

# Lattice cohomology and rational cuspidal curves

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We show a counterexample to a conjecture of Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi on rational cuspidal projective plane curves, formulated in [9]. The counterexample is a tricuspidal curve of degree 8. On the other hand, we show that if the number of cusps is at most 2, then the original conjecture can be deduced from the recent results of Borodzik and Livingston [3] and the computations of [24] by the second author and Román.

We also formulate a weaker conjecture and prove it for all currently known rational cuspidal curves. We make all these identities and inequalities more transparent in the language of lattice cohomologies of certain surgery 3-manifolds.

Finally, we study the behaviour of the semigroup counting function of an irreducible plane curve singularity under blowing up in terms of its multiplicity sequence. As a corollary, we obtain a stability result of the 0th lattice cohomology of certain surgery 3-manifolds with respect to certain manipulation of the multiplicity sequences of the knots.

## 1. Introduction

**1.1.** In [9] Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi formulated a conjecture on the topological types of irreducible singularities of a rational cuspidal projective plane curve  $C \subset \mathbb{CP}^2$ . Recently in [3]

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Borodzik and Livingston, mostly motivated by [9], proved a necessary condition satisfied by the topological types of cusps of rational cuspidal plane curves. (They will be reviewed in Subsections 1.2 and 1.3). Both of them cover some deep connection with low-dimensional topology. Indeed, the conjecture from [9] was motivated by the Seiberg–Witten Invariant Conjecture of Némethi and Nicolaescu [23], which for a normal surface singularity connects the Seiberg–Witten invariant of the link with the geometric genus; while the proof of the main result of Borodzik and Livingston from [3] is based on the properties of the  $d$ -invariant of Heegaard Floer theory. Our goal is to clarify the possible interactions between them by examples and conceptual theoretical explanations. It turns out that this can be done using the theory of lattice cohomology.

In the comparison of the conjecture of [9] and the theorem of [3], the number of cusps plays a crucial role. When there is only one cusp, then the two statements are equivalent; in particular, in the unicuspidal case the theorem of [3] proves the conjecture of [9]. However, in the case of at least two cusps, the connection between the two conditions is less transparent. Although the condition proved in [3] contains equalities, while the original conjecture in [9] contains inequalities and thus it is seemingly ‘less precise’, we will see that it is not a combinatorial corollary of the former one if the number of cusps is at least three.

Nevertheless, after we reformulate all the statements in the language of lattice cohomology (Section 3), we show that for bicuspidal curves the conjecture is implied by the results of Borodzik and Livingston [3] and by the lattice cohomology formulae of [24].

Furthermore, we show that for curves with at least three cusps, the ‘original conjecture’ from [9] is not true, in general.

However, we formulate a weakened version of the conjecture — more in the spirit of the motivation of the Seiberg–Witten Invariant Conjecture, which intended to connect Euler characteristic type invariants instead of cohomology groups. This weaker version, quite surprisingly, turns out to be true for all known rational cuspidal curves, even for those, which have at least three cusps. This is proved in Section 4.

In the final section we present a procedure which makes certain 0th lattice cohomological computations a lot easier: it proves a stability property of the lattice cohomology with respect to some kind of ‘surgery manipulations’ with the multiplicity sequences of the local singularities. These computations are closely related to the lattice cohomological reformulation of the results of [3], and enlarge drastically and conceptually those geometric situations where the output of [3] is valid (showing e.g. that the global analytic

realization of the local cusp types is ‘less’ important among the conditions of the main theorem of [3]). Accordingly, this also shows that the criterion of [3], as a test for the analytic realizability of the rational cuspidal curves, is less restrictive. More precisely, the main result of [3] is a combinatorial condition on the collection of topological types of cusps of existing rational cuspidal projective plane curves. This necessary condition can be applied as a criterion when one wants to classify rational cuspidal curves. It turns out that this criterion is less restrictive when the number of local topological cusp types is larger (see Corollary 5.1.7 and Remark 5.1.8).

## 1.2. Notations and the Conjecture from [9].

Let  $C \subset \mathbb{CP}^2$  be a rational cuspidal curve of degree  $d$  with  $\nu$  cusps (that is, with locally irreducible singularities) at points  $P_1, P_2, \dots, P_\nu$ . By the local *embedded topological* type of the singularity at a point  $P_i$  we mean the homeomorphism type of the algebraic knot  $K_i = C \cap S_i \subset S_i$ , where  $S_i$  is a 3-sphere centered at  $P_i$  with sufficiently small radius. It is completely determined by the *semigroup*  $\Gamma_i \subset \mathbb{Z}_{\geq 0}$  of the plane curve singularity  $(C, P_i)$ , or, equivalently (see (1.2.1)), by the Alexander polynomial  $\Delta_i(t)$  of  $K_i \subset S_i = S^3$ . In our convention  $\Delta_i$  is indeed a polynomial, and it is normalized by  $\Delta_i(1) = 1$ .

The local embedded topological type of an irreducible plane curve singularity can also be characterized by the *multiplicity sequence*, and by the *Newton pairs* as well, see [5, 6].

The *multiplicity sequence*  $[n_1, \dots, n_r]$  is a non-increasing sequence of integers, obtained by noting the consecutive multiplicities of exceptional divisors occurring in the series of blowups during the embedded resolution of the plane curve singularity. For more details, see e.g. [5, §8.4, p. 505]. We will use the short form ‘ $u_n$ ’ for ‘ $u, \dots, u$ ’ ( $n$  copies) in the multiplicity sequences. E.g. we write  $[3_2, 2]$  instead of  $[3, 3, 2]$ .

The *Newton pairs*  $\{(p_k, q_k)\}_{k=1}^s$  are pairs of integers with  $\gcd(p_k, q_k) = 1$ ,  $p_k \geq 2, q_k \geq 1$  and  $p_1 > q_1$ . They are useful when one computes the splice diagram or the Alexander polynomial of the singularity.

By [12],  $\Gamma_i$  and  $\Delta_i$  are related as follows:

$$(1.2.1) \quad \Delta_i(t) = (1 - t) \cdot \sum_{k \in \Gamma_i} t^k.$$

The *delta invariant*  $\delta_i$  of  $(C, P_i)$  is the cardinality  $\#\{\mathbb{Z}_{\geq 0} \setminus \Gamma_i\}$ . Set  $\delta := \delta_1 + \cdots + \delta_\nu$ . The degree-genus formula for singular curves provides the following necessary condition for the existence of a degree  $d$  rational cuspidal curve with cusps of given topological type:

$$(1.2.2) \quad 2\delta = (d-1)(d-2).$$

Consider the product of Alexander polynomials:  $\Delta(t) := \Delta_1(t)\Delta_2(t)\cdots\Delta_\nu(t)$ . There is a unique polynomial  $Q$  for which  $\Delta(t) = 1 + \delta(t-1) + (t-1)^2Q(t)$ . Write

$$(1.2.3) \quad Q(t) = \sum_{j=0}^{2\delta-2} q_j t^j.$$

The definition of  $Q$  was motivated by the expression (1.2.6) below.

For  $\nu = 1$ , using (1.2.1) and properties of  $\Delta_1$ , one shows that (cf. [24, §2])

$$(1.2.4) \quad Q(t) = \sum_{s \notin \Gamma_1} (1 + t + \cdots + t^{s-1}),$$

hence  $q_j = \#\{s \notin \Gamma_1 : s > j\}$  (if  $\nu = 1$ ).

For arbitrary  $\nu$ , the dependence of the coefficients of  $Q$  in terms of  $\Gamma_i$  will be given in (2.1.2). Notice that  $q_0 = \delta$  and  $q_{2\delta-2} = 1$  [24, (2.4.4)]. From the symmetry of  $\Delta$  one also gets

$$(1.2.5) \quad q_{2\delta-2-j} = q_j + j + 1 - \delta \quad \text{for } 0 \leq j \leq 2\delta - 2.$$

Next, set the rational function

$$(1.2.6) \quad R(t) := \frac{1}{d} \sum_{\xi^d=1} \frac{\Delta(\xi t)}{(1-\xi t)^2} - \frac{1-t^{d^2}}{(1-t^d)^3}.$$

In [9, (2.4)] is proved that  $R(t)$  is a symmetric polynomial ( $R(t) = t^{d(d-3)}R(1/t)$ ), and

$$(1.2.7) \quad R(t) = \sum_{j=0}^{d-3} \left( q_{(d-3-j)d} - \frac{(j+1)(j+2)}{2} \right) t^{(d-3-j)d}.$$

Since several conjectures appear in the body of the article it is convenient to give names to them. The conjecture from [9] we wish to discuss will be called the ‘Original Conjecture’.

**Conjecture 1.2.8. (Original Conjecture)** (Fernández de Bobadilla, Lugo, Melle-Hernández, Némethi, [9]) *For any rational cuspidal plane curve  $C \subset \mathbb{CP}^2$  of degree  $d$  the coefficients of  $R(t)$  are non-positive.*

In the body of the paper (Example 1.3.2; Subsections 3.3 and 3.4) we show that for  $\nu \leq 2$  the computations of [24] reduce the conjecture to the statement of [3], and for  $\nu \geq 3$ , in general, it is false, see Example 2.2.4.

The main motivation for the expression  $R(t)$ , and for the formulation of the conjecture was a weaker version of the statement, a comparison of an analytic invariant (the geometric genus  $p_g$ ) and a topological invariant (the Seiberg–Witten invariant of the link) of the superisolated hypersurface singularity associated with  $C$ . The authors of [9] were led to it via the Seiberg–Witten Invariant Conjecture (SWIC) of Némethi and Nicolaescu [23].

More precisely, let  $f_d$  be the homogeneous equation of degree  $d$  of  $C$ , and set a generic homogeneous function  $f_{d+1}$  of degree  $(d+1)$ . Then  $f = f_d + f_{d+1} : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  defines an isolated hypersurface singularity. It is called ‘superisolated’ since a single blowup at the origin resolves  $\{f = 0\}$ , see [14] or [15, §2.2]. The singularity  $\{f = 0\}$  has geometric genus  $p_g = d(d-1)(d-2)/6$ . Let  $M$  denote its oriented link. One shows that it is the surgery manifold  $S^3_{-d}(K)$ , where  $K$  is the connected sum  $\#_i K_i$ . If  $\tilde{X} \rightarrow \{f = 0\}$  is a resolution, we denote the canonical class of  $\tilde{X}$  by  $K_{\text{can}}$  and  $\text{rank } H_2(\tilde{X})$  by  $s_{\tilde{X}}$ . Then  $K_{\text{can}}^2 + s_{\tilde{X}}$  is an invariant of the link (in this case it equals  $-(d-1)(d^2 - 3d + 1)$ ) and one also has ([9])

$$(1.2.9) \quad R(1) = -\text{sw}_{\text{can}}(M) - (K_{\text{can}}^2 + s_{\tilde{X}})/8 - p_g.$$

Here  $\text{sw}_{\text{can}}(M)$  is the Seiberg–Witten invariant of  $M$  associated with the canonical  $\text{Spin}^c$ -structure. Here we adopt the sign convention of later articles, e.g. of [4], which is the opposite of [9]. The integer  $\text{sw}_{\text{can}}(M) + (K_{\text{can}}^2 + s_{\tilde{X}})/8$  is usually called the ‘normalized Seiberg–Witten’ invariant.  $\text{sw}_{\text{can}}(M)$  can be determined (at least) by two ways, the first goes via Turaev torsion (as in [9]), or one can rely on the surgery formula of [4]. In both cases the key term is the sum from (1.2.6). For more details see also §3.1.2 or §3.2.3 below.

The SWIC [23] predicts that for certain singularities  $R(1) = 0$ . Later it turned out that this is not true e.g. for all superisolated singularities [9, 15].

Nevertheless, in all the counterexamples of [9, 15] the inequality  $R(1) \leq 0$  holds. In fact, the computations and examples of [9] suggested that one might expect even the stronger set of inequalities, namely that all the coefficients of  $R(t)$  are non-positive: this fact was formulated in the above (Original) Conjecture 1.2.8. As we will show in this article, this stronger expectation was too optimistic: it fails for certain curves with  $\nu > 2$ . Therefore, it is natural to return to the weaker version motivated by the SWIC, namely to the form  $R(1) \leq 0$ . This is the present reformulated version, what we will call ‘Weak Conjecture’.

