Math. Res. Lett.
Volume 22, Number 3, 675–696, 2015

The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics

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Let $f : X \to Z$ be a separable fibration of relative dimension 1 between smooth projective varieties over an algebraically closed field k of positive characteristic. We prove the subadditivity of Kodaira dimension $\kappa(X) \geq \kappa(Z) + \kappa(F)$, where F is the generic geometric fiber of f, and $\kappa(F)$ is the Kodaira dimension of the normalization of F. Moreover, if $\dim X = 2$ and $\dim Z = 1$, we have a stronger inequality $\kappa(X) \geq \kappa(Z) + \kappa_1(F)$ where $\kappa_1(F) = \kappa(F, \omega_F^o)$ is the Kodaira dimension of the dualizing sheaf ω_F^o.

1. Introduction

The following conjecture due to Iitaka is of fundamental importance in the classification theory of algebraic varieties over $\mathbb{C}$, the field of complex numbers:

Conjecture 1.1 ($C_{n,m}$). *Let $f : X \to Z$ be a surjective morphism of proper, smooth varieties over $\mathbb{C}$, where $n = \dim X$ and $m = \dim Z$. Assuming the generic geometric fibre F of f is connected, then*

$$\kappa(X) \geq \kappa(Z) + \kappa(F).$$

This conjecture is proved in the following cases:

(1) $\dim F = 1, 2$ by Viehweg ([30], [33]);

(2) Z is of general type by Kawamata and Viehweg ([19] Theorem 3, [32]);

(3) $\dim Z = 1$ by Kawamata ([20]);

(4) F has a good minimal model by Kawamata ([21]);

(5) F is of general type by Kollár ([23]);

(6) Z is of maximal Albanese dimension by J. A. Chen and Hacon ([7]);

(7) F is of maximal Albanese dimension by Fujino ([12]);

(8) $n \leq 6$ by Birkar ([3]).

Since the existence of good minimal models of F is proved for certain cases (see [4], [24], [13]), (1,5,7) are special cases of (4).

In this paper, we consider subadditivity of Kodaira dimension of a fibration of relative dimension 1 over an algebraically closed field k of positive characteristic. We say a projective morphism $f: X \to Z$ between two varieties a *fibration*, if $f_*\mathcal{O}_X = \mathcal{O}_Z$. We say a fibration $f: X \to Z$ is *separable* if the field extension $f^*: k(Z) \to k(X)$ between the rational function fields is separable and $k(Z)$ is algebraically closed in $k(X)$. Then general fiber of a separable fibration is geometrically integral ([2, Theorem 7.1]).

Over a field of positive characteristic, the generic geometric fiber of f is possibly not smooth. So a proper definition of Kodaira dimension for singular varieties is needed. We have two attempts (Definitions 2.5 and 2.7).

The main results of the paper are:

Theorem 1.2. *Let $f: X \to Z$ be a separable fibration of relative dimension 1 between smooth projective varieties over an algebraically closed field k of positive characteristic. Then*

$$\kappa(X) \geq \kappa(Z) + \kappa(F), \tag{1.1}$$

where $\kappa(F) = \kappa(\tilde{F}, \omega_{\tilde{F}})$ (see Definition 2.5), F the generic geometric fiber of f, $\tilde{F}$ the normalization of F, and $\omega_{\tilde{F}}$ the canonical line bundle of $\tilde{F}$.

Theorem 1.3. *Let $f: S \to C$ be a fibration between a smooth projective surface S and a smooth projective curve C over an algebraically closed field k of positive characteristic. Then*

$$\kappa(S) \geq \kappa(C) + \kappa_1(F), \tag{1.2}$$

where F is the generic geometric fiber of f, $\kappa_1(F) := \kappa(F, \omega_F^o)$ and ω_F^o is the dualizing sheaf of F (see Definition 2.7).

Theorem 1.3 is essentially known to experts, as Prof. F. Catanese pointed out. For readers' convenience, we shall give a proof in Section 4, by Bombieri-Mumford's classification of surfaces in positive characteristics. One difficulty to generalize the inequality (1.2) to higher dimensional cases is the weak positivity of $f_*\omega^l_{X/Z}$ for $l \gg 0$, which plays an important role in the proof of the known cases of Conjecture $C_{n,m}$ over $\mathbb{C}$, but the positivity fails in positive characteristics (see [28], [34] Theorem 3.6). The semi-positivity of $f_*\omega^l_{X/Z}$ for $l \gg 0$ holds for certain fibrations due to Patakfalvi ([27]).

Let's briefly explain the idea of proof of Theorem 1.2. Following Viehweg's approach ([30]), we modify the fibration $f : X \to Z$ into a family of stable curves (see the commutative diagram (3.1) in Section 3.1), and then study the behavior of the relative canonical sheaves under base changes or alterations. Starting from the fibration $f : X \to Z$, after a base change $\pi : Z' \to Z$, where π is an alteration, we obtain an alteration $X' \to X$ such that, X' and Z' are smooth and the fibration $f' : X' \to Z'$ factors through a stable fibration $f'' : X'' \to Z'$ with X'' having mild singularities. The stableness of the fibration implies that $\kappa(X'', \omega_{X''/Z'}) \geq \kappa(\tilde{F})$ (Theorems 2.12, 2.14). We can attain the goal by showing $\kappa(X'', \omega_{X''/Z'}) \leq \kappa(X', \omega_{X'/Z'}) \leq \kappa(X, \omega_{X/Z})$. The first inequality is due to Proposition 2.2 since X'' has mild singularities. The second one is from carefully comparing $\omega_{X'/Z'}$ and the pull-back of $\omega_{X/Z}$ (see Theorem 2.4).

One difficulty to carry Viehweg's proof of $C_{n,n-1}$ into positive characteristics is the lack of resolution of singularities in positive characteristics. Applying resolution theory, Viehweg shows that ([30, Theorem 5.1]), after replacing $f : X \to Z$ by a birationally equivalent fibration $f_1 : X_1 \to Z_1$, there is a flat base change $Z' \to Z_1$ such that $X_1 \times_{Z_1} Z'$ has mild singularities, and the fibration $X_1 \times_{Z_1} Z' \to Z'$ factors through a stable fibration $f_s : X_s \to Z'$. The advantage of flat base change is that the relative canonical sheaves (or relative dualizing sheaves) are compatible with flat base changes. However, the morphism $Z' \to Z$ constructed in Diagram (3.1) is not flat.

