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Counting rational points on smooth cubic surfaces

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We prove that any smooth cubic surface defined over any number field satisfies the lower bound predicted by Manin's conjecture possibly after an extension of small degree.

1. Introduction

Let K be a number field. Assume $X \subset \mathbb{P}^3_K$ is a smooth cubic surface defined over K for which the set of rational points $X(K)$ is not empty. We are concerned with estimating the number of rational points of bounded height on X. Let $U \subset X$ be the Zariski-open set obtained by removing the lines contained in X, denote by H the exponential Weil height on $\mathbb{P}^3(K)$ and define for all $B \geq 1$ the counting function

$$N_{K,H}(U, B) := \sharp\{\mathbf{x} \in U(K) : H(\mathbf{x}) \leq B\}.$$

Manin's conjecture [9] for smooth cubic surfaces states that

$$N_{K,H}(U, B) \sim cB(\log B)^{\rho_{X,K}-1}, \tag{1.1}$$

as $B \to \infty$, where $\rho_{X,K}$ denotes the rank of the Picard group of X over K and $c = c_{K,H,X}$ is a positive constant which was later interpreted by Peyre [16].

There has been a wealth of results towards this conjecture but it has never been established for a single smooth cubic surface over any number field. There are proofs of Manin's conjecture for certain singular cubic surfaces over $\mathbb{Q}$, e.g. [5], and other number fields [3, 8, 10], but here we will only consider the smooth case. Heath-Brown [13], building upon the work of Wooley [22], proved, using a fibration argument, that if X is a smooth cubic surface defined over $\mathbb{Q}$ that contains 3 rational coplanar

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lines then $N_{\mathbb{Q},H}(U,B) \ll_{X,\epsilon} B^{\frac{4}{3}+\epsilon}$ holds for any $\epsilon > 0$. This result was subsequently extended to arbitrary number fields by Broberg [6] and Browning and Swarbrick Jones [7]. Heath-Brown [14] revisited the subject by proving that a bound of the same order holds for all smooth cubic surfaces defined over $\mathbb{Q}$ subject to a standard conjecture regarding the growth rate of the rank of elliptic curves. Using a generalization of Heath-Brown's determinant method, Salberger [17] was able to prove unconditionally that one has $N_{\mathbb{Q},H}(U,B) \ll_{\epsilon} B^{\frac{12}{7}+\epsilon}$ for arbitrary smooth cubic surfaces defined over $\mathbb{Q}$ and for all $\epsilon > 0$.

Regarding lower bounds, the only available result is due to Slater and Swinnerton-Dyer [19] who used a secant and tangent process to establish that $N_{\mathbb{Q},H}(U,B) \gg_X B(\log B)^{\rho_{X,\mathbb{Q}}-1}$ whenever X has 2 skew lines defined over $\mathbb{Q}$.

Our main result shows that for all smooth cubic surfaces over any number field L, the lower bound predicted by Manin's conjecture has the correct order of magnitude as soon as one passes to a sufficiently large extension of L. Some context for this type of result is provided by the formulation of Manin's conjecture in [2] and by the notion of potential density [21, §3].

Theorem 1.1. *Let X be any smooth cubic surface defined over any number field L. Then there exists an extension K_0 of L with $[K_0 : L] \leq 432$ such that for all number fields $K \supseteq K_0$ we have*

$$N_{K,H}(U,B) \gg B(\log B)^{\rho_{X,K}-1},$$

as $B \to \infty$, where the implicit constant depends at most on X and K.

We hope that the number 432 will serve as a useful benchmark for researchers in the area to compare the strength of other methods with in the future.

Our Theorem 1.1 is a consequence of Theorem 1.2 below, which furthermore provides an explicit description of K_0. One can take K_0 to be any extension of L over which 2 skew lines of X are defined. The fact that there exists such a K_0 with $[K_0 : L] \leq 432 = 27 \cdot 16$ can be proved as follows. Since X contains exactly 27 lines, each of them is defined over an extension of degree at most 27. Since there are 16 complex lines skew to a line ℓ, a further extension of degree at most 16 ensures that a line skew to ℓ is defined.

Theorem 1.2. *Let X be a smooth cubic surface defined over any number field K such that X contains two skew lines defined over K. Then*

$$N_{K,H}(U,B) \gg B(\log B)^{\rho_{X,K}-1},$$

as $B \to \infty$, where the implicit constant depends only on K and X.

Theorem 1.2 is a generalization of Slater and Swinnerton-Dyer's result to arbitrary number fields. Our proof however is entirely different and more conceptual than the one of Slater and Swinnerton-Dyer. It relies on a conic bundle fibration of X and a number field version of the earlier work [20] of the second author which allows us to count rational points on each conic individually.

This result is presented in Section 2, together with our main analytic tool, a variant of Wirsing's theorem. Theorem 1.2 will be proved in Sections 3 and 4.

Throughout this paper, all implied constants are allowed to depend on the cubic surface X and the underlying number field K, unless the contrary is explicitly stated.

2. Preliminaries

We denote the degree of K by n, its discriminant by Δ_K, and its ring of integers by $\mathcal{O}_K$. We write Ω_∞, Ω_0 and Ω_K for the sets of archimedean places, non-archimedean places, and all places of K, respectively. We will write h_K, R_K and μ_K for the class number, regulator and the group of roots of unity in K. Moreover, r_1 (resp. r_2) denotes the number of real (resp. complex) embeddings of K.

