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Artin representations of $\mathbb{Q}$ of dihedral type

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We prove an asymptotic formula for the number of Artin representations of $\mathbb{Q}$ of dihedral type with conductor below a given bound.

The representations described in the title are precisely the two-dimensional *irreducible monomial* Artin representations of $\mathbb{Q}$, and the number of isomorphism classes of such representations of conductor $\leqslant x$ will be denoted $\vartheta^{\mathrm{im}}(x)$. Our aim is to derive an asymptotic formula for $\vartheta^{\mathrm{im}}(x)$. Put

$$\kappa = \frac{\pi}{2} \sum_{K} d_K^{-5/2} \zeta_K(0)^2/\zeta_K(2)^2,$$

where K runs over imaginary quadratic fields (viewed here as subfields of some fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$), d_K is the absolute value of the discriminant of K, and $\zeta_K(s)$ is the Dedekind zeta function of K.

Theorem 1. $\vartheta^{\mathrm{im}}(x) \sim \kappa x^2$.

Equivalently, for a positive integer N, the number of isomorphism classes of two-dimensional irreducible monomial Artin representations of $\mathbb{Q}$ of conductor N is on average κN. By contrast, if we replace "irreducible monomial" by "irreducible primitive" then the average number of isomorphism classes is conjecturally $O(N^{\varepsilon})$ for every $\varepsilon > 0$, and in fact Bhargava and Ghate [2] have proved the remarkable result that the number of octahedral isomorphism classes of prime conductor and odd determinant (thus corresponding to modular forms of weight one and prime level) is on average *bounded.* Other work in this domain has focused on bounds for the number of forms of a given level rather than on asymptotic averages over all levels; cf. Serre [13], Duke [3], Wong [16], Michel and Venkatesh [11], Ellenberg [4], Klüners [9], and Ganguly [6]. In particular, Michel and Venkatesh [11] give the upper bound $O(N^{1/2+\varepsilon})$ for the number of irreducible monomial isomorphism classes of conductor N and fixed determinant. This estimate takes account of Maass forms of eigenvalue 1/4 as well as modular forms of weight one.

More relevant to Theorem 1 than the literature just cited, however, are the asymptotic formulas of Siegel [15] for class numbers of primitive binary quadratic forms. Indeed let $\vartheta^{\mathrm{io}}(x)$ be the number of isomorphism classes of two-dimensional *irreducible orthogonal* – and hence in dimension two automatically monomial – Artin representations of $\mathbb{Q}$ of conductor $\leqslant x$. Thus $\vartheta^{\mathrm{io}}(x)$ counts the representations which are dihedral (in other words, the image is a dihedral group) rather than merely of dihedral type (the image of the associated *projective* representation is a dihedral group). Put $\lambda = \pi/(36\zeta(3)^2)$, where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$. Thus

$$\lambda = 4\pi\zeta_{\mathbb{Q}}(-1)^2/\zeta_{\mathbb{Q}}(3)^2.$$

Siegel's formulas have the following corollary:

Theorem 2. $\vartheta^{\mathrm{io}}(x) \sim \lambda x^{3/2}$.

We emphasize that our "proof" of Theorem 2 is simply a matter of reinterpreting the formulas in [15], and the only reason for including Theorem 2 here at all is to provide an appropriate context for Theorem 1. As for Theorem 1 itself, our proof was inspired by the paper of Goldfeld and Hoffstein [7], and the Dirichlet series $B(s)$ and $C(s)$ appearing in the proof are closely related to $Z_-(s,s)$ and $Z_+(s,s)$ respectively, where $Z_\pm(\rho,w)$ is the two-variable zeta function introduced in [7]. The proof of Theorem 1 will occupy Sections 1 through 6 of this note, and then in Section 7 we will deduce Theorem 2 from [15].

1. Outline of the proof of Theorem 1

Throughout, number fields are regarded as subfields of a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. If K is a number field and ρ is an Artin representation of K then the conductor of ρ is an an integral ideal $\mathfrak{q}(\rho)$ of K, and its absolute norm will be $q(\rho)$. If $K = \mathbb{Q}$ then $q(\rho)$ is the unique positive generator of $\mathfrak{q}(\rho)$, and we refer to $q(\rho)$ itself as the conductor of ρ.

Now consider pairs (K,ξ), where K is a quadratic field and ξ is a one-dimensional Artin representation of K, or in other words a character $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$. Let d_K be the absolute value of the discriminant of K. Since there are only finitely many pairs (K,ξ) such that $d_K q(\xi)$ lies below a given bound, we can put

$$\alpha(x) = \sum_{\substack{(K,\xi) \\ d_K q(\xi) \leqslant x}} 1.$$

We define $\beta(x)$ and $\gamma(x)$ in the same way but with summation restricted to pairs (K, ξ) such that K is imaginary quadratic or real quadratic respectively.

Given a pair (K, ξ) as above, write $\mathrm{ind}_{K/\mathbb{Q}}\xi$ for the Artin representation of $\mathbb{Q}$ induced by ξ. If we put $\rho = \mathrm{ind}_{K/\mathbb{Q}}\xi$, then $q(\rho) = d_K q(\xi)$, and thus $\alpha(x)$ counts pairs (K, ξ) such that $q(\rho) \leqslant x$. Since $\vartheta^{\mathrm{im}}(x)$ counts *isomorphism classes* of *irreducible* such ρ, there are three sources of discrepancy between $\vartheta^{\mathrm{im}}(x)$ and $\alpha(x)$:

(i) There exist pairs (K, ξ) for which $\mathrm{ind}_{K/\mathbb{Q}}\xi$ is reducible. Indeed let η_K be the quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ trivial on $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. A reducible representation ρ is of the form $\mathrm{ind}_{K/\mathbb{Q}}\xi$ precisely when $\rho \cong \chi \oplus \chi\eta_K$ with an arbitrary character χ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Furthermore, the isomorphism class of ρ determines the pair (K, ξ) uniquely, because η_K is the ratio of the direct summands χ and $\chi\eta_K$ in either order, and then η_K determines K while $\xi = \chi|\mathrm{Gal}(\overline{\mathbb{Q}}/K) = \chi\eta_K|\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

(ii) On the other hand, if $\rho = \mathrm{ind}_{K/\mathbb{Q}}\xi$ is irreducible, then there are precisely *two* characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ which induce ρ, namely the direct summands ξ and ξ' in $\rho|\mathrm{Gal}(\overline{\mathbb{Q}}/K) \cong \xi \oplus \xi'$.

(iii) An irreducible ρ can sometimes be induced from more than one quadratic field. We shall see that in this situation ρ can be induced from precisely *three* quadratic fields, of which at least one is real quadratic. Thus in addition to the two pairs (K, ξ) and (K, ξ') corresponding to ρ as in (ii), there are four more: two each for the other two quadratic fields.

A two-dimensional irreducible monomial Artin representation of $\mathbb{Q}$ which can be induced from more that one quadratic field will be referred to as a *Hecke-Shintani representation*, because examples of such representations were noted by Hecke [8] (see pp. 425-426 of the *Math. Werke*) and Shintani [14], p. 158.

