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# Rational curves on quotients of abelian varieties by finite groups

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In [4], it is proved that the quotient of an abelian variety $A$ by a finite order automorphism $g$ is uniruled if and only if some power of $g$ satisfies a numerical condition $0 < \mathrm{age}(g^k) < 1$. In this paper, we show that $\mathrm{age}(g^k) = 1$ is enough to guarantee that $A/\langle g\rangle$ has at least one rational curve.

## 1. Introduction

Let $G$ be a finite group of automorphisms of an abelian variety $A/\mathbb{C}$. It is a classical result [5, II. §1] that $A$ itself cannot contain a rational curve. For $|G| > 1$, there may or may not be rational curves on $A/G$. For general abelian varieties, $\mathrm{Aut}(A) = \pm 1$, and Pirola proved [8] that for $A$ sufficiently general and of dimension at least three, $A/\pm 1$ has no rational curves. At the other extreme, regarding $A = E^n$ as the set of $n+1$-tuples of points on the elliptic curve $E$ which sum to 0, the quotient $A/\Sigma_{n+1}$ of $A$ by the symmetric group $\Sigma_{n+1}$ can be interpreted as the set of effective divisors linearly equivalent to $(n+1)[0]$ and, as such, is just $\mathbb{P}^n$. More generally, Looijenga has shown [7] that the quotient of $E^n$ by the Weyl group of a root system of rank $n$ is a weighted projective space.

Rational curves on $A/G$ over a field $K$ are potentially a source of rational points over $G$-extensions of $K$. For instance, the method [6] for finding pairs $a, b \in \mathbb{Q}^\times$ such that the quadratic twists $E_a$, $E_b$, and $E_{ab}$ all have positive rank amounts to finding a rational curve on $E^3/(\mathbb{Z}/2\mathbb{Z})^2$. Likewise, the theorem of Looijenga cited above gives for each elliptic curve $E$ over a number field $K$ and for each Weyl group $W$, a source of $W$-extensions $L_i$ of $K$ such that the representation of $W$ on each $E(L_i) \otimes \mathbb{Q}$ contains the reflection

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representation. On the other hand, the result of Pirola cited above dims the hope of using geometric methods to show that every abelian variety over a number field $K$ gains rank over infinitely many quadratic extensions of $K$. Thus, it is desirable from the viewpoint of arithmetic to understand when $A/G$ can be expected to have a rational curve over a given field $K$, and to begin with, one would like to know when $A/G$ has a rational curve over $\mathbb{C}$.

Any automorphism $g$ of an abelian variety $A$ defines an invertible linear transformation (also denoted $g$) on $\mathrm{Lie}(A)$. If $g$ is of finite order, there exists a unique sequence of rationals $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n < 1$ such that the eigenvalues of $g$ are $e(x_1), \ldots, e(x_n)$, where $e(x) := e^{2\pi i x}$. We say $g$ is of *type* $(x_1, \ldots, x_n)$. Following Ito and Reid [3] (or Kollár and Larsen [4]), we write $\mathrm{age}(g) = x_1 + \cdots + x_n$. For instance, $\mathrm{age}(g) = 1/2$ for every reflection $g$. The main result of [4] asserts that $A/G$ is uniruled if and only if $0 < \mathrm{age}(g) < 1$ for some $g \in G$. In this paper, we prove that to find a single rational curve in $A/G$, it suffices that $\mathrm{age}(g) \leq 1$.

Since we need only consider the case $\mathrm{age}(g) = 1$, we first classify all types which sum to 1. This requires a combinatorial analysis, which we carried out using a computer algebra system to minimize the risk of an oversight. There are thirty-five cases (see Table 2 below), and our strategy for finding rational curves depends on case analysis. Abelian surfaces play a special role, since here we can use known results on K3 surfaces. It turns out (see Corollary 14 below) that, unlike in the case of threefolds, every quotient of an abelian surface by a non-trivial finite automorphism group has a rational curve.

The other key idea is to find a non-singular projective curve $X$ on which $G$ acts with quotient $\mathbb{P}^1$ and a $G$-equivariant map from $X$ to $A$, or, equivalently, a $G$-homomorphism from the Jacobian variety of $X$ to $A$.