In the proof of Theorem 1.2, we only need to consider a special family of height functions on $\mathbb{P}^2(K)$. Let $\boldsymbol{\lambda} = (\lambda_v)_{v\in\Omega_\infty} \in (0,\infty)^{\Omega_\infty}$. For every $v \in \Omega_K$, and $\mathbf{x} = (x,y,z) \in K_v^3$, let

$$\|\mathbf{x}\|_{\boldsymbol{\lambda},v} := \begin{cases} \max\{|x|_v, \lambda_v |y|_v, |z|_v\} & \text{if } v \in \Omega_\infty \\ \max\{|x|_v, |y|_v, |z|_v\} & \text{if } v \in \Omega_0. \end{cases} \tag{2.1}$$

Here, $|\cdot|_v$ is the unique absolute value on K_v extending the usual absolute value on $\mathbb{Q}_p$, if v lies over the place p of $\mathbb{Q}$. Let $n_v := [K_v : \mathbb{Q}_p]$. We consider heights on $\mathbb{P}^2(K)$ defined by

$$H_{\boldsymbol{\lambda}}((x:y:z)) := \prod_{v\in\Omega_K} \|(x,y,z)\|_{\boldsymbol{\lambda},v}^{n_v}.$$

Let $C \subset \mathbb{P}^2_K$ be a nonsingular conic defined by a ternary quadratic form $Q \in \mathcal{O}_K[x,y,z]$ and assume that $C(K) \neq \emptyset$, which implies that $C \cong \mathbb{P}^1_K$. The heights $H_{\boldsymbol{\lambda}}$ induce heights on $C(K)$ via the embedding $C \subset \mathbb{P}^2_K$. We are interested in estimating the quantity

$$N_{K,H_{\boldsymbol{\lambda}}}(C,B) := \sharp\{\mathbf{x} \in C(K) \,:\, H_{\boldsymbol{\lambda}}(\mathbf{x}) \leq B\}$$

when the underlying quadratic form has the special shape

$$Q = ax^2 + bxy + dxz + eyz + fz^2, \tag{2.2}$$

with $a,b,d,e,f \in \mathcal{O}_K$. It is a simple task to write down an explicit isomorphism between C and $\mathbb{P}^1_K$. Let Π be the matrix

$$\Pi := \begin{pmatrix} b & e & 0 \\ -a & -d & -f \\ 0 & b & e \end{pmatrix},$$

and define

$$\mathbf{q}(u,v) := \Pi \cdot \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}. \tag{2.3}$$

Then the map $(u,v) \mapsto \mathbf{q}(u,v)$ induces an isomorphism $\mathbb{P}^1_K \to C$. To measure the form Q and the height $H_{\boldsymbol{\lambda}}$, we introduce quantities

$$\langle Q \rangle := \prod_{v \in \Omega_K} \max\{|a|_v, |b|_v, |d|_v, |e|_v, |f|_v\}^{n_v}$$
$$\text{and} \quad M_{\boldsymbol{\lambda}} := \prod_{v \in \Omega_\infty} \max\{1, \lambda_v^{-1}\}^{n_v}.$$

The following lemma is a number field version of [20, Prop. 2.1], specialized to the heights $H_{\boldsymbol{\lambda}}$ and with a crude estimation of the error term.

Lemma 2.1. *There exist constants $\beta \in (0,1/2)$ and $\gamma > 0$ which depend at most on K such that whenever $C \subset \mathbb{P}^2_K$ is a nonsingular conic defined by a quadratic form Q as in* (2.2)*, and $\boldsymbol{\lambda} \in (0,\infty)^{\Omega_\infty}$, then*

$$N_{K,H_{\boldsymbol{\lambda}}}(C,B) = c_{K,\boldsymbol{\lambda},C} \cdot B + O\left(B^{1-\beta}(M_{\boldsymbol{\lambda}}\langle Q \rangle)^\gamma\right),$$

for $B \geq 1$. The leading constant $c_{K,\boldsymbol{\lambda},C}$ is positive and is the one predicted by Peyre, and the implied constant in the error term depends only on K.

Of course, Manin's conjecture for conics with respect to arbitrary anticanonical height functions is already known [16], so the novelty of Lemma 2.1 lies in the uniformity of the estimate in the coefficients of the underlying quadratic form.

The proof over $\mathbb{Q}$ in [20] is based on the parameterization of $C(K)$ via $\mathbf{q}$, which reduces the estimation of $N_{K,H_{\boldsymbol{\lambda}}}(C,B)$ to a lattice point counting argument. The same reduction works over arbitrary number fields by considering primitive points with respect to a fixed set of representatives for the ideal classes of $\mathcal{O}_K$ and suitably chosen fundamental domains for the action of the unit group. The resulting lattice point counting problem can then be solved using, for example, the main result from [1]. The special shape of the heights $H_{\boldsymbol{\lambda}}$ enters only here, to ensure definability in an o-minimal structure. Altogether, the passage from $\mathbb{Q}$ to arbitrary number fields in the proof of Lemma 2.1 uses mostly arguments already given in [11], but is straightforward and much simpler. The proof provides explicit values $\beta = 1/(3n)$ and $\gamma = 4$, but we will not give further details here. For the purpose of proving Theorems 1.1 and 1.2 we do not need explicit values for β and γ since any polynomial saving in terms of B and any polynomial dependence on $\langle Q\rangle$ and $M_{\boldsymbol{\lambda}}$ in the error term suffices.