Let $\vartheta^{\mathrm{rm}}(x)$ be the number of isomorphism classes of two-dimensional *reducible monomial* Artin representations of $\mathbb{Q}$ of conductor $\leqslant x$, and let

$\vartheta^{\rm HS}(x)$ be the number of isomorphism classes of Hecke-Shintani representations of conductor $\leqslant x$. It follows from (i), (ii), and (iii) that

$$\alpha(x) = 2\vartheta^{\rm im}(x) + \vartheta^{\rm rm}(x) + 4\vartheta^{\rm HS}(x). \tag{1}$$

On the other hand, we also have

$$\alpha(x) = \beta(x) + \gamma(x) \tag{2}$$

(recall that β and γ counts pairs (K, ξ) with K imaginary and K real respectively). We shall prove the following relations:

$$\beta(x) \sim 2\kappa x^2, \tag{3}$$

$$\gamma(x) = O(x^2/\log x), \tag{4}$$

$$\vartheta^{\rm rm}(x) = O(x(\log x)^3). \tag{5}$$

Since every Hecke-Shintani representation can be induced from a real quadratic field, it follows from (4) that

$$\vartheta^{\rm HS}(x) = O(x^2/\log x). \tag{6}$$

Theorem 1 now follows from (1), (2), (3), (4), (5), and (6).

After an elementary remark about Dirichlet series, we verify (3), (4), and (5) in Sections 3, 4, and 5 respectively. In Section 6 we verify (iii), on which both (1) and (6) depend.

2. A remark about Dirichlet series

Suppose that we are given a sequence of Dirichlet series $D_\nu(s) = \sum_{n\geqslant 1} a_\nu(n)n^{-s}$ for $\nu \geqslant 1$. We say that the series $\sum_{\nu\geqslant 1} D_\nu(s)$ is *formally convergent* if for each n there are only finitely many ν such that $a_\nu(n) \neq 0$. If this condition holds then we can consider the finite sum $a(n) = \sum_{\nu\geqslant 1} a_\nu(n)$, and putting $D(s) = \sum_{n\geqslant 1} a(n)n^{-s}$, we say that $\sum_{\nu\geqslant 1} D_\nu(s)$ is *formally convergent to* $D(s)$. We write $\sum_{\nu\geqslant 1} D_\nu(s) = D(s)$ with the understanding that this equation is merely a formal identity.

Let c be a real number. If the Dirichlet series $D_\nu(s)$ converges (in the analytic sense) for $\Re(s) > c$, then it defines a holomorphic function on the right half-plane H defined by $\Re(s) > c$, and we can consider the series of holomorphic functions $\sum_{\nu\geqslant 1} D_\nu(s)$ on H. As usual, we say that $\sum_{\nu\geqslant 1} D_\nu(s)$ is *normally convergent on compact subsets of H* if for every compact subset

$\Omega \subset H$ there is a sequence of real numbers $M_\nu \geqslant 0$ such that $\sum_{\nu \geqslant 1} M_\nu$ converges and $|D_\nu(s)| \leqslant M_\nu$ for all $\nu \geqslant 1$ and $s \in \Omega$.

Proposition 1. *Suppose that $a_\nu(n) \geqslant 0$ for all $\nu, n \geqslant 1$ and that as a Dirichlet series, $D_\nu(s)$ converges for $s \in H$. If $\sum_{\nu \geqslant 1} D_\nu(s)$ is both formally convergent to $D(s)$ and normally convergent on compact subsets of H, then $D(s)$ converges as a Dirichlet series for $s \in H$ and $\sum_{\nu \geqslant 1} D_\nu(s) = D(s)$ as holomorphic functions.*

Proof. Our hypotheses imply that for $s \in H$ the double sum

$$\sum_{\nu \geqslant 1} \sum_{n \geqslant 1} a_\nu(n) n^{-s}$$

is absolutely convergent, so the order of summation can be reversed. □

3. Imaginary quadratic fields

To begin with let K be an arbitrary number field. Write $\mathcal{O}_K$ for the ring of integers of K and $\mathfrak{q}$ for an arbitrary nonzero ideal of $\mathcal{O}_K$. By class field theory, the number of characters $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$ of conductor $\mathfrak{q}$ equals the number of primitive ray class characters of K of conductor $\mathfrak{q}$. We denote this number $h_K^*(\mathfrak{q})$.

Let $h_K^{\mathrm{nar}}(\mathfrak{q})$ be the narrow ray class number of K to the modulus $\mathfrak{q}$. Then $h_K^{\mathrm{nar}}(\mathfrak{q})$ is the number of ray class characters of K to the modulus $\mathfrak{q}$, or equivalently the number of *primitive* ray class characters of K of conductor *dividing* $\mathfrak{q}$. Hence the usual inclusion-exclusion argument gives

$$h_K^*(\mathfrak{q}) = \sum_{\mathfrak{q}' | \mathfrak{q}} \mu_K(\mathfrak{q}') h_K^{\mathrm{nar}}(\mathfrak{q}/\mathfrak{q}'), \tag{7}$$

where $\mu_K(\mathfrak{q}) = (-1)^t$ if $\mathfrak{q}$ is the product of t distinct prime ideals and $\mu_K(\mathfrak{q}) = 0$ if $\mathfrak{q}$ is divisible by the square of a prime ideal.

Put $U_K = \mathcal{O}_K^\times$, and let $U_K^+(\mathfrak{q})$ be the subgroup of U_K consisting of totally positive units congruent to 1 modulo $\mathfrak{q}$. Also put $\varphi_K(\mathfrak{q}) = |(\mathcal{O}_K/\mathfrak{q})^\times|$, or equivalently,

$$\varphi_K(\mathfrak{q}) = \sum_{\mathfrak{q}' | \mathfrak{q}} \mu_K(\mathfrak{q}') \mathbf{N}(\mathfrak{q}/\mathfrak{q}'). \tag{8}$$

According to a classic formula (cf. [10], p. 127, Theorem 1),

$$h_K^{\mathrm{nar}}(\mathfrak{q}) = 2^{r_1(K)} \cdot h_K \cdot \varphi_K(\mathfrak{q}) / [U_K : U_K^+(\mathfrak{q})], \tag{9}$$

where $r_1(K)$ is the number of real embeddings and h_K the class number of K.

Now suppose that K is an imaginary quadratic field. Then the wide ray class number of K to the modulus $\mathfrak{q}$, which we denote simply $h_K(\mathfrak{q})$, is indistinguishable from $h_K^{\mathrm{nar}}(\mathfrak{q})$, and (9) becomes

$$h_K(\mathfrak{q}) = h_K \cdot \varphi_K(\mathfrak{q}) \cdot (w_K(\mathfrak{q})/w_K), \tag{10}$$

where w_K is the number of roots of unity in K and $w_K(\mathfrak{q})$ is the number of roots of unity congruent to 1 modulo $\mathfrak{q}$. From (8) we have

$$\sum_{\mathfrak{q}} \varphi_K(\mathfrak{q})(\mathbf{N}\mathfrak{q})^{-s} = \zeta_K(s-1)/\zeta_K(s),$$

so multiplying both sides of (10) by $(\mathbf{N}\mathfrak{q})^{-s}$ and summing over $\mathfrak{q}$, we obtain

$$\sum_{\mathfrak{q}} h_K(\mathfrak{q})\ (\mathbf{N}\mathfrak{q})^{-s} = (h_K/w_K)\zeta_K(s-1)/\zeta_K(s) + \mathcal{E}_K(s) \tag{11}$$

with

$$\mathcal{E}_K(s) = (h_K/w_K) \sum_{\mathfrak{q}} \varphi_K(\mathfrak{q})(w_K(\mathfrak{q}) - 1)(\mathbf{N}\mathfrak{q})^{-s}. \tag{12}$$

The error term $\mathcal{E}_K(s)$ is a finite Dirichlet series, because if $w_K(\mathfrak{q}) \neq 1$ then there is a root of unity $\zeta \in K$, $\zeta \neq 1$, such that $\zeta \equiv 1$ modulo $\mathfrak{q}$. Then $\mathfrak{q}$ divides $\zeta - 1$, and consequently $\mathfrak{q}$ divides 2 or 3. In fact if $K \neq \mathbb{Q}(\sqrt{-3})$ then $\mathfrak{q}|2$.