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## 2. Classifying types

If $A = V/\Lambda$, then the Hodge decomposition $\Lambda \otimes \mathbb{C} \cong V \oplus \bar{V}$ respects the action of $\mathrm{Aut}(A)$. Therefore, if $g$ is of finite order with eigenvalues $e(x_1), \ldots,$ $e(x_n)$, then the multiset

$$(*) \qquad \{e(x_1), \ldots, e(x_n), e(-x_1), \ldots, e(-x_n)\}$$

is $\mathrm{Aut}(\mathbb{C})$-stable. By a *type*, we mean a multiset $\{x_1, \ldots, x_n\}$ with $x_i \in [0, 1)$ such that the multiset $(*)$ is $\mathrm{Aut}(\mathbb{C})$-stable. By the *weight* of $\{x_1, \ldots, x_n\}$, we mean the sum $x_1 + \cdots + x_n$, so that $\mathrm{age}(g)$ is the weight of the type of $g$.

A type is *reduced* if 0 does not appear, and the reduced type of a given type is obtained by discarding all copies of 0. The *sum* of types is the union in the sense of multisets. A type which is not the sum of non-zero types is *primitive*. All the elements of a primitive type appear with multiplicity one, and they all have the same denominator. Every type can be realized (not necessarily uniquely) as a sum of primitive types; if the weight of the type is 1, each of the primitive types has weight $\leq 1$, so our first task is to classify primitive types with weight $\leq 1$.

A primitive type $X$ of denominator $n \geq 2$ consists of fractions $a_i/n$ where $0 < a_i < n$, and $(a_i, n) = 1$. Moveover, if $n \geq 3$, for each positive integer $a < n$ prime to $n$, exactly one of $a/n$ and $1 - a/n$ belongs to $X$. If $a < b < n/2$ and $1 - a/n \in X$, then the weight of $X$ exceeds 1 since either $b/n$ or $1 - b/n$ belongs to $X$. Thus, if $0 < a < n/2$ and $1 - a/n \in X$, then $a$ must be the largest integer in $(0, n/2)$ prime to $n$.

**Lemma 1.** *If $\phi(n) > 24$, then*

$$\sum_{x \in S_n} \min(x, n - x) > 2n,$$

*where $S_n$ is the set of positive integers $< n$ and prime to $n$. Moreover, the largest integer $n$ such that $\phi(n) \leq 24$ is 90.*

*Proof.* Note that

$$\min(x, n - x) > \frac{x(n - x)}{n} = \frac{n^2 - x^2 - (n - x)^2}{2n}.$$

In order to prove the first statement, we want to prove if $\phi(n) > 24$, then

$$\sum_{x \in S_n} \left(n^2 - x^2 - (n - x)^2\right) > 4n^2,$$

or equivalently,

$$\phi(n)n^2 - 2\sum_{x \in S_n} x^2 - 4n^2 > 0.$$

By Möbius inversion, one can prove that

$$\sum_{x \in S_n} x^2 = \frac{\phi(n)n^2}{3} + (-1)^{d_n}\frac{\phi(f(n))n}{6},$$

where $f(n)$ denotes the largest squarefree divisor of $n$ and $d_n$ is the number of distinct prime divisors of $n$. Thus, if $\phi(n) > 24$, then $\phi(n) > 24 \geq 12\frac{n}{n-1}$, so $(n-1)\phi(n) - 12n > 0$ and since $\phi(f(n)) \leq \phi(n)$,

$$\phi(n)n^2 - 2\sum_{x\in S_n} x^2 - 4n^2 \geq \Big(\frac{\phi(n)}{3} - 4\Big)n^2 - \frac{\phi(n)n}{3}$$
$$= \frac{n((n-1)\phi(n) - 12n)}{3} > 0,$$

which is the desired inequality.

For the second statement, if $\phi(n) \leq 24$ and $p$ is a prime factor of $n$, then $\phi(p) = p - 1 \leq \phi(n) \leq 24$. Hence $p \leq 23$. Writing

$$n = 2^{n_2}3^{n_3}5^{n_5}7^{n_7}11^{n_{11}}13^{n_{13}}17^{n_{17}}19^{n_{19}}23^{n_{23}},$$

we have

$$0 \leq n_2 \leq 5, 0 \leq n_3 \leq 3, 0 \leq n_5 \leq 2,$$

and $0 \leq n_i \leq 1$ for $7 \leq i \leq 23$. Case analysis now shows $n \leq 90$.