2. Preliminaries and notations

Notations and assumptions:

- $D(X)$ (resp. $D^+(X), D^-(X)$ and $D^b(X)$): the derived category of the (bounded below, bounded above and bounded) complexes of quasi-coherent sheaves on a variety X;
- $\simeq$: quasi-isomorphism between two objects in the derived category;
- Lf^*, Rf_*: the derived functors of f^*, f_* respectively for a morphism $f : X \to Y$;
- $g(C), p_a(C)$: the geometric genus and the arithmetic genus of the curve C.

For basic facts of derived categories, we refer to [17]. Throughout the section, k is an algebraically closed field of arbitrary characteristic, and all varieties are over k. On a variety X, we always identify a sheaf $\mathcal{F}$ with an object in $D(X)$. Let L_1, L_2 be two line bundles on X. We denote $L_1 \geq L_2$, if $L_1 \otimes L_2^{-1}$ has non-zero global sections.

2.1. Duality theory

We list some results on Grothendieck duality theory, part of which appeared in [30, §6]. For details we refer to [15] Chap. III Sec. 8, 10, Chap. V Sec. 9 and Chap. VII Sec. 4.

Let $f : X \to Y$ be a projective morphism between two quasi-projective varieties of relative dimension $r = \dim X/Y$. There exists a functor $f^! : D^+(Y) \to D^+(X)$ such that for $F \in D^-(X)$ and $G \in D^+(Y)$,

$$Rf_* R\mathcal{H}om_X(F, f^! G) \simeq R\mathcal{H}om_Y(Rf_* F, G).$$

Recall that

(a) If $g \circ f : X \to Y \to Z$ is a composite of projective morphisms, then $(g \circ f)^! \simeq f^! \circ g^!$;

(b) For a flat base change $u : Y' \to Y$, there is an isomorphism $v^* f^! = g^! u^*$ where v and g are the two projections of $X \times_Y Y'$;

(c) If $G \in D^b(Y)$ is an object of finite Tor-dimension ([15, p. 97]), there is a functorial isomorphism $f^! F \otimes^L Lf^* G \simeq f^!(F \otimes^L G)$ for $F \in D^+(Y)$ ([15, p. 194]);

(d) We call the bounded complex $\mathcal{K}_{X/Y} := f^! \mathcal{O}_Y$ the *relative dualizing complex*, and $\omega^o_{X/Y} := \mathcal{H}^0(\mathcal{K}_{X/Y}[-r])$ the *relative dualizing sheaf*. In particular if $Y = \operatorname{Spec} k$, then the relative dualizing complex is called the *dualizing complex*, denoted by $\mathcal{K}_X$; and $\omega^o_X := \mathcal{H}^0(\mathcal{K}_X[-r])$ is called the *dualizing sheaf*;

(e) If f is a Cohen-Macaulay morphism ([15, p. 298]), i.e., f is flat and all the fibers are Cohen-Macaulay, then $f^! \mathcal{O}_Y \simeq \omega^o_{X/Y}[r]$ for a quasi-coherent sheaf $\omega^o_{X/Y}$, and $\omega^o_{X/Y}$ is compatible with base change ([15, p. 388]). If moreover f is a Gorenstein morphism, i.e. f is flat and all the fibers are Gorenstein, then $\omega^o_{X/Y}$ is an invertible sheaf ([15, p. 298]).

Proposition 2.1 ([30], Lemma 6.4). *Let $h : X \to S$ be a projective Cohen-Macaulay morphism and $g : Y \to S$ a projective Gorenstein morphism of quasi-projective varieties. Let $f : X \to Y$ be a projective morphism over S. Then $f^!\mathcal{O}_Y \simeq \omega^o_{X/Y}[r]$ and $\omega^o_{X/Y} \simeq \omega^o_{X/S} \otimes (f^*\omega^o_{Y/S})^{-1}$ where $r = \dim X/Y$.*

Proof. Let $n = \dim X/S$ and $m = \dim Y/S$. By (e), $\mathcal{K}_{X/S} \simeq \omega^o_{X/S}[n]$ and $\mathcal{K}_{Y/S} \simeq \omega^o_{Y/S}[m]$ where $\omega^o_{X/S}$ is a sheaf and $\omega^o_{Y/S}$ is an invertible sheaf. By (a) and (c), we have

$$\mathcal{K}_{X/S} \simeq f^!\mathcal{K}_{Y/S} \simeq f^!(\mathcal{O}_Y \otimes^L \omega^o_{Y/S}[m]) \simeq f^!\mathcal{O}_Y \otimes^L f^*\omega^o_{Y/S}[m].$$

So $f^!\mathcal{O}_Y \simeq \omega^o_{X/S} \otimes (f^*\omega^o_{Y/S})^{-1}[r]$. □

Proposition 2.2 ([30], Corollary 6.5). *Let S be a quasi-projective variety, X and Y two projective varieties over S of the same relative dimension n, and $f : X \to Y$ a projective morphism over S. Assume that $\mathcal{K}_{X/S} \simeq \omega^o_{X/S}[n]$, $\mathcal{K}_{Y/S} \simeq \omega^o_{Y/S}[n]$, both $\omega^o_{X/S}$ and $\omega^o_{Y/S}$ are invertible sheaves, and $Rf_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Then there is a natural injection $f^*\omega^o_{Y/S} \to \omega^o_{X/S}$.*

Proof. The duality isomorphism gives $\mathrm{Ext}^i_X(F, f^!G) \to \mathrm{Ext}^i_Y(Rf_*F, G)$ ([15, p. 210]). For $i = 0$, one has

$$\mathrm{Hom}_X(\mathcal{O}_X, f^!\mathcal{O}_Y) \cong \mathrm{Hom}_Y(Rf_*\mathcal{O}_X, \mathcal{O}_Y) \cong \mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_Y).$$

By the assumption and Proposition 2.1, we have $f^!\mathcal{O}_Y \simeq \omega^o_{X/S} \otimes (f^*\omega^o_{Y/S})^{-1}$. Therefore, there is an injection $\mathcal{O}_X \to f^!\mathcal{O}_Y \simeq \omega^o_{X/S} \otimes (f^*\omega^o_{Y/S})^{-1}$, corresponding to $1 \in \mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_Y)$ by the above Proposition. □

Proposition 2.3. *Let $f : X \to S$ and $u : S' \to S$ be two projective morphisms of quasi-projective Gorenstein varieties. Consider the base change*

$$\begin{array}{ccc} X' = X \times_S S' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ S' & \xrightarrow{u} & S \end{array}$$

If either u or f is flat, then $f'^!\mathcal{O}_{S'} \simeq \omega^o_{X'/S'}[n]$ where $n = \dim X/S$ and $\omega^o_{X'/S'} \cong v^*\omega^o_{X/S}$.