We would like to transform (11) into an expression for the Dirichlet series

$$B_K(s) = \sum_{\mathfrak{q}} h_K^*(\mathfrak{q})\ (d_K\mathbf{N}\mathfrak{q})^{-s}. \tag{13}$$

From (7) we see that it suffices to divide both sides of (11) by $d_K^s\zeta_K(s)$:

$$B_K(s) = (h_K/w_K)d_K^{-s}\zeta_K(s-1)/\zeta_K(s)^2 + d_K^{-s}\mathcal{E}_K(s)/\zeta_K(s). \tag{14}$$

Let η_K be the primitive quadratic Dirichlet character determined by K, and replace $\zeta_K(s-1)$ by $\zeta(s-1)L(s-1,\eta_K)$ on the right-hand side of (14).

The result is

$$B_K(s) = \zeta(s-1)L_K(s) + \mathcal{F}_K(s) \tag{15}$$

with

$$L_K(s) = (h_K/w_K)d_K^{-s}L(s-1,\eta_K)/\zeta_K(s)^2 \tag{16}$$

and

$$\mathcal{F}_K(s) = d_K^{-s}\mathcal{E}_K(s)/\zeta_K(s). \tag{17}$$

Here we pause to note that we are in the situation of Propositon 1. Indeed it follows from (13) and (14) that $B_K(s)$ has nonnegative coefficients and converges for $\Re(s) > 2$. Furthermore, given $n \geqslant 1$ there are only finitely many pairs $(K, \mathfrak{q})$ such that $d_K\mathbf{N}\mathfrak{q} = n$, and consequently $\sum_K B_K(s)$ is formally convergent. Put

$$b(n) = \sum_{\substack{(K,\mathfrak{q})\\ d_K\mathbf{N}\mathfrak{q}=n}} h_K^*(\mathfrak{q})$$

and $B(s) = \sum_{n\geqslant 1} b(n)n^{-s}$; then $\sum_K B_K(s)$ is formally convergent to $B(s)$.

An alternative formula for $b(n)$ is

$$b(n) = \sum_{\substack{(K,\xi)\\ d_K q(\xi)=n}} 1,$$

where ξ runs over characters $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{C}^\times$. The latter formula shows that β is the summatory function associated to $B(s)$; in other words, $\beta(x) = \sum_{n\leqslant x} b(n)$. The significance of this remark is that we can determine the asymptotic behavior of $\beta(x)$ by applying a tauberian theorem (cf. [1], p. 154, Theorem 7.7). Indeed (3) is now a consequence of the following statement:

Proposition 2. *The Dirichlet series $B(s) = \sum_{n\geqslant 1} b(n)n^{-s}$ converges in the region $\Re(s) > 2$, whence $B(s)$ is a holomorphic function in this half-plane. Furthermore, $B(s)$ extends to a meromorphic function for $\Re(s) > 7/4$, and the only pole of $B(s)$ in the latter region is a simple pole at $s = 2$ with residue 4κ.*

To prove Proposition 2, it suffices to show that $\sum_K L_K(s)$ and $\sum_K \mathcal{F}_K(s)$ are normally convergent on compact subsets of the region $\Re(s) > 7/4$. For then it follows from (15) that $\sum_K B_K(s)$ is normally convergent on compact subsets of the region $\Re(s) > 2$. Consequently Proposition 1 gives the convergence of $B(s)$ as a Dirichlet series and the equality of holomorphic functions $B(s) = \sum_K B_K(s)$. This equality is in the first instance valid only for $\Re(s) > 2$, but since $\sum_K L_K(s)$ and $\sum_K \mathcal{F}_K(s)$ are holomorphic for $\Re(s) > 7/4$, Proposition 2 follows by analytic continuation, although the residue of $B(s)$ comes out as $\sum_K L_K(2)$. Setting $s = 2$ in (16) and making the substitutions $h_K/w_K = -\zeta_K(0)$ and $L(1,\eta_K) = -2\pi\zeta_K(0)/\sqrt{d_K}$, we obtain the stated value 4κ.

It remains to prove that $\sum_K L_K(s)$ and $\sum_K \mathcal{F}_K(s)$ are normally convergent on compact subsets of the region $\Re(s) > 7/4$.

Lemma 1. $h_K < \sqrt{d_K}\ \log d_K$.

Proof. If K is $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-4})$ then $h_K = 1$ and the inequality is satisfied, so we may assume that $d_K > 4$. Now it is an elementary fact that for a nonprincipal Dirichlet character χ to the modulus q,

$$|L(1,\chi)| < 2 + \log q \tag{18}$$

(cf. [5], p. 262, Théorème 8.2). Taking χ to be η_K and applying Dirichlet's class number formula, we deduce from (18) that

$$\pi h_K/\sqrt{d_K} < 2 + \log d_K. \tag{19}$$

As $d_K > 4 > e$, we have $2 + \log d_K < 3\log d_K$, so the lemma follows from (19). □

Lemma 2. *Fix a compact subset Ω of the region $\Re(s) \geqslant 7/4$. For $s \in \Omega$,*

$$\frac{|L(s-1,\eta_K)|}{|\zeta_K(s)|^2} < c_\Omega\, d_K^{1/4},$$

where c_Ω is a positive constant depending only on Ω, not on K.

Proof. For a nonprincipal Dirichlet character χ to the modulus q and $\Re(s) \geqslant 3/4$,

$$|L(s,\chi)| < 16(q|s|)^{1/4} \tag{20}$$

(cf. [5], p. 260, Théorème 8.1). If $s \in \Omega$ then $\Re(s-1) \geqslant 3/4$, and consequently (20) holds with s replaced by $s-1$. Taking $\chi = \eta_K$, we deduce that if $s \in \Omega$ then

$$|L(s-1,\eta_K)| < b_\Omega\, d_K^{1/4}, \tag{21}$$

where $b_\Omega = \max_{s\in\Omega} 16|s-1|^{1/4}$. On the other hand, for $\Re(s) \geqslant 7/4$, $|\zeta_K(s)|^{-1} \leqslant c_0$ with $c_0 = \prod_p \max((1+p^{-7/4})^2, 1+p^{-7/2})$. Thus we may take $c_\Omega = b_\Omega c_0^2$. □

Suppose now that Ω is a compact subset of the region $\Re(s) > 7/4$. Choose $\varepsilon > 0$ so that Ω is contained in the region $\Re(s) \geqslant 7/4 + \varepsilon$. Using Lemmas 1 and 2 to estimate the right-hand side of (16), we find that for $s \in \Omega$,

$$|L_K(s)| \leqslant \sqrt{d_K}\log d_K \cdot d_K^{-7/4-\varepsilon} \cdot c_\Omega d_K^{1/4}. \tag{22}$$

Since $\sum_K d_K^{-1-\varepsilon}\log d_K$ converges, $\sum_K L_K(s)$ is indeed normally convergent on Ω. As for $\sum_K \mathcal{F}_K(s)$, recall that $w_K(\mathfrak{q}) - 1 \neq 0$ only if $\mathfrak{q}$ divides 2 or 3. Hence the sum over $\mathfrak{q}$ in (12) is bounded by an absolute constant c_1, whence $|\mathcal{E}_K(s)| \leqslant c_1 h_K$. Referring to (17) and Lemma 1, we see that

$$|\mathcal{F}_K(s)| \leqslant c_0 c_1 d_K^{-5/4-\varepsilon}\log d_K,$$

where c_0 is as in the proof of Lemma 2. Again, the normal convergence of $\sum_K \mathcal{F}_K(s)$ on Ω follows.

