma t).$$

Proposition 10 now follows from the above local expression for F, and how the complex structures and symplectic structures are constructed on M_a and M_b, cf. Lemma 9. □

The Guillemin-Lerman-Sternberg wall crossing formula. Consider the effective Hamiltonian action of a k-torus T on a $2n$-dimensional symplectic manifold (M, ω) with a proper momentum map $\mu : M \to \mathfrak{t}^*$. Assume that the list of stabilizer subgroups of the T action on M is finite. Then by Theorem 3, the image of the momentum map $\Delta := \mu(M)$ is a convex rational polyhedral set. In fact, Δ is a union of convex sub-polyhedral set with the property that the interiors of the sub-polyhedral sets are disjoint convex open sets and constitute the set of regular values of μ. These convex sub-polyhedral sets are called the *chambers of* Δ.

For any sub-circle S^1 of T^k, and for any connected component $\mathcal{C}$ of the fixed point submanifold of S^1, the image of $\mathcal{C}$ under the momentum map μ is called an $(k-1)$-dimensional wall, or simply a *wall of* Δ. Moreover, $\mu(\mathcal{C})$ is called an *interior wall of* Δ if $\mu(\mathcal{C})$ is not a subset of the boundary of Δ. Now choose a T^k-invariant inner product on $\mathfrak{t}^*$ to identify $\mathfrak{t}^*$ with $\mathfrak{t}$. Suppose that a is a point on a codimension one interior wall W of Δ, and that v is a normal vector to W such that the line segment $\{a + tv\}$ is transverse to the wall W. For t in a small open interval near 0, write

$$g(t) := \mathrm{DH}_T(a + tv), \tag{5}$$

where DH_T is the Duistermaat-Heckman function of T.

Let S^1 be a circle sitting inside T^k generated by v. Let H be a $(k-1)$-dimensional torus H such that $T^k = S^1 \times H$. Suppose that X is a connected component of the fixed point submanifold of the S^1-action on M such that $\mu(X) = W$. Then X is invariant under the action of H. Moreover, the action of H on X is Hamiltonian. In fact, let $\mathfrak{h}$ be the Lie algebra of H, and let $\pi : \mathfrak{t}^* \to \mathfrak{h}^*$ be the canonical projection map. Then the composite $\mu_H := \pi \circ \mu\,|_X : X \to \mathfrak{h}^*$ is a momentum map for the action of H on X.

By assumption, a is a point in $W = \mu(X)$ such that $\pi(a)$ is a regular value of μ_H. There are two symplectic quotients associated to a: one may reduce M (viewed as an H-space) at $\pi(a)$, and one may reduce X (viewed as a T^k/S^1-space) at a. We denote these symplectic quotients by M_a^* (not to be confused with M_a) and X_a respectively. It follows immediately that

M_a^* inherits a Hamiltonian S^1-action and that X_a is the fixed point submanifold of the S^1-action on M_a^*. Let $\phi : M_a^* \to \mathbb{R}$ be a momentum map for the induced Hamiltonian S^1 action on M_a^*, and let $\alpha \in \mathbb{R}$ be the image of the fixed point submanifold X_a under the momentum map ϕ.

Under the assumption that the symplectic manifold M is compact, Guillemin, Lerman, and Sternberg [13] developed a wall crossing formula which computes the jump in the Duistermaat-Heckman function across the wall $\mu(X)$ of Δ. We claim that the Guillemin-Lerman-Sternberg formula continues to hold in the following non-compact setting.

Theorem 11 (Guillemin-Lerman-Sternberg [13]). *Consider the effective Hamiltonian action of a k-torus T^k on a $2n$-dimensional symplectic manifold (M, ω) with a proper momentum map $\mu : M \to \mathfrak{t}^*$. Assume that the list of stabilizer subgroups of the T action on M is finite. Then the jump of the function $g(t)$ in expression* (5) *when moving across the interior wall W is given by*

$$g_+(t) - g_-(t) \; = \sum \text{volume}(X_a) \left(\prod_{i=1}^{k} \alpha_k^{-1} \right) \frac{t^{k-1}}{(k-1)!},$$

plus an error term of order $\mathcal{O}(t^k)$. Here the α_k's are the weights of the representation of S^1 on the normal bundle of X_a, and the sum is taken over the symplectic quotients of all the connected components X of $M^{S^1} \cap \mu^{-1}(W)$ with respect to the T^k/S^1 action at a.

The essential ingredient in the proof of the Guillemin-Lerman-Sternberg wall crossing formula given in [14, Section 5.3] is an application of the clean version of stationary phase formula on the symplectic quotient M_a^*. We note that for the stationary phase formula to be true, we do not need to assume that the underlying manifold is compact. The formula will hold if the manifold can be covered by finitely many coordinate charts, cf. [16, Ch.14]. In our situation, the symplectic quotient M_a^* may no longer be compact. However, since the momentum map μ is proper, the level set $\phi^{-1}(\alpha)$ is compact. In particular, the clean version of the stationary phase formula holds on an open neighborhood O of $\phi^{-1}(\alpha)$ in M_a^*. (For instance, O can be taken to be the union of finitely many coordinate charts which cover $\phi^{-1}(\alpha)$. Therefore, by applying the stationary phase formula to the open set O, the same argument given in the proof of [14, Theorem 3.5.2] gives Theorem 11.

As a consequence, the same argument given by Graham [12, Section 3] implies the following result.

