On the conjectures of Gross and Leopoldt

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We show that the Leopoldt conjecture implies the so-called Hilbert's theorem 90 for compact modules made from the p-units over the cyclotomic $\mathbb{Z}_p$-extension k_∞^{cyc} of k. And under the generalized Gross conjecture, we show that Hilbert's theorem 90 for the compact modules above is equivalent to the affirmation of the Leopoldt conjecture.

1. Introduction

Let k be a number field and let p be a prime number. The purpose of this paper is to show that two basic conjectures of Gross and Leopoldt are related through arithmetic properties of certain compact modules made from the p-units which is the so-called Hilbert's theorem 90 over the cyclotomic $\mathbb{Z}_p$-extension k_∞^{cyc} of k.

More precisely we prove that under the generalized Gross conjecture, the Leopoldt conjecture is equivalent to the so-called Hilbert's theorem 90 for a closure of the p-units over $k_\infty^{\mathrm{cyc}}/k$. The arithmetic property which is called the universal norm property of the topological closure of the p-units over $k_\infty^{\mathrm{cyc}}/k$ will play a role in connecting two conjectures via class field theory.

We will explain very briefly the general strategy to relate the two conjectures. The Leopoldt conjecture concerns on the ranks of two modules $\mathcal{U}_n(p)$ and $\overline{\Delta(U_n(p))}$ made from the p-units $U_n(p)$ of k_n where k_n is the unique subfield of k_∞^{cyc} of degree p^n over k. Through the weak Leopoldt conjecture and Hilbert's theorem 90 over $k_\infty^{\mathrm{cyc}}/k$, the Leopoldt conjecture reduces to the ranks of the universal norms $\mathcal{U}_n(p)^{\mathrm{univ}}$ and $\overline{\Delta(U_n(p))}^{\mathrm{univ}}$. These universal norm groups finally can be related to the Tate module $T_p(k)$ which is related to the generalized Gross conjecture through infinite class field theory.

In order to state the main theorem, we recall some definitions briefly. For the cyclotomic $\mathbb{Z}_p$-extension k_∞^{cyc}, write $k_\infty^{\mathrm{cyc}} = \bigcup_{n \geq 0} k_n$ with k_n the unique subfield of k_∞^{cyc} of degree p^n over k. Let Γ denote the procyclic group

$G(k_\infty^{\mathrm{cyc}}/k)$ generated by γ and for each $n \geq 0$, let $\Gamma_n = G(k_\infty^{\mathrm{cyc}}/k_n)$ be the unique subgroup of Γ with index p^n. We define the Tate module $T_p(k)$ of k to be

$$T_p(k) = \varprojlim_n G(K_n/k_n)$$

the inverse limit of $G(K_n/k_n)$ where K_n is the maximal abelian unramified p-extension of k_n such that all primes dividing p split completely. One of the equivalent forms of the generalized Gross conjecture can be described via class field theory as the finiteness of the Galois coinvariant

$$T_p(k)_\Gamma := T_p(k)/(\gamma - 1)T_p(k)$$

of the Tate module $T_p(k)$ over $k_\infty^{\mathrm{cyc}}/k$ (cf. Theorem 1.14 of [17]). For a finite set A, let $\sharp(A)$ denote the cardinality of A.

The generalized Gross conjecture for (k, p)**:** Let k be a number field. Then

$$\sharp\,(T_p(k)_\Gamma) < \infty.$$

We will state some equivalent forms of the Leopoldt conjecture. Let δ denote the diagonal embedding

$$\delta : k^\times \longrightarrow \prod_{v|p} k_v^\times, \alpha \mapsto (\alpha, \ldots, \alpha).$$

Let $U_k(p)$ be the p-units of k and let $\overline{\delta(U_k(p))}$ be the topological p-adic closure of $\delta(U_k(p))$ in $\prod_{v|p} k_v^\times$. For each prime v of k dividing p, let $U_v^1 = 1 + \mathfrak{p}_v$ denote the principal units at v.

If we denote by r_1 and r_2 respectively the number of real and complex places of k, then the Leopoldt conjecture is equivalent to the $\mathbb{Z}_p$-rank $\mathrm{rank}_{\mathbb{Z}_p}\big(\overline{\delta(U_k(p))} \cap \prod_{v|p} U_v^1\big)$ of the p-part of $\overline{\delta(U_k(p))}$ is $r_1 + r_2 - 1$ (see Conjecture 7.9 of p. 102 of [21])

$$\mathrm{rank}_{\mathbb{Z}_p}\left(\overline{\delta(U_k(p))} \cap \prod_{v|p} U_v^1\right) = r_1 + r_2 - 1.$$

For a $\mathbb{Z}$-module M, let

$$\widehat{M} = \varprojlim_n M/M^{p^n}$$

denote the p-adic closure of M. Let Δ be the induced map

$$\Delta : k^\times \longrightarrow \prod_{v|p} \widehat{k}_v^\times, \alpha \mapsto (\alpha, \ldots, \alpha).$$

Then the topological closure $\overline{\Delta(U_k(p))}$ of the image $\Delta(U_k(p))$ of $U_k(p)$ under Δ in the group $\prod_{v|p} \widehat{k}_v^\times$ is equal to the image of the induced map

$$\widehat{\Delta} : \mathcal{U}_k(p) \longrightarrow \prod_{v|p} \widehat{k}_v^\times$$

where $\mathcal{U}_k(p) := U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

The Leopoldt conjecture for (k, p)**:** Let k be a number field. Then $\widehat{\Delta}$ induces the following isomorphism

$$\mathcal{U}_k(p) \cong \overline{\Delta(U_k(p))}.$$

Equivalently, the Leopoldt conjecture can be stated in terms of the $\mathbb{Z}_p$-rank of $\overline{\Delta(U_k(p))}$

$$\mathrm{rank}_{\mathbb{Z}_p}(\overline{\Delta(U_k(p))}) = r_1 + r_2 + r - 1$$

where $r = r_k(p)$ denotes the number of primes of k dividing p (cf. Theorem 10.3.6 and Remark of page 634 of [22]). Note that the p-adic closure of $\overline{\delta(U_k(p))}$ in $\prod_{v|p} \widehat{k}_v^\times$ is equal to $\overline{\Delta(U_k(p))}$.

We will define Hilbert's theorem 90 for compact modules over a $\mathbb{Z}_p$-extension k_∞ of k.

Over a $\mathbb{Z}_p$-extension $k_\infty = \bigcup_{n \geq 0} k_n$ of k, let $N_{m,n} = N_{k_m/k_n}$ denote the norm map from k_m to k_n and let $\bar{N}_m = N_{m,0}$ denote the norm map from k_m to the ground field $k_0 = k$.

For any multiplicative subgroup M_n of the intermediate field $k_n^\times$, we define the norm compatible subgroups M_n^{comp} and the universal norm subgroups M_n^{univ} as follows

$$M_n^{\mathrm{comp}} = \pi \left(\varprojlim_{m \geq n} M_m \right), \quad M_n^{\mathrm{univ}} = \bigcap_{m \geq n} N_{k_m/k_n} M_m$$

where the inverse limits are taken with respect to the norm maps and $\pi = \pi_n$ denotes the natural projection from $\varprojlim_{m \geq n} M_m$ to M_n defined as $\pi((b_m)_{m \geq n}) = b_n$. We will define similar notions by tensoring with $\mathbb{Z}_p$.

Let A_n be a compact $\mathbb{Z}_p[G(k_n/k)]$-module with the norm maps for all $m \geq n$,

$$N_{m,n} : A_m \longrightarrow A_n.$$

If we define $A_\infty := \varprojlim_{n \geq 0} A_n$ to be the inverse limits of $\{A_n\}_{n \geq 0}$ with respect to the norm maps $N_{m,n}$, then A_∞ is a $\Lambda := \varprojlim_{n \geq 0} \mathbb{Z}_p[G(k_n/k)]$-module. Let

$$(A_\infty)_\Gamma = A_\infty/(\gamma - 1)A_\infty.$$

Definition 1.1. Let $k_\infty = \bigcup_{n \geq 0} k_n$ be a $\mathbb{Z}_p$-extension of k.

(i) We will say that A_k satisfy Hilbert's theorem 90 over k_∞/k,

$$(A_\infty)_\Gamma \cong A_k^{\mathrm{univ}}.$$

(ii) We will say the generalized Gross conjecture holds for (k_∞, p) if the generalized Gross conjecture holds for (k_n, p) for all $n \geq 0$.

(iii) We will say the Leopoldt conjecture holds for (k_∞, p) if the Leopoldt conjecture holds for (k_n, p) for all $n \geq 0$.

For a Galois extension k'/k and their global p-units $U_{k'}(p)$ and $U_k(p)$ of k' and k respectively, we have the induced norm map $N_{k'/k} : U_p(k') \otimes_\mathbb{Z} \mathbb{Z}_p \longrightarrow U_k(p) \otimes_\mathbb{Z} \mathbb{Z}_p$ such that $N_{k'/k}$ acts trivially on $\mathbb{Z}_p$, i.e., for $a \in k'$ and $\alpha \in \mathbb{Z}_p$,

$$N_{k'/k}(a \otimes \alpha) = N_{k'/k}(a) \otimes \alpha.$$

For the global p-units $U_n(p) = U_{k_n}(p)$ of k_n, write

$$\mathcal{U}_n(p) := U_n(p) \otimes_\mathbb{Z} \mathbb{Z}_p, \quad \mathcal{U}_\infty := \varprojlim_{n \geq 0}(U_n(p) \otimes_\mathbb{Z} \mathbb{Z}_p)$$

with $U_0(p) = U_k(p)$, $\mathcal{U}_0(p) = \mathcal{U}_k(p)$. The group $\mathcal{U}_k(p)^{\mathrm{univ}}$ of universal norm elements is defined as

$$\mathcal{U}_k(p)^{\mathrm{univ}} = \bigcap_{n \geq 0} N_n \mathcal{U}_n(p) = \bigcap_{n \geq 0} N_n(U_n(p) \otimes_\mathbb{Z} \mathbb{Z}_p) = \bigcap_{n \geq 0}(N_n(U_n(p)) \otimes_\mathbb{Z} \mathbb{Z}_p).$$

The projection π induces the following natural map

$$\pi : (\mathcal{U}_\infty)_\Gamma \to \mathcal{U}_k(p)$$

such that the image of π is equal to the norm comparable elements $\mathcal{U}_k(p)^{\mathrm{comp}}$ of $\mathcal{U}_k(p)$ where $(\mathcal{U}_\infty)_\Gamma$ is the coinvariant of $\mathcal{U}_\infty$ by $\Gamma = G(k_\infty/k)$. Then by

an argument of compactness, it follows that $\mathcal{U}_k(p)^{\mathrm{comp}} = \mathcal{U}_k(p)^{\mathrm{univ}}$. We state the main theorem of this paper.