4. Real quadratic fields

Now we repeat the argument with imaginary quadratic fields replaced by real quadratic fields. But an asymptotic formula is out of our reach in this case, so we do not bother with some of the niceties of Section 3.

All along we have regarded number fields as subfields of some fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, but now we fix an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ as well. Thus a real quadratic field K is a subfield of $\mathbb{R}$, and one can speak of the fundamental unit $\epsilon_K > 1$ of K. In fact given any nonzero ideal $\mathfrak{q}$ of $\mathcal{O}_K$, let $\epsilon_K^+(\mathfrak{q}) > 1$ be the fundamental totally positive unit congruent to 1 modulo $\mathfrak{q}$, or in other words, the unique generator of $U_K^+(\mathfrak{q})$ in the interval $(1,\infty)$.

Lemma 3. $\epsilon_K^+(\mathfrak{q}) > ((d_K/4)\mathbf{N}\mathfrak{q})^{1/4}$.

Proof. Write m_K for the largest positive square-free divisor of d_K, so that m_K is $d_K/4$ or d_K according as d_K is 0 or 1 modulo 4. Then $\epsilon_K^+(\mathfrak{q}) = (a + b\sqrt{m_K})/2$ with rational integers $a, b \geqslant 1$ satisfying $a^2 - b^2 m_K = 4$. So $a^2 \geqslant$

$4 + m_K \geqslant 6$, and consequently $a \geqslant 3$. Put $n = (a-2)^2 - b^2 m_K$. Writing $n = (a^2 - b^2 m_K) + 4 - 4a$, we see that $n = 8 - 4a$. Since $a > 2$ we deduce that $n < 0$.

On the other hand, $\mathfrak{q}$ divides $\epsilon_K^+(\mathfrak{q}) - 1$, and therefore $\mathbf{N}\mathfrak{q}$ divides $N_{K/\mathbb{Q}}(\epsilon_K^+(\mathfrak{q}) - 1)$, which is $n/4$. Since $n < 0$ it follows that $-n > 4\mathbf{N}\mathfrak{q}$ or in other words that

$$b^2 m_K \geqslant 4\mathbf{N}\mathfrak{q} + (a-2)^2. \tag{23}$$

Recall once again that $a^2 - b^2 m_K = 4$, whence $a \geqslant \sqrt{4 + m_K}$. Put $c = \sqrt{3} - \sqrt{2}$. Since $\sqrt{4+x} - 2 \geqslant c\sqrt{x}$ for $x \geqslant 2$ we have $a - 2 \geqslant c\sqrt{m_K}$. Thus (23) gives

$$b^2 m_K \geqslant 4\mathbf{N}\mathfrak{q} + c^2 m_K. \tag{24}$$

On the right-hand side of (24) we use the inequality $A^2 + B^2 \geqslant 2AB$ with $A = 2\sqrt{\mathbf{N}\mathfrak{q}}$ and $B = c\sqrt{m_K}$, and then we take the positive square root of both sides:

$$b\sqrt{m_K} \geqslant 2\sqrt{c}(m_K \mathbf{N}\mathfrak{q})^{1/4}. \tag{25}$$

As $a = \sqrt{4 + b^2 m_K} > b\sqrt{m_K}$ we see that (25) also holds (with strict inequality) when the left-hand side is replaced by a. So

$$\epsilon_K^+(\mathfrak{q}) = (a + b\sqrt{m_K})/2 > 2\sqrt{c}(m_K \mathbf{N}\mathfrak{q})^{1/4}. \tag{26}$$

The assertion of the lemma is weaker than (26), because

$$\sqrt{c} = (\sqrt{3} - \sqrt{2})^{1/2} = 0.563770\cdots > 1/2$$

and $4m_K \geqslant d_K$. □

We also need an analogue of Lemmas 1 and 2. As before, write η_K for the primitive quadratic Dirichlet character associated to K.

Lemma 4. $h_K \log \epsilon_K < \sqrt{d_K} \log d_K$.

Proof. If $d_K = 5$ then $h_K = 1$ and $\epsilon_K = (1 + \sqrt{5})/2$, and the inequality is immediate. Henceforth we assume $d_K \geqslant 8$. Applying (18) with $\chi = \eta_K$ and recalling Dirichlet's class number formula, we obtain

$$2h_K \log \epsilon_K/\sqrt{d_K} < 2 + \log d_K.$$

Since $d_K \geqslant 8 > e^2$, we have $2 + \log d_K \leqslant 2\log d_K$, and the lemma follows. □

Lemma 5. *Fix a compact subset Ω of the region $\Re(s) \geqslant 7/4$. For $s \in \Omega$,*

$$|L(s-1, \eta_K)| < c_\Omega d_K^{1/4},$$

where c_Ω is a positive constant depending only on Ω, not on K.

Proof. Follow the proof of Lemma 2 up to (21), and take $c_\Omega = b_\Omega$. □

Our goal is to prove (4). By definition,

$$\gamma(x) = \sum_{\substack{(K,\mathfrak{q}) \\ d_K \mathbf{N}\mathfrak{q} \leqslant x}} h_K^*(\mathfrak{q}),$$

where $(K, \mathfrak{q})$ runs over pairs consisting of a real quadratic field K and a nonzero ideal $\mathfrak{q}$ of $\mathcal{O}_K$. We can apply (9) here: Since $h_K^*(\mathfrak{q}) \leqslant h_K^{\mathrm{nar}}(\mathfrak{q})$ and $\varphi_K(\mathfrak{q}) \leqslant \mathbf{N}\mathfrak{q}$ while

$$[U_K : U_K^+(\mathfrak{q})] = 2(\log \epsilon_K^+(\mathfrak{q}))/(\log \epsilon_K),$$

we obtain

$$\gamma(x) \leqslant 2 \sum_{\substack{(K,\mathfrak{q}) \\ d_K \mathbf{N}\mathfrak{q} \leqslant x}} h_K \mathbf{N}\mathfrak{q} (\log \epsilon_K)/(\log \epsilon_K^+(\mathfrak{q})).$$

Thus Lemmas 3 and 4 give

$$\gamma(x) \leqslant 8 \sum_{\substack{(K,\mathfrak{q}) \\ d_K \mathbf{N}\mathfrak{q} \leqslant x}} \sqrt{d_K} (\log d_K) \mathbf{N}\mathfrak{q} / \log((d_K/4)\mathbf{N}\mathfrak{q}). \tag{27}$$

For $n \geqslant 1$ put

$$c(n) = \sum_{\substack{(K,\mathfrak{q}) \\ d_K \mathbf{N}\mathfrak{q} = n}} \sqrt{d_K} (\log d_K) \mathbf{N}\mathfrak{q}.$$

Since $d_K \geqslant 5$, we see that $c(n) = 0$ for $n \leqslant 4$. We shall prove the estimate

$$\sum_{5 \leqslant n \leqslant x} c(n)/\log(n/4) = O(x^2/\log x), \tag{28}$$

which in conjunction with (27) gives (4).