As usual, the constant $c_{K,\boldsymbol{\lambda},C}$ has an explicit expression of the form

$$c_{K,\boldsymbol{\lambda},C} = \frac{1}{2}\cdot\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{|\mu_K|}\cdot\frac{1}{|\Delta_K|}\cdot\prod_{v\in\Omega_K}\sigma_v, \tag{2.4}$$

with local densities σ_v given as follows. For $v \in \Omega_\infty$, we have

$$\sigma_v = \operatorname{vol}\{(y_1,y_2)\in K_v^2 : \|\mathbf{q}(y_1,y_2)\|_{\boldsymbol{\lambda},v}\le 1\}\cdot\begin{cases}1 & \text{if } v \text{ is real,}\\ 4/\pi & \text{if } v \text{ is complex,}\end{cases} \tag{2.5}$$

where $\operatorname{vol}(\cdot)$ denotes the usual Lebesgue measure on $K_v^2 \cong \mathbb{R}^{2n_v}$. For $v \in \Omega_0$ corresponding to a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we have

$$\sigma_v = 1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2} + \left(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\right)\sum_{d\in\mathbb{N}}\frac{\rho^*_{\mathbf{q}}(\mathfrak{p}^d)}{\mathfrak{N}\mathfrak{p}^d}, \tag{2.6}$$

where, for any ideal $\mathfrak{a}$ of $\mathcal{O}_K$, the function $\rho^*_{\mathbf{q}}(\mathfrak{a})$ is defined as

$$\sharp\{(\sigma,\tau)\in(\mathcal{O}_K/\mathfrak{a})^2 : \sigma\mathcal{O}_K+\tau\mathcal{O}_K+\mathfrak{a}=\mathcal{O}_K,\ \mathbf{q}(\sigma,\tau)\equiv\mathbf{0} \bmod \mathfrak{a}\}. \tag{2.7}$$

The following version of Wirsing's theorem is a straightforward generalization to number fields of [12, Theorem A.5]. Its proof is, mutatis mutandis, the same and therefore omitted.

Lemma 2.2. *Let g be a multiplicative function on nonzero ideals of $\mathcal{O}_K$ that is supported on the set of squarefree ideals. Assume that we have*

$$\sum_{\mathfrak{N}\,\mathfrak{p}\le x} g(\mathfrak{p})\log(\mathfrak{N}\,\mathfrak{p}) = k\log x + O(1) \tag{2.8}$$

for all $x \ge 2$, with $k \ge -1/2$, where the sum runs over nonzero prime ideals $\mathfrak{p}$ and the implied constant is allowed to depend at most on K and g. Assume, moreover, that

$$\prod_{w\le\mathfrak{N}\,\mathfrak{p}<z} (1+|g(\mathfrak{p})|) \ll \left(\frac{\log z}{\log w}\right)^{|k|} \tag{2.9}$$

holds for all $z > w \ge 2$ and that

$$\sum_{\mathfrak{p}} g(\mathfrak{p})^2 \log(\mathfrak{N}\,\mathfrak{p}) < \infty. \tag{2.10}$$

Then

$$\sum_{\mathfrak{N}\,\mathfrak{a}\le x} g(\mathfrak{a}) = c_g(\log x)^k + O((\log x)^{|k|-1}),$$

with a positive constant c_g, where the implied constant depends at most on K and g.

3. Covering the cubic surface with conics

Let K be a number field and $X \subset \mathbb{P}^3_K$ a smooth cubic surface containing two skew lines defined over K. The residual intersection of X with a plane containing the first line generically defines a smooth conic. The second line contained in X intersects each such plane in a point that necessarily lies in the residual conic, thus showing that it is isotropic over K.

The construction we have described does in fact yield a conic bundle morphism. A linear change of variables allows us to assume that the two skew K-lines are given by

$$x_0 = x_1 = 0 \quad \text{and} \quad x_2 = x_3 = 0,$$

whence the cubic form defining X has the shape

$$\begin{aligned} F = {} & a(x_0,x_1)x_2^2 + d(x_0,x_1)x_2x_3 + f(x_0,x_1)x_3^2 \\ & + b(x_0,x_1)x_2 + e(x_0,x_1)x_3, \end{aligned} \tag{3.1}$$

where $a, d, f \in \mathcal{O}_K[x_0, x_1]$ are linear forms and $b, e \in \mathcal{O}_K[x_0, x_1]$ are quadratic forms. Moreover, we can write $F = x_0Q_0 - x_1Q_1$ with quadratic forms $Q_0, Q_1 \in \mathcal{O}_K[x_0, \ldots, x_3]$, and the nonsingularity of X implies that the morphism $\pi : X \to \mathbb{P}^1_K$ given on points by

$$(x_0 : x_1 : x_2 : x_3) \mapsto \begin{cases} (x_0 : x_1) & \text{if } (x_0, x_1) \neq (0,0) \\ (Q_1(\mathbf{x}) : Q_0(\mathbf{x})) & \text{if } (Q_1(\mathbf{x}), Q_0(\mathbf{x})) \neq (0,0) \end{cases}$$