Proposition 12 ([12]). *Under the same assumption as given in Theorem 11, we have that* $g'_+(0) \leqslant g'_-(0)$.

Hodge-Riemann bilinear relations and last step in proof of Theorem 2. Recall that if V is a vector space and $A \subset V$ is a convex open subset, and if $f \colon A \to \mathbb{R}$ is a Borel measurable map that is positive on A almost everywhere, then we say that f is *log-concave* on A if and only if $\log f$ is a concave function on A. Moreover, if f is smooth on A, and if a is a fixed point in A, then a simple calculation shows that f is log-concave on A if and only if

$$f|''_{\{a+tv\}}(a+tv)\cdot f|_{\{a+tv\}}(a+tv) - (f|'_{\{a+tv\}})^2(a+tv) \leqslant 0, \text{ for all } v \in A$$

where $\{a+tv\}$ is a line segment through a, where t lies in some small open interval containing 0.

Proof of Theorem 2. By Proposition 12, to establish the log-concavity of DH_T on $\mu(M)$, it suffices to show that the restriction of $\log \mathrm{DH}_T$ to each connected component of the set of regular values of μ is concave. Let $\mathcal{C}$ be such a component, let $v \in \mathfrak{t}^*$, and let $\{a+tv\}$ be a line segment in $\mathcal{C}$ passing through a point $a \in \mathcal{C}$, where the parameter t lies in some small open interval containing 0. We need to show that $g(t) := \mathrm{DH}_T(a+tv)$ is log-concave, or equivalently, that

$$g''g - (g')^2 \leqslant 0. \tag{6}$$

Since the point a is arbitrary, it suffices to show that Equation (6) holds at $t=0$. By Theorem 4, at $a+tv \in \mathfrak{t}^*$, the Duistermaat-Heckman function is

$$\mathrm{DH}_T(a+tv) = \int_{M_a} \frac{1}{2}(\omega_a + tc)^2.$$

(We identify M_{a+tv} with M_a). Here $M_a = \mu^{-1}(a)/T$ is the symplectic quotient taken at a and c is a closed two-form depending only on $v \in \mathfrak{t}^*$. To prove (6), we must show that

$$\int_{M_a} c^2 \int_{M_a} \omega_a^2 \;\leqslant\; 2\left(\int_{M_a} c\,\omega_a\right)^2. \tag{7}$$

Consider the primitive decomposition of the two-form c as $c = \gamma + s\omega_a$, where s is a real number, and γ is a primitive two-form. By primitivity, we have that $\gamma \wedge \omega_a = 0$. A simple calculation shows that (7) for γ implies (7) for c. But then (7) becomes

$$\int_{M_a} \gamma^2 \leqslant 0. \tag{8}$$

Note that in a symplectic orbifold the subset of orbifold singularities is of codimension greater than or equal to 2, cf. [3, Prop. III.2.20]. In particular, it is of measure zero with respect to the Liouville measure on the symplectic orbifold. Let M_a^{reg} be the complement of the set of orbifold singularities in M_a. It follows that $\int_{M_a} \gamma^2 = \int_{M_a^{\mathrm{reg}}} \gamma^2$. Note that M_a^{reg} is a connected symplectic four-manifold which admits the effective symplectic action of T^2 with symplectic orbits. By Lemma 9, M_a^{reg} is a Kähler manifold. Moreover, ω_a is the Kähler form on M_a^{reg}. In particular, it must be a $(1,1)$-form.

Let $\langle \cdot, \cdot \rangle$ be the metric on the space of $(1,1)$-forms induced by the Kähler metric, and let $*$ be the Hodge star operator induced by the Kähler metric. By Lemma 10, c must be a $(1,1)$-form. As a result, γ must be a real primitive $(1,1)$-form. Applying Weil's identity, cf. [38, Thm. V.3.16], we get that $\gamma = - * \gamma$. Consequently, we have that

$$\int_{M_a^{\mathrm{reg}}} \gamma^2 = - \int_{M_a^{\mathrm{reg}}} \gamma \wedge * \gamma = - \int_{M_a^{\mathrm{reg}}} \langle \gamma, \gamma \rangle \ \leqslant \ 0.$$

This completes the proof of Theorem 2. □

Log-concavity for complexity one Hamiltonian torus actions. We note that the log-concavity of the Duistermatt-Heckman function for complexity one Hamiltonian torus actions can be derived as an immediate consequence of Theorem 2.

Corollary 13. *Suppose that there is an effective Hamiltonian action of an $(n-1)$-dimensional torus T^{n-1} on a compact, connected, $2n$-dimensional symplectic manifold (M, ω). Then the Duistermaat-Heckman function DH_T of the T^{n-1}-action is log-concave.*

Proof. Consider the following action of T^{n-1} on the $(2n+2)$-dimensional product symplectic manifold $M \times T^2$,

$$g \cdot (m, t) = (g \cdot m, t), \ \text{ for all } \ g \in T^{n-1}, \, m \in M, \, t \in T^2. \tag{9}$$

Let $\phi : M \to \mathfrak{t}^*$ be a momentum map for the T^{n-1} action on M. Then

$$\Phi : M \times T^2 \to \mathfrak{t}^*, \ \ (x, t) \mapsto \phi(x), \ \ \forall \, (x, t) \in M \times T^2$$

is a momentum map for the action of T^{n-1} on $M \times T^2$ described above. Note that there is an obvious commuting T^2 symplectic action on M

$$h \cdot (m, t) = (m, h \cdot t), \text{ for all } h \in T^2,\, m \in M,\, t \in T^2, \tag{10}$$

which is effective and which has symplectic orbits. It follows immediately from Theorem 2 that the Duistermaat-Heckman function of the T^{n-1} action on $M \times T^2$ is log-concave. By construction, it is easy to see that the Duistermaat-Heckman function of the T^{n-1} action on $M \times T^2$ coincides with that of the T^{n-1} action on M. □

Remark 14. Our proof of Theorem 2 and Theorem 1 depend on the classification of symplectic four manifolds with a symplectic T^2 action [29], and the Hodge theoretic arguments initiated by Graham [12]. We note that there is a more direct and elementary proof of Corollary 13 which makes no uses of these more advanced techniques.