Consider the Dirichlet series $C(s) = \sum_{n \geqslant 1} c(n) n^{-s}$ and the family of Dirichlet series

$$C_K(s) = \sqrt{d_K}(\log d_K) \sum_{\mathfrak{q}} (\mathbf{N}\mathfrak{q})(d_K \mathbf{N}\mathfrak{q})^{-s}.$$

The sum $\sum_K C_K(s)$ is formally convergent to $C(s)$, and each $C_K(s)$ is a convergent Dirichlet series for $\Re(s) > 2$ because

$$C_K(s) = \sqrt{d_K}(\log d_K) d_K^{-s} \zeta_K(s-1). \tag{29}$$

Furthermore, writing (29) in the form $C_K(s) = \zeta(s-1) M_K(s)$ with

$$M_K(s) = \sum_K \sqrt{d_K}(\log d_K) d_K^{-s} L(s-1, \eta_K),$$

we have $C(s) = \zeta(s-1) \sum_K M_K(s)$, and if Ω is a compact subset of the region $\Re(s) > 7/4$ then we can bound $M_K(s)$ on Ω using Lemma 5 just as we bounded $L_K(s)$ on Ω using Lemma 2 in (22). The upshot is that $\sum_K M_K(s)$ is normally convergent on compact subsets of $\Re(s) > 7/4$. Appealing to Proposition 1 as before, we conclude that $C(s)$ is a convergent Dirichlet series for $\Re(s) > 2$, equal to $\sum_K C_K(s)$ as a holomorphic function. Thus:

Proposition 3. *$C(s)$ extends to a meromorphic function for $\Re(s) > 7/4$ which is holomorphic in this region except for a simple pole at $s = 2$.*

Put $\tilde{\gamma}(x) = \sum_{n \leqslant x} c(n)$. It follows from Proposition 3 that $\tilde{\gamma}(x) \sim cx^2$ for some constant $c > 0$. In particular, $\tilde{\gamma}(x) = O(x^2)$. But Abel summation gives

$$\sum_{5 \leqslant n \leqslant x} c(n)/\log(n/4) = \frac{\tilde{\gamma}(x)}{\log(x/4)} + \int_5^x \frac{\tilde{\gamma}(t)}{t(\log(t/4))^2}\, dt. \tag{30}$$

Since $\tilde{\gamma}(x) = O(x^2)$, the right-hand side of (30) is $O(x^2/\log x)$, proving (28) and hence (4).

5. The reducible case

Let X be the set of characters $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$, H the subset of quadratic characters, and Θ^{rm} the set of isomorphism classes of Artin representations of $\mathbb{Q}$ of the form $\chi \oplus \chi\eta$, with $\chi \in X$ and $\eta \in H$. We define a map $X \times H \to \Theta^{\mathrm{rm}}$ by

$$(\chi, \eta) \mapsto [\chi \oplus \chi\eta], \tag{31}$$

where $[*]$ denotes the isomorphism class of $*$. If (χ, η) and $(\chi'\eta')$ have the same image under (31), then either $\chi' = \chi$ and $\chi'\eta' = \chi\eta$, in which case (χ, η) and (χ', η') are equal, or else $\chi' = \chi\eta$ and $\chi'\eta' = \chi$, in which case $(\chi', \eta') = (\chi\eta, \eta)$. Thus (31) is a two-to-one surjective map.

Now put $N = H \cup \{1\}$, where 1 denotes the trivial character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We consider the Dirichlet series

$$D(s) = \sum_{(\chi,\nu)\in X\times N} (q(\chi)q(\chi\nu))^{-s}$$

and the associated summatory function

$$\delta(x) = \sum_{\substack{(\chi,\nu)\in X\times N \\ q(\chi)q(\chi\nu)\leqslant x}} 1.$$

Writing

$$D(s) = \sum_{(\chi,\eta)\in X\times H} (q(\chi)q(\chi\eta))^{-s} + \sum_{\chi\in X} q(\chi)^{-2s}$$

and passing to the summatory functions corresponding to the three Dirichlet series in this equation, we obtain the relation

$$\delta(x) = 2\vartheta^{\mathrm{rm}}(x) + \theta(x^{1/2}), \tag{32}$$

where $\theta(x)$ is the number of complex-valued characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of conductor $\leqslant x$ and the factor 2 in $2\vartheta^{\mathrm{rm}}(x)$ takes account of the two-to-one map (31).

Proposition 4. *The Dirichlet series $D(s)$ converges for $\Re(s) > 1$, whence $D(s)$ is a holomorphic function in this region. Furthermore, $D(s)$ extends to a meromorphic function for $\Re(s) > 3/4$, and the only pole of $D(s)$ in the latter region is a pole of order 4 at $s = 1$.*

Granting the proposition, we deduce that $\delta(x) \sim cx(\log x)^3$ with $c > 0$. On the other hand, $\theta(x) \sim 18x^2/\pi^4$ (cf. [12], p. 461), whence $\theta(x^{1/2}) \sim 18x/\pi^4$ and in particular $\theta(x^{1/2}) = O(x)$. Thus (32) gives $\vartheta^{\mathrm{rm}}(x) \sim cx(\log x)^3/2$ and (5) follows. It remains to prove Propostion 4.

Let $X_p \subset X$ be the subset of characters unramified outside p and ∞. Every element $\chi \in X$ can be written in a unique way as a product $\prod_p \chi_p$ with $\chi_p \in X_p$ and $\chi_p = 1$ for all but finitely many p. Likewise each $\nu \in N$ has a unique decomposition $\nu = \prod_p \nu_p$ with $\nu_p \in N_p$, where $N_p = N \cap X_p$. It follows that $D(s) = \prod_p D_p(s)$ with

$$D_p(s) = \sum_{(\chi_p, \nu_p) \in X_p \times N_p} (q(\chi_p) q(\chi_p \nu_p))^{-s}. \tag{33}$$

Since we haven't yet checked convergence, the decomposition $D(s) = \prod_p D_p(s)$ should be regarded for the moment as a purely formal identity.

Now fix p and consider the pth Euler factor in (33). To begin with we assume that p is odd. Then $N_p = \{1, \eta_p\}$, where η_p is the unique quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside p and ∞. The following four cases constitute a partition of $X_p \times N_p$:

(i) If $(\chi_p, \nu_p) = (1, 1)$ then $q(\chi_p) q(\chi_p \nu_p) = 1$.

(ii) If $(\chi_p, \nu_p) = (1, \eta_p)$ or (η_p, η_p), then $q(\chi_p) q(\chi_p \nu_p) = p$.

(iii) If $(\chi_p, \nu_p) = (\eta_p, 1)$ then $q(\chi_p) q(\chi_p \nu_p) = p^2$.

(iv) If $\chi_p \notin N_p$ then $q(\chi_p) q(\chi_p \nu_p) = q(\chi_p)^2$.