Theorem 1.2.   (i) *The Leopoldt conjecture for (k,p) implies Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ over $k_\infty^{\mathrm{cyc}}/k$.*

  (ii) *Under the generalized Gross conjecture for $(k_\infty^{\mathrm{cyc}}, p)$, Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$ is equivalent to the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$.*

When k is totally real or a CM-field, we have the following corollary.

Corollary 1.3.   (i) *If k is a totally real field then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to the affirmation of the Gross conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ and Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$.*

  (ii) *If k is a CM-field with a maximal real subfield k^+ then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to the affirmation of the Gross conjecture for $(k_\infty^{+\mathrm{cyc}}, p)$ and Hilbert's theorem 90 for $\overline{\Delta(U_{k_n^+}(p))}$ over $k_\infty^{+\mathrm{cyc}}/k_n^+$ for all $n \geq 0$.*

As special cases of Corollary 1.3, if the number $r_\infty(p)$ (resp. $r_{\infty^+}(p)$) of primes of k_∞^{cyc} (resp. $k_\infty^{+\mathrm{cyc}}$) dividing p is one, then we have more simple descriptions.

  (i) If k is a totally real field with $r_\infty(p) = 1$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$.

  (ii) If k is a CM-field with the maximal real subfield k^+ such that $r_{\infty^+}(p) = 1$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_{k_n^+}(p))}$ over $k_\infty^{+\mathrm{cyc}}/k_n^+$ for all $n \geq 0$.

In §2, we recall the generalized Gross conjecture and the Leopoldt conjecture. In §3, we prove Theorem 1.2 and Corollary 1.3.

2. Conjectures of Gross and Leopoldt

Leopoldt made his conjecture for the units of a totally real field, and Gross made his conjecture for the minus part of the p-units of a CM field. Two conjectures were generalized independently to a number field k. Let $k_\infty =$

$\bigcup_{n\geq 0} k_n$ be a $\mathbb{Z}_p$-extension of k with k_n the unique subfield of k_∞ of degree p^n over k and $G_n = G(k_n/k)$.

Following Iwasawa (see §4 of [13]), we briefly explain how the two conjectures are connected using the cohomologies over $\Gamma = G(k_\infty/k)$. From the seven term exact sequence of Auslander-Brumer and Chase-Harrison-Rosenberg, one can define

$$\psi_S : H^2(\Gamma, U_\infty(S)) \longrightarrow B_S := \ker(H^2(\Gamma, k_\infty^\times) \to H^2(\Gamma, I_\infty))$$

where $U_\infty(S)$ and I_∞ are the group of the $S := \{v|p\}$-units and the group of fractional ideals of k_∞ respectively (see Proposition 2.1 of [5] or Proposition 1 of [13]). Then there are three equivalent conditions;

   i) ψ_S is surjective,

  ii) $Cl_\infty(S)^\Gamma < \infty$,

 iii) $H^1(\Gamma, Cl_\infty(S)) = 0$,

where $Cl_\infty(S)$ is the p-part of the S-ideal class group of k_∞

If k_∞ is equal to the cyclotomic $\mathbb{Z}_p$-extension k_∞^{cyc}, then the generalized Gross conjecture for (k,p) is equivalent to one of the three conditions (i), (ii) and (iii) above. When k is a totally real field, the Leopoldt conjecture, i.e., the uniqueness of a $\mathbb{Z}_p$-extension of k, implies ψ_S is surjective. When k is a CM field and p is odd, there exists a decomposition of ± 1-eigenspaces via the complex conjugation,

$$\psi_S^\pm : H^2(\Gamma, U_\infty(S))^\pm \longrightarrow B_S^\pm.$$

The Leopoldt conjecture implies the surjectivity of the plus part ψ_S^+ as above and the surjectivity of the minus part ψ_S^- is the Gross conjecture. Hence when k is a CM field, the conjectures of Leopoldt and Gross implies the generalized Gross conjecture.

2.1.

We introduce briefly the generalized Gross conjecture with its reformulation in terms of the Tate module. For an arbitrary number field k, let S denote the set of the primes of k dividing p. We denote by $U_k(p)$ the group of S-units of k.

For each finite prime v of k, let k_v^{univ} denote the group of universal norms from the cyclotomic $\mathbb{Z}_p$-extension $k_{v,\infty}^{\mathrm{cyc}}$ of the local field k_v. Let $\log_p$ denote

Iwasawa's p-adic logarithm normalized by $\log_p(p) = 0$. Thus it induces an isomorphism from $\mathbb{Q}_p^\times / p^\mathbb{Z} \mu(\mathbb{Q}_p) = 1 + p\mathbb{Z}_p$ to $p\mathbb{Z}_p$ where $\mu(\mathbb{Q}_p)$ denotes the torsion subgroup of $\mathbb{Q}_p^\times$. It also follows that for each $v|p$,

$$a \in k_v^{\text{univ}} \Leftrightarrow a \in \ker(\log_p \circ N_{k_v/\mathbb{Q}_p}).$$

Let

$$\lambda_k = \sum_{v|p} \log_p \circ N_{k_v/\mathbb{Q}_p} : k^\times \to \oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v \cong \prod_{v|p} k_v^\times / k_v^{\text{univ}}$$

be the map defined by $\lambda_k(a) = \sum_{v|p} \log_p \circ N_{k_v/\mathbb{Q}_p}(a)v$.

If we restrict λ_k to $U_k(p)$ and extend it $U_k(p) \otimes \mathbb{Z}_p$ by $\mathbb{Z}_p$-linearity, then we can obtain a homomorphism g_k of $\mathbb{Z}_p$-modules

$$g_k : U_k(p) \otimes \mathbb{Z}_p \longrightarrow \oplus_{v|p} \log_p \circ N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v.$$

Let $r = r_k(p)$ denote the number of primes of k dividing p. By the product formula of the Artin map, the image of g_k has a $\mathbb{Z}_p$-rank at most $r - 1$ (see page 455 of [6] and page 11 of [17]). Let $d_k(p) \geq 0$ be the nonnegative integer such that

$$d_k(p) = r - 1 - \text{rank} (\text{im}(g_k)).$$

Let r_1 and r_2 denote respectively the number of real and complex primes of k. Notice that by Dirichlet's unit theorem, $\mathbb{Z}_p$-rank rank $(\ker(g_k))$ of the kernel of g_k is given by

$$\text{rank} (\ker(g_k)) = r_1 + r_2 + d_k.$$

The generalized Gross conjecture for (k, p) is then equivalent to

$$d_k(p) = 0.$$

This is an extension of a conjecture of Gross (see [10]) due to Jaulent (see [14] and [17]). In fact, Theorem 1.14 of [17] shows that the condition $d_k(p) = 0$ is equivalent to one of the three conditions (i), (ii) and (iii) defined in the beginning of §2 for $k_\infty = k_\infty^{\text{cyc}}$. For absolute abelian fields, the proof of Gross conjecture is due to Greenberg (see [8]).

Let $k_\infty = k_\infty^{\text{cyc}} = \bigcup_{n \geq 0} k_n$ the cyclotomic $\mathbb{Z}_p$-extension of a number field $k = k_0$ with k_n the unique subfield of k_∞ of degree p^n over k. Let Γ denote

the procyclic group $G(k_\infty/k)$ and for each $n \geq 0$, let $\Gamma_n = G(k_\infty/k_n)$ be the unique subgroup of Γ with index p^n. We recall the Tate module

$$T_p(k) = \varprojlim_n G(K_n/k_n)$$

of k where K_n is the maximal abelian unramified p-extension of k_n such that all primes dividing p split completely.

We now introduce the following equivalence of the generalized Gross conjecture for (k, p)

$$\sharp\, (T_p(k)_\Gamma) < \infty.$$

This follows from the following well known proposition using class field theory (see Proposition of 1.2 of [17]).

$$d_k(p) = 0 \ \text{ if and only if } \ \sharp\, (T_p(k)_\Gamma) < \infty.$$

The generalized Gross conjecture can also be described in terms of the invariant of the Tate module, i.e.,

$$\sharp\, (T_p(k)^\Gamma) < \infty$$

which follows from the identity rank $T_p(k)_\Gamma = $ rank $T_p(k)^\Gamma$. This follows directly from the finiteness rank $T_p(k) < \infty$ (see Theorem 5 of §3 of [12]) and the exact sequence

$$1 \longrightarrow T_p(k)^\Gamma \longrightarrow T_p(k) \xrightarrow{\gamma - 1} T_p(k) \longrightarrow T_p(k)_\Gamma \longrightarrow 1.$$

3. Proof of Theorem 1.2

As explained in the introduction, the general strategy to relate the two conjectures is applying infinite class field theory over $k_\infty^{\mathrm{cyc}} = \bigcup_{n \geq 0} k_n$. The Leopoldt conjecture which concerns on the ranks of two modules $\mathcal{U}_n(p)$ and $\overline{\Delta(U_n(p))}$ reduces to the ranks of the universal norms $\mathcal{U}_n(p)^{\mathrm{univ}}$ and $\overline{\Delta(U_n(p))}^{\mathrm{univ}}$ through the weak Leopoldt conjecture and Hilbert's theorem 90 over $k_\infty^{\mathrm{cyc}}/k$. These universal norm groups finally are related to $T_p(k)$ which can be related to the generalized Gross conjecture through infinite class field theory.