Indeed, in view of Proposition 12, it suffices to show that the Duistermaat-Heckman function is log-concave on each connected component of the regular values of the momentum map. Due to Theorem 4, the Duistermaat-Heckman function is a degree one polynomial when restricted to a connected component of the regular values. Then a simple calculation shows that every degree one polynomial must be log-concave.

5. Proof of Theorem 1

Logarithmic concavity of torus valued functions. We first extend the notion of log-concavity to functions defined on a k-dimensional torus $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$. Consider the covering map

$$\exp : \mathbb{R}^k \to T^k, \quad (t_1, \ldots, t_k) \mapsto (\mathrm{e}^{\mathrm{i}2\pi t_1}, \ldots, \mathrm{e}^{\mathrm{i}2\pi t_k}).$$

Definition 15. We say that a map $f : T^k \to (0, \infty)$ is *log-concave at a point* $p \in T^k$ if there is a point $x \in \mathbb{R}^k$ with $\exp(x) = p$ and an open set $V \subset \mathbb{R}^k$ such that $\exp|_V : V \to \exp(V)$ is a diffeomorphism and such that the logarithm of $\tilde{f} := f \circ \exp$ is a concave function on V. We say that f is *log-concave on* T^k if it is log-concave at every point of T^k.

The log-concavity of the function f does not depend on the choice of V. Indeed, suppose that there are two points $x_1, x_2 \in \mathbb{R}^k$ such that $\exp x_1 = \exp x_2 = p$, and two open sets $x_1 \in V_1 \subset \mathbb{R}^k$ and $x_2 \in V_2 \subset \mathbb{R}^k$ such that both $\exp|_{V_1}: V_1 \to \exp(V_1)$ and $\exp|_{V_2}: V_2 \to \exp(V_2)$ are diffeomorphisms. Set $\tilde{f}_i = f \circ \exp|_{V_i}, \ i = 1, 2$. Then there exists $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ such that

$$\tilde{f}_1(t_1, \ldots, t_k) = \tilde{f}_2(t_1 + n_1, \ldots, t_k + n_k), \text{ for all } \in (t_1, \ldots, t_k) \in V_1.$$

Now let $T^k \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z}$ be a k dimensional torus, and λ the standard length form on $S^1 \cong \mathbb{R}/\mathbb{Z}$. Throughout this section we denote by Θ the $\mathfrak{t}^*$-valued form

$$\lambda \oplus \lambda \oplus \cdots \oplus \lambda \tag{11}$$

on T^k. Clearly, Θ is invariant under the T^k-action on T^k by multiplication.

Consider the action of a k-dimensional torus T^k on a $2n$-dimensional symplectic manifold (M, ω). Let $\mathfrak{t}$ be the Lie algebra of T^k, let $\mathfrak{t}^*$ be the dual of $\mathfrak{t}$, and let Θ be the canonical T^k-invariant $\mathfrak{t}^*$-valued one form given as in (11).

Definition 16. We say that $\Phi : M \to T^k$ is a *generalized momentum map* if $\iota_{X_M}\omega = \langle X, \Phi^*\Theta \rangle$, where $X \in \mathfrak{t}$, and X_M is the vector field on M generated by the infinitesimal action of X on M.

Duistermaat-Heckman densities of torus valued momentum maps . The following fact is a straightforward generalization of a well-known result due to D. McDuff concerning the existence of circle valued momentum maps.

Theorem 17 (McDuff [24]). *Let a k-dimensional torus $T^k \cong \mathbb{R}^k/\mathbb{Z}^k$ act symplectically on the compact symplectic manifold (M, ω). Assume that ω represents an integral cohomology class in $\mathrm{H}^2(M, \mathbb{Z})$. Then there exists a T^k-invariant symplectic form ω' on M that admits a T^k-valued momentum map $\Phi: M \to T^k$.*

When ω is integral and $k = 1$, the 1-form $\iota_X\omega$ is also integral and the map in Theorem 17 is defined as follows. Pick a point $m_0 \in M$, let γ_m be an arbitrary smooth path connecting m_0 to m in M, and define the map $\Phi : M \to S^1$ by

$$\Phi(m) := \left[\int_{\gamma_m} \iota_X\omega \right] \in \mathbb{R}/\mathbb{Z}. \tag{12}$$

One can check that Φ is well-defined.

Remark 18. When $k = 1$, a detailed proof of Theorem 17 may be found in Pelayo-Ratiu [30]. The argument given in [30] extends to actions of higher dimensional tori, as pointed out therein. As noted in [24], the usual symplectic quotient construction carries through for generalized momentum maps.

Now we will need the following result, which is Theorem 3.5 in [33].

