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The multiplicative anomaly of three or more commuting elliptic operators

Victor Castillo-Garate, Eduardo Friedman
and Marius Măntoiu

ζ-regularized determinants are well-known to fail to be multiplicative, so that in general $\det_\zeta(AB) \neq \det_\zeta(A)\det_\zeta(B)$. Hence one is lead to study the n-fold multiplicative anomaly

$$M_n(A_1,\ldots,A_n) := \frac{\det_\zeta\Big(\prod_{i=1}^n A_i\Big)}{\prod_{i=1}^n \det_\zeta(A_i)}$$

attached to n (suitable) operators $A_1,\ldots,A_n$. We show that if the A_i are commuting pseudo-differential elliptic operators, then their joint multiplicative anomaly can be expressed in terms of the pairwise multiplicative anomalies. Namely

$$M_n(A_1,\ldots,A_n)^{m_1+\cdots+m_n} = \prod_{1\le i<j\le n} M_2(A_i,A_j)^{m_i+m_j},$$

where m_j is the order of A_j. The proof relies on Wodzicki's 1987 formula for the pairwise multiplicative anomaly $M_2(A,B)$ of two commuting elliptic operators.

1. Introduction

For an important class of operators A, one can define its ζ-regularized determinant as

$$\det_\zeta(A) := \exp\big(-\tfrac{d}{ds}\zeta_A(s)\big|_{s=0}\big),$$

where

$$\zeta_A(s) := \sum_i \lambda_i^{-s}$$

is the spectral zeta function of A, extended to $s = 0$ by analytic continuation [Se]. Although such determinants have played an important role in mathematical physics, geometry and number theory [El1] [El2] [KV] [JL], it has long been known that they fail to be multiplicative, *i. e.* even for commuting operators $\det_\zeta(AB) \neq \det_\zeta(A)\det_\zeta(B)$, in general.

This phenomenon has lead to the study of the multiplicative (or determinant) anomaly

$$M_2(A,B) := \frac{\det_\zeta(AB)}{\det_\zeta(A)\det_\zeta(B)}.$$

A formula for $M_2(A,B)$ was given by Wodzicki [Wo] [Ka, §6]. He assumed A and B are commuting, positive, invertible, elliptic self-adjoint pseudo-differential operators of positive order acting on the space of smooth sections of a finite-dimensional complex vector bundle E over a compact C^∞-manifold M without boundary. Here we have fixed a Hermitian metric on E and a density on M. Under these assumptions Wodzicki's formula reads [Ka]

$$\log\big(M_2(A,B)\big) = \frac{\operatorname{res}\big(\operatorname{Log}^2(\sigma_{A,B})\big)}{2\operatorname{ord}A\operatorname{ord}B\,(\operatorname{ord}A+\operatorname{ord}B)}, \tag{1}$$

where

$$\sigma_{A,B} := A^{\operatorname{ord}B}B^{-\operatorname{ord}\mathrm{A}},$$

$\operatorname{ord}A$ is the order of A, and res denotes the Wodzicki residue.

Even with Wodzicki's formula, the multiplicative anomaly $M_2(A,B)$ attached to pairs of commuting operators is in general difficult to compute. It has been explicitly computed in terms of special functions only for a handful of cases (see [El2, §2.3] and the references there). Perhaps for this reason the joint multiplicative anomaly

$$M_n(A_1,\ldots,A_n) := \frac{\det_\zeta\Big(\prod_{i=1}^n A_i\Big)}{\prod_{i=1}^n \det_\zeta(A_i)}$$

attached to n commuting operators $A_1,\ldots,A_n$ seems not to have been studied. There is a trivial reduction

$$M_n(A_1,\ldots,A_n) = M_{n-1}(A_1A_2,A_3,\ldots,A_n)M_2(A_1,A_2)$$

which can be unwound inductively into a formula for M_n in terms of M_2's, but it would be hardly practical as all of the A_i's are simultaneously involved in some of the resulting M_2's.