In §§3.1 and 3.2, we recall the condition so-called (HNT) for p-units and respectively Hilbert's theorem 90 over a procyclic extension.

We prove Theorem 1.2 in the separated subsections. In §3.3, we prove that the affirmation of the generalized Gross conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ and

Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k$ for all $n \geq 0$ implies the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$. In §3.4, we prove the first claim of Theorem 1.2 that the affirmation of the Leopoldt conjecture for (k, p) implies Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ over $k_\infty^{\mathrm{cyc}}/k$. This will complete the proof of Theorem 1.2.

In §3.5, we prove Corollary 1.3 of the introduction and explain special cases of the corollary. In §3.6, we explain that results of Jaulent can also be used to prove part of Theorem 1.2.

3.1.

In this subsection, we will define a condition for the p-units which will be called simply as (HNT) over a procyclic extension.

Let K/k be a finite cyclic extension and for a finite set S of primes of k containing all ramified primes in K/k, let $U_k(S)$ be the global S-units of k. Let

$$S' = \{w|v \; ; \; v \in S\}$$

be the set of primes of K lying over each prime $v \in S$. We also let $U_K(S) := U_K(S')$ denote the global S'-units of K. Let

$$J_{K,S} := \prod_{w \notin S'} U_v \times \prod_{w \in S'} k_v^\times$$

be the S-idele group and let

$$C_{K,S} := J_{K,S}/U_K(S)$$

be the S-idele class group where we identify $U_K(S)$ with a subgroup of $J_{K,S}$ via the the diagonal embedding

$$\phi_{K,S} : U_K(S) \longrightarrow J_{K,S}.$$

Since $w \notin S$ is unramified in K/k, the local units U_w is cohomologically trivial for $w \notin S$. Hence by Shapiro's lemma, we have

$$\widehat{H}^i(G(K/k), J_{K,S}) \cong \widehat{H}^i(G(K/k), \prod_{w \in S'} K_w)$$

$$\cong \prod_{v \in S} \widehat{H}^i(G(K/k), \mathrm{Ind}_{G(K/k)}^{G(K_w/k_v)}(K_w))$$

$$\cong \prod_{v \in S} \widehat{H}^i(G(K_w/k_v), K_w^\times)$$

where w is a chosen prime of K above v.

Using the isomorphism above and Hilbert's theorem 90, the exact sequence $1 \longrightarrow U_K(S) \overset{\phi_{K,S}}{\longrightarrow} J_{K,S} \longrightarrow C_{K,S} \longrightarrow 1$ leads to

$$1 \longrightarrow \widehat{H}^{-1}(G_{K/k}, C_{K,S}) \longrightarrow \frac{U_k(S)}{N_{K/k}U_K(S)} \overset{\tilde{\phi}_{K/k,S}}{\longrightarrow} \prod_{v \in S} \widehat{H}^0(G_{K_w/k_v}, K_w^\times)$$

where $N_{K/k}$ is the norm map of K/k and $\tilde{\phi}_{K/k,S}$ is the map induced from the diagonal embedding $\phi_{K,S}$. It follows that

$$\widehat{H}^{-1}(G_{K/k}, C_{K,S}) \cong \ker(\tilde{\phi}_{K/k,S})$$

We let

$$\phi_{K/k,S} : U_k(S) \longrightarrow \prod_{w \in S} \widehat{H}^0(G_{K_w/k_v}, K_w^\times) \cong \prod_{w \in S} k_v^\times / N_{K_w/k_v} K_w^\times$$

denote the composition of the natural projection $\pi : U_k(S) \longrightarrow U_k(S)/N_{K/k}U_K(S)$ and $\tilde{\phi}_{K/k,S} : U_k(S)/N_{K/k}U_K(S) \longrightarrow \prod_{w \in S} \widehat{H}^i(G_{K_w/k_v}, K_w^\times)$. (HNT) for $k^\times$ implies that

$$N_{K/k}U_K(S) \subset \ker(\phi_{K/k,S}) \subset U_k(S) \cap N_{K/k}K^\times.$$

Definition 3.1. For a finite cyclic extension K/k, if $\tilde{\phi}_{K/k,S}$ is injective, i.e.,

$$\ker(\phi_{K/k,S}) = N_{K/k}U_K(S),$$

then we say that $U_k(S)$ satisfies (HNT) over K/k.

We extend the definition into an infinite Galois extension such that the Galois group is an infinite procyclic group, a topological closure of an infinite cyclic group.

For an infinite procyclic extension K/k, let

$$\ker(\phi_{K/k,S}) := \bigcap_L \ker(\phi_{L/k,S}), \quad U_k(S)^{\mathrm{univ}} := \bigcap_L N_{L/k}U_L(S)$$

for all finite cyclic subextensions L of K/k. For a finitely generated $\mathbb{Z}$-module M, let $M/\mathrm{tor}(M)$ be the quotient of M by its torsion $\mathrm{tor}(M)$ and let

$$\mathrm{rank}_{\mathbb{Z}}(M) := \mathrm{rank}_{\mathbb{Z}}(M/\mathrm{tor}(M))$$

be the $\mathbb{Z}$-rank of $M/\mathrm{tor}(M)$.

Definition 3.2. For an infinite procyclic extension K/k, if

$$\operatorname{rank}_{\mathbb{Z}}(\ker(\phi_{K/k,S})) = \operatorname{rank}_{\mathbb{Z}}(U_k(S)^{\operatorname{univ}}),$$

then we say that $U_k(S)$ satisfies (HNT) over K/k.

If we denote by $\ker(\phi_{K/k,S})$ the kernel of $\phi_{K/k,S}$, then we have

$$1 \longrightarrow \ker(\phi_{K/k,S}) \otimes \mathbb{Z}_p \longrightarrow U_k(S) \otimes \mathbb{Z}_p \longrightarrow J_{K,S}^{G(K/k)}/N_{K/k}J_{K,S} \otimes \mathbb{Z}_p$$

by tensoring with $\mathbb{Z}_p$ to the following exact sequence,

$$1 \longrightarrow \ker(\phi_{K/k,S}) \longrightarrow U_k(S) \longrightarrow J_{K,S}^{G(K/k)}/N_{K/k}J_{K,S}.$$

If we denote by $\overline{\phi}_{K/k,S} := \phi_{K/k,S} \otimes 1$ the map induced from $\phi_{K/k,S}$, then the exact sequence above shows that

$$\ker(\overline{\phi}_{K/k,S}) = \ker(\phi_{K/k,S}) \otimes \mathbb{Z}_p.$$

For an infinite procyclic extension K/k, let

$$\ker(\overline{\phi}_{K/k,S}) := \bigcap_{L} \ker(\overline{\phi}_{L/k,S}),$$
$$(U_k(S) \otimes \mathbb{Z}_p)^{\operatorname{univ}} := \bigcap_{L} N_{L/k}(U_L(S) \otimes \mathbb{Z}_p)$$

for all finite cyclic subextension L of K/k. Then (HNT) for $U_k(S)$ implies that

$$N_{K/k}(U_K(S) \otimes \mathbb{Z}_p) \subset \ker(\overline{\phi}_{K/k,S}) \subset (U_k(S) \cap N_{K/k}K^{\times}) \otimes \mathbb{Z}_p.$$

Definition 3.3. For an infinite procyclic extension K/k, if

$$\operatorname{rank}_{\mathbb{Z}_p}(\ker(\overline{\phi}_{K/k,S})) = \operatorname{rank}_{\mathbb{Z}_p}((U_k(S) \otimes \mathbb{Z}_p)^{\operatorname{univ}}),$$

then we say that $U_k(S) \otimes \mathbb{Z}_p$ satisfies (HNT) over K/k.

For a $\mathbb{Z}_p$-extension k_∞/k and for the set S of primes lying over p, we write $\phi_n := \phi_{k_n/k,S}$ for the map

$$\phi_n : U_k(p) \longrightarrow \prod_{v|p} \widehat{H}^0(G_{k_{n,w}/k_v}, k_{n,w}^\times) \cong \prod_{v|p} k_v^\times/N_{k_{n,w}/k_v} k_{n,w}^\times$$

and write

$$\ker(\phi_\infty) := \ker(\phi_{k_\infty,S}) = \bigcap_{n \geq 0} \ker(\phi_n).$$

Similarly, we write $\overline{\phi}_n := \phi_n \otimes 1$ for the map

$$\overline{\phi}_n : U_k(p) \otimes \mathbb{Z}_p \longrightarrow \prod_{v|p} \widehat{H}^0(G_{k_{n,w}/k_v}, k_{n,w}^\times) \cong \prod_{v|p} k_v^\times/N_{k_{n,w}/k_v} k_{n,w}^\times$$

and write

$$\ker(\overline{\phi}_\infty) := \ker(\overline{\phi}_{k_\infty,S}) = \bigcap_{n \geq 0} \ker(\overline{\phi}_n).$$

Remarks. 1. It is well known that the generalized Gross conjecture for (k, p) and Hasse's norm theorem(HNT) for $\mathcal{U}_k(p)$ over $k_\infty^{\mathrm{cyc}}/k$ are equivalent (cf. [9], [18] and [24]). All proofs use the Kuz'min's result on the invariant of the Tate module of k. In fact, independent of this, one can prove it in a more simple and direct way not using the structure of Tate module but using cohomology of the p-idele class group of k.