Theorem 19 (Rochon [33]). *Let a compact connected Lie group G act on a symplectic manifold (M, ω) symplectically. Then the G-action on (M, ω) is Hamiltonian if and only if there exists a symplectic form σ on M such that the G-action on (M, σ) is Hamiltonian.*

It follows from Theorem 19 that in order to show that a symplectic S^1-action on a symplectic manifold (M, ω) is Hamiltonian, it suffices to show it under the assumption that $[\omega]$ is integral, see [4, Remark 2.1].

Definition 20. Suppose that there is an effective symplectic action of a k-dimensional torus T^k on a $2n$-dimensional compact symplectic manifold (M, ω) with a generalized momentum map $\Phi : M \to T^k$. The *Duistermaat-Heckman measure* is the push-froward of the Liouville measure on M by Φ. Its density function is the *Duistermaat-Heckman function.*

Analogous to the case of Hamiltonian actions, we have the following result.

Proposition 21. *Suppose that there is an effective and symplectic action of a k-dimensional torus T^k on a $2n$-dimensional compact symplectic manifold (M, ω) with a generalized momentum map $\Phi : M \to T^k$. Let $(T^k)_{\mathrm{reg}} \subset T^k$ be the set of regular values of Φ. Then*

$$\mathrm{DH}_{T^k}(a) = \int_{M_a} \omega_a^{n-k}, \text{ for all } a \in (T^k)_{\mathrm{reg}},$$

where $M_a := \mu^{-1}(a)/T^k$ is the symplectic quotient taken at a, and ω_a is the reduced symplectic form.

Concluding the proof of Theorem 1.

Proof of Theorem 1. We divide the proof into three steps. Suppose that an $(n-2)$-dimensional torus T^{n-2} acts on a $2n$-dimensional compact connected symplectic manifold (M, ω) with fixed points, and that there is a commuting T^2 symplectic action on M which has a symplectic orbit.

Step 1. By Theorem 17 and Theorem 19, we may assume that there is a generalized momentum map $\Phi : M \to T^{n-2}$. To prove that the action of T^{n-2} is Hamiltonian, it suffices to prove that for any splitting of T^{n-2} of the form $T^{n-2} = S^1 \times H$, where H is a $(n-3)$-dimensional torus, the action of S^1 is Hamiltonian. Let $\pi_1 : T^{n-2} \to S^1$ and $\pi_2 : T^{n-2} \to H$ be the projection maps. Then $\phi := \pi_1 \circ \Phi : M \to S^1$ and $\phi_H := \pi_2 \circ \Phi : M \to H$ are generalized momentum maps with respect to the S^1-action and the H-action on M respectively. Without loss of generality, we may assume that the generalized momentum map ϕ has no local extreme value, cf. [24, Lemma 2]. Let $h \in H$ be a regular value of ϕ_H. Denote by $\mathrm{DH}_{T^{n-2}} : T^{n-2} \to \mathbb{R}$ the Duistermaat-Heckman function of the T^{n-2} action on M. Define

$$f : S^1 \to \mathbb{R}, \quad f(x) := \mathrm{DH}_{T^{n-2}}(x, h).$$

To prove that the action of S^1 is Hamiltonian, it suffices to show that f is a non-constant log-concave function. This is because that if f is log-concave and non-constant, then the generalized momentum map $\phi : M \to S^1$ cannot be surjective, and hence the action of S^1 on M is Hamiltonian.

We will divide the rest of the proof into two steps, and we will use the following notation. Consider the exponential map $\exp : \mathbb{R} \to S^1$, $t \mapsto \mathrm{e}^{\mathrm{i}2\pi t}$. By abuse of notation, for any open connected interval $I = (b, c) \subset \mathbb{R}$ such that $\exp(I) \subsetneq S^1$, we will identify I with its image in S^1 under the exponential map.

Step 2. We first show that if $I_1 = (b_1, c_1) \subsetneq S^1$ is an open connected set consisting of regular values of ϕ, then f is log-concave on I_1. Indeed, suppose that for any $2 \leqslant i \leqslant n-2$, $I_i = (b_i, c_i) \subsetneq S^1$ is an open connected set such that $\mathcal{O} = I_2 \times \cdots \times I_{n-2} \ni h$ consists of regular values of ϕ_H. Then $I_1 \times \mathcal{O} \subsetneq T^{n-2}$ is a connected open set that consists of regular values of $\Phi : M \to T^{n-2}$. Moreover, the action of T^{n-2} on $W = \Phi^{-1}(I_1 \times \mathcal{O})$ is Hamiltonian.

Note that $\Phi_W := \Phi|_W$ is a momentum map for the action of T^{n-2} on W. Clearly, Φ_W is a proper map, and the list of stabilizer subgroups of the T^{n-2} action on W is finite. Consequently, the log-concavity of the function f follows from Theorem 2.

Step 3. We show that if $I_1 = (b_1, c_1) \subsetneq S^1$ is a connected open set which contains a unique critical value a of ϕ, then f is a non-constant log-concave function on I_1. We will use the same notation W as in the Step 2. By definition, to show that f is non-constant log-concave on I_1, it suffices to show that $\tilde{f} = f \circ \exp : \mathbb{R} \to \mathbb{R}$ is non-constant and log-concave on the preimage of I_1 in $\mathbb{R}$ under the exponential map exp. Again, by abuse of notation, we would not distinguish I_1 and its pre-image in $\mathbb{R}$, and we will use the same notation a to denote the pre-image of $a \in I_1 \nsubseteq S^1$ in $\exp^{-1}(I_1)$. To see the existence of such interval I_1, we need the assumption that the fixed point set of T^{n-2} is non-empty. Since the fixed point submanifold of the action of T^{n-2} on M is a subset of M^{S^1}, it follows that M^{S^1} is non-empty. Let X be a connected component of M^{S^1}. Choose $a \in S^1$ to be the image of X under the generalized momentum map ϕ. Then a sufficiently small connected open subset that contains a will contain no other critical values of ϕ.