Proof. If u is flat, then the assertion follows from (b). If f is flat, since both X and S are Gorenstein, all the fibers of f are Gorenstein ([15, Prop. 9.6]). The assertion follows from (e). □

We study the behavior of the relative dualizing sheaf under a non-flat base change.

Theorem 2.4. *Let X, X', Z, Z' be projective varieties. Assume there exists the following commutative diagram*

$$\begin{tikzcd}
X' \arrow[r, "\sigma"] \arrow[d, "h"'] & \bar{X} \hookrightarrow \bar{X}' = X \times_Z Z' \arrow[r, "\pi_1"] \arrow[d, "g'"'] & X \arrow[d, "f"'] \\
Z' \arrow[r, "\mathrm{id}_{Z'}"] & Z' \arrow[r, "\pi"] & Z
\end{tikzcd}$$

where $f : X \to Z$ is a fibration of relative dimension r, $\pi : Z' \to Z$ a generically finite surjective morphism, π_1 and g' the projections, $\bar{X}$ the unique irreducible component of $\bar{X}'$ dominating over X, and $\sigma : X' \to \bar{X}$ a birational morphism.

Assume moreover that f has integral generic geometric fiber, $f^!\mathcal{O}_Z \simeq \omega^o_{X/Z}[r]$, $h^!\mathcal{O}_{Z'} \simeq \omega^o_{X'/Z'}[r]$, and $\omega^o_{X/Z}, \omega^o_{X'/Z'}$ are invertible sheaves on X and X' respectively. Then there exists an effective σ-exceptional divisor E on X' such that

$$\omega^o_{X'/Z'} \leq \sigma^*\pi_1^*\omega^o_{X/Z} + E.$$

Proof. The projective morphism $f : X \to Z$ can be factored through a closed imbedding $i : X \to P$ and a smooth morphism $p : P \to Z$, such that $f = pi$. Let $P' = P \times_Z Z'$ be the base change. Consider the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "\sigma"] \arrow[ddr, "h"'] & \bar{X}' \arrow[r, "\pi_1"] \arrow[d, "j"] & X \arrow[d, "i"] \\
 & P' \arrow[r, "\pi_2"] \arrow[d, "p'"] & P \arrow[d, "p"] \\
 & Z' \arrow[r, "\pi"] & Z
\end{tikzcd}$$

where $g' = p'j$.

We obtain a non-zero composite homomorphism

$$R\sigma_* h^!\mathcal{O}_{Z'} \xrightarrow{\alpha} g'^!\mathcal{O}_{Z'} \xrightarrow{\beta} L\pi_1^* f^!\mathcal{O}_Z\ , \tag{$\spadesuit$}$$

where α is induced by trace map and β is an isomorphism on a nonempty open subset of $\bar{X}$.

Indeed, since $h = g'\sigma$, we have $h^! \simeq \sigma^! g'^!$. The homomorphism α is simply the trace map $R\sigma_*\sigma^! \to \mathrm{id}_{D^+(\bar{X}')}$ applied to $g'^!\mathcal{O}_{Z'}$, which is isomorphic over the locus where $\sigma : X' \to \bar{X}'$ is isomorphic.

The homomorphism β is a base change map, obtained by applying j^{-1} to

$$\begin{aligned}
j_*g'^!L\pi^*\mathcal{O}_Z &\simeq j_*j^!p'^!L\pi^*\mathcal{O}_Z \simeq R\mathcal{H}om_{\mathcal{O}_{P'}}(j_*\mathcal{O}'_X, p'^!L\pi^*\mathcal{O}_Z)\\
&\simeq R\mathcal{H}om_{\mathcal{O}_{P'}}(j_*\pi_1^*\mathcal{O}_X, p'^*L\pi^*\mathcal{O}_Z \otimes \omega^o_{P'/Z'}[d])\\
&\xrightarrow{\gamma} R\mathcal{H}om_{\mathcal{O}_{P'}}(L\pi_2^*i_*\mathcal{O}_X, p'^*L\pi^*\mathcal{O}_Z \otimes \omega^o_{P'/Z'}[d])\\
&\simeq R\mathcal{H}om_{\mathcal{O}_{P'}}(L\pi_2^*i_*\mathcal{O}_X, L\pi_2^*(p^*\mathcal{O}_Z \otimes \omega^o_{P/Z}[d]))\\
&\simeq L\pi_2^*R\mathcal{H}om_{\mathcal{O}_P}(i_*\mathcal{O}_X, p^*\mathcal{O}_Z \otimes \omega^o_{P/Z}[d])\\
&\simeq L\pi_2^*i_*i^!p^!\mathcal{O}_Z \xrightarrow{\delta} j_*L\pi_1^*i^!p^!\mathcal{O}_Z\\
&\simeq j_*L\pi_1^*f^!\mathcal{O}_Z,
\end{aligned}$$

where $d = \dim P/Z$, γ and δ are both given by the base change map $L\pi_2^*i_* \to j_*L\pi_1^*$. If V is an open subvariety of Z' on which π is flat, then $\beta|_{(p'j)^{-1}(V)}$ is an isomorphism.

By taking the 0^{th} cohomology of ♠, we get a nonzero homomorphism of sheaves

$$\sigma_*\omega^o_{X'/Z'} \to \mathcal{H}^0(g'^!\mathcal{O}_{Z'}[-r]) \to \pi_1^*\omega^o_{X/Z}$$

and its pull-back homomorphism

$$\heartsuit : \sigma^*\sigma_*\omega^o_{X'/Z'} \to \sigma^*\pi_1^*\omega^o_{X/Z}.$$

The natural homomorphism $\lambda : \sigma^*\sigma_*\omega^o_{X'/Z'} \to \omega^o_{X'/Z'}$ is surjective outside the σ-exceptional locus, and since $\omega^o_{X'/Z'}$ is a line bundle, we can write the image of λ as $I \otimes \omega^o_{X'/Z'}$ where $I \subset \mathcal{O}_{X'}$ is an ideal. The subvariety defined by I is contained in the exceptional locus of σ. Observing that $\ker\lambda$ is torsion and thus $\mathrm{Hom}_{X'}(\ker\lambda, \sigma^*\pi_1^*\omega^o_{X/Z}) = 0$, applying $\mathrm{Hom}_{X'}(-, \sigma^*\pi_1^*\omega^o_{X/Z})$ to the following exact sequence

$$0 \to \ker\lambda \to \sigma^*\sigma_*\omega^o_{X'/Z'} \to I \otimes \omega^o_{X'/Z'} \to 0$$

we deduce that the homomorphism $\heartsuit : \sigma^*\sigma_*\omega^o_{X'/Z'} \to \sigma^*\pi_1^*\omega^o_{X/Z}$ factors through a homomorphism $I \otimes \omega^o_{X'/Z'} \to \sigma^*\pi_1^*\omega^o_{X/Z}$. We conclude that there is an effective σ-exceptional divisor E such that $\omega^o_{X'/Z'} \le \sigma^*\pi_1^*\omega^o_{X/Z} + E$. $\square$

2.2. Relative canonical sheaves

It is known that the canonical sheaf ω_X coincides with the dualizing sheaf ω_X^o for a smooth projective variety X. For a normal quasi-projective variety X, we define the canonical sheaf $\omega_X := i_*\omega_{X^{sm}}$ where $i : X^{sm} \hookrightarrow X$ is the inclusion of the smooth locus. If X is a normal projective Gorenstein variety, then ω_X is an invertible sheaf coinciding with the dualizing sheaf ω_X^o.