2. Notice that in general the first equality in the following equation does not hold

$$\mathrm{rank}_{\mathbb{Z}_p}(\ker(\overline{\phi}_\infty)) = \mathrm{rank}_{\mathbb{Z}_p}(\ker(\phi_\infty) \otimes \mathbb{Z}_p)$$
$$= \mathrm{rank}_{\mathbb{Z}_p}(U_k(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p).$$

For an abelian field k, since the conjectures of Leopoldt and Gross are true, we can apply the remark after Theorem 1.2. Then when k is abelian and $k_\infty = k_\infty^{\mathrm{cyc}}$, a counterexample to the above equality can be found easily by computing the ranks using Theorem 1.2, and Theorem 2 of [2], i.e.,

$$r_1 + r_2 = \mathrm{rank}_{\mathbb{Z}_p}(\ker(\overline{\phi}_\infty)) > r_2 - r' + 1 = \mathrm{rank}_{\mathbb{Z}}(k^\times)^{\mathrm{univ}}$$
$$\geq \mathrm{rank}_{\mathbb{Z}}(U_k(p)^{\mathrm{univ}}) = \mathrm{rank}_{\mathbb{Z}}(\ker(\phi_\infty))$$

where $r' = r$ or $r/2$ according as the decomposition field at p is real or imaginary.

3.2.

In this subsection, we will recall Hilbert's theorem 90 for compact sub-modules over a $\mathbb{Z}_p$-extension $k_\infty = \bigcup_{n \geq 0} k_n$ of k. Let A_n be a compact $\mathbb{Z}_p[G(k_n/k)]$-module with the norm maps $N_{m,n} : A_m \to A_n$ for all $m \geq n$. If we define $A_\infty^{(n)} := \varprojlim_{s \geq n} A_s$ to be the inverse limits of $\{A_s\}_{s \geq n}$ with respect to the norm maps $N_{s,n}$, then $A_\infty^{(n)}$ is a $\Lambda := \varprojlim_{s \geq n} \mathbb{Z}_p[G(k_s/k_n)]$-module.

Since A_n is compact, we have $A_n^{\mathrm{univ}} = A_n^{\mathrm{comp}}$ and hence we have

$$A_\infty^{(m)} = \varprojlim_{n \geq m} A_n = \varprojlim_{n \geq m} A_n^{\mathrm{univ}} = \varprojlim_{n \geq m} A_n^{\mathrm{comp}}.$$

As in the introduction, we say that A_k satisfy Hilbert's theorem 90 over k_∞/k if the following isomorphisms hold for all $n \geq 0$,

$$(A_\infty)_\Gamma \cong A_k^{\mathrm{univ}}$$

where $(A_\infty)_\Gamma = A_\infty/(\gamma - 1)A_\infty$.

Note that Hilbert's theorem 90 for each finite level of compact group A_n will show that

$$\frac{A_\infty^{(n)}}{\varprojlim_{s \geq n}((\gamma^{p^n} - 1)A_s)} \cong A_n^{\mathrm{univ}}.$$

In fact, by taking inverse limits in the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & (\gamma^{p^n} - 1)A_t & \longrightarrow & A_t & \xrightarrow{N_t} & N_t(A_t) & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle N_{t,s}} & & \downarrow{\scriptstyle N_{t,s}} & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & (\gamma^{p^n} - 1)A_s & \longrightarrow & A_s & \xrightarrow{N_s} & N_s(A_s) & \longrightarrow & 1
\end{array}
$$

we have the exact sequence

$$1 \longrightarrow \varprojlim_{s \geq n}((\gamma^{p^n} - 1)A_s) \longrightarrow A_\infty^{(n)} \longrightarrow \bigcap_{s \geq n} N_s(A_s) = A_n^{\mathrm{univ}} \longrightarrow 1$$

where inverse limit is exact over compact groups.

Since we have the natural inclusion

$$(\gamma^{p^n} - 1)A_\infty^{(n)} \subset \varprojlim_{s \geq n}((\gamma^{p^n} - 1)A_s),$$

Hilbert's theorem 90 for A_n over k_∞/k_n for all $n \geq 0$ will be satisfied if Hilbert's theorem 90 for A_n is satisfied for all $n \geq 0$, and if the above inclusion becomes equality

$$(\gamma^{p^n} - 1)A_\infty^{(n)} = \varprojlim_{s \geq n}((\gamma^{p^n} - 1)A_s).$$

Notice that the equality above is neither obvious nor easy to determine in general settings. We will show that the modules studied here satisfy the equality.

Let B_n be a compact $\mathbb{Z}_p[G(k_n/k)]$-module with norm maps defined as above and let $f_n : A_n \to B_n$ be the Galois equivariant map so that the following diagram is commutative

$$
\begin{array}{ccc}
A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} \\
N_{n+1,n} \downarrow & & \downarrow N_{n+1,n} \\
A_n & \xrightarrow{f_n} & B_n.
\end{array}
$$

Note that if we further assume that there is a Λ-module isomorphism

$$A_\infty \cong B_\infty$$

such that A_n and B_n also satisfy Hilbert's theorem 90 over k_∞/k for all $n \geq 0$, then we have $\mathbb{Z}_p[G(k_n/k)]$-module isomorphism

$$A_n^{\mathrm{univ}} \cong (A_\infty)_{\Gamma_n} \cong (B_\infty)_{\Gamma_n} \cong B_n^{\mathrm{univ}} \quad \text{for all } n \geq 0.$$

For the global p-units $U_n(p)$ of k_n, let $A_n = \mathcal{U}_n(p) = U_n(p) \otimes \mathbb{Z}_p$ and let $f_n = \overline{\Delta}_n$ be the map

$$\overline{\Delta}_n : \mathcal{U}_n(p) \longrightarrow \prod_{v|p} \widehat{k}_{n,v}^\times.$$

This is the $\mathbb{Z}_p$-linear extension of the diagonal embedding

$$\Delta_n : U_n(p) \longrightarrow \prod_{v|p} \widehat{k}_{n,v}^\times, \quad \alpha \mapsto (\alpha, \dots, \alpha)$$

where $\widehat{k}_{n,v}^\times = \varprojlim_n k_{n,v}^\times / k_{n,v}^{\times p^n}$ denotes the p-adic completion of the local field $k_{n,v}^\times$. Let $B_n = \overline{U}_n(p) := \overline{\Delta}_n(\mathcal{U}_n(p))$ be the image. Then $\overline{U}_n(p)$ is the topological closure of $U_n(p)$ under Δ_n in $\prod_{v|p} \widehat{k}_{n,v}^\times$.

We have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{U}_{n+1}(p) & \xrightarrow{\ \overline{\Delta}_{n+1}\ } & \overline{U}_{n+1}(p) \\
\scriptstyle{N_{n+1,n}}\big\downarrow & & \big\downarrow\scriptstyle{N_{n+1,n}} \\
\mathcal{U}_n(p) & \xrightarrow{\ \overline{\Delta}_n\ } & \overline{U}_n(p).
\end{array}
$$

Write

$$
\mathcal{U}_\infty^{(n)} := \varprojlim_{s \geq n} \mathcal{U}_n(p), \quad \overline{U}_\infty^{(n)} := \varprojlim_{s \geq n} \overline{U}_s(p), \quad \overline{\Delta}^{(n)} := \prod_{s \geq n} \overline{\Delta}_s
$$

and

$$
\mathcal{U}_\infty := \mathcal{U}_\infty^{(0)}, \quad \overline{U}_\infty := \overline{U}_\infty^{(0)}, \quad \overline{\Delta} := \overline{\Delta}^{(0)}.
$$

The isomorphism of two inverse limits above is known as the weak Leopoldt conjecture for k_∞/k and p,

$$
\overline{\Delta} : \mathcal{U}_\infty \cong \overline{U}_\infty.
$$

It is well known that the weak Leopoldt conjecture is true for the cyclotomic $\mathbb{Z}_p$-extension $k_\infty^{\mathrm{cyc}}/k$ for any number field k and p (cf. §3 of Ch 10 of [22]).

3.3.

In this subsection, we show that the affirmation of the generalized Gross conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ and Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$ implies the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$. We start with the basic lemmas without proofs which are well known.

Lemma 3.4. $\mathcal{U}_k(p)^{\mathrm{univ}} = \mathcal{U}_k(p)^{\mathrm{comp}}.$

For the local field k_v which is the completion of k at a finite place v, let $k_{v,\infty}$ be the corresponding $\mathbb{Z}_p$-extension of k_v. By the definition, we have $k_v^{\mathrm{comp}} = \pi(\varprojlim_n k_{v,n}^\times)$ where $k_{v,n}$ is the subfield of $k_{v,\infty}$ of degree p^n over k_v. Write $k^{\mathrm{loc}} = k^\times \bigcap_v k_v^{\mathrm{comp}}$ for the set of elements which are locally norm

comparable at all finite places of k

$$k^{\mathrm{loc}} = \{\alpha \in k^\times \,|\, \text{there is } (\alpha_{v,n}) \in \varprojlim_n k_{v,n}$$

$$\text{such that } \alpha_{v,0} = \alpha \text{ for all finite places } v\}.$$

By Hasse's local-global norm theorem, k^{loc} is equal to the group k^{univ} of the universal norms of $k^\times$. Since it is well known that k^{univ} is contained in the group of p-units (see Proposition 2.1 of [2]), we obtain the following lemma.

Lemma 3.5. $k^{\mathrm{loc}} = k^{\mathrm{univ}} = U_k(p) \bigcap_{n \geq 0} N_n k_n^\times.$

Let L be the maximal abelian unramified p-extension of k_∞ where all primes dividing p split completely. Then, obviously,

$$G(L/k_\infty) \cong T_p(k)$$

and L is Galois over k with the following exact sequence

$$1 \longrightarrow G(L/k_\infty) \longrightarrow G(L/k) \longrightarrow \Gamma \longrightarrow 1.$$

Let L_0 denote the maximal abelian subextension of L over k. Thus we have

(1) $$G(L_0/k_\infty) \cong T_p(k)_\Gamma$$

and the exact sequence

(2) $$1 \longrightarrow G(L_0/k_\infty) \longrightarrow G(L_0/k) \longrightarrow \Gamma \longrightarrow 1.$$

Let F_0 denote the maximal abelian extension of k containing k_∞ and unramified outside p, such that F/k_∞ splits completely at all primes dividing p. Let H denote the maximal unramified abelian extension of k such that all primes dividing p split completely and K the maximal p-extension of k in H.