Note that X is a symplectic submanifold of M which is invariant under the action of H, and that there are two symplectic manifolds associated to the point $(a, h) \in T^k = S^1 \times H$: the symplectic quotient of M (viewed as an H-space) at h, and the symplectic quotient of X (viewed as T^k/S^1-space) at (a, h). We will denote these two quotient spaces by M_h and X_h respectively. An immediate calculation shows that the dimension of M_h is six. Since the action of S^1 commutes with that of H, by Lemma 6 and Remark 7, the action of S^1 preserves the level set of the generalized momentum map ϕ_H. Consequently, there is an induced action of S^1 on M_h. Since $\phi : M \to S^1$ has a constant value on any H orbit in M, it descends to a generalized momentum map $\tilde{\phi} : M_h \to S^1$. It is easy to see that X_h consists of the critical points of the generalized momentum map $\tilde{\phi} : M_h \to S^1$. Moreover, since ϕ has no local extreme values in M, it follows that $\tilde{\phi}$ has no local extreme value in M_h. Thus for dimensional reasons, X_h can only be of codimension four or six. Now observe that the commuting symplectic action of T^2 on M descends to a symplectic action on M_h which commutes with the action of S^1 on M^{S^1}. It follows that X_h is invariant under the induced action of T^2 on M_h and so must contain a T^2 orbit. As a result, X_h must have codimension four.

Applying Theorem 11 to the non-compact Hamiltonian manifold W, we get that

$$\tilde{f}_+(a + t) - \tilde{f}_-(a - t) \;=\; \sum_{X_h} \frac{\mathrm{vol}\,(X_h)}{(d-1)! \prod_j \alpha_j} t^{d-1} + \mathcal{O}(t^d).$$

In the above expression, X_h runs over the collection of all connected components of $M_h^{S^1}$ that sit inside $\phi^{-1}(a)$, the α_j's are the weights of the representation of S^1 on the normal bundle of X_h, and d is half of the real dimension of X_h. By our previous work, $d = 2$ for any X_h; moreover, the two non-zero weights of X_h must have opposite signs. So the jump in the derivative is strictly negative, i.e., $\tilde{f}'_+(a) - \tilde{f}'_-(a) < 0$. □

6. Examples and consequences of main results

Applications of Theorems 1 and 2. Our proof of Theorem 1 depends on Lemma 9. We note that the proof of Lemma 9 builds on the classification result of symplectic T^2 action on symplectic four manifolds, cf. [29] and [7], as well as Hodge theoretic methods initiated by Graham [12]. However, we have a more direct and elementary proof in the following special case without using any of these big machineries. This in particular yields a quick proof of the main result of the article [18] (see Section 1 therein).

Theorem 22 (Kim [18]). *Suppose that* (M, ω) *is a compact, connected symplectic* $2n$*-dimensional manifold. Then every effective symplectic action of an* $(n-1)$*-dimensional torus* T^{n-1} *on* (M, ω) *with non-empty fixed point set is Hamiltonian.*

Proof. We prove the equivalent statement that if (N, ω) is $(2n-2)$-dimensional compact, connected symplectic manifold, then every symplectic action of an $(n-2)$-dimensional torus T^{n-2} on (N, ω) with non-empty fixed point set is Hamiltonian.

Indeed, suppose that there's a symplectic action of an $(n-2)$-dimensional torus on a $(2n-2)$-dimensional compact connected symplectic manifold N. By Theorem 17 and Theorem 19, without loss of generality we assume that there is a generalized momentum map $\phi_N : N \to T^{n-2}$. Consider the action of T^{n-2} on the product symplectic manifold $Z := N \times T^2$ as given in Equation (9), where T^2 is equipped with the standard symplectic structure. Define $\Phi : N \times T^2 \to T^{n-2}$, $(x, t) \mapsto \phi_N(x)$ for all $x \in N$, $t \in T^2$. Then Φ is a generalized momentum map for the symplectic action of T^{n-2} on Z. Moreover, there is an effective commuting symplectic action of T^2 on M with symplectic orbits given as in Equation (10). To show that the action of T^{n-2} on N is Hamiltonian, it suffices to show that the action of T^{n-2} on Z is Hamiltonian.

Using the same notation as in the proof of Theorem 1, we explain that we can modify the argument there to give an elementary proof that the action of T^{n-2} on Z is Hamiltonian. We first note that both Step 1 and Step 3 in the proof of Theorem 1 do not use Lemma 9 or Hodge theory, and that the same argument there applies to the present special case. Lemma 9 and Hodge theory are used in Step 2 of the proof of Theorem 1 to show that f is log-concave on I_1. However, in the current situation, we can use the elementary argument that we explained in Remark 14 to establish the log-concavity of the function f on I_1. This yields an elementary proof of Theorem 22. □

The following result [24, Proposition 2] is a special case of Theorem 22.