**Conjecture 1.2.10. (Weak Conjecture)**

$$R(1) \leq 0, \quad \text{that is, } p_g \geq -\mathbf{sw}_{\text{can}}(M) - (K_{\text{can}}^2 + s_{\tilde{X}})/8.$$

Its analytic interpretation is the following. For several analytic structures of normal surface singularities (e.g. for rational, minimally elliptic, weighted homogeneous, splice quotient)  $R(1) = 0$ , that is, the geometric genus equals the topological invariant  $-\mathbf{sw}_{\text{can}}(M) - (K_{\text{can}}^2 + s_{\tilde{X}})/8$  of the link. The above conjecture predicts that for superisolated singularities, though  $p_g$  might be different than this topological invariant, it cannot be smaller. This, reinterpreted in terms of the projective curve  $C$  produces serious restrictions on the topology of local singularities and the degree  $d$ .

### 1.3. The counting functions of the semigroups

The goal of this subsection is to relate the above two conjectures with a theorem of Borodzik and Livingston. This will be done via the semigroups of the local cusps.

In fact, instead of the semigroup  $\Gamma_i$  we will often use its ‘counting function’  $k \mapsto H_i(k)$ ,

$$(1.3.1) \quad H_i(k) := \#\{s \in \Gamma_i : s < k\}.$$

From analytic point of view,  $H_i(k)$  is the coefficient of  $t^k$  in the Hilbert function of the local singularity  $(C, P_i)$ , associated with the filtration given by its normalization, though this point of view will not be used in this note.

**Example 1.3.2. (The case  $\nu = 1$ )** In this case  $q_j = \#\{s \notin \Gamma_1 : s > j\}$ , cf. (1.2.4) or [24, §2]. By the symmetry of  $\Gamma_1$  (that is,  $s \in \Gamma_1$  if and only if

$2\delta - 1 - s \notin \Gamma_1$ ) one also has

$$(1.3.3) \quad q_{2\delta-2-k} = H_1(k+1) \quad \text{for } k = 0, \dots, 2\delta - 2.$$

Hence, the  $q$ -coefficient needed in (1.2.7) is  $q_{(d-3-j)d} = \#\{s \in \Gamma_1 : s \leq jd\} = H_1(jd+1)$ .

Furthermore, the coefficients of  $R(t)$  from equation (1.2.7) can be reinterpreted geometrically by Bézout's theorem as follows (for details see [9, Proposition 3.2.1]). The dimension of the vector space  $V$  of homogeneous polynomials  $h$  of degree  $j$  in three variables is  $(j+1)(j+2)/2$ . Fix  $j < d$ . The number of conditions for  $h \in V$  to have with  $C$  at  $P_1$  intersection multiplicity  $> jd$  is  $\#\{s \in \Gamma_1 : s \leq jd\}$ . Hence,  $H_1(jd+1) < (j+1)(j+2)/2$  would imply the existence of a curve with equation  $\{h=0\}$ , which would contradict Bézout's theorem. Therefore, if  $C \subset \mathbb{CP}^2$  is a rational unicuspidal curve of degree  $d$ , then the counting function  $H_1$  of the local topological type of its singularity for each  $j = 0, 1, \dots, d-3$  satisfies

$$(1.3.4) \quad q_{(d-3-j)d} = H_1(jd+1) \geq \frac{(j+1)(j+2)}{2}.$$

In particular, this inequality and (1.2.7) show that for  $\nu = 1$  the Original Conjecture 1.2.8 is equivalent to the vanishing of  $R(t)$ . Furthermore, they are also equivalent with the Weak Conjecture 1.2.10, since if  $R(1) \leq 0$  then necessarily  $R(t) = 0$ .

**1.3.5.** For arbitrary  $\nu$ , in terms of our present notation, the above inequality (1.3.4) provided by Bézout theorem transforms into the following general form.

**Lemma 1.3.6.** [9, Proposition 3.2.1] *Let  $C \subset \mathbb{CP}^2$  be a rational cuspidal curve of degree  $d$  with  $\nu$  cusps. Then the counting functions  $H_i$  ( $i = 1, \dots, \nu$ ) of the local singularities satisfy*

$$(1.3.7) \quad \min_{j_1+j_2+\dots+j_\nu=jd+1} \{H_1(j_1) + H_2(j_2) + \dots + H_\nu(j_\nu)\} \geq \frac{(j+1)(j+2)}{2}$$

for each  $j = 0, 1, \dots, d-3$ .

This inequality was improved by Borodzik and Livingston.

**Theorem 1.3.8.** (Borodzik, Livingston [3, Theorem 5.4]) *With the notations of Lemma 1.3.6, in (1.3.7), in fact, one has equality. Namely, for each*

$j = 0, 1, \dots, d - 3$  one has

$$\min_{j_1+j_2+\dots+j_\nu=jd+1} \{H_1(j_1) + H_2(j_2) + \dots + H_\nu(j_\nu)\} = \frac{(j+1)(j+2)}{2}.$$

It is convenient to reformulate the identity as follows, cf. [3, §5.3]. Consider any two functions  $H_1$  and  $H_2$  defined on integers and bounded from below. Then we define their ‘infimal convolution’, denoted by  $H_1 \diamond H_2$ , in the following way:

$$(H_1 \diamond H_2)(j) = \min_{j_1+j_2=j} \{H_1(j_1) + H_2(j_2)\}.$$

Then from the counting functions  $\{H_i\}_{i=1}^\nu$  we construct

$$(1.3.9) \quad H := H_1 \diamond H_2 \diamond \dots \diamond H_\nu.$$

Since the operator  $\diamond$  is associative and commutative, the above function  $H$  is well-defined. Then the statement of Theorem 1.3.8 says that for all  $j = 0, 1, \dots, d - 3$  one has

$$(1.3.10) \quad H(jd + 1) = \frac{(j+1)(j+2)}{2}.$$

It is clear that for  $\nu = 1$  (when  $H = H_1$ ) this result implies the Original Conjecture 1.2.8, hence the Weak Conjecture 1.2.10 as well. By Lemma 1.3.6, in fact, the statements of the Original Conjecture 1.2.8 and Theorem 1.3.8 are equivalent, provided that  $\nu = 1$ .

In the last point of [3, Remark 5.5] the authors ask about the relation of the two statements for  $\nu \geq 2$ . We will completely clarify this relation in the next two sections. Namely, we will prove the following:

**Theorem 1.3.11.** *Consider  $\nu$  topological types of plane curve singularities and set  $\delta$  for the sum of their delta invariants. Let  $H$  be the function defined in (1.3.9) and  $q_k$  be the coefficients of the polynomial as in the Formula (1.2.3). Then the following assertions hold:*

- 1) *If  $\nu = 2$ , then  $q_{2\delta-2-k} \leq H(k+1)$ ,  $k = 0, 1, \dots, 2\delta - 2$ . Therefore, for bicuspidal curves Theorem 1.3.8 implies the Original Conjecture 1.2.8.*
- 2) *If  $\nu \geq 3$ , then the above inequality does not hold in general, not even for  $k = jd$  ( $j = 0, 1, \dots, d - 3$ ), where  $d$  is the degree of a cuspidal curve with  $\nu$  cusps of the given local singularity types.*



In [3, Example 6.16], the difficulty of obtaining bounds on the number of cusps of a cuspidal curve is briefly discussed, by presenting a list of 5 local topological types such that, among other criteria, Theorem 1.3.8 can not obstruct the existence of a corresponding hypothetical cuspidal curve. In fact, we will show that Theorem 1.3.8 alone can not provide any restrictions on the number of cusps of projective curves (see Remark 5.1.8 (b)).

The key point is that the equality (1.3.10) depends only on the infimal convolution  $H$  and many different combinations of semigroup counting functions can have the same infimal convolution. In fact, in Section 5 we will prove the following:

**Theorem 1.3.12.** *Let  $H$  be the semigroup counting function of a singularity with multiplicity sequence  $[n_1, n_2, \dots, n_r]$ . For any  $n \geq 2$ , denote by  $H_{[n]}$  the semigroup counting function of the singularity with multiplicity sequence  $[n]$ . Then*

$$H = H_{[n_1]} \diamond H_{[n_2]} \diamond \cdots \diamond H_{[n_r]}.$$

We end this section by the following symmetry property of  $H$ , an analogue of (1.2.5).

**Lemma 1.3.13.**  $H(2\delta - 2 - j + 1) = H(j + 1) - j - 1 + \delta$  for every  $j \in \mathbb{Z}$ .

*Proof.* By the symmetry of each semigroup one gets for each counting function  $H_i(j_i) = H_i(2\delta_i - j_i) + j_i - \delta_i$  for any  $j_i \in \mathbb{Z}$ . Then use the definition of  $H$ .  $\square$

## 2. Combinatorial comparison of the Original Conjecture 1.2.8 and Theorem 1.3.8 of Borodzik and Livingston

### 2.1. Reformulation of the Original Conjecture 1.2.8.

Conjecture 1.2.8 and the coefficients in equation (1.2.7) resemble the identity (1.3.10). Let us emphasize the difference.

We start in both cases with the counting functions  $H_i$ . In the Borodzik–Livingston theorem one has to take the ‘infimal convolution’  $H = H_1 \diamond H_2 \diamond \cdots \diamond H_\nu$  and (1.3.10) says that  $H(jd + 1)$  equals  $(j + 1)(j + 2)/2$  under the assumption of realizability.

On the other hand, in the Original Conjecture 1.2.8 first one determines  $\Delta_i$  from  $H_i$  by (1.2.1) and (1.3.1). Then one takes the product of all  $\Delta_i$ , and

finally one takes the coefficients of  $Q$ , which is compared with  $(j+1)(j+2)/2$ .

Next, we make explicit these last steps and provide the combinatorial formula for  $q_j$ .

Define sequences  $\{h_j^{(i)}\}_{j=0}^\infty$  by  $h_j^{(i)} := H_i(j+1)$  (notice the shift by one). For any sequence  $a = \{a_j\}_{j=0}^\infty$  denote by  $\partial a$  its *difference sequence*, i.e.  $(\partial a)_j = a_j - a_{j-1}$  with the convention that the ‘ $(-1)$ st element’ of a sequence is always zero, i.e.  $a_{-1} = 0$ . Similarly, we will denote by  $\Sigma a$  the *sequence of partial sums*, i.e.  $(\Sigma a)_j = a_0 + \cdots + a_j$ . Of course,  $\Sigma \partial a = a$  and  $\partial \Sigma a = a$  for any sequence  $a$ .

By (1.2.1) and (1.3.1), the coefficient  $c_j^{(i)}$  of  $t^j$  in  $\Delta_i(t)$  can be written as  $c_j^{(i)} = (\partial \partial h^{(i)})_j$ .