Let S_v denote the kernel of the Artin map $k_v^\times \longrightarrow G(k_\infty/k)$. Then, the class field theory says that S_v is the group of universal norm elements of $k_v^\times$ over k_∞/k. By the class field theory, we have the following isomorphism

$$G(F_0/k) \cong J_k/A$$

where $A = \overline{k^\times \prod_{v \nmid p} U_v \prod_{v|p} S_v}$ is the topological p-adic closure of $k^\times \prod_{v \nmid p} U_v \cdot \prod_{v|p} S_v$ (see the last line of page 455 in [6] or the equation 3.4 of page 280

in [18]). There is exact sequence for the $G(H/k)$

$$1 \longrightarrow k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} k_v^{\times}/A \longrightarrow J_k/A \longrightarrow G(H/k) \longrightarrow 1$$

induced from $1 \longrightarrow k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} k_v^{\times} \longrightarrow J_k \longrightarrow G(H/k) \longrightarrow 1$.

Let V_p^n be the subgroup of $\prod_{v\mid p} U_v$ which are $\equiv 1 \bmod p^n$

$$V_p^n = \left\{ a \in \prod_{v\mid p} U_v \middle| \ a \equiv 1 \bmod p^n \right\}.$$

Then the groups $V_p^n k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v$ forms a fundamental system of neighborhoods of the topological p-adic closure A of $k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v$ and its closure is given as follows (see the proof of Theorem 7.8 of [21])

$$\overline{k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v} = \bigcap_n \left(V_p^n k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v \right).$$

The first term $G(F_0/H) \cong k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} k_v^{\times}/\overline{k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v}$ of the short exact sequence above leads to the following exact sequence (see p. 456 of [6] or page 280 of [18])

$$1 \longrightarrow X \longrightarrow \prod_{v\mid p}(k_v^{\times}/S_v) \longrightarrow G(F_0/H) \longrightarrow 1$$

where

$$X = \prod_{v\mid p} S_v \overline{\delta(U_k(p))}/\prod_{v\mid p} S_v$$

denotes the topological p-adic closure of the image $U_k(p) \longrightarrow \prod_{v\mid p}(k_v^{\times}/S_v)$.

In fact, we see that the first term $k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} k_v^{\times}/\overline{k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v}$ is isomorphic to

$$\prod_{v\mid p} k_v^{\times}/\prod_{v\mid p} k_v^{\times} \cap \overline{k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v} = \prod_{v\mid p} k_v^{\times}/\prod_{v\mid p} k_v^{\times} \cap \bigcap V_p^n k^{\times} \prod_{v\nmid p} U_v \prod_{v\mid p} S_v)$$

$$= \prod_{v\mid p} k_v^{\times}/\prod_{v\mid p} S_v(\prod_{v\mid p} k_v^{\times} \cap \bigcap V_p^n k^{\times} \prod_{v\nmid p} U_v)$$

Since

$$\prod_{v|p} S_v \left(\bigcap V_p^n \left(\prod_{v|p} k_v^\times \cap k^\times \prod_{v\nmid p} U_v \right) \right) = \prod_{v|p} S_v \left(\bigcap V_p^n (\delta(U_k(p))) \right)$$

$$= \prod_{v|p} S_v \overline{\delta(U_k(p))}$$

the first term above is isomorphic to

$$\prod_{v|p} (k_v^\times / S_v \overline{\delta(U_k(p))}) \cong \prod_{v|p} (k_v^\times / S_v) / \prod_{v|p} (S_v \overline{\delta(U_k(p))} / S_v).$$

From the identities above, the Artin map induces the exact sequence above. Moreover, it was shown from this that $G(F_0/H)$ is a p-group (see p. 456 of [6]), i.e.,

$$G(F_0/H) \cong G(L_0/K).$$

Let r be the number of primes of k dividing p. By the assumption of Gross conjecture, the $\mathbb{Z}_p$-rank of $G(F_0/H)$ is equal to one since the $\mathbb{Z}_p$-rank of $G(L_0/k_\infty) \cong T_p(k)_\Gamma$ is zero. Then, since $G(K/k)$ is finite, the rank of the topological p-adic closure $X(p)$ of X is given by

$$(3) \qquad\qquad \text{rank}\,(X(p)) = r - 1.$$

It follows from the isomorphism

$$X = \prod_{v|p} S_v \overline{\delta(U_k(p))} / \prod_{v|p} S_v \cong \overline{\delta(U_k(p))} / \overline{\delta(U_k(p))} \cap \prod_{v|p} S_v$$

that there exists a short exact sequence for $\overline{\delta(U_k(p))} \cap \prod_{v|p} S_v$

$$1 \longrightarrow \overline{\delta(U_k(p))} \cap \prod_{v|p} S_v \longrightarrow \overline{\delta(U_k(p))} \longrightarrow X \longrightarrow 1.$$

Note that the topological p-adic closure $\overline{\delta(U_k(p))}^\wedge$ of $\overline{\delta(U_k(p))}$ in $\prod_{v|p} \widehat{k}_v^\times$ is equal to $\overline{\Delta(U_k(p))}$,

$$\overline{\delta(U_k(p))}^\wedge = \overline{\Delta(U_k(p))}.$$

We can compare the $\mathbb{Z}_p$-ranks of the topological p-adic closures of each terms above

$$\text{rank}\,(\overline{\Delta(U_k(p))}) = \text{rank}\,(\Omega) + \text{rank}\,(X(p))$$

where Ω denotes the topological p-adic closure of $\overline{\delta(U_k(p))} \cap \prod_{v|p} S_v$.

We may assume that k_∞/k is totally ramified at primes dividing p since the Leopoldt conjecture descends to a subfield (see Corollary 10.3.11 of [22]), i.e., the validity of the Leopoldt conjecture for a number field K implies the validity of the Leopoldt conjecture for any subfield of K. We will use the same notation v for the prime of k_n lying over p as that of k.

It follows from §3.1 that

$$U_k(p) \cap \prod_{v|p} N_{n_v} k_{n,v}^\times = \ker(\phi_n)$$

and that δ induces $\tilde{\delta}$

$$1 \longrightarrow U_k(p)/\ker(\phi_n) \xrightarrow{\tilde{\delta}} \prod_{v|p} k_v^\times / N_{n_v} k_{n,v}^\times$$

where $k_{n,v}$ denotes the completion of k_n at a prime dividing v and N_{n_v} denotes the norm map from $k_{n,v}$ to k_v. By taking topological closure inside $\prod_{v|p} N_{n_v} k_{n,v}^\times$ to the above identity, we have

$$\overline{\left(\delta(U_k(p) \cap \prod_{v|p} N_{n_v} k_{n,v}^\times)\right)} = \overline{\delta(U_k(p))} \cap \prod_{v|p} N_{n_v} k_{n,v}^\times = \overline{\delta(\ker(\phi_n))}.$$

This leads to

$$\overline{\delta(U_k(p))} \cap \prod_{v|p} S_v = \bigcap_n \overline{(\delta(U_k(p))} \cap \prod_{v|p} N_{n_v} k_{n,v}^\times)$$

$$= \bigcap_n \overline{\delta(\ker(\phi_n))}$$

By the generalized Gross conjecture, it follows that

$$(\overline{\delta(\ker(\phi_n))} : \overline{\delta(N_n(U_n))}) < (\delta(\ker(\phi_n)) : \delta(N_n(U_n))) < \infty$$

independently of $n \geq 0$ and that

$$\operatorname{rank} \overline{\delta(U_k(p))} \cap \prod_{v|p} S_v = \operatorname{rank} \bigcap_n \overline{\delta(N_n(U_n))} = \operatorname{rank} \bigcap_n N_n(\overline{\delta(U_n)})$$

$$= \operatorname{rank} \overline{\delta(U_k(p))}^{\text{univ}}.$$

By taking closures, the inclusion above results in

$$\text{rank } \Omega = \text{rank } (\overline{\delta(U_k(p))} \cap \prod_{v|p} S_v)^\wedge = \text{rank } (\overline{\delta(U_k(p))}^{\text{univ}})^\wedge$$

$$= \text{rank } \overline{\Delta(U_k(p))}^{\text{univ}}.$$

Here, the identity $(\overline{\delta(U_k(p))}^{\text{univ}})^\wedge = \overline{\Delta(U_k(p))}^{\text{univ}}$ follows from the the following argument using the assumption that k_∞/k is totally ramified at all primes dividing p.

By the local Kronecker-Weber theorem, the intermediate field $k_{v,n}/k_v$ is contained in a composite field $k_{v,n}^{(f)} L_m$ of the field L_m of the π^m-division points of Lubin-Tate extensions and the unramified extension $k_{v,n}^{(f)}$ of degree f(see the proof of Corollary 7.7 of [21]).