Corollary 23 (McDuff [24]). *If* (M, ω) *is* 4*-dimensional compact, connected symplectic manifold, then every effective symplectic* S^1*-action on* (M, ω) *with non-empty fixed point set is Hamiltonian.*

Examples. Next we give some more examples to which our theorems apply.

Example 24. The Kodaira-Thurston manifold is the symplectic manifold $\mathrm{KT} := (\mathbb{R}^2 \times T^2)/\mathbb{Z}^2$ (see [32, Section 2.4]). This is the case in Theorem 1 which is "trivial" since $n-2=0$, and hence $T^{n-2} = \{e\}$ (in fact this manifold is non-Kähler, and as such it does not admit any Hamiltonian torus action, of any non-trivial dimension). Nevertheless the KT example serves to illustrate a straightforward case which admits the symplectic transversal symmetry, and it may be used to construct lots of examples satisfying the assumptions of the theorem, eg. $\mathrm{KT} \times S^2$.

Example 25. This example is a generalization of Example 24. Theorem 1 covers an infinite class of symplectic manifolds with Hamiltonian S^1-actions, for instance

$$M := (\mathbb{CP}^1)^2 \times \mathbb{T}^2/\mathbb{Z}_2, \tag{13}$$

where S^1 acts Hamiltonianly on the left factor $(\mathbb{CP}^1)^2$, and $\mathbb{T}^2$ acts symplectically with symplectic orbits on the right factor; the quotient is taken with respect to the natural diagonal action of $\mathbb{Z}_2$ (by the antipodal action on a circle of $\mathbb{T}^2$, and by rotation by 180 degrees about the vertical axes of the sphere $S^2 \simeq \mathbb{CP}^1$), and the action on $(\mathbb{CP}^1)^2 \times \mathbb{T}^2$ descends to an action on M. Any product symplectic form upstairs also descends to the quotient.

Example 26. The $2n$-dimensional symplectic manifolds with $\mathbb{T}^n$-actions in Theorem 1 are a subclass of those classified in [29], which, given that $\mathbb{T}^2$ is 2-dimensional, are of the form we describe next. Let Z be any good $(2n-2)$-dimensional orbifold, and let $\widetilde{\Sigma}$ be its orbifold universal cover. Equip Z with any orbifold symplectic form, and $\widetilde{\Sigma}$ with the induced symplectic form. Equip $\mathbb{T}^2$ with any area form. Equip $\widetilde{Z} \times \mathbb{T}^2$ with the product symplectic form, and the symplectic $\mathbb{T}^2$-action with symplectic orbits by translations on the right most factor. This symplectic $\mathbb{T}^2$-action descends to a symplectic $\mathbb{T}^2$-action with symplectic orbits on

$$M_Z := \widetilde{Z} \times_{\pi_1^{\mathrm{orb}}(Z)} \mathbb{T}^2 \tag{14}$$

considered with the induced product symplectic form. An a priori intractable question is: *give an explicit list of manifolds M_Z which admit a Hamiltonian T^{n-2}-action*. Examples of such M_Z, and hence fitting in the statement of Theorem 1, may be constructed in all dimensions, eg. the manifold in (13). Theorem 1 says that any commuting symplectic $\mathbb{T}^{n-2}$ action on (14) *which has fixed points*, must be Hamiltonian.

7. Appendix: actions with symplectic orbits

This section is a review of [29, Sections 2, 3], with a new remark at the end. Let (X, σ) be a connected symplectic manifold equipped with an effective symplectic action of a torus T for which there is at least one T-orbit which is a $\dim T$-dimensional symplectic submanifold of (X, σ) (this implies that all are). *We do not assume that X is necessarily compact.* Then there exists a unique non-degenerate antisymmetric bilinear form $\sigma^{\mathfrak{t}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ on the Lie algebra $\mathfrak{t}$ of T such that $\sigma_x(u_X(x), v_X(x)) = \sigma^{\mathfrak{t}}(u, v)$, for every $u, v \in \mathfrak{t}$, and every $x \in X$. One can check that the stabilizer subgroup T_x of the T-action at every point $x \in X$ is a finite group.

Orbit space X/T. As usual, X/T denotes the orbit space of the T-action. Let $\pi \colon X \to X/T$ the canonical projection. The space X/T is provided with the maximal topology for which π is continuous; this topology is Hausdorff. Because X is connected, X/T is connected. Let $k := \dim X - \dim T$. By the tube theorem of Koszul (see eg. [10, Theorem B24]), for each $x \in X$ there exists a T-invariant open neighborhood U_x of the T-orbit $T \cdot x$ and a T--equivariant diffeomorphism Φ_x from U_x onto the associated bundle $T \times_{T_x} D_x$, where D_x is an open disk centered at the origin in $\mathbb{R}^k \cong \mathbb{C}^{k/2}$ and T_x acts by linear transformations on D_x. The action of T on $T \times_{T_x} D_x$ is induced by the action of T by translations on the left factor of $T \times D_x$. Because Φ_x is a T-equivariant diffeomorphism, it induces a homeomorphism $\widehat{\Phi}_x$ on the quotient $\widehat{\Phi}_x \colon D_x/T_x \to \pi(U_x)$, and there is a commutative diagram

$$\begin{array}{ccccc} T \times D_x & \xrightarrow{\pi_x} & T \times_{T_x} D_x & \xrightarrow{\Phi_x} & U_x \\ \uparrow i_x & & \downarrow p_x & & \downarrow \pi|_{U_x} \\ D_x & \xrightarrow{\pi'_x} & D_x/T_x & \xrightarrow{\widehat{\Phi}_x} & \pi(U_x) \end{array}$$

where π_x, π'_x, p_x are the canonical projection maps, and $i_x(d) := (\mathrm{e}, d)$ is the inclusion map. Let $\phi_x := \widehat{\Phi}_x \circ \pi'_x$. The collection of charts $\widehat{\mathcal{A}} := \{(\pi(U_x), D_x, \phi_x, T_x)\}_{x \in X}$ is an orbifold atlas for X/T. We call $\mathcal{A}$ the class of atlases equivalent to the orbifold atlas $\widehat{\mathcal{A}}$. We denote the orbifold X/T endowed with the class $\mathcal{A}$ by X/T, and the class $\mathcal{A}$ is assumed.