Consequently

$$D_p(s) = 1 + 2p^{-s} + p^{-2s} + 2(p-3)p^{-2s} + 2 \sum_{k \geqslant 2} \psi(p^k) p^{-2ks}, \tag{34}$$

where $\psi(p^k) = (p-1)^2 p^{(k-2)}$. Note that if $k \geqslant 2$ then $\psi(p^k)$ is the number of elements of X_p of conductor p^k, while $p - 3$ is the number of elements of $X_p \smallsetminus N_p$ of conductor p. The 2 in $2(p-3)$ and $2\psi(p^k)$ is $|N_p|$.

The case $p = 2$ is similar but more tedious. Let η_4 denote the quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of conductor 4, and let η_8^+ and η_8^- denote respectively the even and odd quadratic characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of conductor 8. We have the following division into cases:

(i) If $(\chi_2, \nu_2) = (1, 1)$ then $q(\chi_2) q(\chi_2 \nu_2) = 1$.

(ii) If $(\chi_2, \nu_2) = (1, \eta_4)$ or (η_4, η_4), then $q(\chi_2) q(\chi_2 \nu_2) = 4$.

(iii) If $(\chi_2, \nu_2) = (1, \eta)$ or (η, η) with $\eta = \eta_8^\pm$ then $q(\chi_2)q(\chi_2\nu_2) = 8$.

(iv) If $(\chi_2, \nu_2) = (\eta_4, 1)$ then $q(\chi_2)q(\chi_2\nu_2) = 16$.

(v) If $(\chi_2, \nu_2) = (\eta, \eta\eta_4)$ or (η_4, η) with $\eta = \eta_8^\pm$ then $q(\chi_2)q(\chi_2\nu_2) = 32$.

(vi) If $(\chi_2, \nu_2) = (\eta, 1)$ or (η, η_4) with $\eta = \eta_8^\pm$ then $q(\chi_2)q(\chi_2\nu_2) = 64$.

(vii) If $\chi_2 \notin N_2$ then $q(\chi_2)q(\chi_2\nu_2) = q(\chi_2)^2$.

Note that $\chi_2 \notin N_2$ if and only if $q(\chi_2)$ is divisible by 16. Consequently

$$\begin{aligned} D_2(s) = 1 + 2 \cdot 4^{-s} + 4 \cdot 8^{-s} + 16^{-s} + 4 \cdot 32^{-s} \\ + 4 \cdot 64^{-s} + 4 \sum_{k \geqslant 4} \psi(2^k) 2^{-2ks}, \end{aligned} \tag{35}$$

where $\psi(2^k) = 2^{k-2}$.

We are ready to verify the convergence of $D(s)$ for $\Re(s) > 1$. Say $\Re(s) \geqslant 1 + \varepsilon$, where $\varepsilon > 0$. Since $\psi(p^k) < p^k$, elementary estimates show that the series (34) and (35) converge and that $D_p(s) = 1 + O(p^{-1-\varepsilon})$, where the implicit constant does not depend on p or s (or even on ε). It follows that as an Euler product and Dirichlet series, $D(s)$ converges for $\Re(s) > 1$.

Now consider the larger right half-plane $\Re(s) \geqslant 3/4 + \varepsilon$. The series (34) and (35) still converge in this region, and for p odd, (34) gives

$$D_p(s) = 1 + 2p^{-s} + 2p^{1-2s} + O(p^{-3/2-2\varepsilon}) + O(p^{-1-4\varepsilon}), \tag{36}$$

Here $O(p^{-3/2-2\varepsilon})$ and $O(p^{-1-4\varepsilon})$ are bounds for $-5p^{-2s}$ and

$$2 \sum_{k \geqslant 2} \psi(p^k) p^{-2ks}$$

respectively, but we shall henceforth assume that $\varepsilon < 1/4$, so the term $O(p^{-3/2-2\varepsilon})$ can be absorbed by $O(p^{-1-4\varepsilon})$. Once again the implicit constant is independent of p and s.

It follows from (36) that if p is sufficiently large then $|D_p(s) - 1| < 1$, whence $D_p(s)$ is in the domain of the principal branch of the logarithm. Thus

$$\log D_p(s) = 2p^{-s} + 2p^{1-2s} + O(p^{-1-4\varepsilon}). \tag{37}$$

The right-hand side of (37) represents only the linear term of the relevant Taylor series, but for p sufficiently large we have

$$\sum_{k\geqslant 2}(2p^{-s}+2p^{1-2s}+O(p^{-1-4\varepsilon}))^k = O(p^{-1-4\varepsilon}),$$

so the higher-order terms can be absorbed by the term $O(p^{-1-4\varepsilon})$.

On the other hand, the Euler factor at p of the function

$$E(s)=\frac{\zeta(s)^2\zeta(2s-1)^2}{\zeta(2s)^2\zeta(4s-2)^2} \tag{38}$$

is the function $E_p(s)=(1+p^{-s})^2(1+p^{1-2s})^2$. In particular, for $\Re(s)\geqslant 3/4+\varepsilon$ and p sufficiently large, we have $|E_p(s)-1|<1$ and

$$\log E_p(s) = 2p^{-s}+2p^{1-2s}+O(p^{-1-4\varepsilon}). \tag{39}$$

Let p_0 be a prime such that both (37) and (39) are in force for $p\geqslant p_0$. Then

$$\prod_{p\geqslant p_0} D_p(s)/E_p(s) = e^{f(s)} \tag{40}$$

where $f(s)$ is holomorphic for $\Re(s)>3/4+\varepsilon$. So (40) gives an analytic continuation of the left-hand side to this region. Now for any given p, and in particular for $p<p_0$, the Euler factors $D_p(s)$ are holomorphic in the half-plane $\Re(s)>3/4+\varepsilon$ and do not vanish at $s=1$, while the Euler factors $E_p(s)$ are holomorphic and nonvanishing throughout the region $\Re(s)>3/4+\varepsilon$. Hence if we put $g(s)=\prod_{p\leqslant p_0} D_p(s)/E_p(s)$, then (40) becomes

$$D(s)=g(s)e^{f(s)}E(s) \tag{41}$$

with $f(s)$ and $g(s)$ holomorphic for $\Re(s)>3/4+\varepsilon$ and $g(1)\neq 0$. Since ε can be chosen arbitrarily small, Proposition 4 follows from (38), (41), and the familiar analytic properties of $\zeta(s)$, from which those of $E(s)$ follow.

6. Hecke-Shintani representations

If G is a group and H and H' are distinct subgroups of G, both of index 2, then $(G/H)\times(G/H')$ is the Klein four-group and $H\cap H'$ is the kernel of the natural map $G\to(G/H)\times(G/H')$, so there is a third subgroup of index 2 in G containing $H\cap H'$. Underlying assertion (iii) of Section 1 is an elementary remark about this situation:

Proposition 5. *Let ρ be a faithful two-dimensional irreducible complex representation of a finite group G, and suppose that ρ can be induced from two different subgroups of index 2 in G, say H and H'. Then ρ can be induced from precisely three subgroups of index 2 in G, namely the three index-two subgroups containing $H \cap H'$.*

Proof. Let H'' denote the third subgroup of index 2 in G containing $H \cap H'$. We must show that ρ is induced from H'' but not from any index-two subgroup of G besides H, H', and H''.