Let $f : X \to Y$ be a projective morphism between two quasi-projective varieties of relative dimension r. We define the *relative canonical sheaf* $\omega_{X/Y}$ as the relative dualizing sheaf $\omega_{X/Y}^o$. In particular if X and Y are smooth projective varieties, then $\omega_{X/Y}$ is a line bundle linearly equivalent to $\omega_X \otimes (f^*\omega_Y)^{-1}$ by Proposition 2.1, which coincides with the usual definition.

2.3. Kodaira dimension

Let X be a projective variety, L a line bundle on X and $N(L)$ the set of all positive integers m such that $|mL| \neq \emptyset$. For an integer $m \in N(L)$, let $\Phi_{|mL|}$ be the rational map defined by the linear system $|mL|$. The L-dimension $\kappa(X, L)$ is defined as

$$\kappa(X, L) = \begin{cases} -\infty & \text{if } N(L) = \emptyset \\ \max\{\dim \Phi_{|mL|}(X) | m \in N(L)\}, & \text{if } N(L) \neq \emptyset \end{cases}$$

If X is a smooth projective variety over k, the Kodaira dimension $\kappa(X) := \kappa(X, \omega_X)$, where ω_X is the canonical sheaf of X.

Definition 2.5 ([29] Chap II, Definition 6.5, [26] Example 2.1.5). Let X be a projective variety. We say that X' is a *smooth model* of X, if X' is smooth projective and X' is birational to X. If X has a smooth model X', then the Kodaira dimension $\kappa(X)$ is defined by $\kappa(X) := \kappa(X')$.

Remark 2.6. 1) Resolution of singularities in a positive characteristic field hasn't been settled yet, so the existence of a smooth model is not clear. However, if $\dim X \leq 3$, smooth models of X always exist ([9], [10]).

2) If smooth models of X exist, then the definition of Kodaira dimension is independent of choice of the smooth models.

Another way to define Kodaira dimension is to use the dualizing sheaf.

Definition 2.7. Let X be a projective variety with the invertible dualizing sheaf ω_X^o. Then $\kappa_1(X) := \kappa(X, \omega_X^o)$.

Remark 2.8. 1) For a fibration $f : X \to Z$ between smooth quasi-projective varieties, the dualizing sheaf of a general fiber and the generic fiber is a line bundle.

2) If X is Gorenstein and projective, then ω_X^o is an invertible sheaf. If X is smooth, then the dualizing sheaf coincides with the canonical sheaf, thus $\kappa_1(X) = \kappa(X)$.

Example 2.9. Let C' be a cuspidal projective curve in $\mathbb{P}_k^2$. Then $\kappa(C') = -\infty$ and $\kappa_1(C') = 0$. For a projective curve C over k, $\kappa_1(C) \geq \kappa(C)$.

2.4. Covering theorem

The following theorem is needed in the sequel.

Theorem 2.10 ([18] Theorem 10.5). *Let $f : X \to Y$ be a proper surjective morphism between smooth complete varieties. If D is a Cartier divisor on Y and E an effective f-exceptional divisor on X. Then*

$$\kappa(X, f^*D + E) = \kappa(Y, D).$$

2.5. Stable fibrations

Definition 2.11 (Delinge and Mumford). A flat family of nodal curves $f : X \to S$ together with sections $s_i : S \to X, i = 1, \dots, n$ with image schemes $S_i = s_i(S)$ is called a family of *n-pointed stable curves* (or an *n-pointed stable fibration*) over S of genus g if

1) S_i are mutually disjoint, and S_i disjoint from the non-smooth locus $\mathrm{Sing}(f)$;
2) all the geometric fibers have arithmetic genus g;
3) the sheaf $\omega_{X/S}(\sum_i S_i)$ is f-ample.

In case $n = 0$ we simply call these stable curves (rather than stable 0-pointed curves).

For an n-pointed stable fibration $f : X \to S$ and a base change $S' \to S$, the map $X \times_S S' \to S'$ is again an n-pointed stable fibration. Since f is a Gorenstein morphism, we know that $f^!\mathcal{O}_S[-1] \simeq \omega_{X/S}$ is an invertible sheaf, which is compatible with base change.

2.6. Relative canonical sheaf of stable fibrations

We recall the following result due to Keel for a fibration of stable curves.

Theorem 2.12 ([22] Theorem 0.4). *Let S be a normal projective variety and $f : X \to S$ a stable fibration with fibers being curves of arithmetic genus $g \geq 2$. Then*

(i) $\omega_{X/S}$ *is a semi-ample line bundle if* char $k > 0$*;*

(ii) *if the natural map $S \to \bar{M}_g$ is finite, then $\omega_{X/S}$ is big.*

Only (ii) is needed in the sequel. We sketch a proof of (ii) by the argument of [22, Proposition 4.8].

Proof of (ii). By taking a local basis of $f_*\omega_{X/S}$ and a proper choice of g positive integers, one can define the relative Wronskian section (see [25]), which is independent of the choices of the basis. Thus the Wronskian section defines a non-zero global section of the sheaf $\omega^N_{X/S} \otimes f^* \det(f_*\omega_{X/S})^{-1}$ for some $N > 0$, because the non-smooth locus of f is of codimension ≥ 2. Then we can write that

$$\omega^N_{X/S} \sim f^*H + Z$$

where $H = \det(f_*\omega_{X/S})$ is a big line bundle on S ([8, 2.2]) and Z is the effective divisor on X defined by the wronskian section. Therefore, $\omega_{X/S}$ is big ([6, Lemma 2.5]). □

Remark 2.13. One point of the proof above is the positivity of $\det(f_*\omega_{X/S})$. The semi-positivity of $f_*\omega^l_{X/S}, l \gg 0$ for fibrations with sharply F-pure fibers (stable curves are sharply F-pure) is proved by Patakfalvi ([27, Theorems 1.5, 1.6, Corollary 1.8]). It is possible that analogous results to Theorem 2.12 hold for this type of fibrations. To prove the bigness, it suffices to prove that $f^* \det(f_*\omega^l_{X/S}) \leq \omega^N_{X/S}$ for some $l \gg 0$ and $N \gg 0$. However, the above proof does not apply if the fibers are of higher dimension, because the Wronskian section is not defined.