We show that there is a simple formula expressing $M_n(A_1,\ldots,A_n)$ in terms of the individual $M_2(A_i,A_j)$.

Theorem. *Suppose $A_1, \ldots, A_n$ are n commuting, positive, invertible, elliptic self-adjoint pseudo-differential operators of positive order acting on the smooth sections of a finite-dimensional vector bundle E over a compact manifold M without boundary. Then, their joint multiplicative anomaly M_n is defined for $n \geq 2$ and can be expressed in terms of the pairwise multiplicative anomalies M_2 as*

$$M_n(A_1, \ldots, A_n)^{m_1+\cdots+m_n} = \prod_{1\leq i<j\leq n} M_2(A_i, A_j)^{m_i+m_j}, \tag{2}$$

where m_i is the order of A_i.

Our proof uses some elementary identities involving the operator $\mathrm{Log}^2(\sigma_{A,B})$ appearing inside the Wodzicki residue in (1). A special case of the above theorem was proved in [CGF].

The theorem reduces the calculation of M_n to that of M_2. In fact, we can also reduce to M_k for any integer k between 2 and n.

Corollary. *For $2 \leq k \leq n$, and letting $C^p_q := \frac{p!}{q!(p-q)!}$, we have*

$$M_n(A_1, \ldots, A_n)^{(m_1+\cdots+m_n)C^{n-2}_{k-2}} = \prod_{1\leq i_1<\cdots<i_k\leq n} M_k(A_{i_1}, \ldots, A_{i_k})^{m_{i_1}+\cdots+m_{i_k}}.$$

We shall prove the corollary at the end of the next section.

2. Proof

Let M be a compact smooth C^∞-manifold provided with a 1-density, and let E/M be a finite-dimensional complex vector bundle over M endowed with a Hermitian metric. Let A be a pseudo-differential operator acting on the C^∞-sections of E/M. We can extend A to a possibly unbounded operator on the Hilbert space of square-integrable sections of E/M. We shall say that A satisfies Wodzicki's hypothesis if A is a positive, invertible, elliptic self-adjoint pseudo-differential operator of positive order. Then we can define the spectral zeta function

$$\zeta_A(s) := \sum_i \lambda_i^{-s}, \qquad (\mathrm{Re}(s) > m/\mathrm{ord}\, A)$$

where λ_i runs over the (positive) eigenvalues of A and the real branch of log is used to define the complex powers [Se]. The spectral zeta function admits a meromorphic continuation to $\mathbb{C}$, regular at $s=0$, so we can define the ζ-regularized determinant of A by

$$\det_\zeta(A) := \exp\bigl(-\,\zeta'_A(0)\bigr).$$

If $A_1,\ldots,A_n$ are n commuting operators satisfying Wodzicki's hypothesis, their product $A := A_1A_2\cdots A_n$ also satisfies it, so we can define

$$\delta_n = \delta_n(A_1,\ldots,A_n) := -\zeta'_A(0) + \sum_{i=1}^{n}\zeta'_{A_i}(0).$$

The joint multiplicative anomaly of the A_i is then

$$M_n = M_n(A_1,\ldots,A_n) := \exp(\delta_n) = \frac{\det_\zeta(A_1A_2\cdots A_n)}{\det_\zeta(A_1)\det_\zeta(A_2)\cdots\det_\zeta(A_n)}.$$

We will prove the relation (2) between M_n and the various M_2's by induction on n. For this our main tools will be the trivial reduction formula

$$\delta_n(A_1,\ldots,A_n) = \delta_{n-1}(A_1A_2,\ldots,A_n) + \delta_2(A_1,A_2), \tag{3}$$

and Wodzicki's formula[1]

$$\delta_2(A,B) = \frac{\operatorname{res}\bigl(\operatorname{Log}^2(\sigma_{A,B})\bigr)}{2\operatorname{ord}A\operatorname{ord}B\,(\operatorname{ord}A+\operatorname{ord}B)}, \tag{4}$$

where

$$\sigma_{A,B} := A^{\operatorname{ord}B}B^{-\operatorname{ord}\mathrm{A}},$$

$\operatorname{ord}A$ is the order of A, and res denotes the Wodzicki residue. In fact, we shall need to know nothing about the Wodzicki residue beyond the fact that it is linear. Instead, we shall rely on some simple properties of the operator Log acting on commuting self-adjoint operators.