By taking the ramification indices and linear disjointness over k_v of L_m and $k_{v,n}^{(f)}$ into account, we see that $k_{v,n}$ is contained in L_m. Since the prime element π_v is a norm element of a prime element λ_m of L_m (see Theorem 7.4 of [21]), π_v is the norm of the prime element $N_{L_m/k_{v,n}} \lambda_m$ of $k_{v,n}$. It follows that any prime element π_v dividing p is a universal norm element over $k_{v,\infty}/k_v$. Since the p-adic closure $\widehat{k_v}$ of $k_v^\times = \pi_v^{\mathbb{Z}} \oplus U_v$ is given by

$$\widehat{k_v} = \pi_v^{\mathbb{Z}_p} \oplus U_v^1$$

where U_v^1 is the principal unit at v, we have

$$(\widehat{k_v})^{\text{univ}} = \pi_v^{\mathbb{Z}_p} \oplus (U_v^1)^{\text{univ}}$$

using universal norm property of π_v. Since the norm map is continuous, it follows that $(U_v^1)^{\text{univ}}$ is a closed subgroup of $U_v^1 = \widehat{U_v}$ and that $((U_v^1)^{\text{univ}})^\wedge = (U_v^1)^{\text{univ}}$, i.e.,

$$\widehat{U_v^{\text{univ}}} = (\mu_q \times (U_v^1)^{\text{univ}})^\wedge = (U_v^1)^{\text{univ}} = \widehat{U_v}^{\text{univ}}$$

where $U_v = \mu_q \times U_v^1$ for $q = \#(U_v/U_v^1)$. It follows that the p-adic closure $\widehat{(k_v^{\text{univ}})}$ of k_v^{univ} is given by

$$\widehat{(k_v^{\text{univ}})} = (\pi_v^{\mathbb{Z}} \oplus U_v^{\text{univ}})^\wedge = \pi_v^{\mathbb{Z}_p} \oplus (U_v^1)^{\text{univ}} = (\widehat{k_v})^{\text{univ}}.$$

Note that the difference between $\overline{\delta(U_k(p))} \subset \prod k_v^\times$ and $\overline{\Delta(U_k(p))} \subset \prod \widehat{k_v^\times}$ occurs only in the cyclic groups generated by each prime π_v dividing p, one

$\pi_v^{\mathbb{Z}}$ and the other $\pi_v^{\mathbb{Z}_p}$ which are, by the assumption, already norm compara-
ble, and its torsion subgroups in unit-parts which are also norm comparable.
Hence the closure $(\overline{\delta(U_k(p))}^{\text{univ}})^\wedge$ is equal to $\overline{\Delta(U_k(p))}^{\text{univ}}$ as was claimed.
It follows from Theorems 3.6 and 3.7 of §3.4 that

$$\text{rank } \mathcal{U}_k(p)^{\text{univ}} = \text{rank } (U_k'(p) \otimes \mathbb{Z}_p)^{\text{univ}} = \text{rank } (\mathcal{U}_\infty')_\Gamma = r_1 + r_2.$$

Since the weak Leopoldt conjecture is true for the cyclotomic $\mathbb{Z}_p$-extension
of any number field, we have (see Corollary 10.3.24 and Theorem 10.3.25
of [22])

$$\overline{\Delta(U_\infty(p))} := \varprojlim \overline{\Delta(U_n(p))} \cong \varprojlim (\mathcal{U}_n(p)) = \mathcal{U}_\infty.$$

The above isomorphism and Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ lead to
the following lower bound of rank (Ω)

$$
\begin{aligned}
\text{rank } (\Omega) \ \geq \ \text{rank } (\overline{\Delta(U_k(p))}^{\text{univ}}) &= \text{rank } (\overline{\Delta(U_\infty(p))})_\Gamma \\
&= \text{rank } (\mathcal{U}_\infty)_\Gamma \\
&= \text{rank } \mathcal{U}_k(p)^{\text{univ}} \\
&= r_1 + r_2.
\end{aligned}
$$

Hence it results in the Leopoldt conjecture

$$
\begin{aligned}
\text{rank } (\overline{\Delta(U_k(p))}) &= \text{rank } (X(p)) + \text{rank } (\Omega) \\
&\geq r - 1 + r_1 + r_2.
\end{aligned}
$$

This shows that if we assume that k_∞/k is totally ramified, then the gener-
alized Gross conjecture and Hilbert's theorem 90 imply the Leopoldt conjec-
ture. Since for an arbitrary number field k, there exists a sufficiently large n
such that k_∞/k_n is totally ramified at all primes dividing p (cf. §§3.4 and 6.2
of [12]), we have shown that for such n, the assumption implies the validity
of the Leopoldt conjecture for k_n. Since the Leoplodt conjecture descends
to a subfield, we obtain the validity of the Leopoldt conjecture for k from
k_n. This completes the proof. $\qquad \square$

3.4.

In this subsection, we will show that the Leopoldt conjecture for (k, p)
implies Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ over k_∞^{cyc}/k.

For the p-units $U_n(p)$ of k_n, let $U'_n(p) = U_n(p)/\mu(U_n(p))$ be the quotient of $U_n(p)$ by its torsion $\mu(U_n(p))$, i.e., the $\mathbb{Z}$-free part of $U_n(p)$ and let

$$\mathcal{U}'_\infty = \varprojlim(U'_n(p) \otimes \mathbb{Z}_p)$$

denote the inverse limit of $U'_n(p) \otimes \mathbb{Z}_p$ with respect to the norm maps. The rank of $\mathcal{U}'_\infty$ as a Γ-module is computed in the following theorem which is Theorem 7.2 of [18]. He proves this over the cyclotomic $\mathbb{Z}_p$-extension of k. This is reproved in a generalized form by Greither (see Theorem of [9]).

In Theorem 3.7, we give a proof of the rank of $(U'_k(p) \otimes \mathbb{Z}_p)^{\text{univ}}$ for any $\mathbb{Z}_p$-extension of k using a result of Iwasawa together with an argument of compactness and hence different from those of Kuz'min and Greither. Note that the proof of Greither uses the direct limits rather than the inverse limits of ours. Even if it seems to be dual each other, the proof given below we think is more simple and direct.

Theorem 3.6. *Let k be a number field and let r_1 and r_2 be the number of real and complex places of k. Then $\mathcal{U}'_\infty$ is a free Γ-module of rank $r_1 + r_2$.*

Proof. See the proof of Theorem 7.2 of [18]. Since the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_p$-extension of a number field, we can find another proof from Corollary 10.3.24 and Theorem 11.3.11 of [22].

In fact, the statement (ii) of Theorem 11.3.11 of *loc.cit* holds with $S'_\infty = S^{cd} = \varnothing$ for the cyclotomic $\mathbb{Z}_p$-extension over any number field (cf. Remark and Corollary 11.3.12 of *loc.cit*).   □

Theorem 3.7. *Let $k_\infty = \bigcup_{n \geq 0} k_n$ be an arbitrary $\mathbb{Z}_p$-extension of k. Then*

$$\text{rank } (\mathcal{U}'_\infty)_\Gamma = \text{rank } (U'_k(p) \otimes \mathbb{Z}_p)^{\text{univ}} = r_1 + r_2.$$

Proof. For a $\mathbb{Z}[G(k_n/k)]$-module M, we have

$$1 \longrightarrow H^{-1}(G(k_n/k), M) \longrightarrow M/(\gamma - 1)M \xrightarrow{N_n} N_n(M) \longrightarrow 1.$$

Since the $\mathbb{Z}$-module $\mathbb{Z}_p$ is flat (see (iii) of Theorem 3 of §3.4 of [4]), we have

$$N_n(M) \otimes_\mathbb{Z} \mathbb{Z}_p = N_n(M \otimes_\mathbb{Z} \mathbb{Z}_p)$$
$$(M/(\gamma - 1)M) \otimes \mathbb{Z}_p = (M \otimes \mathbb{Z}_p)/(\gamma - 1)(M \otimes \mathbb{Z}_p).$$

By tensoring $\mathbb{Z}_p$, the exact sequence leads to

$$1 \longrightarrow H^{-1}(G(k_n/k), M) \otimes \mathbb{Z}_p$$
$$\longrightarrow M \otimes \mathbb{Z}_p/(\gamma - 1)(M \otimes \mathbb{Z}_p) \overset{N_n \otimes 1}{\longrightarrow} N_n(M \otimes \mathbb{Z}_p) \longrightarrow 1.$$

This shows the following isomorphism

$$H^{-1}(G(k_n/k), M) \otimes \mathbb{Z}_p \cong H^{-1}(G(k_n/k), M \otimes \mathbb{Z}_p).$$

By putting $M = k_n^\times$ and using Hilbert's theorem 90 for $k_n^\times$, the above isomorphism leads to

$$\ker(N_n \otimes 1) = (\gamma - 1)(k_n^\times \otimes \mathbb{Z}_p).$$

Hence we have the following commutative diagram

$$1 \longrightarrow ((\gamma - 1)(k_{n+1}^\times \otimes \mathbb{Z}_p)) \cap \mathcal{U}_{n+1}(p)^{\mathrm{univ}} \longrightarrow \mathcal{U}_{n+1}(p)^{\mathrm{univ}}$$

$$N_{n+1,n} \Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow N_{n+1,n}$$

$$1 \longrightarrow ((\gamma - 1)(k_n^\times \otimes \mathbb{Z}_p)) \cap \mathcal{U}_n(p)^{\mathrm{univ}} \longrightarrow \mathcal{U}_n(p)^{\mathrm{univ}}$$

$$\overset{N_{n+1} \otimes 1}{\longrightarrow} \mathcal{U}_k(p)^{\mathrm{univ}} \longrightarrow 1$$

$$\Big\downarrow id$$

$$\overset{N_n \otimes 1}{\longrightarrow} \mathcal{U}_k(p)^{\mathrm{univ}} \longrightarrow 1$$

where the surjectiveness of each rows follows from

$$\mathcal{U}_n(p)^{\mathrm{univ}} = \mathcal{U}_n(p)^{\mathrm{comp}}$$

which is Lemma 3.4. Write

$$V_n = ((\gamma - 1)(k_n^\times \otimes \mathbb{Z}_p)) \cap \mathcal{U}_n(p).$$

By taking inverse limits with respect to the norm maps and using exactness of inverse limit over compact groups, we have

$$1 \longrightarrow \varprojlim_n V_n \longrightarrow \mathcal{U}_\infty \longrightarrow \mathcal{U}_k(p)^{\mathrm{univ}} \longrightarrow 1.$$

Lemma 3.8. $\varprojlim_n V_n = (\gamma - 1)\mathcal{U}_\infty.$

Proof. To prove Lemma 3.8, we need the following result of Iwasawa.