is well defined. The fibre $\pi^{-1}(s : t)$ is the residual conic in the plane $\Lambda_{(s:t)}$ defined by $tx_0 - sx_1 = 0$. For any choice of (s, t), it is isomorphic to the plane conic $C_{(s,t)}$ defined by the quadratic form

$$Q_{(s,t)} := a(s,t)x^2 + d(s,t)xz + f(s,t)z^2 + b(s,t)xy + e(s,t)yz = 0 \tag{3.2}$$

via the isomorphism $\phi_{(s,t)} : \mathbb{P}^2_K \to \Lambda_{(s:t)}$ given by

$$(x : y : z) \mapsto (sy : ty : x : z).$$

The discriminant locus of π is given by the quintic binary form

$$\Delta(s,t) := (ae^2 - bde + fb^2)(s,t),$$

which is separable owing to the nonsingularity of X (see [18, II.6.4, Proposition 1]). This confirms that the resultant

$$W_0 := \operatorname{Res}(b(s,t), e(s,t))$$

must be in $\mathcal{O}_K \smallsetminus \{0\}$, since the square of any common divisor of $b(s,t)$ and $e(s,t)$ divides $\Delta(s,t)$.

Clearly, each $C_{(s,t)}$ contains the rational point $(0 : 1 : 0)$, which is tantamount to the conic bundle morphism having a section defined over K. By a standard argument (see, e.g., the paragraph following (1.6) in [4]), we have

$$\rho_{X,K} = 2 + r, \tag{3.3}$$

where $r = r(X, K)$ is the number of split singular fibres above closed points of $\mathbb{P}^1_K$. Since the section meets exactly one component of every singular fibre, we see that all singular fibres are split. Consequently r equals the number of irreducible factors of $\Delta(s,t)$ in $K[s,t]$.

Using the conic fibration described above, we can reduce counting points on X to counting points on the fibres $\pi^{-1}(s:t)$ as follows:

$$N_{K,H}(U,B) = \sum_{(s:t)\in\mathbb{P}^1(K)} N_{K,H}(\pi^{-1}(s:t)\cap U, B).$$

Let $\mathcal{G}$ be a fundamental domain for the action of $\mathcal{O}_K^\times$ on $(K^\times)^2$ with the property that

$$\max\{|s|_v, |t|_v\} \ll \max\{|s|_w, |t|_w\} \ll \max\{|s|_v, |t|_v\} \tag{3.4}$$

holds for all $v, w \in \Omega_\infty$ and all $(s,t) \in \mathcal{G}$. We can construct such a fundamental domain using, for example, the method from [15, Section 4]. Define the set

$$\mathcal{B}(x) := \left\{ (s,t) \in \mathcal{O}_K^2 \cap \mathcal{G} \ : \ \begin{array}{l} H((s:t)) \le x, \\ s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K, \\ \pi^{-1}(s:t) \text{ is nonsingular} \end{array} \right\}, \tag{3.5}$$

where $H((s:t))$ is the usual exponential Weil height on $\mathbb{P}^1(K)$. For the purpose of acquiring a lower bound it is sufficient to restrict the summation to points $(s:t)$ with representatives in $\mathcal{B}(B^\delta)$, for $\delta := \beta/(2(1+\gamma))$. Then $N_{K,H}(U,B)$ is larger than

$$\sum_{(s,t)\in\mathcal{B}(B^\delta)} N_{K,H}(\pi^{-1}(s:t)\cap U, B) = \sum_{(s,t)\in\mathcal{B}(B^\delta)} N_{K,H}(\pi^{-1}(s:t), B) + O(B^{2\delta}),$$

by Schanuel's theorem, since every nonsingular conic contains at most 54 points lying on lines in X.

We use the isomorphism $\phi_{(s,t)}$ defined above to identify $\pi^{-1}(s:t)$ with the plane conic $C_{(s,t)}$ given by (3.2). The height H on $\pi^{-1}(s:t)$ is pulled back to the height $H \circ \phi_{(s,t)} = H_{\boldsymbol{\lambda}}$ on $C_{(s,t)}(K)$, with $\lambda_v := \max\{|s|_v, |t|_v\}$

for all $v \in \Omega_\infty$, making the succeeding equality apparent,

$$N_{K,H}(\pi^{-1}(s:t), B) = N_{K,H_{\boldsymbol{\lambda}}}(C_{(s,t)}, B).$$

Clearly, $\langle Q_{(s,t)} \rangle \ll H((s:t))^2$, and due to (3.4) we have $M_{\boldsymbol{\lambda}} \ll 1$. Lemma 2.1 therefore reveals that

$$N_{K,H}(\pi^{-1}(s:t), B) = c(s,t)B + O(B^{1-\beta} H((s:t))^{2\gamma}),$$

with an explicit formula for $c(s,t) := c_{K,\boldsymbol{\lambda},C_{(s,t)}}$ given below Lemma 2.1. Our choice of δ implies that

$$N_{K,H}(U, B) \gg B\ \mathfrak{S}(B^\delta) + O(B), \tag{3.6}$$

where

$$\mathfrak{S}(x) := \sum_{(s,t)\in\mathcal{B}(x)} c(s,t).$$

Our last undertaking is to show that the quantity $\mathfrak{S}(B^\delta)$, the sum of the Peyre constants of the smooth conic fibres, provides the logarithmic factors appearing in Theorem 1.2.