We begin by noting that $\delta_2(A,B)$ can be expressed in terms of δ_2 of two operators having the same order. Namely,

$$(\operatorname{ord}A\,+\,\operatorname{ord}B)\,\delta_2(A,B) = 2\delta_2(A^{\operatorname{ord}B},B^{\operatorname{ord}A}). \tag{5}$$

The proof is immediate from Wodzicki's formula (4) and the linearity of Wodzicki's residue.

[1] Wodzicki has not published his proof, although he kindly sketched it to us in a letter. A proof can be found in [Ok, p. 726].

The next calculation will be the main step in our inductive proof of the Theorem stated in §1.

Lemma. *Let $A_1, A_2, \dots, A_n$ be n commuting operators, $n \geq 3$, all satisfying Wodzicki's hypotheses, and set $m_i := \operatorname{ord} A_i$. Then*

$$\sum_{1\leq i<j\leq n} \delta_2(A_i^{m_j}, A_j^{m_i}) = \sum_{j=2}^{n-1} \delta_2\big((A_1A_2)^{m_{j+1}}, A_{j+1}^{m_1+m_2}\big) + \sum_{3\leq i<j\leq n} \delta_2(A_i^{m_j}, A_j^{m_i}) + \frac{m_1+\dots+m_n}{2}\delta_2(A_1, A_2). \tag{6}$$

Proof. Since

$$\sum_{1\leq i<j\leq n} \delta_2(A_i^{m_j}, A_j^{m_i}) - \sum_{3\leq i<j\leq n} \delta_2(A_i^{m_j}, A_j^{m_i}) = \sum_{j=2}^{n} \delta_2(A_1^{m_j}, A_j^{m_1}) + \sum_{j=3}^{n} \delta_2(A_2^{m_j}, A_j^{m_2}),$$

it suffices to prove

$$\sum_{j=2}^{n-1} \delta_2\big((A_1A_2)^{m_{j+1}}, A_{j+1}^{m_1+m_2}\big) + \frac{m_1+\dots+m_n}{2}\delta_2(A_1, A_2) = \sum_{j=2}^{n} \delta_2(A_1^{m_j}, A_j^{m_1}) + \sum_{j=3}^{n} \delta_2(A_2^{m_j}, A_j^{m_2}). \tag{7}$$

We first consider $n = 3$. Then (7) reads

$$\delta_2\big((A_1A_2)^{m_3}, A_3^{m_1+m_2}\big) + \frac{m_1+m_2+m_3}{2}\delta_2(A_1, A_2) = \delta_2(A_1^{m_2}, A_2^{m_1}) + \delta_2(A_1^{m_3}, A_3^{m_1}) + \delta_2(A_2^{m_3}, A_3^{m_2}). \tag{8}$$

Using (5), after some simple cancellations we find that to prove (8) we must prove

$$(m_1+m_2+m_3)\delta_2(A_1A_2, A_3) + m_3\delta_2(A_1, A_2) = (m_1+m_3)\delta_2(A_1, A_3) + (m_2+m_3)\delta_2(A_2, A_3). \tag{9}$$