For a semi-stable elliptic fibration, we have the following result. We give a proof for the readers' convenience. One can also refer to [31, Section 9.3] for a proof.

Theorem 2.14. *Let S be a smooth surface and $f : S \to C$ a semi-stable elliptic fibration over a smooth curve C. Then $\kappa(S, \omega_{S/C}) \geq 0$, and the equality is attained if and only if f is isogenous to a product.*

Proof. First note that $f_*\omega_{S/C}$ is a line bundle such that $\omega_{S/C} \sim f^*f_*\omega_{S/C}$. We have that $\omega_S \sim f^*(f_*\omega_{S/C} \otimes \omega_C)$, then it follows that $c_1(S)^2 = 0$ and

$$\begin{aligned}\chi(S,\omega_S) &= \chi(C, f_*\omega_{S/C} \otimes \omega_C) - \chi(C, R^1 f_*\omega_{S/C} \otimes \omega_C) \\ &= \deg(f_*\omega_{S/C}) + \deg(\omega_C) + \chi(C, \mathcal{O}_C) - \chi(C, \omega_C) \\ &= \deg(f_*\omega_{S/C})\end{aligned} \tag{2.1}$$

where the second equality is from Riemann-Roch formula and $R^1 f_*\omega_{S/C} = \mathcal{O}_C$, and the third equality is due to $\deg(\omega_C) = 2g(C) - 2$ and $\chi(C, \mathcal{O}_C) = -\chi(C, \omega_C) = 1 - g(C)$.

Denote by t the number of the singular fibers. Then $c_2(S) = t$ since Euler number of smooth fibers is 0, and of singular fibers is 1. Applying Noether's formula $12\chi(S, \omega_S) = 12\chi(S, \mathcal{O}_S) = c_1(S)^2 + c_2(S)$, we get that $12\deg(f_*\omega_{S/C}) = t \geq 0$, hence $\kappa(S, \omega_{S/C}) \geq 0$.

If $t > 0$, then $\kappa(S, \omega_{S/C}) > 0$.

If $t = 0$, then all the fibers are smooth. We conclude that $f : S \to C$ is isogenous to a product by moduli theory of curves, hence $\kappa(S, \omega_{S/C}) = 0$.

Therefore, $\kappa(S, \omega_{S/C}) \geq 0$, and the equality is attained if and only if $t = 0$, then we are done. □

3. Proof of Theorem 1.2

Notations are as in Theorem 1.2. We can assume that the normalization of the generic geometric fiber of f has genus $g \geq 1$.

3.1. Stable reduction

Applying de Jong's idea of alterations, we have the following commutative diagram.

(3.1)
$$\begin{tikzcd} U \arrow[d, "h"'] & X'' = U \times_M Z' \arrow[l, "\rho_1"'] \arrow[d, "f''"'] & X' \arrow[l, "\rho'"'] \arrow[r, "\sigma"] \arrow[d, "f'"'] & \bar{X}' = X \times_Z Z' \arrow[r, "\pi_1"] \arrow[d, "g'"'] & X \arrow[d, "f"'] \\ M & Z' \arrow[l, "\rho"'] & Z' \arrow[l, "\mathrm{id}_{Z'}"'] \arrow[r, "\mathrm{id}_{Z'}"] & Z' \arrow[r, "\pi"] & Z \end{tikzcd}$$

where

1) $\pi : Z' \to Z$ is an alteration (see Definition 5.1) where Z' is smooth;

2) $\sigma : X' \to \bar{X}'$ is a birational morphism onto the strict transformation of X under $Z' \to Z$ (see Definition 5.2), and X' is smooth;

3) $\rho' : X' \to X''$ is a birational morphism such that $R\rho'_* \mathcal{O}_{X'} \cong \mathcal{O}_{X''}$;

4) $h : U \to M$ with M normal and projective, is a family of stable curves if $g \geq 2$, or a family of 1-point stable curves if $g = 1$. Moreover, the natural morphism $M \to \bar{M}_g$ if $g \geq 2$ (or $M \to \bar{M}_{1,1}$ if $g = 1$) is a finite morphism.

We state the idea of the construction of the diagram. For detailed construction, see Appendix 5. After a finite purely inseparable extension $Z_1 \to Z$, general fibers of $f_1 : X_1 \to Z_1$ are smooth, where X_1 is the normalization of $X \times_Z Z_1$. Then flattening f_1, we get a flat family $f_2 : X_2 \to Z_2$ of curves. By stable reduction, we get a family of stable curves, which is the pull-back of a family of stable curves $h : U \to M$, where the natural map $M \to \bar{M}_g$ is finite. In the process, some singularities may appear. So we need to carefully control the singularities of each steps.

Claim 3.1. *If $g = 1$, then $\kappa(U, \omega_{U/M}) = \dim M$; if $g > 1$, then $\kappa(U, \omega_{U/M}) = \dim U$.*

Proof. If $g = 1$, then $\dim M = 0$ or 1. It is trivial, if $\dim M = 0$. If $\dim M = 1$, then U is a surface with at worst rational double singularities. Let $\tilde{U} \to U$ be a minimal resolution. Then $\tilde{U} \to M$ is a semi-stable elliptic fibration. We get $\kappa(\tilde{U}, \omega_{\tilde{U}/M}) = 1$, by Theorem 2.14. Thus $\kappa(U, \omega_{U/M}) = 1$.

If $g > 1$, then $\omega_{U/M}$ is big by Theorem 2.12. Thus $\kappa(U, \omega_{U/M}) = \dim U$. $\square$

Claim 3.2. $\kappa(X, \omega_{X/Z}) \geq \kappa(U, \omega_{U/M})$.