The coefficient sequence of a polynomial product is the usual convolution of coefficient sequences of the factors. Hence, the coefficient  $c_j$  of  $t^j$  in  $\Delta(t)$  is

$$c_j = \sum_{j_1 + \cdots + j_\nu = j} c_{j_1}^{(1)} \cdots c_{j_\nu}^{(\nu)}.$$

Denoting the convolution of two sequences  $a = \{a_j\}_{j=0}^\infty$  and  $b = \{b_j\}_{j=0}^\infty$  by  $a * b$ , i.e.  $(a * b)_j = \sum_{k=0}^j a_k b_{j-k}$ , we get  $c_j = (\partial \partial h^{(1)} * \cdots * \partial \partial h^{(\nu)})_j$ . Let us define:

$$(2.1.1) \quad F(j) := (\Sigma \Sigma (\partial \partial h^{(1)} * \cdots * \partial \partial h^{(\nu)}))_j.$$

If  $A(t) = \sum_j a_j t^j$  and  $B(t) = \sum_j b_j t^j$  satisfy  $A(t) = A(1) + (t-1)B(t)$ , then  $(\Sigma a)_j = A(1) - b_j$ . This applied twice for  $\Delta$  gives  $(\Sigma \Sigma c)_j = j+1 - \delta + q_j$ . Hence, the definition of  $Q$  and (1.2.5) provides

$$(2.1.2) \quad q_{2\delta-2-j} = (\Sigma \Sigma (\partial \partial h^{(1)} * \cdots * \partial \partial h^{(\nu)}))_j = F(j) \\ \text{for } 0 \leq j \leq 2\delta - 2.$$

**Example 2.1.3.** Let us exemplify these steps. Consider three singularities given by Newton pairs  $(3, 4)$ ,  $(2, 5)$  and  $(2, 3)$ , respectively. The sum of delta invariants is  $\delta = \delta_1 + \delta_2 + \delta_3 = 3 + 2 + 1 = 6$ . The Alexander polynomials

are

$$\begin{aligned}\Delta_1(t) &= \frac{(t-1)(t^{12}-1)}{(t^3-1)(t^4-1)} = 1 - t + t^3 - t^5 + t^6, \\ \Delta_2(t) &= \frac{(t-1)(t^{10}-1)}{(t^2-1)(t^5-1)} = 1 - t + t^2 - t^3 + t^4, \\ \Delta_3(t) &= \frac{(t-1)(t^6-1)}{(t^2-1)(t^3-1)} = 1 - t + t^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\Delta(t) &= \Delta_1(t)\Delta_2(t)\Delta_3(t) \\ &= 1 + 6(t-1) \\ &\quad + (t-1)^2(6 + 3t + 5t^2 + 2t^3 + 3t^4 + t^5 + 2t^6 + 2t^8 - t^9 + t^{10}),\end{aligned}$$

that is,

$$Q(t) = 6 + 3t + 5t^2 + 2t^3 + 3t^4 + t^5 + 2t^6 + 2t^8 - t^9 + t^{10}.$$

Recall that the semigroups are generated over  $\mathbb{Z}_{\geq 0}$  by the two elements of the corresponding Newton pairs. That is, the values  $H_i(k)$  of the semigroup counting functions for  $k = 0, 1, 2, \dots$  are  $0, 1, 1, 1, 2, 3, 3, 4, 5, \dots$ ;  $0, 1, 1, 2, 2, 3, 4, \dots$  and  $0, 1, 1, 2, 3, \dots$ , respectively ( $i = 1, 2, 3$ ).

| $i$      | 1 |   |   |   |   |   |   | 2 |   |   |   |   | 3 |   |   |
|----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 |
| $H_i(k)$ | 0 | 1 | 1 | 1 | 2 | 3 | 3 | 0 | 1 | 1 | 2 | 2 | 0 | 1 | 1 |

The sequences  $h^{(i)}$  and their difference sequences are as follows.

| $i$                        | 1 |    |   |   |   |    |   | 2 |    |   |    |   | 3 |    |   |
|----------------------------|---|----|---|---|---|----|---|---|----|---|----|---|---|----|---|
| $h^{(i)}$                  | 1 | 1  | 1 | 2 | 3 | 3  | 4 | 1 | 1  | 2 | 2  | 3 | 1 | 1  | 2 |
| $\partial h^{(i)}$         | 1 | 0  | 0 | 1 | 1 | 0  | 1 | 1 | 0  | 1 | 0  | 1 | 1 | 0  | 1 |
| $\partial\partial h^{(i)}$ | 1 | -1 | 0 | 1 | 0 | -1 | 1 | 1 | -1 | 1 | -1 | 1 | 1 | -1 | 1 |

Notice that one can read off the coefficients of the corresponding Alexander polynomials from the last row. Then one can compute the convolution  $p = \partial\partial h^{(1)} * \partial\partial h^{(2)} * \partial\partial h^{(3)}$ , then twice the sequence of partial sums to obtain the values of  $F$  (compare the last row of  $F$  to the coefficients of  $Q$  above):

| $p$              | 1 | 1  | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
|------------------|---|----|---|---|---|---|---|---|---|---|---|
| $\Sigma p$       | 1 | 1  | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| $\Sigma\Sigma p$ | 1 | -1 | 2 | 0 | 2 | 1 | 3 | 2 | 5 | 3 | 6 |

One also can compute the infimal convolution  $H = H_1 \diamond H_2 \diamond H_3$ . To summarize, we get

|          |   |    |   |   |   |   |   |   |   |   |    |
|----------|---|----|---|---|---|---|---|---|---|---|----|
| $k$      | 0 | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $H(k+1)$ | 1 | 1  | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 6  |
| $F(k)$   | 1 | -1 | 2 | 0 | 2 | 1 | 3 | 2 | 5 | 3 | 6  |

Now we can reformulate the inequalities of the Original Conjecture 1.2.8 as follows.

**Conjecture 2.1.4. (Original Conjecture, alternative form)**

Let  $C \subset \mathbb{CP}^2$  be a rational cuspidal curve of degree  $d$  with  $\nu$  cusps of given topological types (in particular,  $d(d-3) = 2\delta - 2$ ). Set  $F(j) := (\Sigma \Sigma(\partial \partial h^{(1)} * \dots * \partial \partial h^{(\nu)}))_j$ , where  $h_j^{(i)} = H_i(j+1)$ , and  $H_i$  is the semigroup counting function of the  $i$ -th singularity. Then

$$(2.1.5) \quad F(jd) \leq \frac{(j+1)(j+2)}{2} \quad \text{for all } j = 0, 1, \dots, d-3.$$

The Weak Conjecture is obtained by taking sum.

**Conjecture 2.1.6. (Weak Conjecture, first alternative form)**

Under the conditions and with the notation of Conjecture 2.1.4,

$$\sum_{j=0}^{d-3} F(jd) \leq \sum_{j=0}^{d-3} \frac{(j+1)(j+2)}{2} = \frac{d(d-1)(d-2)}{6}.$$

## 2.2. Examples and counterexamples.

Let us summarize the situation. Starting from the semigroups of  $\nu$  local singularities we can define integral functions  $H$  and  $F$  depending only on the local topological types of the singularities.

**Definition 2.2.1.** If the sum of delta invariants of the local singularity types,  $\delta$ , is of form  $2\delta = (d-1)(d-2)$  for some integer  $d$ , we say that these  $\nu$  local topological types are *candidates* to be the  $\nu$  singularities of a rational cuspidal plane curve of degree  $d$ .

If such a curve exists then Theorem 1.3.8 of [3] prescribes ‘each  $d$ -th value’ of  $H$  by  $H(jd+1) = (j+1)(j+2)/2$ . Furthermore, the Original Conjecture 2.1.4 would give an upper bound on ‘each  $d$ -th value’ of  $F$ , namely  $F(jd) \leq (j+1)(j+2)/2$ .

As we already mentioned, if  $\nu = 1$  then  $F(k) = H(k + 1)$  for each  $k \in \mathbb{Z}_{\geq 0}$  (not just for  $k \in d \cdot \mathbb{Z}_{\geq 0}$ ), and the theorem implies the conjecture (and the inequalities are equalities).

However, for  $\nu > 1$  the values  $F(k)$  and  $H(k + 1)$  become different. If one starts to play with *two* singularities, one can notice that  $F(k) \leq H(k + 1)$  seems to be true for every integer  $k \geq 0$ , not just for the multiples of  $d$ . Later, using lattice cohomology interpretations, we will prove that this is indeed true for  $\nu = 2$ , cf. §3.4. See also [19] for an elementary proof.

With these facts in mind, it is tempting to conjecture that maybe *the inequality  $F(k) \leq H(k + 1)$  is always true — even independently of  $d$  as a property of local singularity types* — which would be an interesting, completely combinatorial statement making the Original Conjecture 1.2.8 a corollary of Theorem 1.3.8. But, for  $\nu \geq 3$  there is no such relation between functions  $F$  and  $H$ , as we will demonstrate next.

**Example 2.2.2.** Take  $\nu = 3$ , and assume that all local singularities are ‘simple’ cusps, that is cusps with multiplicity sequence [2] (or, equivalently, with one Newton pair  $(2, 3)$ , or with semigroup  $\langle 2, 3 \rangle = \{0, 2, 3, 4, \dots\}$ ). Then the functions  $F$  and  $H$  are as follows:

|                   |   |    |    |   |   |
|-------------------|---|----|----|---|---|
| $k$               | 0 | 1  | 2  | 3 | 4 |
| $H(k + 1)$        | 1 | 1  | 2  | 2 | 3 |
| $F(k)$            | 1 | −1 | 3  | 0 | 3 |
| $H(k + 1) - F(k)$ | 0 | 2  | −1 | 2 | 0 |

Notice that for  $k = 2$  the desired inequality  $F(k) \leq H(k + 1)$  fails. Hence the inequality  $F(k) \leq H(k + 1)$  cannot be true for *any integer*  $k$ . (By the way, this collection of local cusp types can be realized on a rational tricuspidal curve of degree four, cf. Proposition 4.1.1.)

**Example 2.2.3.** (Example 2.1.3 revisited.) Consider now a tricuspidal rational projective plane curve of degree  $d = 5$  such that each of its singularities has one Newton pair, namely  $(3, 4)$ ,  $(2, 5)$  and  $(2, 3)$ , already considered in Example 2.1.3. For its realizability see Proposition 4.1.1. As we already have shown the function values are as follows:

|                   |   |    |    |   |   |   |   |   |    |   |    |
|-------------------|---|----|----|---|---|---|---|---|----|---|----|
| $k$               | 0 | 1  | 2  | 3 | 4 | 5 | 6 | 7 | 8  | 9 | 10 |
| $H(k + 1)$        | 1 | 1  | 1  | 2 | 2 | 3 | 3 | 4 | 4  | 5 | 6  |
| $F(k)$            | 1 | −1 | 2  | 0 | 2 | 1 | 3 | 2 | 5  | 3 | 6  |
| $H(k + 1) - F(k)$ | 0 | 2  | −1 | 2 | 0 | 2 | 0 | 2 | −1 | 2 | 0  |

Since the data are provided by an existing curve of degree 5, the  $H(jd + 1)$ -values at  $jd = k = 0, 5, 10$  are the corresponding triangular numbers (1, 3, 6, respectively), as predicted by Theorem 1.3.8. Notice also that again, the desired inequality  $F(k) \leq H(k + 1)$  fails at  $k = 2, 8$ . However, the inequalities needed for the Original Conjecture 1.2.8 corresponding to  $k = 0, 5, 10$  (multiples of  $d$ ) are true. (This example appears in [9], supporting Conjecture 1.2.8.) So one could still hope in the inequality  $F(k) \leq H(k + 1)$  in the case of existing curves and for  $k \in d \cdot \mathbb{Z}$ , where  $d$  is the degree of the curve.

**Example 2.2.4. (Counterexample to the Original Conjecture 1.2.8)**

Consider three semigroups given by two generators each as follows:  $\Gamma_1 = \langle 6, 7 \rangle$ ,  $\Gamma_2 = \langle 2, 9 \rangle$  and  $\Gamma_3 = \langle 2, 5 \rangle$ . These are semigroups of plane curve singularities characterized by multiplicity sequences [6],  $[2_4]$  and  $[2_2]$ , respectively. There exists a rational tricuspidal curve of degree  $d = 8$  with three singularities exactly of this topological type (cf. Proposition 4.1.1). The values of functions  $H$  and  $F$  are as follows — we are interested here only in the values at the multiples of  $d$ :

|                   |   |     |    |     |    |     |    |     |    |     |    |
|-------------------|---|-----|----|-----|----|-----|----|-----|----|-----|----|
| $k$               | 0 | ... | 8  | ... | 16 | ... | 24 | ... | 32 | ... | 40 |
| $H(k + 1)$        | 1 | ... | 3  | ... | 6  | ... | 10 | ... | 15 | ... | 21 |
| $F(k)$            | 1 | ... | 4  | ... | 5  | ... | 9  | ... | 16 | ... | 21 |
| $H(k + 1) - F(k)$ | 0 | ... | -1 | ... | 1  | ... | 1  | ... | -1 | ... | 0  |

Of course, as the data are realized by a curve, Theorem 1.3.8 is satisfied: note the triangular numbers in the second row. The condition  $\sum_{j=0}^{d-3} F(jd) \leq \sum_{j=0}^{d-3} (j + 1)(j + 2)/2$  asked by the first alternative version of the Weak Conjecture 2.1.6 is also satisfied, in fact, by equality: this can be seen immediately by summation of the last row of the above table.