□

**Proposition 2.** *There are* 28 *primitive types with weight* $\leq 1$*:*

| # | n | primitive types | weight |
|---|---|---|---|
| 1 | 2 | 1/2 | 1/2 |
| 2 | 3 | 1/3 | 1/3 |
| 3 | | 2/3 | 2/3 |
| 4 | 4 | 1/4 | 1/4 |
| 5 | | 3/4 | 3/4 |
| 6 | 5 | 1/5, 2/5 | 3/5 |
| 7 | | 1/5, 3/5 | 4/5 |
| 8 | 6 | 1/6 | 1/6 |
| 9 | | 5/6 | 5/6 |
| 10 | 7 | 1/7, 2/7, 3/7 | 6/7 |
| 11 | | 1/7, 2/7, 4/7 | 1 |
| 12 | 8 | 1/8, 3/8 | 1/2 |
| 13 | | 1/8, 5/8 | 3/4 |
| 14 | 9 | 1/9, 2/9, 4/9 | 7/9 |
| 15 | | 1/9, 2/9, 5/9 | 8/9 |
| 16 | 10 | 1/10, 3/10 | 2/5 |
| 17 | | 1/10, 7/10 | 4/5 |

| # | n | primitive types | weight |
|---|---|---|---|
| 18 | 12 | 1/12, 5/12 | 1/2 |
| 19 | | 1/12, 7/12 | 2/3 |
| 20 | 14 | 1/14, 3/14, 5/14 | 9/14 |
| 21 | | 1/14, 3/14, 9/14 | 13/14 |
| 22 | 15 | 1/15, 2/15, 4/15, 7/15 | 14/15 |
| 23 | | 1/15,2/15, 4/15, 8/15 | 1 |
| 24 | 16 | 1/16, 3/16, 5/16, 7/16 | 1 |
| 25 | 18 | 1/18, 5/18, 7/18 | 13/18 |
| 26 | | 1/18, 5/18, 11/18 | 17/18 |
| 27 | 20 | 1/20, 3/20, 7/20, 9/20 | 1 |
| 28 | 24 | 1/24, 5/24, 7/24, 11/24 | 1 |

Table 1

*Proof.* For $n \geq 3$, the weight of a primitive type of denominator $n$ is at least

$$\sum_{\{x\in S_n | x<n/2\}} \frac{x}{n} \geq \frac{1}{2n} \sum_{x\in S_n} \min(x, n-x).$$

By Lemma 1, it suffices to carry out an exhaustive search up to $n = 90$. □

**Lemma 3.** *There are* 35 *types with* age 1 *of automorphisms given in Table 2 below.*

| # | n | types | notes |
|---|---|---|---|
| 1 | 2 | 1/2, 1/2 | Prop. 6 |
| 2 | 3 | 1/3, 1/3, 1/3 | Thm. 12 |
| 3 | | 1/3, 2/3 | Prop. 7 |
| 4 | 4 | 1/4, 1/4, 1/4, 1/4 | Thm. 12 |
| 5 | | 1/4, 1/4, 2/4 | $g^2 \to \#1$ |
| 6 | | 1/4, 3/4 | $g^2 \to \#1$ |
| 7 | 6 | 1/6, 1/6, 1/6, 1/6, 1/6, 1/6 | Thm. 12 |
| 8 | | 1/6, 1/6, 1/6, 1/6, 2/6 | Thm. 12 |
| 9 | | 1/6, 1/6, 1/6, 3/6 | $g^2 \to \#2$ |
| 10 | | 1/6, 1/6, 4/6 | $g^3 \to \#1$ |
| 11 | | 1/6, 5/6 | $g^3 \to \#1$ |
| 12 | | 1/6, 2/6, 3/6 | $g^3 \to \#1$ |
| 13 | | 1/6, 1/6, 2/6, 2/6 | $g^3 \to \#1$ |
| 14 | 7 | 1/7, 2/7, 4/7 | Cor. 10 |