Proof. Since X'' is from the base change $Z' \to M$ and $h : U \to M$ is a stable fibration, we have $\omega_{X''/Z'} \simeq \rho_1^* \omega_{U/M}$. Since $R\rho'_* \mathcal{O}_{X'} \simeq \mathcal{O}_{X''}$, there is a natural injection $\rho'^* \omega_{X''/Z'} \hookrightarrow \omega_{X'/Z'}$ by Proposition 2.1. Applying Theorem 2.4, there exists an effective σ-exceptional divisor E on X' such that $\omega_{X'/Z'} \leq \sigma^* \pi_1^* \omega_{X/Z} + E$. Then applying the covering theorem (Theorem 2.10), we have

$$\begin{aligned}
(3.2) \qquad \kappa(X, \omega_{X/Z}) &= \kappa(X', \sigma^* \pi_1^* \omega_{X/Z} + E) \\
&\geq \kappa(X', \omega_{X'/Z'}) \\
&\geq \kappa(X', \rho'^* \omega_{X''/Z'}) = \kappa(X', \rho'^* \rho_1^* \omega_{U/M}) = \kappa(U, \omega_{U/M})
\end{aligned}$$

$\square$

3.2. Proof of Theorem 1.2

We can assume that $\kappa(Z) \geq 0$ and the genus $g \geq 1$. Then

$$\kappa(X, \omega_X) \geq \kappa(X, f^*\omega_Z) = \kappa(Z, \omega_Z)$$

since $\omega_X = f^*\omega_Z \otimes \omega_{X/Z}$ and $\kappa(X, \omega_{X/Z}) \geq 0$ by the two claims above. To prove the theorem, it suffices to prove $\kappa(X) > \kappa(Z)$ if $g \geq 2$. We argue by contradiction.

Assume now, $g \geq 2$, and $\kappa(X) = \kappa(Z)$. Let $\Phi_{|\omega_X^m|}$ (resp. $\Phi_{|\omega_Z^m|}$) be the rational map induced by $|\omega_X^m|$ (resp. $|\omega_Z^m|$) for a sufficiently divisible positive integer m. Let X_m (resp. Z_m) be the closure of the image of $\Phi_{|\omega_X^m|}$ (resp. $\Phi_{|\omega_Z^m|}$). Since $\omega_X = \omega_{X/Z} \otimes f^*\omega_Z$, we have $|\omega_{X/Z}^m| + |f^*\omega_Z^m| \subseteq |\omega_X^m|$. Notice that $\kappa(X) = \kappa(Z) \geq 0$ and $\kappa(X, \omega_{X/Z}) \geq 0$. There is a natural injection $l : f^*H^0(Z, \omega_Z^m) \to H^0(X, \omega_X^m)$ induced by tensoring with a nonzero section $s \in H^0(X, \omega_{X/Z}^m)$. The injection l gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi_{|\omega_X^m|}"] \arrow[d, "f"'] & X_m \arrow[d, dashed, "p"] \\ Z \arrow[r, dashed, "\Phi_{|\omega_Z^m|}"] & Z_m \end{tikzcd}$$

where p is the projection induced by l. The rational map p is generically finite and dominant, since $\kappa(X) = \kappa(Z)$. Hence, a general fiber of f is contracted by $\Phi_{|\omega_X^m|}$.

On the other hand, by the proof of Claim 3.2, we have inclusions of linear systems

$$\begin{aligned} \rho'^*\rho_1^*|\omega_{U/M}^m| &\subseteq \rho'^*|\rho_1^*\omega_{U/M}^m| = \rho'^*|\omega_{X''/Z'}^m| \\ &\subseteq |\rho'^*\omega_{X''/Z'}^m| \subseteq |\omega_{X'/Z'}^m| \subseteq |\sigma^*\pi_1^*\omega_{X/Z}^m(mE)|. \end{aligned}$$

Let Ψ'_m (resp. Ψ_m) be the rational map induced by $|\sigma^*\pi_1^*\omega_{X/Z}^m(mE)|$ (resp. $|\omega_{X/Z}^m|$). There is an injection $l' : (\pi_1 \circ \sigma)^*H^0(X, \omega_{X/Z}^m) \to H^0(X',$

$\sigma^*\pi_1^*\omega_{X/Z}^m(mE))$, and it gives a commutative diagram

$$\begin{tikzcd} Z' \arrow[d, "\pi"'] & X' \arrow[l, "f'"'] \arrow[d, "\pi_1\sigma"'] \arrow[r, dashed, "\Psi'_m"] & Y'_m \arrow[d, dashed, "p'"] \\ Z & X \arrow[l, "f"'] \arrow[r, dashed, "\Psi_m"] & Y_m \end{tikzcd}$$

where Y'_m (resp. Y_m) is the closure of the image of Ψ'_m (resp. Ψ_m), p' is the rational map induced by l'. By Thm. 2.10, $\kappa(X, \omega_{X/Z}) = \kappa(X', \sigma^*\pi_1^*\omega_{X/Z} + E)$, so $\dim Y'_m = \dim Y_m$. Thus the map p' is generically finite and dominant. Since $\omega_{U/M}$ is big, Ψ'_m does not contract a general fiber of f'. So Ψ_m does not contract a general fiber of f. Contradiction!

Remark 3.3. By Claim 3.2, we have $\kappa(X, \omega_{X/Z}) \geq \kappa(U, \omega_{U/M})$. Note that $\dim M = \mathrm{Var}(f)$ where $\mathrm{Var}(f)$ is the variation of f (cf. [33], [21]). So Conjecture $C'_{n,n-1}$ holds (cf. [30, 1.6]). If $\kappa(Z) \geq 0$, we have a more optimistic inequality $\kappa(X) \geq \kappa(F) + \max\{\kappa(Z), \mathrm{Var}(f)\}$, i.e., Conjecture $C^+_{n,n-1}$ holds (cf. [33]).

Remark 3.4. The separableness assumption of f is necessary in our proof. If f is not separable, we can also get Diagram (3.1) by Appendix 5. Note that Step 1 in Appendix 5 is necessary. The strict transform $\bar{X} \subset X \times_Z Z'$ is a non-reduced scheme, and X' actually is a resolution of the reduction of $\bar{X}$. However, in this situation, the homomorphism α in the proof of Thm. 2.4 is possibly zero.

4. Proof of Theorem 1.3

Proposition 4.1 ([2] Corollary 7.3). *Let $f : X \to Y$ be a dominant morphism from an irreducible nonsingular variety X of dimension at least 2 to an irreducible curve Y, such that the rational function field $k(Y)$ is algebraically closed in $k(X)$. Then the fiber $f^{-1}(y)$ is geometrically integral for all but a finite number of closed points $y \in Y$.*

Definition 4.2 ([2] Definition 7.6). A morphism $f : S \to C$ is *elliptic* if f is minimal, $f_*\mathcal{O}_S = \mathcal{O}_C$, and almost all the fibres of f (i.e., except finitely many closed fibers) are nonsingular elliptic curves.

A morphism $f : S \to C$ if called *quasi-elliptic* if f is minimal, $f_*\mathcal{O}_S = \mathcal{O}_C$ and almost all the fibers of f are singular integral curves of arithmetic genus one.