However, for  $j = 1$  and  $j = 4$  the inequality  $F(jd) \leq (j + 1)(j + 2)/2$  fails, hence *this is a counterexample to the Original Conjecture 1.2.8*.

This example was not checked in [9], as it was not clear at that time that the number of cusps was crucial. Some series with  $\nu = 1$  were checked and other examples as well (also with  $\nu \geq 3$ ), but only up to degree 7 (note that a complete classification of cuspidal curves exists only up to degree 6). As we will see later in Remark 4.1.3 the smallest degree where the Original Conjecture 1.2.8 fails among currently known rational cuspidal curves is exactly 8.

This example also shows that the inequalities of the Original Conjecture 1.2.8 are *not combinatorial consequences* of the equalities of Theorem 1.3.8. Moreover, the inequalities  $F(k) \leq H(k+1)$  are not true in general, not even for existing curves of degree  $d$  and setting  $k \in d \cdot \mathbb{Z}$ .

**Example 2.2.5.** The following example will show that the Weak Conjecture (version 2.1.6 of it) is *not a combinatorial consequence* of the equalities of Theorem 1.3.8 either.

Consider three semigroups given by their generators:  $\Gamma_1 = \langle 3, 5 \rangle$ ,  $\Gamma_2 = \langle 2, 3 \rangle$ ,  $\Gamma_3 = \langle 2, 3 \rangle$ . The corresponding multiplicity sequences are  $[3, 2]$ ,  $[2]$ ,  $[2]$ , respectively. The sum of delta invariants is  $4 + 1 + 1 = (5 - 1)(5 - 2)/2$ , so these three topological types of local singularities are possible candidates for three cusps of a tricuspidal rational projective plane curve of degree  $d = 5$ . Since the complex projective quintics are completely classified (see e.g. [18] or [16, Chapter 6]), it is known that such a curve does not exist. However, Theorem 1.3.8 does not exclude the existence of this curve, as the values  $H(k+1)$  are again 1, 3, 6 at  $k = 0, 5, 10$ , respectively.

| $k$             | 0 | 1  | 2  | 3 | 4 | 5  | 6 | 7 | 8  | 9 | 10 |
|-----------------|---|----|----|---|---|----|---|---|----|---|----|
| $H(k+1)$        | 1 | 1  | 1  | 2 | 2 | 3  | 3 | 4 | 4  | 5 | 6  |
| $F(k)$          | 1 | -1 | 2  | 1 | 0 | 4  | 1 | 3 | 5  | 3 | 6  |
| $H(k+1) - F(k)$ | 0 | 2  | -1 | 1 | 2 | -1 | 2 | 1 | -1 | 2 | 0  |

On the other hand, the inequality of the Original Conjecture 1.2.8 fails at  $k = 5$ . (Recall however that there does not exist a quintic with the above singularities and that Conjecture 1.2.8 can not be used to show this fact, as it already failed in Example 2.2.4.) But, additionally, for this candidate the Weak Conjecture 2.1.6 also fails.

Therefore, if the Weak Conjecture 2.1.6 would be proved — independently of the classification of projective curves —, it would provide *an independent tool for checking whether a given collection of local topological singularity types can be realized as the collection of cusp types of a rational cuspidal projective plane curve*.

### 3. Lattice cohomological interpretation

**3.1.** Now we show the lattice-cohomological meaning of the values of functions  $H$  and  $F$ . From this point of view, it will be obvious that, on one hand, for  $\nu = 2$  inequalities  $F(k) \leq H(k+1)$  hold, but on the other hand, it is odd to expect such a relation for  $\nu \geq 3$ . The necessary computations

are done in [24], but in that note they were not analyzed from the present point of view.

### 3.1.1. The lattice cohomology of a weight function.

For the definition of lattice cohomology, see [20]. There is a detailed description in Section 3 of [24] as well. In short, the construction is the following.

Usually one starts with a lattice  $\mathbb{Z}^s$  with fixed base elements  $\{E_i\}_i$ . This automatically provides a cubical decomposition of  $\mathbb{R}^s = \mathbb{Z}^s \otimes \mathbb{R}$ : the 0-cubes are the lattice points  $l \in \mathbb{Z}^s$ , the 1-cubes are the ‘segments’ with endpoints  $l$  and  $l + E_i$ , and more generally, a  $q$ -cube  $\square = (l, I)$  is determined by a lattice point  $l \in \mathbb{Z}^s$  and a subset  $I \subset \{1, \dots, s\}$  with  $\#I = q$ , and it has vertices at the lattice points  $l + \sum_{j \in J} E_j$  for different  $J \subset I$ .

One also takes a weight function  $w : \mathbb{Z}^s \rightarrow \mathbb{Z}$  bounded below, and for each cube  $\square = (l, I)$  one takes  $w(\square) := \max\{w(v), v \text{ vertex of } \square\}$ . Then, for each integer  $n \geq \min(w)$  one considers the simplicial complex  $S_n$ , the union of all the cubes of any dimension with  $w(\square) \leq n$ . Then the *lattice cohomology associated with  $w$*  is  $\{\mathbb{H}^q(\mathbb{Z}^s, w)\}_{q \geq 0}$ , defined by  $\mathbb{H}^q(\mathbb{Z}^s, w) := \bigoplus_{n \geq \min(w)} H^q(S_n, \mathbb{Z})$ . Each  $\mathbb{H}^q$  is graded (by  $n$ ) and it is a  $\mathbb{Z}[U]$ -module, where the  $U$ -action consists of the restriction maps induced by the inclusions  $S_n \hookrightarrow S_{n+1}$ . Similarly, one defines the *reduced cohomology associated with  $w$*  by  $\mathbb{H}_{\text{red}}^q(\mathbb{Z}^s, w) := \bigoplus_{n \geq \min(w)} \tilde{H}^q(S_n, \mathbb{Z})$ . In all our cases  $\mathbb{H}_{\text{red}}^q(\mathbb{Z}^s, w)$  has finite  $\mathbb{Z}$ -rank. The normalized Euler characteristic of  $\mathbb{H}^*(\mathbb{Z}^s, w)$  is  $\text{eu } \mathbb{H}^* := -\min(w) + \sum_{q \geq 0} (-1)^q \text{rank}_{\mathbb{Z}} \mathbb{H}_{\text{red}}^q$ . Formally, we also set  $\text{eu } \mathbb{H}^0 := -\min(w) + \text{rank}_{\mathbb{Z}} \mathbb{H}_{\text{red}}^0$ .

### 3.1.2. The lattice cohomology of a negative definite plumbing graph.

Let  $G$  be a connected negative definite plumbing graph, and let  $(\cdot, \cdot)$  be the negative definite intersection form associated with it. In this note we assume that in all cases  $G$  is a tree, and all the genus decorations of the vertices are zero, that is, the corresponding plumbed 3-manifold  $M = M(G)$  is a rational homology sphere. We set  $L$  for the lattice generated by elements  $E_v$  indexed by the vertices of the graph, which is naturally endowed with the form  $(\cdot, \cdot)$ . Set  $L' := \text{Hom}(L, \mathbb{Z})$  regarded as a subgroup of  $L \otimes \mathbb{Q}$ . Then  $L'/L$  is identified with  $H_1(M(G), \mathbb{Z})$ , and  $\text{Spin}^c(M)$  is an  $L'/L$ -torsor (and in this note it will be identified with  $L'/L$ ). The canonical  $\text{Spin}^c$ -structure corresponds to the class of zero. Let  $\mathcal{S}'$  be the anti-nef cone  $\{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$ .



Let  $K_{\text{can}} \in L'$  be the canonical characteristic element defined by the adjunction formulae  $(K_{\text{can}}, E_v) = -(E_v, E_v) - 2$  for any vertex  $v$ . Moreover, for any class  $\sigma \in L'/L$  let  $l'_\sigma$  be the unique minimal element of  $\mathcal{S}'$  whose class is  $\sigma$ . Then one defines the weight function

$$\chi_\sigma : L \rightarrow \mathbb{Z}, \quad \chi_\sigma(l) := -(l, l + K_{\text{can}} + 2l'_\sigma)/2.$$

To each  $\text{Spin}^c$ -structure  $\sigma \in L'/L$  of  $M$  one considers the lattice  $L$  with the weight function  $\chi_\sigma$ . This pair determines the lattice cohomology  $\mathbb{H}^*(M, \sigma)$  and the whole lattice cohomology package as in §3.1.1. It depends only on the 3-manifold  $M$  and  $\sigma \in \text{Spin}^c(M)$ . For more details see e.g. [20, 24].

For the canonical  $\text{Spin}^c$ -structure  $\sigma = [0]$ ,  $l'_\sigma = 0$ , and the corresponding lattice cohomology is denoted by  $\mathbb{H}_{\text{can}}^*$ .

In this language the invariant  $K_{\text{can}}^2 + s_{\widehat{X}}$  considered in (1.2.9) is the sum of  $(K_{\text{can}}, K_{\text{can}})$  and the number of vertices  $s$ . For a different interpretation of  $\text{eu } \mathbb{H}^*$  see Remark 3.2.3.

### 3.1.3. The lattice cohomology of the surgery 3-manifold $S_{-d}^3(K)$ .

In [24] the authors computed lattice cohomologies of the following surgery 3-manifold. The input consists of  $\nu$  local topological plane curve singularity types, as in our case above, and a positive integer  $d$ . The surgery 3-manifold is  $S_{-d}^3(K)$ , the manifold obtained by a  $(-d)$ -surgery along the connected sum  $K$  of knots of the given plane curve singularities ( $K = K_1 \# \cdots \# K_\nu \subset S^3$ ). For motivation see Subsection 1.2. However, we do not assume here that  $(d-1)(d-2) = 2\delta$ .

$S_{-d}^3(K)$  is a plumbed 3-manifold represented by a negative definite plumbing graph whenever  $d > 0$ , hence the above constructions run. Moreover,  $L'/L = \mathbb{Z}_d$ , accordingly we parametrize the  $\text{Spin}^c$ -structures by  $a \in \{0, \dots, d-1\}$  ( $\sigma = [a] \in \mathbb{Z}_d$ ), and for each of them one considers the weight function  $\chi_\sigma$ . For details see [24].

However, in [24], the lattice cohomologies  $\mathbb{H}^*(S_{-d}^3(K), a)$  are not computed by the definition presented above, but by a powerful reduction machinery, called ‘lattice reduction’, cf. [13, 24]. This allows to express the cohomology modules in a lattice (in fact, in a ‘rectangle’) of rank  $\nu$ , and its newly defined weight function is determined directly from the semigroups (or counting functions) of the given local topological singularity types. The needed facts are summarized in the proof of the next theorem and in Theorem 3.1.7. For more details see again [24].

The main result of this section is the following theorem.

**Theorem 3.1.4.** *Assume that  $K = \#_i K_i$ , where  $\{K_i\}_i$  are algebraic knots. Define functions  $H$  and  $F$  as in previous sections. Assume that  $d$  is any positive integer. Then*

$$(3.1.5) \quad \text{eu } \mathbb{H}^0(S_{-d}^3(K), a) = \sum_{\substack{j \equiv a \pmod{d} \\ 0 \leq j \leq 2\delta-2}} (H(j+1) + \delta - 1 - j),$$

$$(3.1.6) \quad \text{eu } \mathbb{H}^*(S_{-d}^3(K), a) = \sum_{\substack{j \equiv a \pmod{d} \\ 0 \leq j \leq 2\delta-2}} (F(j) + \delta - 1 - j).$$

*Proof.* We will recall several needed statements from [24].

Let  $f_i$  be the local equation of  $(C, P_i)$ , and let  $m_i$  be the multiplicity along the unique  $(-1)$ -irreducible exceptional divisor of the pullback of  $f_i$  in the minimal good embedded resolution of  $(C, P_i)$ . It is a topological invariant, and  $m_i > 2\delta_i$ .