Lemma 3.9 (Iwasawa). *The order $\#(H^1(G(k_n/k), U_n(p)))$ is bounded independently of n.*

Proof. This follows immediately from [12]. More precisely, since every primes outside p are unramified over any $\mathbb{Z}_p$-extension k_∞/k, Proposition 3 of [12] and the five term exact sequence of Hochschild-Serre spectral sequence lead to the proof of Lemma 3.9 (see Corollary 2.4.2 of [22] and Proposition 13.2 of [25]). $\qquad\square$

Notice that from Hilbert's theorem 90,

$$H^1(G(k_n/k), U_n(p)) \cong H^{-1}(G(k_n/k), U_n(p)) = \frac{((\gamma - 1)k_n^\times) \cap U_n(p)}{(\gamma - 1)U_n(p)}.$$

We claim that for $s \geq n$, the following inclusion map induced from the identity map is injective

$$\frac{((\gamma - 1)k_n^\times) \cap U_n(p)}{(\gamma - 1)U_n(p)} \longrightarrow \frac{((\gamma - 1)k_s^\times) \cap U_s(p)}{(\gamma - 1)U_s(p)}.$$

For an $(\gamma - 1)\alpha \in (\gamma - 1)k_n^\times \cap U_n(p)$, let

$$(\gamma - 1)\alpha = (\gamma - 1)e$$

for some $e \in U_s(p)$. Then $a\alpha = e \in U_s(p) \cap k_n^\times = U_n(p)$ for some $a \in k^\times$. Hence

$$(\gamma - 1)\alpha = (\gamma - 1)(a\alpha) \in (\gamma - 1)U_n(p).$$

which leads to the injection of the natural map induced from the inclusion

$$1 \longrightarrow \frac{((\gamma - 1)k_n^\times) \cap U_n(p)}{(\gamma - 1)U_n(p)} \longrightarrow \frac{((\gamma - 1)k_s^\times) \cap U_s(p)}{(\gamma - 1)U_s(p)}.$$

By Lemma 3.9, we have the following lemma.

Lemma 3.10. *For all sufficiently large $s \geq n \gg 0$, the inclusion map induces an isomorphism*

$$\frac{((\gamma - 1)k_n^\times) \cap U_n(p)}{(\gamma - 1)U_n(p)} \xrightarrow{\cong} \frac{((\gamma - 1)k_s^\times) \cap U_s(p)}{(\gamma - 1)U_s(p)} < \infty.$$

Since $H^{-1}(G(k_n/k), U_n(p)) = (\gamma - 1)k_n^\times \cap U_n(p)/(\gamma - 1)U_n(p)$ is a p-group and the norm map $N_{s,n}$ is the p^{s-n} power map for all sufficiently large $s \geq n$ by Lemma 3.10, we have

$$\varprojlim \left(\frac{(\gamma - 1)k_n^\times \cap U_n(p)}{(\gamma - 1)U_n(p)} \right) = 1.$$

This results in the following corollary.

Corollary 3.11. $\varprojlim((\gamma - 1)k_n^\times \cap U_n(p)) = \varprojlim((\gamma - 1)U_n(p))$.

Since $\mathbb{Z}_p$ is flat over $\mathbb{Z}$, we have

$$1 \to (\gamma - 1)U_n(p) \otimes \mathbb{Z}_p \longrightarrow ((\gamma - 1)k_n^\times \cap U_n(p)) \otimes \mathbb{Z}_p$$
$$\longrightarrow \left(\frac{(\gamma - 1)k_n^\times \cap U_n(p)}{(\gamma - 1)U_n(p)} \right) \otimes \mathbb{Z}_p \to 1.$$

By taking inverse limits with respect to the norm maps over compact groups, we have

$$1 \to \varprojlim((\gamma - 1)U_n(p) \otimes \mathbb{Z}_p) \to \varprojlim(((\gamma - 1)k_n^\times \cap U_n(p)) \otimes \mathbb{Z}_p)$$
$$\to \varprojlim \left(\left(\frac{(\gamma - 1)k_n^\times \cap U_n(p)}{(\gamma - 1)U_n(p)} \right) \otimes \mathbb{Z}_p \right) \to 1.$$

Since $H^{-1}(G(k_n/k), U_n(p))$ is a p-group, it follows from Corollary 3.11 that

$$\varprojlim \left(\left(\frac{(\gamma - 1)k_n^\times \cap U_n(p)}{(\gamma - 1)U_n(p)} \right) \otimes \mathbb{Z}_p \right) = \varprojlim \left(\frac{(\gamma - 1)k_n^\times \cap U_n(p)}{(\gamma - 1)U_n(p)} \right) = 1$$

and hence the exact sequence leads to the following isomorphism

$$\varprojlim V_n = \varprojlim(((\gamma - 1)k_n^\times \cap U_n(p)) \otimes \mathbb{Z}_p) = \varprojlim((\gamma - 1)U_n(p) \otimes \mathbb{Z}_p)$$

where we used

$$((\gamma - 1)k_n^\times \cap U_n(p)) \otimes \mathbb{Z}_p = ((\gamma - 1)k_n^\times \otimes \mathbb{Z}_p) \cap (U_n(p) \otimes \mathbb{Z}_p) = V_n$$

since $\mathbb{Z}_p$ is a flat over $\mathbb{Z}$ (see Remark 1 of §2.6, Chapter I of [3]).

The exact sequence $1 \longrightarrow U_k(p) \longrightarrow U_n(p) \xrightarrow{\gamma-1} (\gamma-1)U_n(p) \longrightarrow 1$ leads to

$$1 \longrightarrow U_k(p) \otimes \mathbb{Z}_p \longrightarrow U_n(p) \otimes \mathbb{Z}_p \xrightarrow{\gamma-1} (\gamma-1)U_n(p) \otimes \mathbb{Z}_p \longrightarrow 1.$$

By taking inverse limits over compact groups, we have

$$(\gamma-1)\mathcal{U}_\infty \cong \varprojlim((\gamma-1)U_n(p) \otimes \mathbb{Z}_p).$$

We obtain the proof of the claim

$$\varprojlim V_n \cong (\gamma-1)\mathcal{U}_\infty$$

and thereby complete the proof of Lemma 3.8. $\qquad\square$

Lemma 3.8 and Theorem 3.6 complete the proof of Theorem 3.7. $\qquad\square$

From the weak Leopoldt conjecture for (k,p), Theorems 3.6 and 3.7 lead to

$$\text{rank}(\varprojlim \overline{\Delta(U_n(p))})_\Gamma = r_1 + r_2.$$

It follows from the natural surjective map $(\varprojlim \overline{\Delta(U_n(p))})_\Gamma \to \overline{\Delta(U_k(p))}^{\text{univ}} \to 1$ that $\overline{\Delta(U_k(p))}^{\text{univ}}$ has $\mathbb{Z}_p$-rank at most $r_1 + r_2$. From the Leopoldt conjecture for (k,p), $\overline{\Delta}_k$ induces an injection $1 \to (\mathcal{U}_k(p))^{\text{univ}} \to \overline{\Delta(U_k(p))}^{\text{univ}}$ which shows from Theorem 3.7 that $\overline{\Delta(U_k(p))}^{\text{univ}}$ has $\mathbb{Z}_p$-rank at least $r_1 + r_2$ and hence exactly $r_1 + r_2$. It follows that

$$\overline{\Delta(U_k(p))}^{\text{univ}} \cong (\mathcal{U}_k(p))^{\text{univ}} \cong (\mathcal{U}_\infty)_\Gamma \cong (\varprojlim \overline{\Delta(U_n(p))})_\Gamma$$

which is Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$. This completes the proof of Theorem 1.2. $\qquad\square$

Remark. We notice from the proof of Theorem 1.2 that if k_∞^{cyc}/k is totally ramified then the affirmation of the Leopoldt conjecture for (k,p) is equivalent to the affirmation of Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ over k_∞^{cyc}/k.

<div align="center">

3.5.

</div>

In this subsection we prove Corollary 1.3 of the introduction and explain special cases of the corollary. We recall Corollary 1.3 of the introduction.

Corollary 3.12. (i) *If k is a totally real field then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to the affirmation of the Gross conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ and Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$.*

(ii) *If k is a CM-field with a maximal real subfield k^+ then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to the affirmation of the Gross conjecture for $(k_\infty^{+\mathrm{cyc}}, p)$ and Hilbert's theorem 90 for $\overline{\Delta(U_{k_n^+}(p))}$ over $k_\infty^{+\mathrm{cyc}}/k_n^+$ for all $n \geq 0$.*

Proof. It is well known that the Leopoldt conjecture implies the generalized Gross conjecture when the base field k is totally real. In fact, we immediately recover this from the equations (1) and (2) in the proof of Theorem 1.2 since the Leopoldt conjecture is equivalent to the claim that k has only one $\mathbb{Z}_p$-extension when k is totally real. Hence from Theorem 1.2, we prove the first claim of the corollary.

For a CM-field k with its maximal real subfield k^+, it is also well known that the affirmation of the Leopoldt conjecture for (k, p) is equivalent to the affirmation of the Leopoldt conjecture for (k^+, p) (see Corollary 10.3.11 of [22]). Note that for each $n \geq 0$, the intermediate field k_n^+ of $k_\infty^{+\mathrm{cyc}} = k^+\mathbb{Q}_\infty^{\mathrm{cyc}}$ is totally real. Hence from Theorem 1.2 and the first claim of the corollary, we can prove the second claim of the corollary. This completes the proof of the corollary. $\qquad\square$

When the number $r_k(p)$ of primes of k dividing p is one, we have the following lemma which is well known. We prove this using a remark of Iwasawa.