| # | n | types | notes |
|---|---|---|---|
| 15 | 8 | 1/8, 2/8, 5/8 | $g^4 \to \#1$ |
| 16 | | 1/8, 3/8, 4/8 | $g^4 \to \#1$ |
| 17 | | 1/8, 1/8, 3/8, 3/8 | Thm. 12 |
| 18 | | 1/8, 2/8, 2/8, 3/8 | $g^4 \to \#1$ |
| 19 | 10 | 1/10, 2/10, 3/10, 4/10 | $g^5 \to \#1$ |
| 20 | 12 | 4/12, 2/12, 1/12, 5/12 | $g^6 \to \#1$ |
| 21 | | 4/12, 3/12, 3/12, 2/12 | $g^6 \to \#1$ |
| 22 | | 6/12, 1/12, 5/12 | $g^6 \to \#1$ |
| 23 | | 3/12, 3/12, 1/12, 5/12 | $g^4 \to \#3$ |
| 24 | | 2/12, 2/12, 2/12, 1/12, 5/12 | $g^6 \to \#1$ |
| 25 | | 1/12, 1/12, 5/12, 5/12 | Thm. 12 |
| 26 | | 4/12, 1/12, 7/12 | $g^6 \to \#1$ |
| 27 | | 2/12, 2/12, 1/12, 7/12 | $g^6 \to \#1$ |
| 28 | | 3/12, 3/12, 2/12, 2/12, 2/12 | $g^6 \to \#1$ |
| 29 | 15 | 1/15, 2/15, 4/15, 8/15 | Cor. 11 |
| 30 | 16 | 1/16, 3/16, 5/16, 7/16 | Cor. 9 |
| 31 | 20 | 1/20, 3/20, 7/20, 9/20 | Cor. 9 |
| 32 | 24 | 1/24, 5/24, 7/24, 11/24 | Cor. 9 |
| 33 | | 8/24, 4/24, 3/24, 9/24 | $g^{12} \to \#1$ |
| 34 | | 3/24, 9/24, 2/24, 10/24 | $g^{12} \to \#1$ |
| 35 | | 4/24, 4/24, 4/24, 3/24, 9/24 | $g^{12} \to \#1$ |

Table 2

*Proof.* Let $[a_i]$ be a formal variable representing the $i$th primitive type in Table 1, and let $w_i$ denote the weight of the $i$th type. A monomial $\prod a_i^{m_i}$ stands for a sum of primitive types in which the $i$th type appears $m_i$ times. The g.c.d. of the denominators of the $w_i$ is 5040. Let $y = x^{1/5040}$, so $(1 - [a_i]x^{w_i})^{-1}$ is a power series in $y$ for every $i$. By MAPLE 13, the coefficient of $y^{5040}$ in the product

$$\prod_{i=1}^{28}(1 - [a_i]x^{w_i})^{-1} = \prod_{i=1}^{28}(1 - [a_i]y^{w_i \cdot 5040})^{-1},$$

is $a1^2 + a2^3 + a2\ a3 + a4\ a5 + a8\ a9 + a4\ a13 + a6\ a16 + a4^2 a1 + a28 + a27 + a23 + a24 + a11 + a19\ a2 + a19\ a8^2 + a18\ a1 + a18\ a4^2 + a18\ a8^3 + a18\ a12 + a18^2 + a12\ a1 + a12\ a4^2 + a12\ a8^3 + a12^2 + a8^2 a2^2 + a8^2 a3 + a8^3 a1 + a8^3 a4^2 + a8^4 a2 + a8^6 + a4^4 + a18\ a8\ a2 + a12\ a8\ a2 + a8\ a1\ a2 + a8\ a2\ a4^2$.

Each monomial in this sum corresponds to an entry in Table 2. □

## 3. Rational curves in $A/\langle g\rangle$

In this section we explain how to find rational curves on $A/\langle g\rangle$ in each case in Table 2.