We consider the lattice points in the rank- $\nu$  multirectangle  $R := [0, m_1] \times \cdots \times [0, m_\nu]$ . We denote them by  $\mathbf{x} = (x_1, \dots, x_\nu)$ , and we also write  $|\mathbf{x}| := \sum_{i=1}^\nu x_i$ .

For any  $a$  with  $0 \leq a \leq d-1$  we set the weight function on  $R$  by

$$w_a(\mathbf{x}) = \sum_{i=1}^\nu H_i(x_i) + \min\{0, 1 + a - |\mathbf{x}|\}.$$

It is convenient to define another weight function too, which is independent of  $d$  and  $a$ :

$$\begin{aligned} W(\mathbf{x}) &= \sum_{i=1}^\nu \#\{s \notin \Gamma_i : s \geq x_i\} = \sum_{i=1}^\nu (\delta_i - x_i + H_i(x_i)) \\ &= \delta - |\mathbf{x}| + \sum_{i=1}^\nu H_i(x_i). \end{aligned}$$

For any  $j \geq 0$  denote the ‘diagonal hyperplanes’ of the multirectangle by

$$T_j := \{\mathbf{x} \in R : |\mathbf{x}| = j+1\}.$$

Note that  $T_j = \emptyset$  whenever  $j > M := m_1 + \cdots + m_\nu$ .

Next, as in [24], we define lattice cohomologies on the ‘diagonal’ sets  $T_j$  as well, considering the cohomologies of the intersection of simplicial level sets of the lattice rectangle and the  $(\nu-1)$ -dimensional hyperplane of  $T_j$ , i.e.  $\mathbb{H}_{\text{red}}^q(T_j, W) := \oplus_{n \geq \min(W)} \widetilde{H}^q(S_n \cap T_j, \mathbb{Z})$  (and similarly for the non-reduced

version; cf. [24, (6.1.10)]), where the simplicial complex (level set)  $S_n$  is the union of all cubes  $\square$  with  $W(\square) \leq n$ .

**Theorem 3.1.7.** ([24], formulae (6.1.15) and (6.1.16)) *For any  $d > 0$  one has:*

$$(3.1.8) \quad \text{eu } \mathbb{H}^0(S_{-d}^3(K), a) = \sum_{\substack{j \equiv a \pmod{d} \\ 0 \leq j \leq M}} \min W|_{T_j},$$

$$(3.1.9) \quad \text{eu } \mathbb{H}^*(S_{-d}^3(K), a) = - \sum_{\substack{j \equiv a \pmod{d} \\ 0 \leq j \leq M}} \text{eu } \mathbb{H}^*(T_j, W).$$

Clearly  $\min W|_{T_j} = \delta - j - 1 + H(j + 1)$ , which equals  $H(2\delta - 1 - j)$  by Lemma 1.3.13, thus it is zero for  $j \not\leq 2\delta - 2$ . Hence the identity (3.1.5) follows.

Next, fix some  $j \geq 0$ , and apply Theorem 3.1.7 for an auxiliary large  $D > M$  (substituted for  $d$ ), and for  $a = j$ . By [24, Proposition 5.3.4, Corollary 5.3.7, Theorem 6.1.6 e)], for such  $D > M$ , one has

$$(3.1.10) \quad \mathbb{H}^*(S_{-D}^3(K), j) \cong \mathbb{H}^*([0, m_1] \times \cdots \times [0, m_\nu], w_j).$$

Moreover, by [24, Proposition 7.1.3], for  $D > M$  the normalized Euler characteristic of this cohomology can be compared with the coefficients of the polynomial  $Q$ . Namely,

$$(3.1.11) \quad q_j = \text{eu } \mathbb{H}^*([0, m_1] \times \cdots \times [0, m_\nu], w_j).$$

Then (3.1.9), (3.1.10) and (3.1.11) combined give  $q_j = -\text{eu } \mathbb{H}^*(T_j, W)$  for any  $j$ . Notice that  $q_j = 0$  if  $j$  is not in the interval  $[0, 2\delta - 2]$ , and for these values by (1.2.5) one also has  $q_j = q_{2\delta-2-j} + \delta - j - 1$ , which equals  $F(j) + \delta - j - 1$  by (2.1.2). Hence (3.1.9) (now applied with the original  $d$ ) implies (3.1.6).  $\square$

**Remark 3.1.12.** In fact, the integer  $d$ , the sum of delta invariants  $\delta$  and the function  $H$  completely determine the whole  $\mathbb{H}^0$  as a graded  $\mathbb{Z}[U]$ -module (and not just its Euler characteristic). For this fact, we refer to [24, Lemma 6.1.1, Theorem 6.1.6].

**Corollary 3.1.13.** *Assume that  $d(d-3) = 2\delta - 2$ , cf. (1.2.2). Then*

$$\begin{aligned} \mathrm{eu} \mathbb{H}^0(S_{-d}^3(K), a) &= \sum_{\substack{j \equiv -a \pmod{d} \\ 0 \leq j \leq 2\delta-2}} H(j+1), \\ \mathrm{eu} \mathbb{H}^*(S_{-d}^3(K), a) &= \sum_{\substack{j \equiv -a \pmod{d} \\ 0 \leq j \leq 2\delta-2}} F(j). \end{aligned}$$

The value  $a = 0$  corresponds to the canonical  $\mathrm{Spin}^c$ -structure. Denote the corresponding lattice cohomology of  $S_{-d}^3(K)$  by  $\mathbb{H}_{\mathrm{can}}^*(S_{-d}^3(K))$ . Then the above identities read as:

$$(3.1.14) \quad \mathrm{eu} \mathbb{H}_{\mathrm{can}}^0(S_{-d}^3(K)) = \sum_{0 \leq j \leq d-3} H(jd+1),$$

$$(3.1.15) \quad \mathrm{eu} \mathbb{H}_{\mathrm{can}}^*(S_{-d}^3(K)) = \sum_{0 \leq j \leq d-3} F(jd).$$

*Proof.* Use the symmetry properties (1.2.5) and (1.3.13). □

### 3.2. Reformulations of Theorem 1.3.8 and the Weak Conjecture 1.2.10

Because of the inequalities (implied by Bézout's theorem) of Lemma 1.3.6 from [9, Proposition 3.2.1], Theorem 1.3.8 of Borodzik and Livingston is true if and only if the corresponding sums over  $j$  are equal. This combined with (3.1.14) provides the following equivalent form.

**Theorem 3.2.1. (Equivalent form of Theorem 1.3.8 of Borodzik and Livingston)** *For a link  $M = S_{-d}^3(K)$  of a superisolated surface singularity corresponding to a rational cuspidal projective plane curve of degree  $d$  we have:*

$$\mathrm{eu} \mathbb{H}_{\mathrm{can}}^0(S_{-d}^3(K)) = d(d-1)(d-2)/6.$$

This form is also present in the recent article [25, Example 2.4.3 (a) and Section 3]; cf. also [8, Theorem 8.9] for the unicuspidal case.

Next, using (3.1.15) we give an equivalent formulation of the Weak Conjecture 1.2.10 in terms of lattice cohomology, see also its alternative version, Conjecture 2.1.6.

**Conjecture 3.2.2. (Weak Conjecture, second alternative form)**

For a link  $M = S^3_{-d}(K)$  of a superisolated surface singularity corresponding to a rational cuspidal projective plane curve of degree  $d$  we have:

$$\mathrm{eu} \, \mathbb{H}_{\mathrm{can}}^*(S^3_{-d}(K)) \leq d(d-1)(d-2)/6.$$

Alternatively, in the light of the previous theorem:

$$\mathrm{eu} \, \mathbb{H}_{\mathrm{can}}^*(S^3_{-d}(K)) \leq \mathrm{eu} \, \mathbb{H}_{\mathrm{can}}^0(S^3_{-d}(K)).$$

In this context, the Weak Conjecture 2.1.6 is much more natural than the Original Conjecture 1.2.8 which would require the validity of  $F(jd) \leq (j+1)(j+2)/2$  for every single  $j = 0, 1, \dots, d-3$ , i.e. an inequality for the lattice cohomological Euler characteristic of each diagonal set  $T_{jd}$ .

**Remark 3.2.3.** (a) **(Connection with the Seiberg–Witten invariant)**

Let  $G$  be a connected negative definite plumbing graph,  $M = M(G)$  the corresponding plumbed 3-manifold, and  $\sigma \in \mathrm{Spin}^c(M)$ , cf. §3.1.2. In [21] it is proved that  $\mathrm{eu} \, \mathbb{H}^*(M, \sigma)$  equals the normalized Seiberg–Witten invariant of  $M$  associated with the  $\mathrm{Spin}^c$ -structure  $\sigma$ . In particular, for the canonical  $\mathrm{Spin}^c$ -structure, one has  $\mathrm{eu} \, \mathbb{H}_{\mathrm{can}}^*(M) = -\mathrm{sw}_{\mathrm{can}}(M) - (K_{\mathrm{can}}^2 + s)/8$ , compatibly with the Weak Conjecture 1.2.10.

(b) **(Connection with the Heegaard Floer homology)**

Let  $HF^+(M, \sigma)$  denote the Heegaard Floer homology of a plumbed 3-manifold  $M$  associated with a connected negative definite graph  $G$  and  $\sigma \in \mathrm{Spin}^c(M)$ . Then one has a graded  $\mathbb{Z}[U]$ -module isomorphism  $HF^+(M, \sigma) = \mathcal{T}_{d(M, \sigma)}^+ \oplus HF^+(M, \sigma)_{\mathrm{red}}$ , where the reduced Heegaard Floer homology  $HF^+(M, \sigma)_{\mathrm{red}}$  has a finite  $\mathbb{Z}$ -rank and an absolute  $\mathbb{Z}_2$ -grading (*even*, *odd*),  $\mathcal{T}_{d(M, \sigma)}^+$  is isomorphic to  $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$  with shifted grading such that  $d(M, \sigma)$  is the minimal value among  $\mathbb{Z}$ -gradings of its elements and  $d(M, \sigma)$  denotes the  $d$ -invariant, see [29]. Then

$$\mathrm{sw}(M, \sigma) = \mathrm{rank}_{\mathbb{Z}} HF_{\mathrm{red}, \mathrm{even}}^+(M, \sigma) - \mathrm{rank}_{\mathbb{Z}} HF_{\mathrm{red}, \mathrm{odd}}^+(M, \sigma) - d(M, \sigma)/2.$$

In particular, via part (a),  $\mathrm{eu} \, \mathbb{H}^*(M, \sigma)$  can also be interpreted as the normalized Euler characteristic of the Heegaard Floer homology.

In fact, in [20] the second author conjectured that up to a degree shift by  $d(M, \sigma)$

$$\begin{cases} HF_{\mathrm{red}, \mathrm{even}}^+(-M, \sigma) = \bigoplus_{q \text{ even}} \mathbb{H}_{\mathrm{red}}^q(M, \sigma) \\ HF_{\mathrm{red}, \mathrm{odd}}^+(-M, \sigma) = \bigoplus_{q \text{ odd}} \mathbb{H}_{\mathrm{red}}^q(M, \sigma). \end{cases}$$

If  $M = S^3_{-d}(K)$  as above, and  $\nu \leq 2$ , then the conjecture is true: if  $\nu = 1$  then the graph is ‘almost rational’, hence the statement follows from [20], if  $\nu = 2$  then one can use [28]. In these cases, in fact,  $HF^+_{red,even}(-M) = \mathbb{H}^0_{red}(M)$ ,  $HF^+_{red,odd}(-M) = \mathbb{H}^1_{red}(M)$ . Furthermore,  $\mathbb{H}^q_{red}(M) = 0$  for  $q \geq 2$  since the graph has only  $\nu \leq 2$  ‘bad vertices’, cf. [22, 24]. Nevertheless, if  $\nu \geq 3$  then  $\mathbb{H}^3$  in principle can be nontrivial, hence  $\mathbb{H}^0$  cannot be determined from the Heegaard Floer homology: the additional  $\mathbb{Z}$ -grading of the lattice cohomology is a finer invariant.