In view of Wodzicki's formula (4), we compute

$$\begin{aligned}
&\frac{m_1+m_2+m_3}{2(m_1+m_2)m_3(m_1+m_2+m_3)}\,\mathrm{Log}^2\big((A_1A_2)^{m_3}A_3^{-m_1-m_2}\big)\\
&\quad+\frac{m_3}{2m_1m_2(m_1+m_2)}\,\mathrm{Log}^2\big(A_1^{m_2}A_2^{-m_1}\big)\\
&=\frac{1}{2(m_1+m_2)}\Bigg(\frac{\big(m_3(\mathrm{Log}\,A_1+\mathrm{Log}\,A_2)-(m_1+m_2)\mathrm{Log}\,A_3\big)^2}{m_3}\\
&\quad+\frac{m_3\big(m_2\mathrm{Log}\,A_1-m_1\mathrm{Log}\,A_2\big)^2}{m_1m_2}\Bigg)\\
&=\frac{(m_3\mathrm{Log}\,A_1-m_1\mathrm{Log}\,A_3)^2}{2m_1m_3}+\frac{(m_3\mathrm{Log}\,A_2-m_2\mathrm{Log}\,A_3)^2}{2m_2m_3}\\
&\text{[compare coefficients of } \mathrm{Log}\,A_i\,\mathrm{Log}\,A_j\ (1\le i,j\le 3) \text{ on both sides above]}\\
&=\frac{m_1+m_3}{2m_1m_3(m_1+m_3)}\mathrm{Log}^2\big(A_1^{m_3}A_3^{-m_1}\big)\\
&\quad+\frac{m_2+m_3}{2m_2m_3(m_2+m_3)}\mathrm{Log}^2\big(A_2^{m_3}A_3^{-m_2}\big).
\end{aligned}$$

If we now apply the Wodzicki residue to the above equation, formula (4) and linearity of the residue yield (8). This proves the case $n=3$.

We can now complete the proof of the lemma by induction on n. Comparing (7) for n and $n+1$, we find that the inductive step amounts to

$$\begin{aligned}
&\delta_2((A_1A_2)^{m_{n+1}},A_{n+1}^{m_1+m_2})+\frac{m_{n+1}}{2}\delta_2(A_1,A_2)\\
&=\delta_2(A_1^{m_{n+1}},A_{n+1}^{m_1})+\delta_2(A_2^{m_{n+1}},A_{n+1}^{m_2}).
\end{aligned}$$

Using (5), we see that the above is exactly (9), with A_3 replaced by A_{n+1}. □

We now prove the theorem stated in §1, in the equivalent form

$$\delta_n(A_1,\ldots,A_n)=\sum_{1\le i<j\le n}\frac{m_i+m_j}{m_1+\cdots+m_n}\delta_2(A_i,A_j). \tag{10}$$

We again proceed by induction on n. For $n=2$ both sides of (10) are trivially equal, so we suppose $n\ge 3$. By the inductive hypothesis and the equal-orders formula (5), we have

$$\begin{aligned}
&\delta_{n-1}(A_1A_2,A_3,\ldots,A_n)\\
&=2\frac{\sum_{j=2}^{n-1}\delta_2\big((A_1A_2)^{m_{j+1}},A_{j+1}^{m_1+m_2}\big)+\sum_{3\le i<j\le n}\delta_2(A_i^{m_j},A_j^{m_i})}{(m_1+m_2)+\cdots+m_n}.
\end{aligned}$$

Substituting this into the trivial reduction formula (3) we find

$$\begin{aligned}
&\delta_n(A_1, A_2, \dots, A_n)\\
&= \frac{2}{m_1+m_2+\cdots+m_n}\left(\sum_{j=2}^{n-1}\delta_2\big((A_1A_2)^{m_{j+1}}, A_{j+1}^{m_1+m_2}\big)\right.\\
&\quad\left. + \sum_{3\le i<j\le n}\delta_2(A_i^{m_j}, A_j^{m_i}) + \frac{m_1+\cdots+m_n}{2}\delta_2(A_1, A_2)\right)\\
&= \frac{2}{m_1+m_2+\cdots+m_n}\sum_{1\le i<j\le n}\delta_2(A_i^{m_j}, A_j^{m_i}) \qquad \text{[use (6)]}\\
&= \frac{1}{m_1+m_2+\cdots+m_n}\sum_{1\le i<j\le n}(m_i+m_j)\delta_2(A_i, A_j) \quad \text{[use (5)]},
\end{aligned}$$

which concludes the proof of (10). □

We now prove the corollary stated at the end of §1. It suffices to prove, for $2 \le k \le n$,

$$\begin{aligned}
(11) \qquad &(m_1+\cdots+m_n)C_{k-2}^{n-2}\delta_n(A_1, \dots, A_n)\\
&= \sum_{1\le i_1<\cdots<i_k\le n}(m_{i_1}+\cdots+m_{i_k})\delta_k(A_{i_1}, \dots, A_{i_k}).
\end{aligned}$$