**Lemma 4.** *If $A/\langle g^n\rangle$ has a rational curve for some positive integer $n$, then $A/\langle g\rangle$ has a rational curve.*

*Proof.* The morphism $A/\langle g^n\rangle \to A/\langle g\rangle$ is finite, so the image of a rational curve is again a rational curve. □

**Proposition 5.** *Let $A$ be an abelian variety and $g$ an automorphism of finite order. Suppose that for every abelian variety $B$ and finite-order automorphism $h \in \mathrm{Aut}(B)$ whose type is the reduced type of $g$, $B/\langle h\rangle$ has a rational curve. Then $A/\langle g\rangle$ has a rational curve.*

*Proof.* Let $B$ denote the image of $1 - g$ acting on $A$. Then $B$ is an abelian subvariety of $A$, and $g$ restricts to an automorphism $h$ of $B$ whose type is the reduced type of $g$. As $B/\langle h\rangle \subset A/\langle g\rangle$ has a rational curve, the same is true of $A/\langle g\rangle$. □

The following proposition is well known.

**Proposition 6.** *If $A$ is an abelian surface, then $A/\pm 1$ has a rational curve.*

*Proof.* Resolving the 16 singularities of $A/\pm 1$, we obtain a K3 surface with Picard number $\geq 16 \geq 5$. By work of Bogomolov and Tschinkel [1], any such surface is either elliptic or has infinite automorphism group and in either case has infinitely many rational curves, all but finitely many of which lie on $A/\pm 1$. □

Note that Proposition 6 covers not only case #1 in Table 2 but twenty other cases as well, namely those (indicated in the "notes" column) for which the reduced type of some power of $g$ is $(1/2, 1/2)$.

**Proposition 7.** *Let $A \cong V/\Lambda$ be an abelian surface with an automorphism $g$ of type $(1/3, 2/3)$. Then $A/\langle g\rangle$ contains a rational curve.*

*Proof.* Let $G = \langle g \rangle$ and $X = A/G$. Regarding $1 - g$ as an isogeny of $A$, the number of fixed points of $g$ is

$$\deg(1-g) = \#\ker(1-g) = \det(1-g|\Lambda) = 3^2 = 9.$$

These are singularities of type $A_2$, since under $(x, y) \mapsto (\omega x, \omega^2 y)$ where $\omega$ is a cube root of unity, the invariants are generated by $X = x^3, Y = y^3, Z = xy$, and so

$$\mathbb{C}[[x, y]]^G = \mathbb{C}[[X, Y, Z]]/(XY - Z^3).$$

This is isomorphic to $\mathbb{C}[[x, y, z]]/(x^2 + y^2 + z^3)$, which has a Du Val singularity of type $A_2$ (see [9, Ch.4, 4.2]).

Consider the minimal resolution $f : Y \to X$, for which the 9 exceptional divisors $Y_i$ each consists of two projective lines $D_{i,1}$ and $D_{i,2}$ intersecting at one point. The canonical divisor of $Y$ is $K_Y = f^*K_X = 0$. Hence $Y$ is a K3-surface of Picard number $\geq 18$, and again by [1], we deduce that $X$ has infinitely many rational curves.. □

**Theorem 8.** *Let $(X, x_0)$ be a pointed non-singular projective curve of genus $\geq 2$, $B$ the Jacobian variety of $X$, and $h$ an automorphism of $(X, x_0)$ such that $X/\langle h \rangle$ is a rational curve. Suppose $h$ and $h^{-1}$ have disjoint types regarded as automorphisms of $X$. Let $A$ be an abelian variety with a finite order automorphism $g$ whose type is contained in that of $h$. Then there is a rational curve on $A/\langle g \rangle$.*

*Proof.* The map $x \mapsto [x] - [x_0]$ defines an $\langle h \rangle$-equivariant closed immersion $i \colon X \to B$ and therefore realizes the rational curve $X/\langle h \rangle$ as a subvariety of $B/\langle h \rangle$.

It suffices to prove that there exists a surjective homomorphism $p \colon B \to A$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & B & \xrightarrow{p} & A \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & B & \xrightarrow{p} & A \end{array}$$

commutes. Indeed, $p(i(X))$ generates $A$ and is therefore 1-dimensional, and it follows that the map $X/\langle h \rangle \to A/\langle h \rangle$ is non-constant. The left square commutes by construction, so it suffices to consider the right square.