4. The proof of Theorem 1.2

For each place v of K, let $\sigma_v(s,t)$ be as in (2.5),(2.6), with the parameterizing functions $\mathbf{q} = \mathbf{q}_{(s,t)}$ defined as in (2.3) for the quadratic form $Q_{(s,t)}$, and the norms $\|\cdot\|_{\boldsymbol{\lambda},v}$ as in (2.1), with $\lambda_v = \max\{|s|_v, |t|_v\}$. Let ζ_K be the Dedekind zeta function of K and ϕ_K be Euler's totient function for nonzero ideals of $\mathcal{O}_K$. Moreover, for nonzero ideals $\mathfrak{a}$ of $\mathcal{O}_K$, we define the multiplicative function

$$\phi_K^\dagger(\mathfrak{a}) := \prod_{\mathfrak{p}\mid\mathfrak{a}} \left(1 + \frac{1}{\mathfrak{N}\mathfrak{p}}\right),$$

where the product extends over all prime ideals $\mathfrak{p}$ dividing $\mathfrak{a}$. Clearly,

$$\frac{1}{\zeta_K(2)} \le \frac{\phi_K^\dagger(\mathfrak{a})\phi_K(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \le 1$$

holds for all $\mathfrak{a}$.

Lemma 4.1 (The non-archimedean densities). *Let η be any positive constant and suppose $s, t \in \mathcal{O}_K$ fulfill $s\mathcal{O}_K + t\mathcal{O}_K = \mathcal{O}_K$. Then we have*

$$\prod_{v \in \Omega_0} \sigma_v(s,t) \geq \frac{1}{\zeta_K(2)} \sum_{\substack{\mathfrak{N}\,\mathfrak{a} \leq B^\eta \\ \mathfrak{a} | \Delta(s,t) \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}} \right)^2.$$

Proof. Let $\rho^*_{(s,t)}(\mathfrak{a}) := \rho^*_{\mathbf{q}_{(s,t)}}(\mathfrak{a})$ as in (2.7). Expanding the Euler product present in the lemma reveals its equality to

$$\frac{1}{\zeta_K(2)} \sum_{\mathfrak{a}} \frac{\rho^*_{(s,t)}(\mathfrak{a})}{\phi^\dagger_K(\mathfrak{a})\,\mathfrak{N}\,\mathfrak{a}} \geq \frac{1}{\zeta_K(2)} \sum_{\substack{\mathfrak{N}\,\mathfrak{a} \leq B^\eta \\ \mathfrak{a} | \Delta(s,t) \\ \mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K}} \frac{\rho^*_{(s,t)}(\mathfrak{a})}{\phi^\dagger_K(\mathfrak{a})\,\mathfrak{N}\,\mathfrak{a}}.$$

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ with $\mathfrak{a} \mid \Delta(s,t)$ and $\mathfrak{a} + W_0\mathcal{O}_K = \mathcal{O}_K$. We proceed to show that $\rho^*_{(s,t)}(\mathfrak{a}) \geq \phi_K(\mathfrak{a})$. Since s^3W_0 and t^3W_0 can be expressed as linear combinations over $\mathcal{O}_K$ of $b(s,t)$ and $e(s,t)$, we acquire the validity of $b(s,t)\mathcal{O}_K + e(s,t)\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$. For every $\lambda \in \mathcal{O}_K/\mathfrak{a}$ with $\lambda\mathcal{O}_K + \mathfrak{a} = \mathcal{O}_K$, let $u := \lambda e(s,t)$ and $v := -\lambda b(s,t)$. Then $\mathbf{q}_{(s,t)}(u,v) \equiv 0 \pmod{\mathfrak{a}}$, and thus $\rho^*_{(s,t)} \geq \phi_K(\mathfrak{a})$. □

Lemma 4.2 (The archimedean densities). *Suppose that s and t satisfy the assumption of Lemma 4.1. Then we have*

$$\prod_{v \in \Omega_\infty} \sigma_v(s,t) \gg \frac{1}{H((s:t))^2}.$$

Proof. The estimates

$$\begin{aligned} &|b(s,t)|_v, |e(s,t)|_v \ll \max\{|s|_v, |t|_v\}^2 \quad \text{and} \\ &|a(s,t)|_v, |d(s,t)|_v, |f(s,t)|_v \ll \max\{|s|_v, |t|_v\} \end{aligned}$$

hold for each place $v \in \Omega_\infty$. Hence, all $(y_1, y_2) \in K_v^2$ satisfying

$$|y_1|_v, |y_2|_v \ll \max\{|s|_v, |t|_v\}^{-1},$$

with a suitably small implied constant, fulfills $\left\|\mathbf{q}_{(s,t)}(y_1,y_2)\right\|_{\boldsymbol{\lambda},v} \leq 1$. We therefore get that

$$\prod_{v \in \Omega_\infty} \sigma_v(s,t) \gg \prod_{v \in \Omega_\infty} \max\{|s|_v, |t|_v\}^{-2n_v} = H((s:t))^{-2}.$$