Set $\omega := \{1, \dots n\}$. For a subset $\gamma = \{i_1, \dots, i_\ell\} \subset \omega$ of cardinality $\#\gamma = \ell$, set

$$\mu(\gamma) := (m_{i_1}+\cdots+m_{i_\ell})\delta_\ell(A_{i_1}, \dots, A_{i_\ell}),$$

which is unambiguously defined due to the symmetry of δ_ℓ for commuting operators. In this notation, (11) amounts to

$$C_{k-2}^{n-2}\,\mu(\omega) = \sum_{\beta\subset\omega, \#\beta=k}\mu(\beta).$$

Our main theorem can be rewritten as

$$(12) \qquad \mu(\beta) = \sum_{\alpha\subset\beta, \#\alpha=2}\mu(\alpha).$$

We have, using (12),

$$\sum_{\beta\subset\omega,\#\beta=k}\mu(\beta) = \sum_{\beta\subset\omega,\#\beta=k}\ \sum_{\alpha\subset\beta,\#\alpha=2}\mu(\alpha) = C_{k-2}^{n-2}\sum_{\alpha\subset\omega,\#\alpha=2}\mu(\alpha),$$

since every two-element subset $\alpha \subset \omega = \{1, \ldots, n\}$ is contained in exactly C_{k-2}^{n-2} subsets $\beta \subset \omega$ of cardinality k. We conclude the proof by again using (12), with $\beta = \omega$. $\square$

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References

[CGF] V. Castillo-Garate and E. Friedman, *Discrepancies of products of zeta-regularized products*. Math. Review Letters, **19** (2012), 199–212.

[El1] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*. Springer, Berlin, (1995).

[El2] E. Elizalde, *Zeta function regularization in Casimir effect calculations and J. S. Dowker's contribution*. Int. J. Mod. Phys. Conf. Ser., **14** (2012), 57–72.

[JL] J. Jorgenson and S. Lang, *Basic Analysis of Regularized Series and Products*. Lecture Notes in Mathematics **1564**, Springer, Berlin, (1993).

[Ka] C. Kassel, *Le résidu non commutatif (d'après M. Wodzicki)*. Asterisque, **177** (1989), 199–229; Séminaire N. Bourbaki 1988-89, exp. n^o 708, available at `wwww.numdam.org`.

[KV] M. Kontsevich and S. Vishik, *Geometry of determinants of elliptic operators*. Functional Analysis on the Eve of the 21st Century, Vol. 1 (New Brunswick, NJ, 1993), 173–197, Birkhäuser Boston (1995).

[Ok] K. Okikiolu, *The multiplicative anomaly for determinants of elliptic operators*. Duke Math. J., **79** (1995), 723–750.

[Se] R. Seeley, *Complex powers of an elliptic operator*. Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), 288–307, Amer. Math. Soc., Providence, (1967).

[Wo] M. Wodzicki, *Noncommutative residue, Chapter I*. K-Theory, Arithmetic and Geometry (Y. Manin, ed.), Lecture Notes in Math. **1289**, 320–399, Springer, Berlin, (1987).

Departamento de Matemática, P. Universidad Católica de Chile
Casilla 306, Santiago 22, Chile
E-mail address: vacastil@uc.cl

Departamento de Matemáticas, Universidad de Chile
Casilla 653, Santiago, Chile
E-mail address: friedman@uchile.cl

Departamento de Matemáticas, Universidad de Chile
Casilla 653, Santiago, Chile
E-mail address: mantoiu@uchile.cl

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