**Remark 3.2.4.** We wish to emphasize that it is essential that in the above Weak Conjecture, alternative form 3.2.2, we talk about the lattice cohomologies corresponding to the *canonical*  $\text{Spin}^c$ -structure *only*. Using formulae of Corollary 3.1.13 one can check easily that for superisolated singularity link  $M$  coming from the existing curve of Example 2.2.4 choosing  $\text{Spin}^c$ -structure corresponding to  $a = 4$  we have  $\text{eu } \mathbb{H}^*(M, a = 4) = 45 > \text{eu } \mathbb{H}^0(M, a = 4) = 42$ . Also, for many curves from series (1) in Proposition 4.1.1 of the next section, one can find  $\text{Spin}^c$ -structures for which the inequality fails, e.g.  $\text{eu } \mathbb{H}^*(M(C_{4,1}), a = 2) = 3 > \text{eu } \mathbb{H}^0(M(C_{4,1}), a = 2) = 2$ , cf. Example 2.2.2.

**Remark 3.2.5.** It is well-known that the link  $M$  of a superisolated singularity corresponding to a projective plane curve is a rational homology sphere ( $\mathbb{Q}HS^3$ ) if and only if the curve is rational and cuspidal, see [8, §7.1] and the references therein.

The fact that  $M$  is  $\mathbb{Q}HS^3$  is probably also essential in the above conjecture. To see this, consider the following example. It is not hard to see by a construction using Cremona transformations that there exists a rational projective curve  $C$  of degree  $d = 5$  with three singular points which are of the following type. One singularity is a simple transversal self-intersection: a reducible  $A_1$ -singularity. The other two are locally irreducible singularities, with multiplicity sequences  $[3, 2]$ , resp.  $[2]$ , alternatively, with Newton pairs  $(3, 5)$ , resp.  $(2, 3)$ . From the embedded resolution graphs of plane curve singularities, it is easy to construct the plumbing graph of the link  $M$  of the corresponding superisolated singularity. (Due to the locally reducible singularity, it has one cycle, so it is not a tree.) One checks that for this link  $\text{eu } \mathbb{H}^*_{\text{can}}(M) = 11 > \text{eu } \mathbb{H}^0_{\text{can}}(M) = 10$ . Note that although  $M$  is not a rational homology sphere, we can still speak about the corresponding lattice cohomologies with the same definition as in the  $\mathbb{Q}HS^3$  case.

### 3.3. Proof of the second alternative form 3.2.2 of the Weak Conjecture for $\nu = 2$ .

First we recall that  $\mathbb{H}^q(S_{-d}^3(K), a) = 0$  for any  $q \geq \nu$ . This follows from the fact that the non-compact simplicial subcomplexes  $S_n$  of  $\mathbb{R}^\nu$  (in the reduced lattices) have no nonzero cohomologies  $H^q(S_n, \mathbb{Z})$  for  $q \geq \nu$ ; or just apply [13] or [22, §6.2.1]. Then, for  $\nu = 2$ , we have  $\text{eu } \mathbb{H}^*(S_{-d}^3(K), a) = \text{eu } \mathbb{H}^0(S_{-d}^3(K), a) - \text{rank}_{\mathbb{Z}} \mathbb{H}^1(S_{-d}^3(K), a)$ , hence the second alternative form transforms into  $\text{rank}_{\mathbb{Z}} \mathbb{H}_{\text{can}}^1(S_{-d}^3(K)) \geq 0$ , which is certainly true.

Notice also that for  $\nu \geq 3$  similar argument does not work. From this point of view, it is even more surprising that in all the known cases, the Weak Conjecture holds, cf. Section 4.

### 3.4. Proof of the alternative form 2.1.4 of the Original Conjecture for $\nu = 2$ .

In fact, essentially by the same argument, in case of  $\nu = 2$  one can prove the Original Conjecture 2.1.4 as well. Using formula (1.3.10) of Theorem 1.3.8 and comparing it with (2.1.5), it is enough to prove that for  $\nu = 2$  the inequality  $F(k) \leq H(k+1)$  holds for *any*  $0 \leq k \leq 2\delta - 2$ . This inequality is purely combinatorial, completely independent of the parameter  $d$ , and has nothing to do with the realizability of cusp types on an existing rational projective curve (neither with the validity or failure of equalities (1.3.10)).

Indeed, similarly as in the proof of Theorem 3.1.4, set  $D > 2\delta - 2$ . Then, by (3.1.5) and (3.1.6), the inequality  $F(k) \leq H(k+1)$  turns into  $\text{eu } \mathbb{H}^*(S_{-D}^3(K), k) \leq \text{eu } \mathbb{H}^0(S_{-D}^3(K), k)$  which is again true, since due to the vanishing following from the reduction principle, the difference is the only summand  $\text{rank}_{\mathbb{Z}} \mathbb{H}_{\text{red}}^1(S_{-D}^3(K), k) \geq 0$ .

For a different, elementary proof, see [19].

*Proof of Theorem 1.3.11.* For part (1), see the previous argument in Subsection 3.4. For part (2), see Example 2.2.4.  $\square$

## 4. Verifying the Weak Conjecture 3.2.2 for known curves with $\nu \geq 3$

**4.1.** In this section we show that the Weak Conjecture 3.2.2 is true for all rational cuspidal curves with at least three cusps currently known (by the authors). For the list of such curves we refer to [17, §2.4.5], [30, Conjecture 4].

There are three infinite series of tricuspidal curves; one is a two-parameter family, the other two series have one parameter each (the curve degree).

There are two ‘sporadic’ curves not contained in any of the three series. Both of them is of degree 5; one is tricuspidal, the other has four cusps (this curve is conjectured to be the only rational cuspidal curve with more than three cusps).

The data for the curves with three cusps in the three infinite series are as follows (we present the multiplicity sequences and, for the convenience, also the Newton pairs of the cusp types):

**Proposition 4.1.1.** (Flenner, Zaidenberg and Fenske; see [10, §3.5], [11, §1.1], [7], cf. also [17, §2.4.5], [30])

*The following rational cuspidal curves exist:*

- 1) A curve  $C_{d,u}$  (with  $d \geq 4$  and  $1 \leq u \leq d-3$ ) is of degree  $d$  and has the following cusp types:  
 $[d-2]$ , alternatively  $(d-2, d-1)$   
 $[2_u]$ , alternatively  $(2, 2u+1)$   
 $[2_{d-2-u}]$ , alternatively  $(2, 2d-2u-3)$ .  
(Here it is enough to take  $u \leq \lfloor \frac{d-2}{2} \rfloor$  or  $\lceil \frac{d-2}{2} \rceil \leq u$ , as  $C_{d,u} = C_{d,d-2-u}$ .)
- 2) A curve  $D_l$  (with  $l \geq 1$ ) is of degree  $d = 2l+3$  and has the following cusp types:  
 $[2l, 2_l]$ , alternatively  $(l, l+1)(2, 1)$  (case  $l=1$  degenerates to one Newton pair)  
 $[3_l]$ , alternatively  $(3, 3l+1)$   
 $[2]$ , alternatively  $(2, 3)$ .
- 3) A curve  $E_l$  (with  $l \geq 1$ ) is of degree  $d = 3l+4$  and has the following cusp types:  
 $[3l, 3_l]$ , alternatively  $(l, l+1)(3, 1)$  (case  $l=1$  degenerates to one Newton pair)  
 $[4_l, 2_2]$ , alternatively  $(2, 2l+1)(2, 1)$   
 $[2]$ , alternatively  $(2, 3)$ .
- 4) A tricuspidal curve of degree  $d = 5$  with cusp types  $[2_2]$ ,  $[2_2]$ ,  $[2_2]$ .
- 5) A curve of degree  $d = 5$  with four cusps of type  $[2_3]$ ,  $[2]$ ,  $[2]$ ,  $[2]$ .

**Theorem 4.1.2.** Let  $H$  and  $F$  be the functions as defined in Formulae (1.3.9) and (2.1.1) corresponding to the singularities of the curves given in the previous Proposition 4.1.1. Then we have:



1)

$$\sum_{j=0}^{d-3} H(jd+1) - F(jd) = \begin{cases} l(l-1), & \text{if } d = 2l+1; \\ (u-l)(u-l+1), & \text{if } d = 2l, u \geq l-1. \end{cases}$$

In particular,  $\sum_{j=0}^{d-3} (H(jd+1) - F(jd)) \geq 0$  in all cases.

2)

$$\sum_{j=0}^{d-3} H(jd+1) - F(jd) = \begin{cases} 4p(3p-1)+2, & \text{if } l = 3p-1; \\ 4p(3p-1), & \text{if } l = 3p; \\ 12p(p+1)+2, & \text{if } l = 3p+1. \end{cases}$$

3)

$$\sum_{j=0}^{d-3} H(jd+1) - F(jd) = \begin{cases} 60p^2 - 2p, & \text{if } l = 4p; \\ 60p^2 + 46p + 10, & \text{if } l = 4p+1; \\ 60p^2 + 62p + 16, & \text{if } l = 4p+2; \\ 60p^2 + 100p + 42, & \text{if } l = 4p+3. \end{cases}$$

4)

$$\sum_{j=0}^{d-3} H(jd+1) - F(jd) = 6.$$

5)

$$\sum_{j=0}^{d-3} H(jd+1) - F(jd) = 8.$$

In particular, the Weak Conjecture 3.2.2 is satisfied in each case.

*Proof.* Since in each case we know explicitly the three singularities, we also know explicitly the product  $\Delta(t)$  of their Alexander polynomials. Therefore, it is convenient to work with Formula (1.2.6), as (1.2.7) reads as:

$$R(1) = \text{eu } \mathbb{H}_{\text{can}}^* - \text{eu } \mathbb{H}_{\text{can}}^0 = \sum_{j=0}^{d-3} F(jd) - H(jd+1).$$

Recall also Theorem 3.2.1 and Formulae (3.1.14), (3.1.15).

We do not give the computations here, just present the form of the polynomial  $\Delta(t) = \Delta_1(t)\Delta_2(t)\Delta_3(t)$  in terms of the parameters in each case.

1)

$$\Delta^{(C)}(t) = \frac{(t-1)(t^{(d-2)(d-1)}-1)}{(t^{d-2}-1)(t^{d-1}-1)} \frac{(t-1)(t^{2(2u+1)}-1)}{(t^2-1)(t^{2u+1}-1)} \cdot \frac{(t-1)(t^{2(2d-2u-3)}-1)}{(t^2-1)(t^{2d-2u-3}-1)}.$$

2)

$$\Delta^{(D)}(t) = \frac{(t-1)(t^{2l(l+1)}-1)(t^{2+4l(l+1)}-1)}{(t^{2l}-1)(t^{2(l+1)}-1)(t^{1+2l(l+1)}-1)} \cdot \frac{(t-1)(t^{3(3l+1)}-1)}{(t^3-1)(t^{3l+1}-1)} \frac{(t-1)(t^6-1)}{(t^2-1)(t^3-1)}.$$

3)

$$\Delta^{(E)}(t) = \frac{(t-1)(t^{3l(l+1)}-1)(t^{3+9l(l+1)}-1)}{(t^{3l}-1)(t^{3(l+1)}-1)(t^{1+3l(l+1)}-1)} \cdot \frac{(t-1)(t^{4(2l+1)}-1)(t^{2+8(2l+1)}-1)}{(t^4-1)(t^{2(2l+1)}-1)(t^{1+4(2l+1)}-1)} \frac{(t-1)(t^6-1)}{(t^2-1)(t^3-1)}.$$

Then, in each case, use formula (1.2.6) with the corresponding  $d$  to obtain the result.