□

By (2.4), Lemma 4.1 and Lemma 4.2, we obtain

$$\mathfrak{S}(B^\delta) \gg \sum_{(s,t)\in\mathcal{B}(B^\delta)} \frac{1}{H((s:t))^2} \sum_{\substack{\mathfrak{N}\,\mathfrak{a}\le B^\eta \\ \mathfrak{a}|\Delta(s,t) \\ \mathfrak{a}+W_0\mathcal{O}_K=\mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}}\right)^2. \tag{4.1}$$

We observe that, apart from the condition $(s,t)\in\mathcal{G}$ from (3.5), every expression involving (s,t) in the above formula is invariant under scalar multiplication of (s,t) by units in $\mathcal{O}_K^\times$. Hence, we may replace $\mathcal{G}$ by another fundamental domain $\mathcal{H}$, which will enable us to continue our estimation of $\mathfrak{S}(x)$. We obtain a fundamental domain $\mathcal{H}_0$ for the action of $\mathcal{O}_K^\times$ on $(K\otimes_{\mathbb{Q}}\mathbb{R})^\times$ by making use of the embedding $K^\times \to (K\otimes_{\mathbb{Q}}\mathbb{R})^\times = \prod_{v\in\Omega_\infty} K_v^\times$ as well as the construction in [15, Section 4] for the trivial distance functions

$$N_v : K_v \to [0,\infty)\ ,\ s\mapsto |s|_v\,.$$

The norm $N : K\to\mathbb{Q}$ extends to $K\otimes_{\mathbb{Q}}\mathbb{R}\to\mathbb{R}$ in an obvious way. The sets $\mathcal{H}_0(T) := \{s\in\mathcal{H}_0 : |N(s)|\le T\}$ clearly satisfy $\mathcal{H}_0(T) = T^{1/n}\mathcal{H}_0(1)$, and by [15, Lemma 3], the set $\mathcal{H}_0(1)$ is bounded with Lipschitz-parameterizable boundary. This enables us to perform lattice point counting arguments in the sets $\mathcal{H}_0(T)$ and their translates, via [15, Lemma 2] for example. We choose $\mathcal{H} := (\mathcal{H}_0\cap K)\times K^\times \subset (K^\times)^2$ as our fundamental domain for the action of $\mathcal{O}_K^\times$ on K^2.

Partitioning into congruence classes modulo $\mathfrak{a}$ yields

$$\mathfrak{S}(B^\delta) \gg \sum_{\substack{\mathfrak{N}\,\mathfrak{a}\le B^\eta \\ \mathfrak{a}+W_0\mathcal{O}_K=\mathcal{O}_K}} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}}\right)^2 \sum_{\substack{(\sigma,\tau) \bmod \mathfrak{a} \\ \sigma\mathcal{O}_K+\tau\mathcal{O}_K+\mathfrak{a}=\mathcal{O}_K \\ \mathfrak{a}|\Delta(\sigma,\tau)}} G_{\sigma,\tau}(B^\delta,\mathfrak{a}), \tag{4.2}$$

where

$$G_{\sigma,\tau}(x,\mathfrak{a}) := \sum_{\substack{(s,t)\in(\mathcal{O}_K\cap\mathcal{H}_0)\times\mathcal{O}_K \\ s\mathcal{O}_K+t\mathcal{O}_K=\mathcal{O}_K \\ (s,t)\equiv(\sigma,\tau) \bmod \mathfrak{a} \\ H((s:t))\le x \\ C_{(s,t)} \text{ nonsingular}}} \frac{1}{H((s:t))^2}.$$

Lemma 4.3 (Lattice point counting). *Let $\sigma\mathcal{O}_K+\tau\mathcal{O}_K+\mathfrak{a}=\mathcal{O}_K$. Then*

$$G_{\sigma,\tau}(x,\mathfrak{a}) \gg \frac{\log x}{\mathfrak{N}\,\mathfrak{a}\ \phi_K(\mathfrak{a})\ \phi_K^\dagger(\mathfrak{a})} + O\left(x^{-\frac{1}{2n}}\log x\right).$$

Proof. The discriminant $\Delta(s,t)$ is a quintic form whence the conic $C_{(s,t)}$ is singular for (s,t) lying on one of at most 5 lines through the origin in K^2. Hence, there exists a constant $0<\alpha<1$, depending only on F and K, such that $C_{(s,t)}$ is nonsingular whenever $s,t\neq 0$ and $|t|_v<\alpha\,|s|_v$ holds for all $v\in\Omega_\infty$. Observe that for such (s,t) with $s\mathcal{O}_K+t\mathcal{O}_K=\mathcal{O}_K$ we have $H((s:t))=|N(s)|$. This shows that

$$G_{\sigma,\tau}(x,\mathfrak{a})\gg \sum_{\substack{(s,t)\in(\mathcal{O}_K\cap\mathcal{H}_0)\times\mathcal{O}_K\\ s\mathcal{O}_K+t\mathcal{O}_K=\mathcal{O}_K\\ (s,t)\equiv(\sigma,\tau)\bmod \mathfrak{a}\\ x^{1/2}\le|N(s)|\le x\\ 0<|t|_v<\alpha|s|_v\forall v\in\Omega_\infty}} |N(s)|^{-2} =: G(x),\ \text{say.}$$