First we show that ρ is induced from H''. Since ρ is faithful, so is $\rho|H$, and as $\rho|H$ is the direct sum of two characters of H, it gives an embedding of H into the diagonal subgroup of $\mathrm{GL}_2(\mathbb{C})$. Hence H is abelian, and by the same reasoning, H' is abelian. Let h and h' be representatives for the nonidentity cosets of $H \cap H'$ in H and H' respectively; then h and h' commute with every element of $H \cap H'$, whence hh' does too. As hh' represents the nonidentity coset of $H \cap H'$ in H'' we conclude that H'' is abelian. But H'' is not central, because G is nonabelian whereas a group which is cyclic modulo a central subgroup is abelian. It follows that $\rho|H''$ is the direct sum of two distinct characters of H'' and consequently that ρ is induced from H''.

The argument just given also shows that $H \cap H'$ is contained in the center of G. Indeed $H \cap H'$ is abelian, and h and h' commute with every element of $H \cap H'$. Since h, h', and hh' are representatives for the nonidentity cosets of $H \cap H'$ in G, it follows that $H \cap H'$ is central.

Now let J be an arbitrary index-two subgroup of G from which ρ is induced. Then J contains the center of G: for otherwise the nonidentity coset of J in G is represented by some element of the center, say z, and conjugation by z does not interchange the two distinct characters of J occurring in $\rho|J$. Thus J is an index-two subgroup of G containing the center of G and in particular containing $H \cap H'$. Consequently J is H, H', or H''. □

We return to assertion (iii) of Section 1. Suppose that ρ is a Hecke-Shintani representation, so that $\rho \cong \mathrm{ind}_{K/\mathbb{Q}}\xi$ and $\rho \cong \mathrm{ind}_{K'/\mathbb{Q}}\xi'$ with one-dimensional Galois characters ξ and ξ' of distinct quadratic fields K and K'. Let K'' be the third quadratic field contained in KK'. Then Proposition 5 implies that $\rho \cong \mathrm{ind}_{K''/\mathbb{Q}}\xi''$ with some Galois character ξ'' of K''. At least one of K, K', and K'' is real, and assertion (iii) of Section 1 follows.

Example. (Cf. Hecke [8].) Put $L = \mathbb{Q}(\sqrt{-4}, 12^{1/4})$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. Then G is isomorphic to the dihedral group of order 8. Hence up to isomorphism G has a unique two-dimensional irreducible representation ρ, and ρ can be induced from $\mathbb{Q}(\sqrt{-4})$, $\mathbb{Q}(\sqrt{-12})$, and $\mathbb{Q}(\sqrt{12})$.

7. Siegel's formulas

Earlier it was convenient to speak of ray class groups, but henceforth an adelic language will be more efficient. We denote the group of ideles of a number field K by $\mathbb{A}_K^\times$, and we view idele class characters as characters of $\mathbb{A}_K^\times$ trivial on $K^\times \subset \mathbb{A}_K^\times$.

Let K denote a quadratic field and $q > 0$ a rational integer. The number of finite-order idele class characters of K of conductor dividing $q\mathcal{O}_K$ which are trivial on $\mathbb{A}_\mathbb{Q}^\times$ will be denoted $h_{K/\mathbb{Q}}^{\mathrm{nar}}(q)$ and referred to as the *narrow* (or *strict*) *ring class number of* K *to the modulus* q. If K is imaginary then the superscript "nar" and the words "narrow" or "strict" can be omitted. We also write $h_{K/\mathbb{Q}}^*(q)$ for the number of finite-order idele class characters of K of conductor *equal to* $q\mathcal{O}_K$ which are trivial on $\mathbb{A}_\mathbb{Q}^\times$. Underlying these definitions is the elementary fact that the conductor of a finite-order idele class character of K which is trivial on $\mathbb{A}_\mathbb{Q}^\times$ has the form $q\mathcal{O}_K$ with $q \in \mathbb{Z}$.

The formulas of Siegel referred to in the introduction are

$$\sum_{\substack{d_K q^2 \leqslant x \\ K \text{ imaginary}}} h_{K/\mathbb{Q}}(q) = \pi x^{3/2}/(18\zeta(3)) \quad + \quad O(x \log x) \tag{42}$$

([15], p. 671, formula (22)) and

$$\sum_{\substack{d_K q^2 \leqslant x \\ K \text{ real}}} h_{K/\mathbb{Q}}^{\mathrm{nar}}(q) \log \epsilon_{K,q}^+ = \pi^2 x^{3/2}/(18\zeta(3)) \quad + \quad O(x \log x) \tag{43}$$

([15], p. 667, formula (2)), where $\epsilon_{K,q}^+$ is the fundamental totally positive unit of the order $\mathcal{O}_{K,q} = \mathbb{Z} + q\mathcal{O}_K$. Strictly speaking, in [15] the role of $h_{K/\mathbb{Q}}^{\mathrm{nar}}(q)$ and $h_{K/\mathbb{Q}}(q)$ is played by the narrow class number of primitive binary quadratic forms of discriminants $d_K q^2$ and $-d_K q^2$ respectively, but it is a standard remark that these coincide with $h_{K/\mathbb{Q}}^{\mathrm{nar}}(q)$ and $h_{K/\mathbb{Q}}(q)$ (cf. [12], p. 472, Proposition 5.7).

The connection between Siegel's formulas and $\vartheta^{\mathrm{io}}(x)$ arises as follows. An Artin representation ρ of $\mathbb{Q}$ is dihedral – in other words, two-dimensional, irreducible, and orthogonal – if and only if $\rho \cong \mathrm{ind}_{K/Q}\xi$ for some quadratic field K and some character ξ of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ satisfying two conditions: first, the order of ξ is $\geqslant 3$, and second, $\xi \circ \mathrm{tran}_{K/\mathbb{Q}} = 1$, where 1 is the trivial character and $\mathrm{tran}_{K/\mathbb{Q}}$ is the transfer from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^{\mathrm{ab}}$ to $\mathrm{Gal}(K/\mathbb{Q})^{\mathrm{ab}}$. Using class field theory, we identify ξ with an idele class character $\hat{\xi}$ of finite order $\geqslant 3$, and then the condition $\xi \circ \mathrm{tran}_{K/\mathbb{Q}} = 1$ coincides with the previous condition $\hat{\xi}|\mathbb{A}_{\mathbb{Q}}^{\times} = 1$. Furthermore, the map sending the pair (K, ξ) to the isomorphism class of $\mathrm{ind}_{K/\mathbb{Q}}\xi$ is two-to-one; indeed the inverse image of this isomorphism class consists precisely of the pairs (K, ξ) and (K, ξ^{-1}). Lest this last assertion appear to conflict with the existence of Hecke-Shintani representations, note that ξ is required to have order $\geqslant 3$ (thus of the 6 pairs (K, ξ) corresponding to a dihedral Hecke-Shintani representation, we deduce that exactly 4 have the property that ξ is quadratic). In any case, it follows from the preceding remarks that

$$2\vartheta^{\mathrm{io}}(x) = \sum_{\substack{(K,q) \\ d_K q^2 \leqslant x}} (h^*_{K/\mathbb{Q}}(q) - e(K, q)), \tag{44}$$

where $e(K, q)$ is the number of idele class characters of K of conductor $q\mathcal{O}_K$ and order $\leqslant 2$ which are trivial on $\mathbb{A}_{\mathbb{Q}}^{\times}$.