Finally, check the conjecture for the two exceptional curves. The tricuspidal one in (4) of degree  $d = 5$  has cusp types  $[2_2], [2_2], [2_2]$ . The numerical values of functions  $F$  and  $H$  are as follows:

|                 |   |    |    |    |    |    |    |    |    |   |    |
|-----------------|---|----|----|----|----|----|----|----|----|---|----|
| $k$             | 0 | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9 | 10 |
| $H(k+1)$        | 1 | 1  | 2  | 2  | 3  | 3  | 4  | 4  | 5  | 5 | 6  |
| $F(k)$          | 1 | -1 | 3  | -3 | 6  | -3 | 7  | -1 | 6  | 3 | 6  |
| $H(k+1) - F(k)$ | 0 | 2  | -1 | 5  | -3 | 6  | -3 | 5  | -1 | 2 | 0  |

Hence  $\text{eu } \mathbb{H}_{\text{can}}^0 - \text{eu } \mathbb{H}_{\text{can}}^* = 0 + 6 + 0 > 0$ .

The single known rational cuspidal curve with four cusps in (5) has degree  $d = 5$ . The cusp types are  $[2_3], [2], [2], [2]$ . The detailed data:

|                 |   |    |    |    |    |    |    |    |    |   |    |
|-----------------|---|----|----|----|----|----|----|----|----|---|----|
| $k$             | 0 | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9 | 10 |
| $H(k+1)$        | 1 | 1  | 2  | 2  | 3  | 3  | 4  | 4  | 5  | 5 | 6  |
| $F(k)$          | 1 | -2 | 5  | -5 | 8  | -5 | 9  | -3 | 8  | 2 | 6  |
| $H(k+1) - F(k)$ | 0 | 3  | -3 | 7  | -5 | 8  | -5 | 7  | -3 | 3 | 0  |

Hence  $\text{eu } \mathbb{H}_{\text{can}}^0 - \text{eu } \mathbb{H}_{\text{can}}^* = 0 + 8 + 0 > 0$ .  $\square$

**Remark 4.1.3.** We present a table containing the detailed data  $H(jd+1) - F(jd)$ ,  $j = 0, \dots, d-3$  for the first few members of the first series:

| Curve     | Degree  | Cusp types          | $H(jd+1) - F(jd)$ |    |   |   |    |   |   | $\text{eu } \mathbb{H}^0 - \text{eu } \mathbb{H}^*$ |
|-----------|---------|---------------------|-------------------|----|---|---|----|---|---|---|
| $C_{4,1}$ | $d = 4$ | $[2], [2], [2]$     | 0                 | 0  |   |   |    |   |   | 0   |
| $C_{5,1}$ | $d = 5$ | $[3], [2_2], [2]$   | 0                 | 2  | 0 |   |    |   |   | 2   |
| $C_{6,1}$ | $d = 6$ | $[4], [2_3], [2]$   | 0                 | 0  | 0 | 0 |    |   |   | 0   |
| $C_{6,2}$ | $d = 6$ | $[4], [2_2], [2_2]$ | 0                 | 0  | 0 | 0 |    |   |   | 0   |
| $C_{7,1}$ | $d = 7$ | $[5], [2_4], [2]$   | 0                 | 3  | 0 | 3 | 0  |   |   | 6   |
| $C_{7,2}$ | $d = 7$ | $[5], [2_3], [2_2]$ | 0                 | 3  | 0 | 3 | 0  |   |   | 6   |
| $C_{8,1}$ | $d = 8$ | $[6], [2_5], [2]$   | 0                 | 0  | 1 | 1 | 0  | 0 |   | 2   |
| $C_{8,2}$ | $d = 8$ | $[6], [2_4], [2_2]$ | 0                 | -1 | 1 | 1 | -1 | 0 |   | 0   |
| $C_{8,3}$ | $d = 8$ | $[6], [2_3], [2_3]$ | 0                 | -1 | 1 | 1 | -1 | 0 |   | 0   |
| $C_{9,1}$ | $d = 9$ | $[7], [2_6], [2]$   | 0                 | 3  | 1 | 4 | 1  | 3 | 0 | 12  |
| $C_{9,2}$ | $d = 9$ | $[7], [2_5], [2_2]$ | 0                 | 4  | 0 | 4 | 0  | 4 | 0 | 12  |
| $C_{9,3}$ | $d = 9$ | $[7], [2_4], [2_3]$ | 0                 | 4  | 0 | 4 | 0  | 4 | 0 | 12  |

We see that the smallest degree where the Original Conjecture 1.2.8 fails is degree 8. The general pattern for larger  $d$ 's seems to be that the conjecture fails only at even degrees and when  $d-3 > u > 1$  (so it still seems to be true when the degree is odd or the degree is even and  $u = 1$  or  $u = d-3$ ).

Computations suggest that the other two series satisfy even the Original Conjecture 1.2.8.

## 5. A combinatorial surgery formula for $\mathbb{H}^0(S_{-d}^3(K))$

Let  $[n_1, n_2, \dots, n_r]$  be a multiplicity sequence of a singularity. We omit the 1's at the end of the multiplicity sequences, i.e. we define the multiplicity sequence as a sequence of multiplicities occurring in consecutive blowups resulting in *smooth* exceptional divisors and strict transform, but not necessarily in normal crossings of exceptional divisors and strict transform; in particular,  $n_r > 1$ .

Denote the counting function of the semigroup of a singularity with multiplicity sequence  $[n_1, n_2, \dots, n_r]$  by  $H_{[n_1, n_2, \dots, n_r]}$ .

First we prove the formula

$$H_{[n_1, n_2, \dots, n_r]} = H_{[n_1]} \diamond H_{[n_2]} \diamond \dots \diamond H_{[n_r]}$$

from Theorem 1.3.12.

*Proof of Theorem 1.3.12.* Due to the obvious associativity of the infimal convolution, for the statement of the theorem it is enough to show that

$$H_{[n_1, n_2, \dots, n_r]} = H_{[n_1]} \diamond H_{[n_2, \dots, n_r]}.$$

This will be proved in the next subsection (as Proposition 5.2.4).  $\square$

### 5.1. Dependence of $\mathbb{H}^0(S^3_{-d}(K))$ and $\text{eu } \mathbb{H}^0(S^3_{-d}(K))$ on the multiplicity sequences

In this section we present an effective way to compute  $\text{eu } \mathbb{H}^0(S^3_{-d}(K), a)$ , where the setting is as in Section 3 and in [24], i.e.  $K = \#_{i=1}^\nu K_i$ ,  $d$  is an arbitrary positive integer, and  $a$  stands for a  $\text{Spin}^c$ -structure, hence  $a \in \{0, \dots, d-1\}$ . In this discussion we prefer to fix the integers  $d$ ,  $a$  and  $\delta$ . Hence, by (3.1.5),  $\text{eu } \mathbb{H}^0(S^3_{-d}(K))$  (and, by Remark 3.1.12 the  $\mathbb{Z}[U]$ -module  $\mathbb{H}^0(S^3_{-d}(K))$  as well) is completely determined by the infimal convolution  $H = H_1 \diamond \dots \diamond H_\nu$ . In this section we focus on the dependence of  $H$  on the multiplicity sequences of plane curve singularities corresponding to knots  $K_i$ .

**5.1.1.** Let  $[n_1^{(1)}, \dots, n_{r_1}^{(1)}], \dots, [n_1^{(\nu)}, \dots, n_{r_\nu}^{(\nu)}]$  be the multiplicity sequences of the local singularities.

Then the sum of delta invariants of the singularities is

$$(5.1.2) \quad \delta = \sum_{i=1}^\nu \sum_{j=1}^{r_i} \frac{n_j^{(i)}(n_j^{(i)} - 1)}{2}.$$

We will show that the infimal convolution  $H = H_1 \diamond \dots \diamond H_\nu$  depends only on the *multiset* of multiplicities  $\{\{n_1^{(1)}, \dots, n_{r_1}^{(1)}\}, \dots, \{n_1^{(\nu)}, \dots, n_{r_\nu}^{(\nu)}\}\}$ . By a multiset we mean a set, where the same element might be repeated and we keep track the number of appearances; hence a multiset with integer entries basically is an element of the group ring  $\mathbb{Z}[\mathbb{Z}]$ .

**Theorem 5.1.3.** *Assume we are given two collections of plane curve singularity types with their multiplicity sequences:*

$$[n_1^{(1)}, \dots, n_{r_1}^{(1)}], \dots, [n_1^{(\nu)}, \dots, n_{r_\nu}^{(\nu)}]$$

and

$$[\bar{n}_1^{(1)}, \dots, \bar{n}_{\bar{r}_1}^{(1)}], \dots, [\bar{n}_1^{(\bar{\nu})}, \dots, \bar{n}_{\bar{r}_{\bar{\nu}}}^{(\bar{\nu})}].$$

Denote the counting functions of their semigroups by  $H_1, \dots, H_\nu$  and  $\bar{H}_1, \dots, \bar{H}_{\bar{\nu}}$ , respectively.

If

$$\{\{n_1^{(1)}, \dots, n_{r_1}^{(1)}, \dots, n_1^{(\nu)}, \dots, n_{r_\nu}^{(\nu)}\}\} = \{\{\bar{n}_1^{(1)}, \dots, \bar{n}_{\bar{r}_1}^{(1)}, \dots, \bar{n}_1^{(\bar{\nu})}, \dots, \bar{n}_{\bar{r}_{\bar{\nu}}}^{(\bar{\nu})}\}\}$$

as multisets, then

$$H_1 \diamond \dots \diamond H_\nu = \bar{H}_1 \diamond \dots \diamond \bar{H}_{\bar{\nu}}.$$

*Proof.* This is an immediate corollary of Theorem 1.3.12.  $\square$

**Corollary 5.1.4.** *Assume that  $K = K_1 \# \dots \# K_\nu$  and  $\bar{K} = \bar{K}_1 \# \dots \# \bar{K}_{\bar{\nu}}$  are connected sums of algebraic knots with summands as above. If the collections of numbers coming from multiplicity sequences corresponding to the algebraic knots are equal as multisets, then*

$$\mathbb{H}^0(S_{-d}^3(K), a) \cong \mathbb{H}^0(S_{-d}^3(\bar{K}), a)$$

for any integer  $d > 0$  and any  $\text{Spin}^c$ -structure  $a \in \{0, \dots, d-1\}$ .

The same is true for  $\text{eu } \mathbb{H}^0$  as well.

*Proof.* Use (5.1.2) and Remark 3.1.12 (resp. formula (3.1.5)).  $\square$

**Remark 5.1.5.** Note that a similar statement is not true for  $\mathbb{H}^q$  ( $q \geq 1$ ), not even for the numerical value  $\text{eu } \mathbb{H}^*$ ; see e.g. the links of superisolated singularities corresponding to the curves (4), (5) of Proposition 4.1.1. The two curves have the same degree, the same multiset of multiplicities, but different  $F$ -functions and different lattice cohomologies with the canonical  $\text{Spin}^c$ -structure as well.

**Remark 5.1.6.** It is quite surprising that from the point of view of the zeroth lattice cohomology only the collection of multiplicities ‘put together’ is important. This fact makes a lot easier to compute  $\mathbb{H}^0(S_{-d}^3(K), a)$  in many

cases. We can view this result in the following way as well: the zeroth lattice cohomology shows a stability with respect to the ‘combinatorial surgery’ of moving multiplicity numbers from one multiplicity sequence to another. We illustrate this by a simple example from [9, table in Section 2.3].

There exist cuspidal curves of degree 5 with the following cusp data (we present the multiplicity sequences):

- $[3], [2_3]$  ( $\nu = 2$ )
- $[3, 2], [2_2]$  ( $\nu = 2$ )
- $[3], [2_2], [2]$  ( $\nu = 3$ ).

Of course the corresponding surgery manifolds have  $\mathbb{H}^0(S_{-d}^3(K), 0)$  as prescribed by Theorem 3.2.1. From Corollary 5.1.4 we see immediately without any computation that these manifolds must have identical zeroth lattice cohomology and not only for the canonical  $\text{Spin}^c$ -structure  $a = 0$ , but for the other values  $a = 1, \dots, 4$  as well.

Notice that curves of degree 5 with the following cusp data do not exist:

- $[3, 2], [2], [2]$  ( $\nu = 3$ )
- $[3], [2], [2], [2]$  ( $\nu = 4$ )

However, the corresponding surgery manifolds also have  $\mathbb{H}^0(S_{-d}^3(K), a)$  as above, and to see this we do not need any further computations, since it is obvious from the multiset of multiplicities.