Using Möbius inversion to remove the coprimality condition, we see that

$$G(x)=\sum_{\substack{\mathfrak{N}\,\mathfrak{d}\le x\\ \mathfrak{d}+\mathfrak{a}=\mathcal{O}_K}}\mu_K(\mathfrak{d})\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ s\equiv\sigma\bmod\mathfrak{a}\\ x^{1/2}\le|N(s)|\le x}}|N(s)|^{-2}\sum_{\substack{t\in\mathfrak{d}\\ t\equiv\tau\bmod\mathfrak{a}\\ 0<|t|_v<\alpha|s|_v\forall v\in\Omega_\infty}}1\ .$$

The condition $\mathfrak{d}+\mathfrak{a}=\mathcal{O}_K$ comes from $\sigma\mathcal{O}_K+\tau\mathcal{O}_K+\mathfrak{a}=\mathcal{O}_K$. The sum over t is just counting ideal-lattice points in a translated "box", and their number is well known to be

$$\frac{c_K\alpha^n\,|N(s)|}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})}+O\left(\left(\frac{\alpha^n\,|N(s)|}{\mathfrak{N}(\mathfrak{a}\mathfrak{d})}\right)^{(n-1)/n}+1\right),$$

with a positive constant c_K depending only on K (see, for example, the proof of [11, Lemma 7.1]). Hence,

$$\begin{aligned}G(x)=\frac{c_K\alpha^n}{\mathfrak{N}\,\mathfrak{a}}&\sum_{\substack{\mathfrak{N}\,\mathfrak{d}\le x\\ \mathfrak{d}+\mathfrak{a}=\mathcal{O}_K}}\frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\,\mathfrak{d}}\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ s\equiv\sigma\bmod\mathfrak{a}\\ x^{1/2}\le|N(s)|\le x}}\frac{1}{|N(s)|}\\ &+O\left(\sum_{\mathfrak{N}\,\mathfrak{d}\le x}\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ x^{1/2}\le|N(s)|\le x}}\frac{1}{|N(s)|^2}\right)\\ &+O\left(\frac{1}{\mathfrak{N}\,\mathfrak{a}^{(n-1)/n}}\sum_{\mathfrak{N}\,\mathfrak{d}\le x}\frac{1}{\mathfrak{N}\,\mathfrak{d}^{(n-1)/n}}\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ x^{1/2}\le|N(s)|\le x}}\frac{1}{|N(s)|^{1+1/n}}\right).\end{aligned}$$

The sums over s in the error terms are taken over principal ideals of $\mathcal{O}_K$ contained in $\mathfrak{d}$. For any $a>0$, we have

$$\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ x^{1/2}\le|N(s)|\le x}}\frac{1}{|N(s)|^{1+a}}\ll\sum_{\substack{\mathfrak{b}\in[\mathfrak{d}^{-1}]\\ \mathfrak{N}\,\mathfrak{b}\ge x^{1/2}\,\mathfrak{N}\,\mathfrak{d}^{-1}}}\frac{1}{\mathfrak{N}(\mathfrak{b}\mathfrak{d})^{1+a}}\ll\frac{1}{\mathfrak{N}\,\mathfrak{d}\;x^{a/2}}.$$

This shows that both error terms in the above expression for $G(x)$ are of size $\ll x^{-1/(2n)}\log x$. Using the nice properties of our fundamental domain $\mathcal{H}_0$ and [15, Lemma 2], we see that

$$\begin{aligned}&\sharp\{s\in\mathfrak{d}\cap\mathcal{H}_0\,:\,s\equiv\sigma\bmod\mathfrak{a},\ |N(s)|\le x\}\\ &=\frac{c_K'x}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})}+O\left(\left(\frac{x}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})}\right)^{(n-1)/n}+1\right),\end{aligned}$$

with a positive constant c_K' depending only on K. Together with the Abel sum formula we are thus provided with the asymptotic formula

$$\sum_{\substack{s\in\mathfrak{d}\cap\mathcal{H}_0\\ s\equiv\sigma\bmod\mathfrak{a}\\ x^{1/2}\le|N(s)|\le x}}\frac{1}{|N(s)|}=\frac{c_K'}{\mathfrak{N}(\mathfrak{d}\mathfrak{a})}\log(x)+O\left(x^{-1/(2n)}\right),$$

from which it is immediately apparent that

$$G(x)\gg\frac{\log x}{\mathfrak{N}\,\mathfrak{a}^2}\sum_{\substack{\mathfrak{N}\,\mathfrak{d}\le x\\ \mathfrak{d}+\mathfrak{a}=\mathcal{O}_K}}\frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\,\mathfrak{d}^2}+O\left(x^{-1/(2n)}\log x\right).$$