Writing $B = \mathrm{Lie}(B)/\Lambda_B$ and $A = \mathrm{Lie}(A)/\Lambda_A$, the goal is to find a surjective $\mathbb{C}[t]$-linear map $\phi \colon \mathrm{Lie}(B) \to \mathrm{Lie}(A)$ (where $t$ acts as $g$ on $\mathrm{Lie}(A)$

and as $h$ on $\mathrm{Lie}(B)$) such that $\phi(\Lambda_B) \subset \Lambda_A$. If $\psi$ is a surjective $\mathbb{C}[t]$-linear map $\mathrm{Lie}(B) \to \mathrm{Lie}(A)$ such that $\psi(\Lambda_B \otimes \mathbb{Q}) = \Lambda_A \otimes \mathbb{Q}$, then we can define $\phi := n\psi$ for $n$ a sufficiently divisible positive integer. It suffices, therefore, to find $\psi$ with the desired properties.

As the type of $A$ is a subset of the type of $B$, there exists a surjective $\mathbb{Q}[t]$-linear map $T\colon \Lambda_B \otimes \mathbb{Q} \to \Lambda_A \otimes \mathbb{Q}$. Extending scalars to $\mathbb{C}$, $T \otimes 1$ maps $\Lambda_B \otimes \mathbb{C} = \mathrm{Lie}(B) \oplus \overline{\mathrm{Lie}(B)}$ to $\Lambda_A \otimes \mathbb{C} = \mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)}$. The type of $g$ acting on $\overline{\mathrm{Lie}(A)}$ is the same as the type of $g^{-1}$ acting on $\mathrm{Lie}(A)$ and therefore disjoint from the type of $g$ acting on $\mathrm{Lie}(A)$, and the same is true for the type of $h$ acting on $\mathrm{Lie}(B)$. As $\mathrm{Lie}(B)$ and $\mathrm{Lie}(A)$ are direct sums of certain $t$-eigenspaces of $\Lambda_B \otimes \mathbb{C}$ and $\Lambda_A \otimes \mathbb{C}$ respectively and as the spectrum of $t$ acting on $\mathrm{Lie}(A)$ is the intersection of the spectra of $t$ acting on $\mathrm{Lie}(B)$ and on $\Lambda_A \otimes \mathbb{C}$, it follows that $T \otimes 1$ maps $\mathrm{Lie}(B)$ to $\mathrm{Lie}(A)$. The restriction of $\psi$ to $\mathrm{Lie}(B)$ is therefore the desired map. □

**Corollary 9.** *Let $n \geq 3$ be a positive integer, and let $m = \lceil n/2 \rceil - 1$. If $A$ is an abelian variety and $g \in \mathrm{Aut}(A)$ is an automorphism of order $n$ whose type is contained in $\{1/n, 2/n, \ldots, m/n\}$, then $A/\langle g \rangle$ has a rational curve.*

*Proof.* Let $X$ denote the non-singular projective hyperelliptic curve of genus $m$ which contains the affine curve $x^2 = y^n + 1$.

The order-$n$ automorphism $h(x, y) = (x, e(1/n)y)$ extends to an automorphism of $(X, (1, 0))$ and therefore defines an automorphism of $B := \mathrm{Jac}(X)$. The Lie algebra $\mathrm{Lie}(B)$ can be identified with the space $H^0(X, \Omega_X)$ of holomorphic differential forms on $X$, which has a basis

$$\left\{ \frac{dy}{y} x, \frac{xy\, dy}{x}, \ldots, \frac{y^{m-1}\, dy}{x} \right\}.$$

Therefore, the type of $h$ acting on $B$ is $\{1/n, 2/n, \ldots, m/n\}$, which is disjoint from the type of $h^{-1}$. On the other hand, $B/\langle h \rangle$ contains the rational curve $X/\langle h \rangle$. Thus, Theorem 8 applies. □

**Corollary 10.** *If $A$ is an abelian 3-fold and $g$ is an automorphism of $A$ of type $(1/7, 2/7, 4/7)$, then $A/\langle g \rangle$ has a rational curve.*