Let S be a smooth surface and $f : S \to C$ be an elliptic fibration or a quasi-elliptic fibration. Since C is a smooth curve, we obtain a decomposition $R^1 f_* \mathcal{O}_S = \mathcal{L} \oplus \mathcal{T}$, where $\mathcal{L}$ is an invertible sheaf and $\mathcal{T}$ is a torsion sheaf on C.

Theorem 4.3 (Canonical bundle formula, [5] Theorem 2). *Let $S \to C$ be a relatively minimal elliptic fibration or a quasi-elliptic fibration from a smooth surface. Then*

$$\omega_S \cong f^*(\omega_C \otimes \mathcal{L}^{-1}) \otimes \mathcal{O}_S(\sum_{i=1}^{r} a_i P_i),$$

where

1) $m_i P_i = F_{b_i}$ $(i = 1, \ldots, r)$ *are all the multiple fibers of* f,
2) $0 \leq a_i < m_i$,
3) $a_i = m_i - 1$ *if* F_{b_i} *is not an exceptional fiber, and*
4) $\deg(\mathcal{L}^{-1} \otimes \omega_C) = 2p_a(C) - 2 + \chi(\mathcal{O}_S) + l(\mathcal{T})$, *where* $l(\mathcal{T})$ *is the length of* $\mathcal{T}$.

Note that the condition $f_* \mathcal{O}_S = \mathcal{O}_C$ implies that $k(C)$ is algebraically closed in $k(S)$. So if $f : S \to C$ is a fibration, then f is a separable fibration by Proposition 4.1.

We apply classification of surfaces by Bombieri and Mumford.

Theorem 4.4 ([2] Corollary 10.22). *If S is a smooth surface with $\kappa(S) \geq 0$, then the birational isomorphism class of S contains a unique smooth minimal model X (i.e. there are no (-1)-curves on X).*

Theorem 4.5 ([2] Theorem 12.8, Proposition 13.8). *Let S be a minimal surface with $\kappa(S) = -\infty$.*

1) If $h^1(S, \mathcal{O}_S) = 0$ then S is isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ with $n \neq 1$.

2) If $h^1(S, \mathcal{O}_S) \geq 1$ then the image C of the Albanese map is a smooth curve. Moreover, there exists a rank two vector bundle $\mathcal{E}$ on C such that $alb_X : X \to C$ is isomorphic to $\mathbb{P}(\mathcal{E}) \to C$.

Proof of Theorem 1.3. We can assume that the genus $g(C) \geq 1$.

After running relative minimal model program, we can assume that $f : S \to C$ does not contain (-1)-curve on each fiber. If there is a horizontal (-1)-curve, then it will be mapped onto C which has genus $g(C) \geq 1$. Contradiction! Therefore, we can assume that S is minimal and $g(C) \geq 1$.

There are four cases: $\kappa(S) = -\infty, 0, 1, 2$.

1) $\kappa(S) = 2$. Since $\kappa_1(F)$ and $\kappa(C)$ are at most 1, theorem holds.

2) $\kappa(S) = 1$. Since S is minimal, ω_S is semi-ample. Consider the Iitaka fibration on ω_S, which is a morphism $g : S \to T$ with general fiber G. Then G is an elliptic curve or a singular curve with arithmetic genus 1, and the self-intersection number $G^2 = 0$.

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "f"'] & T \\ C & \end{tikzcd}$$

If G is mapped onto C by f, then $g(C) \leq 1$. So $g(C) = 1$ and $\kappa(C) = 0$. Then $\kappa(S) \geq \kappa_1(F) + \kappa(C)$.

If G is mapped into a point of C. Notice that $G^2 = 0$, then G is a multiple of a fiber of f. Then general fiber of f is an elliptic curve, or a singular curve with arithmetic genus 1. Since f is a flat morphism, $\kappa_1(F) = 0$. So again we have $\kappa(S) \geq \kappa_1(F) + \kappa(C)$.

3) $\kappa(S) = 0$. Since S is minimal and $\kappa(S) = 0$, we have $\omega_S \sim_{\mathbb{Q}} 0$. For a general fiber F' of f, by adjunction formula, $(\omega_S + F')|_{F'} = \omega^o_{F'} \sim_{\mathbb{Q}} 0$. Then f is either an elliptic fibration or a quasi-elliptic fibration. By Theorem 4.3, $\deg f_*\omega_S^n \geq n(2p_a(C) - 2) + n \cdot \chi(\mathcal{O}_S)$. By Riemann-Roch formula

$$\chi(C, f_*\omega_S^n) = \deg f_*\omega_S^n + (1 - p_a(C)) \geq (2n - 1)(p_a(C) - 1) + n \cdot \chi(\mathcal{O}_S).$$

Since $\kappa(S) = 0$, we have $\chi(\mathcal{O}_S) \geq 0$ ([5, the table above Theorem 5]) and $\chi(C, f_*\omega_S^n) \leq h^0(C, f_*\omega_S^n) \leq 1$. Therefore, $p_a(C) = 1$ and $\kappa(S) \geq \kappa_1(F) + \kappa(C)$.

4) $\kappa(S) = -\infty$. By Theorem 4.5, S admits a $\mathbb{P}^1$-fibration $h : S \to C'$, where C' is a smooth curve.

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "f"'] & C' \\ C & \end{tikzcd}$$

Let H be a general fiber of h, then H is a smooth rational curve and the self intersection number $H^2 = 0$. If H is horizontal of f, then $f(H) = C$.

Contradicts to the assumption $g(C) \geq 1$. Therefore, general fiber of f is $\mathbb{P}^1$. So $\kappa(S) \geq \kappa_1(F) + \kappa(C)$. $\square$

5. Appendix: Alterations and stable reduction

In the section, we discuss the construction of the diagram (3.1) in Section 3.1. Let $f : X \to Z$ be a separable fibration between smooth projective varieties over k of relative dimension 1. We assume that the normalization of the generic geometric fiber of f has genus $g \geq 1$. Then f can be altered into a stable fibration due to de Jong ([11]) . We sketch the construction by [1, Sections 3 and 4]. First we introduce some definitions.

Definition 5.1 (de Jong). A morphism of varieties $f : Y \to X$ is called a *modification* if it is proper and birational. The morphism f is called an *alternation* if it is proper, surjective and generically finite.

Definition 5.2 ([1] Definition 3.1). Let $f : X \to Z$ be a fibration between two varieties, $Z' \to Z$ a proper surjective morphism, where Z' is a variety. The fiber product $\bar{X}' := X \times_Z Z'$ has a unique irreducible component dominant over X, which is the Zariski closure $X' := \overline{X \times_Z \eta} \subset X \times_Z Z'$ of the generic fiber, where η is the generic point of Z'. We call X' the *strict transform* of X under the base change $Z' \to Z$.