Lemma 3.13. *For a number field k, if $r_k(p) = 1$, then the generalized Gross conjecture holds for (k, p).*

Proof. Let $u(k_\infty/k)$ be the set of primes of k which are ramified in k_∞/k. Then it follows from the elementary Iwasawa theory and the class field theory that $u(k_\infty/k)$ is a nonempty subset of the set of primes of k dividing p. By the assumption of the lemma, the cardinality $\sharp(u(k_\infty/k))$ is equal to one for any $\mathbb{Z}_p$-extension k_∞ of k. We recall the observation of Iwasawa that a $\mathbb{Z}_p$-extension k_∞ of k such that $\sharp(u(k_\infty/k))$ is minimum in the family of all $\mathbb{Z}_p$-extensions of k must satisfy one of the equivalent conditions (i),(ii) and (iii) of §2 (see p. 804 of [13]). By taking $k_\infty = k_\infty^{\mathrm{cyc}}$, we see that the generalized Gross conjecture holds for (k, p). This completes the proof of the lemma. $\qquad\square$

Remarks. 1. For every $\mathbb{Z}_p$-extension of k, there exists an integer $n(k) \geq 0$ such that every prime which ramifies in $k_\infty/k_{n(k)}$ is totally ramified (see Lemma 13.3 of [25]). Hence note that for all $n \geq n(k)$, the number $r_\infty(p)$ of primes of k_∞ lying over p is equal to that of k_n, i.e., $r_\infty(p) = r_{k_n}(p)$.

2. As special cases of Corollary 3.12, if the real subfields in (i) and (ii) of Corollary 3.12 have only one prime lying over p, then Lemma 3.13 provides us with more simple descriptions.

(i) If k is a totally real field with $r_\infty(p) = 1$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$.

(ii) If k is a CM-field with the maximal real subfield k^+ such that $r_{\infty+}(p) = 1$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_{k_n^+}(p))}$ over $k_\infty^{+\mathrm{cyc}}/k_n^+$ for all $n \geq 0$.

3. Now let k be an arbitrary number field which is not necessarily totally real with $r_\infty(p) = 1$. We notice that due to Lemma 3.13, if $r_\infty(p) = 1$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is still equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$. This statement can also be recovered in the proof of Theorem 1.2. In fact the proof of Theorem 1.2 use the generalized Gross conjecture essentially only for (3) in the proof of Theorem 1.2 which is automatically satisfied when $r_\infty(p) = 1$.

4. Finally let k be an arbitrary number field which is not necessarily totally real with $r_\infty(p) > 1$. In this case, we may apply a result of [24] with the assumption that k is Galois containing μ_p and $r_\infty(p) = 2$. In this case the result shows that the generalized Gross conjecture holds for $(k_\infty^{\mathrm{cyc}}, p)$. In fact if k is Galois containing μ_p with $r_\infty(p) = 2$, then it follows that for each $n \geq 0$, k_n is also Galois containing μ_p and $r_{k_n}(p) = 2$. Hence if k is Galois containing μ_p with $r_\infty(p) = 2$, then the affirmation of the Leopoldt conjecture for $(k_\infty^{\mathrm{cyc}}, p)$ is equivalent to Hilbert's theorem 90 for $\overline{\Delta(U_n(p))}$ over $k_\infty^{\mathrm{cyc}}/k_n$ for all $n \geq 0$.

3.6.

We are informed that results of Jaulent (cf. [15] and [16]) can also be used to recover a part of Theorem 1.2 using results of [18] and [24]. Under the assumption of the generalized Gross conjecture for $(K_\infty^{\mathrm{cyc}}, p)$, we will show that the Leopoldt conjecture for $(K_\infty^{\mathrm{cyc}}, p)$ implies the condition of Hilbert's theorem 90 for $\overline{\Delta(U_k(p))}$ over $k_\infty^{\mathrm{cyc}}/k$.

We use the logarithmic description introduced by Jaulent in [15] to compute here the $\mathbb{Z}_p$-rank of Ω. Let r_1, r_2 and s be the numbers of real places, complex places and respectively places dividing p in K. We use the following definitions from [15].

(i) μ_K the p-subgroup of the group of roots of unity in K;

(ii) $\mathcal{E}'_K = \mathbb{Z}_p \otimes_{\mathbb{Z}} E'_K$ the p-adification of the group of p-units $E'_K \cong \mu_K \mathbb{Z}^{r_1+r_2+s-1}$;

(iii) $\widetilde{\mathcal{E}}_K$ the subgroup of logarithmic units of K;

(iv) $\mu_K^{loc} = (\mathbb{Z}_p \otimes K^\times) \cap \prod \mu_{\mathfrak{p}}$;

(v) $\mathcal{R}_p = \prod_{\mathfrak{p}|p} \mathcal{R}_{\mathfrak{p}}$ the p-adic compactification of the semi-local product $\prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$;

(vi) $\mu_p = \prod_{\mathfrak{p}|p} \mu_{\mathfrak{p}}$ the torsion subgroup of $\mathcal{R}_p$.

From the proposition 1.1 of [24], it follows that the Γ-invariant $T_p(K)^\Gamma$ of the Tate module $T_p(K)$ of K is given by

$$T_p(K)^\Gamma = \frac{\widetilde{\mathcal{E}}_K}{(U_K(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}.$$

If we denote by $\delta_G(K)$ the defect of the generalized Gross conjecture(GGC) of K, i.e.,

$$\delta_G(K) := \mathrm{rank}_{\mathbb{Z}_p} T_p(K)$$

then it follows from Theorem 3.7 that

$$\widetilde{\mathcal{E}}_K \cong \mu_K \mathbb{Z}_p^{r_1+r_2+\delta_G(K)}.$$

This is in fact Proposition 3.4 of [15].

Since μ_K^{loc} is the kernel of the semi-localization homomorphism

$$s_p : \widetilde{\mathcal{E}}_K \longrightarrow \mathcal{R}_p/\mu_p,$$

we can write

$$\mu_K^{loc} \cong \mu_K \mathbb{Z}_p^{\delta_L(K)}$$

where $\delta_L(K)$ denotes the defect of the Leopoldt conjecture(GLC) of K (cf. Scolie 2.13 of [16]) and hence

$$s_p(\widetilde{\mathcal{E}}_K) \cong \widetilde{\mathcal{E}}_K/\mu_K^{loc} \cong \mathbb{Z}_p^{r_1+r_2-(\delta_L(K)-\delta_G(K))}$$

where $s_p(\widetilde{\mathcal{E}}_K)$ is just the group Ω. Hence we obtain immediately the following statement. If one assumes $rk_{\mathbb{Z}_p}(\Omega) = rk_{\mathbb{Z}_p}(s_p(\widetilde{\mathcal{E}}_K)) = r_1 + r_2$, then it follows that

$$\delta_L(K) = \delta_G(K).$$

In special this implies that (GLC) $\Longleftrightarrow$ (GGC).

Moreover under the assumption (GGC) for $(K_\infty^{\mathrm{cyc}}, p)$, we will show that (GLC) for $(K_\infty^{\mathrm{cyc}}, p)$ implies the condition of Hilbert's theorem 90(H90) for $\overline{\Delta(U_k(p))}$ over $k_\infty^{\mathrm{cyc}}/k$. Put $G_n := G(K_n/K)$. By Proposition 2 of [9], the generalized Gross conjecture for $(K_\infty^{\mathrm{cyc}}, p)$ implies that the quotients $\widetilde{\mathcal{E}}_{K_n}/\mu_{K_n}$ are free over the group algebra $\mathbb{Z}_p[G_n]$ of rank $r_1 + r_2$. Thus, in this case, the logarithmic groups $\widetilde{\mathcal{E}}_{K_n}$ have trivial cohomology.

It follows from the weak Leopoldt conjecture over $K_\infty^{\mathrm{cyc}}/K$ that the defect $\mu_{K_n}^{loc}/\mu_{K_n}$ of the Leopoldt conjecture stabilizes in $K_\infty^{\mathrm{cyc}}/K$.

Hence the exact sequence

$$1 \longrightarrow \mu_{K_n}^{loc} \longrightarrow \widetilde{\mathcal{E}}_{K_n} \longrightarrow s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow 1$$

and the well known fact that $H^i(G_n, \mu_{K_n}) = 1$ for all $i \geq 0$ result in the following isomorphisms

$$H^1(G_n, s_p(\widetilde{\mathcal{E}}_{K_n})) \cong H^2(G_n, \mu_{K_n}^{loc}) = H^2(G_n, \mu_{K_n}^{loc}/\mu_{K_n}) \cong (\mathbb{Z}_p/p^n\mathbb{Z}_p)^{\delta_L(K_\infty^{\mathrm{cyc}})}.$$

It follows that

$$H^1(G_n, s_p(\widetilde{\mathcal{E}}_{K_n})) = 1 \Longleftrightarrow \delta_L(K_n) = 0.$$

By taking inverse limits to the exact sequence of compact modules obtained from $H^1(G_n, s_p(\widetilde{\mathcal{E}}_{K_n})) = 1$,

$$1 \longrightarrow (\gamma_n - 1)s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow N_n s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow 1$$

it follows that

$$1 \longrightarrow \varprojlim(\gamma_n - 1)s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow \overline{\Delta(U_\infty(p))} \longrightarrow \overline{\Delta(U_K(p))}^{\mathrm{univ}} \longrightarrow 1.$$

The exact sequence $1 \longrightarrow \ker(\gamma - 1) \longrightarrow s_p(\widetilde{\mathcal{E}}_{K_n}) \xrightarrow{\gamma - 1} (\gamma - 1)s_p(\widetilde{\mathcal{E}}_{K_n}) \longrightarrow 1$ leads to

$$(\gamma - 1)\overline{\Delta(U_\infty(p))} \cong \varprojlim((\gamma - 1)s_p(\widetilde{\mathcal{E}}_{K_n}))$$

since $\varprojlim \ker(\gamma - 1) = 1$. Hence we have

$$\delta_L(K_n) = 0 \text{ for all } n \geq 0 \Longleftrightarrow H^1(G_n, s_p(\widetilde{\mathcal{E}}_{K_n})) = 1 \text{ for all } n \geq 0 \Longrightarrow (H90).$$

For the reverse direction $(H90) \implies \delta_L(K_n) = 0$, we need to apply again §3.3.

Acknowledgments

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