*Proof.* Let $X$ denote the Klein quartic:

$$X : x^3 y + y^3 z + z^3 x = 0,$$

$x_0 = (1 : 0 : 0)$, and $B$ the Jacobian of $X$. The self-map $h(x : y : z) = (\zeta_7 x : \zeta_7^4 y : \zeta_7^2 z)$ of $(X, x_0)$ belongs to the automorphism group $\mathrm{PSL}_2(\mathbb{F}_7)$ of $X$

which acts non-trivially on the Jacobian variety $B$ and therefore on $\mathrm{Lie}(B) = H^0(X, \Omega_X)$. Conjugating $h$ by the cyclic permutations of $(x, y, z)$, we see that $h$ is conjugate to $h^2$ and $h^4$ in $\mathrm{Aut}(X)$, and therefore the type of $h$ is invariant under multiplication by 2 (mod 1). It is therefore $(1/7, 2/7, 4/7)$ or $(3/7, 5/7, 6/7)$, and replacing $h$ by $h^{-1}$ if necessary, we may assume that it is the former. □

We remark that $B/\langle h\rangle$ in the proof of Corollary 10 has appeared in the literature; it is known to have a Calabi-Yau resolution [2, Example 6.3].

**Corollary 11.** *If $A$ is an abelian variety and $g$ an automorphism such that the type of $A$ is contained in* $(1/15, 2/15, 3/15, 4/15, 8/15, 9/15)$*, then $A/\langle g\rangle$ has a rational curve.*

*Proof.* Let $X$ be the non-singular projective curve which has a (singular, affine) model $X'\colon y^{15} = x^2(x-1)$. This is singular only at $(0, 0)$, and the inverse image of this singularity under the normalization map $X \setminus \{P_\infty\} \to X'$ is a single point $P_0 \in X$. Let $x_0$ denote the unique point of $X$ mapping to the non-singular point $(1, 0)$. The automorphism $h\colon (x, y) \mapsto (x, e(1/15)y)$ of the affine curve induces an automorphism of $(X, x_0)$ of order 15. As $15y^{14}dy = (3x^2 - 2x)dx$, any differential form $\frac{x^m y^n dy}{3x^2-2x}$, $m, n \geq 0$, is holomorphic except possibly at $P_0$ and $P_\infty$. One checks that

$$\frac{dy}{3x-2}, \frac{ydy}{3x-2}, \frac{y^2dy}{3x-2}, \frac{y^3dy}{3x-2}, \frac{y^7dy}{3x^2-2x}, \frac{y^8dy}{3x^2-2x}$$

are all holomorphic, and their eigenvalues under $h$ are $e(1/15)$, $e(2/15)$, $e(3/15)$, $e(4/15)$, $e(8/15)$, $e(9/15)$ respectively. Applying the Riemann-Hurwitz theorem to the map $X \to \mathbb{P}^1$ given by $y$, we see that $X$ is of genus 6, and therefore, that these differential forms form a basis of $\mathrm{Lie}(\mathrm{Jac}(B)) = H^0(X, \Omega_X)$. □

We recall [10, II §5] that if $F$ is a totally imaginary number field of degree $2n$ over $\mathbb{Q}$, a *CM-type* on $K$ is a set $\Phi$ of $n$ embeddings of $F$ in $\mathbb{C}$ such that no two elements of $\Phi$ are complex conjugates of one another. If $A$ is an abelian variety of dimension $n$, $\mathrm{End}(A) \otimes \mathbb{Q}$ is a totally imaginary number field $F$ of degree $2n$ over $\mathbb{Q}$, and the representation of $F$ on $\mathrm{Lie}(A)$ is a direct sum of the elements of $\Phi$, then we say that $A$ is *of type* $\Phi$.

For every CM-type $\Phi$, there exists an abelian variety of dimension $n$ of type $\Phi$ [10, II Theorem 3], and conversely, all such abelian varieties are isogenous to one another [10, II Theorem 2].