Step 1. Since $f : X \to Z$ is a separable fibration, by [11, Lemma 2.8], there is a finite purely inseparable extension $Z_1 \to Z$, where Z_1 is a projective variety, such that the normalization X_1 of $X \times_Z Z_1$ is smooth over the generic point of Z_1. Denote the fibration by $f_1 : X_1 \to Z_1$.

Step 2. There exists a modification $Z_2 \to Z_1$, such that the morphism $f_2 : X_2 \to Z_2$ is flat by Flattening Lemma ([1, Lemma 3.4]), where X_2 is the strict transform of Z_2 under $X_1 \to Z_1$.

Step 3. There exists a separable finite morphism $Z_3 \to Z_2$ such that the fibration $f_3 : X_3 \to Z_3$ has $l \geq 3$ distinct sections $s_i : Z_3 \to X_3, i = 1, 2, \ldots, l$ and the sections s_i intersect every irreducible component of the geometric fiber of f_3 more than 2 points ([1, Lemmas 4.6, 4.9]), where X_3 is the strict transform of Z_3 under $X_2 \to Z_2$.

Step 4. There exists an alteration $Z_4 \to Z_3$ such that there exists a family of l-pointed stable curves $f_4 : X_4 \to Z_4$ ([1, Sections 4.6-4.9]). Here we need the assumption $l \geq 3$, so that we can apply Three Point Lemma ([1, Lemma 4.10]).

Step 5. From $f_4 : X_4 \to Z_4$, we can obtain a family of stable curves $\bar{f}_4 : \bar{X}_4 \to Z_4$ if general fibers of are of genus $g \geq 2$ (respectively, a family of

1-pointed stable curves if $g = 1$) by "contraction" ([1, Section 3.7]). Indeed, first deleting l sections if general fibers of f_4 have genus $g \geq 2$ (respectively deleting $l - 1$ sections if $g = 1$), then there is a relative contraction map $X_4 \to \bar{X}_4$ over Z_4 by contracting all the "non-stable components" of fibers (regular rational curves containing not enough singularities and marked points).

By moduli theory of curves (cf. [1, Section 13]), the moduli of the stable curves of genus g if $g > 1$ (1-pointed stable curves if $g = 1$) with level-m structure is a fine moduli space for $m \geq 3$. Thus, there is a universal family of stable curves $U_g^{(m)} \to \bar{M}_g^{(m)}$ (or 1-pointed stable curves $U_{1,1}^{(m)} \to \bar{M}_{1,1}^{(m)}$), and a natural finite surjective morphism $\bar{M}_g^{(m)} \to \bar{M}_g$ (or $\bar{M}_{1,1}^{(m)} \to \bar{M}_{1,1}$).

For simplicity, we only consider the case $g \geq 2$. The family $\bar{f}_4 : \bar{X}_4 \to Z_4$ induces a natural morphism $Z_4 \to \bar{M}_g$. After a finite surjective base change $Z_4' \to Z_4$, we have a natural map $Z_4' \to \bar{M}_g^{(m)}$, and denote its image by M and the pulled back universal family by $U \to M$. Here we can assume both Z_4' and M are normal.

$$
\begin{tikzcd}
 & & U \arrow[r] \arrow[d] & U_g^{(m)} \arrow[d] \\
 & Z_4' \arrow[r] \arrow[d] & M \arrow[r] & M_g^{(m)} \arrow[d] \\
\bar{X}_4 \arrow[r] & Z_4 \arrow[rr] & & \bar{M}_g
\end{tikzcd}
$$

Notice that the two fiber products $\bar{X}_4 \times_{Z_4} Z_4' \to Z_4'$ and $U \times_M Z_4' \to Z_4'$ are two stable fibrations having isomorphic generic geometric fibers. By [1, Lemma 3.19], there is a finite extension $Z_4'' \to Z_4'$ such that $\bar{X}_4 \times_{Z_4} Z_4'' \to Z_4''$ and $U \times_M Z_4'' \to Z_4''$ are isomorphic families of stable curves.

Step 6. Let $Z_5 \to Z_4''$ be an alteration satisfying

1) Z_5 is smooth, and

2) the locus $\Sigma \subset Z_5$, over which the fibration $f_5 : X_5 := X_4 \times_{Z_4} Z_5 \to Z_5$ is not smooth, is a simple normal crossing divisor ([1, Section 4.10]).

Then

$$\bar{X}_5 := \bar{X}_4 \times_{Z_4} Z_5 \cong U \times_M Z_5 \to Z_5$$

is a family of stable curves which is from $f_5 : X_5 \to Z_5$ by contracting all the "non-stable component" of fibers.

The variety X_5 has mild singularities. That is, if x is a singular point of X_5, then the complete local ring of X_5 at x can be described as:

$$k[[u,v,t_1,\dots,t_{n-1}]]/(uv - t_1^{n_1}\cdots t_r^{n_r})$$

where $n = \dim X_5$, $t_1,\dots,t_{d-1}$ is a regular system of parameters of $f_5(x) \in Z_5$ such that Σ coincides on a neighborhood with the zero locus of $t_1\cdots t_r$ for some $r \le n-1$ (cf. [1, Section 4.11]).

There is a resolution of singularities $\mu : X_5' \to X_5$ by blowing up (constructed in [1, Section 4.11]). We can check that $R\mu_*\mathcal{O}_{X_5'} \cong \mathcal{O}_{X_5}$. For the composite morphism $\nu : X_5' \to \bar{X}_5$, we have $R\nu_*\mathcal{O}_{X_5'} \cong \mathcal{O}_{\bar{X}_5}$, because only rational curves are contracted by the morphism $X_5 \to \bar{X}_5$.

Step 7. In conclusion, letting $Z' \to Z$ be the composite base change $Z_5 \to Z$, $X'' = \bar{X}_5$ and $X' = X_5'$ introduced above, then we obtain the commutative diagram (3.1) in Section 3.1.

Acknowledgements

Part of the work is done during the first author visiting DPMMS at University of Cambridge. He thanks Caucher Birkar's invitation, DPMMS at Cambridge for stimulating research environment, and the financial support from AMSS, Chinese Academy of Sciences. He also thanks many useful comments and discussions from people in the seminar organized by Birkar. The authors thank Yi Gu, Lingguang Li, Hiromu Tanaka, Jinsong Xu and Ze Xu for many useful discussions. They would like to thank Osamu Fujino for the comments about the history of the Iitaka conjecture. They are grateful to Weizhe Zheng for simplifying the arguments of Theorem 2.4 in the early version of the paper. The first author is supported by NSFC (No. 11201454 and No. 11231003). The second author is supported by NSFC (No. 11401358).

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Received January 15, 2014