Finally, the obvious estimate

$$\sum_{\substack{\mathfrak{N}\,\mathfrak{d}\le x\\ \mathfrak{d}+\mathfrak{a}=\mathcal{O}_K}}\frac{\mu_K(\mathfrak{d})}{\mathfrak{N}\,\mathfrak{d}^2}=\frac{\mathfrak{N}\,\mathfrak{a}}{\zeta_K(2)\phi_K(\mathfrak{a})\phi_K^\dagger(\mathfrak{a})}+O\left(\frac{1}{x}\right)$$

allows us to complete the proof of the lemma. □

For any binary form $g\in\mathcal{O}_K[u,v]$, we define the multiplicative function $\varrho_g^*(\mathfrak{a})$ on non–zero ideals of $\mathcal{O}_K$ by

$$\sharp\{(\sigma,\tau)\in(\mathcal{O}_K/\mathfrak{a})^2,\sigma\mathcal{O}_K+\tau\mathcal{O}_K+\mathfrak{a}=\mathcal{O}_K,\ g(\sigma,\tau)\equiv 0\bmod\mathfrak{a}\}$$

and note that its value is trivially bounded by $\mathfrak{N}\,\mathfrak{a}^2$. From the estimate (4.2) with $\eta := \delta/(7n)$ and Lemma 4.3, we obtain

$$\mathfrak{S}(B^\delta) \gg \sum_{\substack{\mathfrak{N}\,\mathfrak{a} \leq B^{\delta/(7n)} \\ a+W_0\mathcal{O}_K=\mathcal{O}_K}} \frac{\varrho^*_\Delta(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}^2}\left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}}\right)^2 + O(1). \tag{4.3}$$

The following lemma is proved via an application of Wirsing's theorem and its validity implies that of Theorem 1.2.

Lemma 4.4. *For $x \geq 1$,*

$$\sum_{\substack{\mathfrak{N}\,\mathfrak{a} \leq x \\ \mathfrak{a}+W_0\mathcal{O}_K=\mathcal{O}_K}} \frac{\varrho^*_\Delta(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}^2}\left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}}\right)^2 \gg (\log x)^r.$$

Proof. The form Δ factors as $a\Delta(s,t) = \prod_{i=1}^r \Delta_i(s,t)$ over K for an appropriate value of $a = a(K,F) \in \mathcal{O}_K$ and irreducible forms $\Delta_i \in \mathcal{O}_K[s,t]$. For $1 \leq i \leq r$ with $\Delta_i(1,0) \neq 0$, let $\delta_i(x) := \Delta_i(x,1) \in \mathcal{O}_K[x]$. We moreover define for any polynomial $g \in \mathcal{O}_K[x]$ and any ideal $\mathfrak{a}$ of $\mathcal{O}_K$,

$$\tau_g(\mathfrak{a}) := \sharp\{s \in \mathcal{O}_K/\mathfrak{a} : g(s) \equiv 0 \bmod \mathfrak{a}\},$$

and we subsequently let

$$\tau_i(\mathfrak{a}) := \begin{cases} \tau_{\delta_i}(\mathfrak{a}) & \text{if } \Delta_i(1,0) \neq 0, \\ \tau_x(\mathfrak{a}) = 1 & \text{if } \Delta_i(1,0) = 0, \end{cases}$$
$$\text{and} \quad a_i := \begin{cases} \Delta_i(1,0) & \text{if } \Delta_i(1,0) \neq 0, \\ 1 & \text{if } \Delta_i(1,0) = 0. \end{cases}$$

The asymptotic relationships

$$\sum_{\mathfrak{N}\,\mathfrak{p} \leq x} \frac{\tau_i(\mathfrak{p})}{\mathfrak{N}\,\mathfrak{p}} = \log\log x + O(1)$$
$$\text{and} \quad \sum_{\mathfrak{N}\,\mathfrak{p} \leq x} \frac{\tau_i(\mathfrak{p})\log(\mathfrak{N}\,\mathfrak{p})}{\mathfrak{N}\,\mathfrak{p}} = \log x + O(1) \tag{4.4}$$

follow from Landau's prime ideal theorem applied to $K(\theta_i)$, where θ_i is a root of the irreducible polynomial δ_i. Since Δ is separable, all resultants

$\mathrm{Res}(\Delta_i, \Delta_j)$ are nonzero. Whence, upon introducing

$$W = W_F := aW_0 \prod_{i \neq j} \mathrm{Res}(\Delta_i, \Delta_j) \prod_{i=1}^{r} a_i \in \mathcal{O}_K \setminus \{0\},$$

the equality

$$\varrho_\Delta^*(\mathfrak{p}) = (\mathfrak{N}\,\mathfrak{p} - 1) \sum_{i=1}^{r} \tau_i(\mathfrak{p})$$

is rendered valid for each nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, coprime to W. This fact, along with (4.4), reveals that the multiplicative function defined by

$$g(\mathfrak{a}) := \begin{cases} \varrho_\Delta^*(\mathfrak{a})\phi_K(\mathfrak{a})^2\,\mathfrak{N}\,\mathfrak{a}^{-4} & \text{if } \mathfrak{a} + W\mathcal{O}_K = \mathcal{O}_K \text{ and } \mathfrak{a} \text{ squarefree,} \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the assumptions of Lemma 2.2 with $k = r$. We therefore get that there exists $c_g > 0$ such that

$$\sum_{\substack{\mathfrak{N}\,\mathfrak{a} \leq x \\ a + W\mathcal{O}_K = \mathcal{O}_K \\ \mathfrak{a} \text{ squarefree}}} \frac{\varrho_\Delta^*(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}^2} \left(\frac{\phi_K(\mathfrak{a})}{\mathfrak{N}\,\mathfrak{a}} \right)^2 = c_g (\log x)^r + O((\log x)^{r-1}),$$

an estimate which concludes our proof. □

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