Flat connection. The collection $\Omega = \{\Omega_x\}_{x \in X}$ of subspaces $\Omega_x \subset \mathrm{T}_x X$, where Ω_x is the σ_x-orthogonal complement to $\mathrm{T}_x(T \cdot x)$ in $\mathrm{T}_x X$, for every $x \in X$, is a smooth distribution on X. The projection mapping $\pi \colon X \to X/T$ is a smooth principal T-orbibundle for which Ω is a T-invariant flat connection. Let $\mathcal{I}_x$ be the maximal integral manifold of the distribution Ω. The inclusion $i_x \colon \mathcal{I}_x \to X$ is an injective immersion between smooth manifolds and the composite $\pi \circ i_x \colon \mathcal{I}_x \to X/T$ is an orbifold covering map. Moreover, there exists a unique 2-form ν on X/T such that $\pi^*\nu|_{\Omega_x} = \sigma|_{\Omega_x}$ for every $x \in X$. The form ν is symplectic, and so the pair $(X/T, \nu)$ is a connected symplectic orbifold.

Model for $\boldsymbol{X/T}$. We define the space that we call the *T-equivariant symplectic model* $(X_{\mathrm{model},\,p_0}, \sigma_{\mathrm{model}})$ *of* (X, σ) *based at a regular point* $p_0 \in X/T$ as follows.

i) The space $X_{\mathrm{model},\,p_0}$ is $X_{\mathrm{model},\,p_0} := \widetilde{X/T} \times_{\pi_1^{\mathrm{orb}}(X/T,\,p_0)} T$, where the space $\widetilde{X/T}$ denotes the orbifold universal cover of the orbifold X/T based at a regular point $p_0 \in X/T$, and the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X/T, p_0)$ acts on the Cartesian product $\widetilde{X/T} \times T$ by the diagonal action $x\,(y, t) = (x \star y^{-1}, \mu(x) \cdot t)$, where $\star \colon \pi_1^{\mathrm{orb}}(X/T, p_0) \times \widetilde{X/T} \to \widetilde{X/T}$ denotes the natural action of $\pi_1^{\mathrm{orb}}(X/T, p_0)$ on $\widetilde{X/T}$, and $\mu :$ $\pi_1^{\mathrm{orb}}(X/T, p_0) \to T$ denotes the monodromy homomorphism of Ω.

ii) The symplectic form σ_{model} is induced on the quotient by the product symplectic form on the Cartesian product $\widetilde{X/T} \times T$. The symplectic form on $\widetilde{X/T}$ is defined as the pullback by the orbifold universal covering map $\widetilde{X/T} \to X/T$ of ν. The symplectic form on the torus T is the unique T-invariant symplectic form σ^T determined by $\sigma^{\mathfrak{t}}$.

iii) The action of T on the space $X_{\mathrm{model},\,p_0}$ is the action of T by translations which descends from the action of T by translations on the right factor of the product $\widetilde{X/T} \times T$.

In this definition we are implicitly using that $\widetilde{X/T}$ is a smooth manifold and X/T is a good orbifold, which is proven by analogy with [29].

Model of (X, σ) with T-action. In [29], the following theorem was shown (see Theorem 3.4.3).

Theorem 27 ([29]). *Let (X, σ) be a compact connected symplectic manifold equipped with an effective symplectic action of a torus T, for which at least one, and hence every T-orbit is a $\dim T$-dimensional symplectic submanifold of (X, σ). Then (X, σ) is T-equivariantly symplectomorphic to its T-equivariant symplectic model based at any regular point $p_0 \in X/T$.*

The purpose of this appendix was to point out that Theorem 27 with the word "compact" removed from the statement holds. We refer to [29] for a proof in the case that X is compact, which works verbatim in the case that X is not compact.

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References

[1] A. Adem, J. Leida and Y. Ruan, *Orbifolds and Stringy Topology.* Cambridge Tracts in Mathematics, No. 171, 2007.

[2] M. Atiyah, *Convexity and commuting Hamiltonians.* Bull. London Math. Soc., **14** (1982), 1–15.

[3] M. Audin, *Torus Actions on Symplectic Manifolds.* Second revised edition. Progress in Mathematics 93, Birkhäuser Verlag, Basel, 2004.

[4] Y. Cho, T. Hwang and D. Y. Suh, *Semifree Hamiltonian circle actions on 6-dimensonal symplectic manifolds with non-isolated fixed point set.* `arXiv:1005.0193`.

[5] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment.* Bull. Soc. Math. France, **116** (1988), 315-39.

[6] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space.* Invent. Math., **69** (1982), 259–268.