**Theorem 12.** *If $A$ is an abelian variety and $g$ is an automorphism of finite order such that $g$ and $g^{-1}$ have disjoint types and the type of $g$ is a sum of primitive types at least one of which has weight less than* 1*, then $A/\langle g\rangle$ has a rational curve.*

*Proof.* For every primitive type, there exists an abelian variety $B_i$ with complex multiplication and an automorphism $h_i$ of $B_i$ with the given type. Indeed, the primitive types of denominator $n$ are in natural correspondence with CM-types on $\mathbb{Q}(\zeta_n) = \mathbb{Q}(e(1/n))$. Any CM-type $\Phi$ on $\mathbb{Q}(\zeta_n)$, $n \geq 3$ defines an embedding $\mathbb{Q}(\zeta_n) \to \mathbb{C}^{\phi(n)/2}$. The image of $\mathbb{Z}[\zeta_n]$ defines a lattice $\Lambda \subset \mathbb{C}^{\phi(n)/2}$, and the quotient $\mathbb{C}^{\phi(n)/2}/\Lambda$ is a complex torus with a natural action of $\mathbb{Z}/n\mathbb{Z}$ of the type associated with $\Phi$. The quotient $\mathbb{C}^{\phi(n)/2}/\Lambda$ admits a polarization [10, II 6 Theorem 4], so there exists a pair $(B_i, h_i)$ as claimed. If $\mathrm{age}(h_1) < 1$, then by [4], $B_1/\langle g_1\rangle$ has a rational curve. If $A = A_1 \times \cdots \times A_m$, and $g = (g_1, \ldots, g_m)$ is a finite order automorphism of $A$ which stabilizes each factor, then $A_1/\langle g_1\rangle \subset A/\langle g\rangle$, so $A/\langle g\rangle$ has a rational curve. The theorem now follows from Theorem 8. □

To summarize, we have the following theorem:

**Theorem 13.** *Let $A$ be an abelian variety with a nontrivial automorphism $g$ of finite order such that* $\mathrm{age}(g) \leq 1$*. Then $A/\langle g\rangle$ contains a rational curve.*

**Corollary 14.** *Let $A$ be an abelian variety of dimension $n$ with a nontrivial automorphism $g$ of finite order. If* $\dim(\ker(1-g)) \geq n-2$ *(i.e. the codimension of the fixed subspace of $A$ under $g$ is less than or equal to* 2*), then the quotient $A/\langle g\rangle$ contains a uniruled hypersurface.*

*Proof.* Since $\dim(\ker(1-g)) \geq n-2$, $B := \mathrm{im}(1-g)$ is an abelian variety of dimension $n - \dim(\ker(1-g)) \leq 2$. Let $h$ denote the restriction of $g$ to $B$. As $\mathrm{age}(h) + \mathrm{age}(h^{-1}) \geq 2$, we may assume without loss of generality that $\mathrm{age}(h) \leq 1$, so $B/\langle h\rangle$ has a rational curve $Z$ by Theorem 13. Let $C$ denote the identity component of $\ker(1-g)$, which is an abelian subvariety of dimension $\dim\ker(1-g)$ on which $g$ acts trivially. The addition morphism $B \times C \to A$ is an isogeny and respects the action of $\langle g\rangle$. We therefore obtain a finite morphism from

$$Z \times C \subset (B/\langle g\rangle) \times C \cong (B \times C)/\langle g\rangle$$

to $A/\langle g\rangle$. The image of an $n-1$-dimensional ruled variety under a finite morphism is a uniruled hypersurface. □

**Corollary 15.** *Let $E$ be an elliptic curve. If $W$ is a Weyl group of simple roots of rank $n \geq 3$ acting on $E^n$ and $W^+$ is an index 2-subgroup of $W$, then the quotient $E^n/W^+$ contains a rational curve.*

*Proof.* $W$ is generated by reflections $s_j$ of simple roots. Since $W^+$ is an index 2-subgroup of $W$, there exist two reflections $s_1$ and $s_2$ such that $s_1 s_2 \in W^+$. Then for each $i$, $\ker(1 - s_i)$ has codimension 1 and their intersection has codimension $\leq 2$. Since $\ker(1 - s_1 s_2)$ contains the intersection of $\ker(1 - s_1)$ and $\ker(1 - s_2)$, this follows from Corollary 14. □

We conclude with a question:

**Question 16.** *If $n \geq 3$ is an integer, $A$ is any abelian variety of dimension $\phi(n)/2$, and $g$ is an automorphism of order $n$ of $A$, must $A/\langle g \rangle$ always have a rational curve?*

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