To apply Siegel's formulas, it is convenient to write (44) in the form

$$2\vartheta^{\mathrm{io}}(x) = \sum_{\substack{d_K q^2 \leqslant x \\ K \text{ imaginary}}} h^*_{K/\mathbb{Q}}(q) + \sum_{\substack{d_K q^2 \leqslant x \\ K \text{ real}}} h^*_{K/\mathbb{Q}}(q) - \sum_{d_K q^2 \leqslant x} e(K, q), \tag{45}$$

where the third sum runs over both real and imaginary quadratic fields. Call the three sums on the right-hand side I_x, II_x, and III_x, so that

$$2\vartheta^{\mathrm{io}}(x) = \mathrm{I}_x + \mathrm{II}_x - \mathrm{III}_x. \tag{46}$$

We shall see that II_x and III_x are $o(x^{3/2})$ and that

$$\mathrm{I}_x \sim \pi x^{3/2}/(18\zeta(3)^2), \tag{47}$$

whence Theorem 2 follows from (46).

First consider II_x. Since $h^*_{K/\mathbb{Q}}(q) \leqslant h^{\mathrm{nar}}_{K/\mathbb{Q}}(q)$ and $\epsilon^+_{K,q} > q\sqrt{d}/2$, we have

$$\mathrm{II}_x \leqslant \sum_{5 \leqslant n \leqslant x} \frac{1}{\log(\sqrt{n}/2)} \sum_{\substack{d_K q^2 = n \\ K \text{ real}}} h^{\mathrm{nar}}_{K/\mathbb{Q}}(q) \log \epsilon^+_{K,q}. \tag{48}$$

Let $w(n)$ be the inner sum on the right-hand side, so that (48) becomes

$$\mathrm{II}_x \leqslant \sum_{5 \leqslant n \leqslant x} w(n)/\log(\sqrt{n}/2).$$

Also put $\omega(t) = \sum_{5 \leqslant n \leqslant t} w(n)$. Then Abel summation gives

$$\sum_{5 \leqslant n \leqslant x} w(n)/\log(\sqrt{n}/2) = \omega(x)/\log(\sqrt{x}/2) + \int_5^x \frac{\omega(t)}{2t(\log(\sqrt{t}/2))^2} dt.$$

Since $\omega(x)$ coincides with the left-hand side of (43), we deduce that

$$\text{II}_x = O(x^{3/2}/\log x). \tag{49}$$

Note that we have used (43) only as an upper bound, not as an asymptotic equality.

Next consider III_x. We claim that

$$\sum_{d_K q^2 \leqslant x} e(K, q) \leqslant \vartheta^{\text{rm}}(x), \tag{50}$$

whence

$$\text{III}_x = O(x(\log x)^3) \tag{51}$$

by (5). To verify (50), let $\hat{\xi}$ be an idele class character of K counted by $e(K, q)$, so that $\hat{\xi}^2 = 1$, $\hat{\xi}|\mathbb{A}_{\mathbb{Q}}^{\times} = 1$, and $\mathfrak{q}(\hat{\xi}) = q\mathcal{O}_K$. Let ξ be the character of $\text{Gal}(\overline{\mathbb{Q}}/K)$ corresponding to $\hat{\xi}$. It suffices to show that the representation $\rho = \text{ind}_{K/\mathbb{Q}}\xi$ is reducible, because then the pair (K, ξ) – hence also the pair $(K, \hat{\xi})$ – is uniquely determined by the isomorphism class of ρ (recall (i) of Section 1). Let σ be the nonidentity element of $\text{Gal}(K/\mathbb{Q})$, viewed as acting on $\mathbb{A}_K^{\times}$. Since $\hat{\xi}|\mathbb{A}_{\mathbb{Q}}^{\times} = 1$, we have $\hat{\xi}(x^{\sigma+1}) = 1$ for all $x \in \mathbb{A}_K^{\times}$, but then $\hat{\xi}(x^{\sigma-1}) = 1$ also because $\hat{\xi}^2 = 1$. It follows that $\hat{\xi}$ is trivial on the kernel of the idelic norm $N_{K/Q} : \mathbb{A}_K^{\times} \to \mathbb{A}_{\mathbb{Q}}^{\times}$, whence $\hat{\xi} = \hat{\chi} \circ N_{K/\mathbb{Q}}$ for some idele class character $\hat{\chi}$ of $\mathbb{Q}$. The corresponding character χ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then satisfies $\chi|\text{Gal}(\overline{\mathbb{Q}}/K) = \xi$, and consequently $\rho = \text{ind}_{K/\mathbb{Q}}\xi$ is reducible.

It remans to deal with I_x. Define Dirichlet series $H(s)$ and $H^*(s)$ by

$$H(s) = \sum_{\substack{K \text{ imaginary} \\ q \geqslant 1}} h_{K/\mathbb{Q}}(q)(d_K q^2)^{-s}$$

and

$$H^*(s) = \sum_{\substack{K \text{ imaginary} \\ q \geqslant 1}} h^*_{K/\mathbb{Q}}(q)(d_K q^2)^{-s}$$

Since $h_{K/\mathbb{Q}}(q) = \sum_{r|q} h^*_{K/\mathbb{Q}}(r)$, it follows that

$$H(s) = H^*(s)\zeta(2s). \tag{52}$$

The significance of (52) is that I_x is the summatory function associated to $H^*(s)$.

Proposition 6. *The Dirichlet series $H^*(s)$ converges for $\Re(s) > 3/2$, whence $H^*(s)$ is a holomorphic function in this region. Furthermore, $H^*(s)$ extends to a meromorphic function for $\Re(s) > 1$ which is holomorphic except for a simple pole at $s = 3/2$ with residue $\pi/(12\zeta(3)^2)$.*

Granting the proposition, we deduce the validity of (47), and then Theorem 2 follows from (46), (47), (49), and (51). It remains to prove Proposition 6.

The argument is standard, given (42). For $n \geqslant 1$ put

$$h(n) = \sum_{\substack{d_K q^2 = n \\ K \text{ imaginary}}} h_{K/\mathbb{Q}}(q), \tag{53}$$

so that $H(s) = \sum_{n\geqslant 1} h(n)n^{-s}$. (Note by the way that there is at most one summand on the right-hand side of (53), because K is determined by n via the equality $K = \mathbb{Q}(\sqrt{-n})$.) Let $\eta(t) = \sum_{n\leqslant t} h(n)$. Then $\eta(x)$ coincides with the left-hand side of (42), and

$$\sum_{n\leqslant x} h(n)n^{-s} = \eta(x)x^{-s} + s\int_1^x \frac{\eta(t)}{t^{s+1}}\,dt.$$

by Abel summation. Thus (42) gives

$$\sum_{n\leqslant x} h(n)n^{-s} = \frac{\pi s(1 - x^{-(s-3/2)})}{18\zeta(3)(s-3/2)} + x^{3/2-s}O(1) + s\int_1^x \frac{O(\log t)}{t^s}\,dt, \tag{54}$$

where the $O(*)$ terms do not depend on s. Letting x go to infinity, we obtain first the convergence of $H(s)$ as a Dirichlet series for $\Re(s) > 3/2$ and then the continuation of $H(s)$ to a meromorphic funtion for $\Re(s) > 1$ which is holomorphic in this region apart from a simple pole at $s = 3/2$ with residue $\pi/(12\zeta(3))$. Proposition 6 now follows from (52) and the known analytic properties of $\zeta(s)$.

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