[7] J. J. Duistermaat and Á. Pelayo, *Complex structures on 4-manifolds with symplectic 2-torus actions.* Intern. J. of Math., **206** (2011), 449–463.

[8] K. E. Feldman, *Hirzebruch genera of manifolds supporting a Hamiltonian circle action (Russian).* Uspekhi Mat. Nauk, **56**(5) (2001), 187–188; translation in "Russian Math. Surveys", **56** (5) (2001), 978–979.

[9] T. Frankel, *Fixed points and torsion on Kähler manifolds.* Ann. Math., **70** (1959), 1–8.

[10] V. Ginzburg, V. Guillemin and Y. Karshon, *Moment Maps, Cobordisms and Hamiltonian Group Actions.* Appendix J by Maxim Braverman. Mathematical Surveys and Monographs, 98. American Mathematical Society, Providence, RI, 2002. viii+350 pp.

[11] L. Godinho, *Semifree symplectic circle actions on 4-orbifolds.* Trans. Amer. Math. Soc., **358** (2006), 4919–4933.

[12] W. Graham, *Logarithmic convexity of push-forward measures.* Invent. Math., **123** (1996), 315–322.

[13] V. Guillemin, E. Lerman and S. Sternberg, *On the Kostant multiplicity formula.* J. Geom. Phys., **5** (1988), 721–750.

[14] V. Guillemin, E. Lerman and S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams.* Cambridge University Press, 1996, ISBN-10: 0521443237.

[15] V. Guillemin and S. Sternberg, *Convexity properties of the momentum mapping.* Invent. Math., **67** (1982), 491–513.

[16] V. Guillemin and S. Sternberg, *Semi-Classical Analysis.* International Press, 2013, monograph available at `http://math.mit.edu/~vwg/semiclassGuilleminSternberg.pdf`.

[17] Y. Karshon, *Example of a non-log-concave Duistermaat-Heckman measure*. Math. Res. Lett., **3** (1996), 537–540.

[18] M. Kyu Kim, *Frankel's theorem in the symplectic category*. Trans. Amer. Math. Soc., **358** (2006), 4367–4377.

[19] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, *Nonabelian convexity by symplectic cuts*. Topology, **37** (1998), 245–259.

[20] P. Li and K. Liu, *Some remarks on circle action on manifolds*. Math. Res. Letters, **18** (2011), 435–446.

[21] Y. Lin, *Examples of non-Kähler Hamiltonian circle manifolds with the strong Lefschetz property*. Adv. Math., **208** (2007), 699–709.

[22] Y. Lin, *The log-concavity conjecture for the Duistermaat-Heckman measure revisited*. International Mathematics Research Notices IMRN (2008), Vol. 2008, article ID rnn027, 19 pages, `doi:10.1093/imrn/rnn027`.

[23] R. Mazzeo, Á. Pelayo and T. S. Ratiu, *L^2-cohomology and complete Hamiltonian manifolds*. J. Geom. Phys., **87** (2015), 305–313.

[24] D. McDuff, *The momentum map for circle actions on symplectic manifolds*. J. Geom. Phys., **5** (1988), 149–160.

[25] I. Moerdijk, *Orbifolds as groupoids: an introduction*. Orbifolds in mathematics and physics (Madison,WI, 2001), 205–222, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.

[26] A. Okounkov, *Brunn-Minkowski inequality for multiplicities*. Invent. Math., **125** (1996), 405–411.

[27] A. Okounkov, *Log-concavity of multiplicities with application to characters of* $\mathrm{U}(\infty)$. Adv. Math., **127** (1997), 258–282.

[28] A. Okounkov, *Multiplicities and Newton Polytopes*. Kirillov's seminar on representation theory, 231–244, Amer. Math. Soc. Transl.: Ser. 2, 181, Amer. Math. Soc., Providence, RI, 1998.

[29] Á. Pelayo, *Symplectic actions of* 2*-tori on* 4*-manifolds*. Mem. AMS, **204** (2010), no. 959.

[30] Á. Pelayo and T. Ratiu, *Circle valued momentum maps for symplectic periodic flows*. L'Enseign. Math., **58** (2012), 205–219.

[31] Á. Pelayo and S. Tolman, *Fixed points of symplectic periodic flows*. Erg. Theory and Dyn. Systems, **31** (2011), 1237–1247.

[32] Á. Pelayo and S. Vũ Ngọc, *Symplectic theory of completely integrable Hamiltonian systems*. Bull. Amer. Math. Soc., **48** (2011), 409–455.

[33] F. Rochon, *Rigidity of Hamiltonian actions*. Canad. Math. Bull., **46** (2003), 277–290.

[34] I. Satake, *On a generalization of a notion of manifold*. Proc. Natl. Acad. Sci., USA, **42** (1956), 359–363.

[35] W. P. Thurston, *Three-dimensional Geometry and Topology*. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp.

[36] S. Tolman and J. Weitsman, *On semi-free symplectic circle actions with isolated fixed points*. Topology, **39** (2000), 299–309.

[37] A. Weinstein, *Symplectic V-manifolds, periodic orbits of Hamiltonian systems and the volume of certain Riemannian manifolds*. Comm. Pure. Appl. Math., **30** (1997), 265–271.

[38] R. O. Wells, Jr., *Differential Analysis on Complex Manifolds*. Graduate Text in Mathematics, **65** (1980), Springer Verlag, New York.

[39] D. Yan, *Hodge structure on symplectic manifolds*. Adv. Math., **120** (1996), 143–154.

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