Notice that these manifolds are all different: one can construct their plumbing graphs from the embedded resolution graphs of the plane curve singularities and the Dehn surgery coefficient  $d$  (see [24, §2.3]), then observe that the plumbing graphs are all different and they are in reduced form (see [26, §4]). Alternatively, one can also compute  $\text{eu } \mathbb{H}^*$  by (3.1.6) from the Alexander polynomials of the given singularities to distinguish some of these manifolds.

In general, we have the following statement.

**Corollary 5.1.7.** *Assume that we have local singularity types which are candidates to be the singularities of a rational cuspidal plane curve of degree  $d$  in the sense of Definition 2.2.1, and they satisfy the necessary condition given by Theorem 1.3.8. Then any other collection of local singularity types, such that the multiset of the occurring multiplicities is the same as in the case of the original collection of singularities, as a new candidate satisfies the necessary condition given by Theorem 1.3.8 as well.*

**Remark 5.1.8.** (a) This result enlarges the applicability of the criterion provided by Borodzik–Livingston Theorem 1.3.8 drastically: if we reorganize the multiplicity numbers of a rational cuspidal curve candidate (by keeping the multiset), then the new combinatorial candidate satisfies the output of the Borodzik–Livingston Theorem 1.3.8 if and only if the original candidate satisfied it (regardless of the algebraic realizability).

This shows that although in the Borodzik–Livingston Theorem 1.3.8 the algebraic realizability is important, in reality it matters ‘less’, and presumably it can be replaced by a much weaker assumption. E.g., one could possibly require only the smooth realizability of the curve (near the singular points a smooth model of the singular local embeddings, otherwise a smooth embedding). In fact, analyzing the proof of [3], only this data is used: the fact that the smooth boundary  $\partial T$  of a tubular neighbourhood  $T$  of  $\{f_d = 0\}$  in the projective plane  $\mathbb{CP}^2$  bounds a rational homology ball, namely  $\mathbb{CP}^2 \setminus T$ , the determination of the  $\text{Spin}^c$ -structures of these manifolds, and properties of the  $d$ -invariant of  $\partial T$ .

It would be interesting to prove that two candidates with equivalent data in the sense of Theorem 5.1.3 can/cannot be simultaneously smoothly embedded in  $\mathbb{CP}^2$ .

(b) As there exist rational cuspidal curves with arbitrarily long multiplicity sequences (even in the unicuspidal case, see e.g. Orevkov’s curves in [27]), the above corollary also shows that Theorem 1.3.8 cannot provide any restriction on the number of cusps of rational cuspidal curves. (It is conjectured that the number of cusps is always less than five, i.e.  $\nu \leq 4$ , see e.g. [30]. A result of Tono shows that  $\nu \leq 8$ , see [32], cf. also [3, Example 6.16].)

## 5.2. The behaviour of the counting function under the blowup.

The goal of this subsection is to prove Proposition 5.2.4, thus completing the proof of Theorem 1.3.12 and consequently, Theorem 5.1.3. The notations in this subsection are completely independent of the other parts of this article.

Let  $\Gamma_1$  and  $\Gamma_2$  be semigroups of plane curve singularities. We will assume that  $\Gamma_1$  is a single blowup of  $\Gamma_2$ . Let  $H_1(i)$  and  $H_2(i)$  be the corresponding counting functions, i.e.  $H_\ell(i) = \#\{s \in \Gamma_\ell : s < i\}$ . Our goal is to compare  $H_1$  and  $H_2$ .

Denote by  $m$  the *multiplicity* of the second singularity, i.e.  $m = \min\{s \in \Gamma_2 : 0 < s\}$ . The *Apéry set* of a numerical semigroup with respect to one of its elements is a standard invariant commonly used in semigroup theory. It consists of the smallest elements of the semigroup from each (nonempty) residue class modulo the given element. We consider the Apéry set of  $\Gamma_2$

with respect to  $m$ , that is,  $\text{Ap}(m, \Gamma_2) = \{b_0, b_1, \dots, b_{m-1}\}$ , where  $0 = b_0 < b_1 < \dots < b_{m-1}$ . It is a complete residual system modulo  $m$ , and by the definition, for each  $i$  ( $0 \leq i \leq m-1$ ) we have  $b_i \in \Gamma_2$  but  $b_i - m \notin \Gamma_2$ .

The definition guarantees that  $\Gamma_2 = \text{Ap}(m, \Gamma_2) + m \cdot \mathbb{Z}_{\geq 0}$ . In fact, for every element  $s \in \Gamma_2$  there exist *uniquely*  $j \in \mathbb{Z}$  and  $u \in \mathbb{Z}$  such that  $s = b_j + mu$ ,  $0 \leq j \leq m-1$ ,  $0 \leq u$ .

**Lemma 5.2.1.** *If  $\Gamma_1$  is the blowup of  $\Gamma_2$ , then  $m \in \Gamma_1$  as well.*

*Proof.* The strict transform of the singular curve after the blowup and the reduced exceptional divisor of the blowup have intersection multiplicity  $m$ .  $\square$

Therefore, we can consider the Apéry set with respect to  $m$  of  $\Gamma_1$  as well:  $\text{Ap}(m, \Gamma_1) = \{a_0, a_1, \dots, a_{m-1}\}$  is a complete residual system mod  $m$  such that  $a_i \in \Gamma_1$  but  $a_i - m \notin \Gamma_1$  for all  $0 \leq i \leq m-1$ , and  $0 = a_0 < a_1 < \dots < a_{m-1}$ . Again,  $\Gamma_1 = \text{Ap}(m, \Gamma_1) + m \cdot \mathbb{Z}_{\geq 0}$ , i.e. for any  $s \in \Gamma_1$  there exist unique  $j, u \in \mathbb{Z}$  such that  $s = a_j + mu$ ,  $0 \leq j \leq m-1$ ,  $0 \leq u$ .

**Proposition 5.2.2.** *([1, Lemme 2], [2, Proposition 2.3]) The blowup of a semigroup is described by the two Apéry sets with respect to the original multiplicity  $m$  in the following way:*

$$a_j + jm = b_j \quad (j = 0, \dots, m-1).$$

**Remark 5.2.3.** The previous proposition implies that ‘the order is preserved’, i.e. if  $\Gamma_2$  is a semigroup of a plane curve singularity which has multiplicity  $m$  and the *ordered* Apéry set with respect to this multiplicity is  $\text{Ap}(m, \Gamma_2) = \{b_0, b_1, \dots, b_{m-1}\}$  with  $0 = b_0 < b_1 < \dots < b_{m-1}$ , then the series of inequalities  $0 = b_0 - 0 \cdot m < b_1 - m < b_2 - 2m < \dots < b_{m-1} - (m-1)m$  must be satisfied. This is a nontrivial necessary condition for an algebraic numerical semigroup to be a semigroup of a plane curve singularity.

Let  $\Gamma_{[m]}$  be the semigroup of the plane curve singularity with multiplicity sequence  $[m]$ , it is generated as a semigroup by  $m$  and  $m+1$ . Denote its counting function by  $H_{[m]}$ .

The counting functions  $H_1$ ,  $H_2$  and  $H_{[m]}$  are related as follows.

**Proposition 5.2.4.** *For all  $l \geq 0$  one has*

$$H_2(l) = \min_{0 \leq j \leq l} \{H_1(l-j) + H_{[m]}(j)\}.$$



*Proof.* We need to prove that for all  $l \geq 0$ , and for all  $j$  with  $0 \leq j \leq l$  one has  $H_2(l) - H_1(l - j) \leq H_{[m]}(j)$ ; furthermore, that for all  $l \geq 0$  equality holds for some  $j$ .

It will be useful to view the semigroups as unions of ‘layers’ according to the Apéry sets. Namely, for  $i = 0, 1, \dots, m - 1$  set  $\Gamma_1^{(i)} := a_i + m\mathbb{Z}_{\geq 0}$ . Then  $\Gamma_1 = \sqcup_{i=0}^{m-1} \Gamma_1^{(i)}$  as a *disjoint* union. Similarly, set  $\Gamma_2^{(i)} := b_i + m\mathbb{Z}_{\geq 0}$ , hence  $\Gamma_2 = \sqcup_{i=0}^{m-1} \Gamma_2^{(i)}$  as a *disjoint* union. Then

$$H_1(l - j) = \sum_i \#\{s \in \Gamma_1^{(i)} : s < l - j\}, \quad H_2(l) = \sum_i \#\{s \in \Gamma_2^{(i)} : s < l\}.$$

By Proposition 5.2.2 we get that the  $i$ -th layer  $\Gamma_2^{(i)}$  of the semigroup  $\Gamma_2$  just has to be shifted to the left by  $im$  to get the  $i$ -th layer  $\Gamma_1^{(i)}$  of the semigroup  $\Gamma_1$ . Hence,

$$H_2(l) = \sum_i \#\{s \in \Gamma_1^{(i)} : s < l - im\}.$$

Now for a fixed  $l$  the difference which has to be (sharply) bounded from above can be written as a difference of set-cardinalities, the sets being differences of subsets of the semigroup (layers of)  $\Gamma_1$ :

$$H_2(l) - H_1(l - j) = \#\{A_{j,l}\} - \#\{B_{j,l}\},$$

where  $A_{j,l} = \sqcup_{i=0}^{m-1} A_{j,l}^{(i)}$  and  $B_{j,l} = \sqcup_{i=0}^{m-1} B_{j,l}^{(i)}$  as disjoint unions, with

$$\begin{aligned} A_{j,l}^{(i)} &= \{s \in \Gamma_1^{(i)} : l - j \leq s < l - im\}, \\ B_{j,l}^{(i)} &= \{s \in \Gamma_1^{(i)} : l - im \leq s < l - j\}. \end{aligned}$$

(Note that for all  $i$ , at least one of  $A_{j,l}^{(i)}$  and  $B_{j,l}^{(i)}$  is empty.)

Hence, we need to prove that  $\#\{A_{j,l}\} - \#\{B_{j,l}\} \leq H_{[m]}(j)$ , and for each  $l$  equality holds for some  $j = 0, 1, \dots, l$ .

The inequality follows from  $\#\{B_{j,l}\} \geq 0$  and  $\#\{A_{j,l}\} \leq H_{[m]}(j)$ , where the second inequality is not straightforward.

First we check it for the multiples of  $m$ , i.e. for  $j$ ’s of form  $j = \omega m$ :

$$\begin{aligned} (5.2.5) \quad \#\{A_{\omega m, l}\} &= \sum_{i=0}^{m-1} \#\{A_{\omega m, l}^{(i)}\} \leq \sum_{i=0}^{m-1} \max\{\omega - i, 0\} \\ &= \sum_{i=0}^{\omega} \min\{i, m\} = H_{[m]}(\omega m). \end{aligned}$$

This is true because  $\#\{A_{\omega m, l}^{(i)}\} = \#\{r \in \mathbb{Z}_{\geq 0} : l - \omega m \leq a_i + rm < l - im\} \leq \max\{\omega - i, 0\}$ . (This upper bound is valid even if  $\omega m > l$ .)

For  $j$ 's not of the form  $\omega m$ , write  $j = \omega m + \gamma$  with  $0 < \gamma < m$ .

First assume that  $\gamma \leq \omega + 1$ . In this case,  $H_{[m]}((\omega + 1)m) = H_{[m]}(\omega m) + \omega + 1$  and  $H_{[m]}(\omega m + \gamma) = \min\{\gamma, \omega + 1\} + H_{[m]}(\omega m)$ . Observe also that  $0 \leq \#\{A_{j+1, l}\} - \#\{A_{j, l}\} \leq 1$  (for  $\#\{A_{j+1, l}^{(i)}\} - \#\{A_{j, l}^{(i)}\} \in \{0, 1\}$ , and except for at most one  $i$ , the difference is 0, as elements of  $A_{j, l}^{(i)}$  for different  $i$ 's have different residues modulo  $m$ ). Therefore, on one hand,

$$\#\{A_{\omega m + \gamma}\} \leq \#\{A_{\omega m}\} + \gamma \leq H_{[m]}(\omega m) + \gamma,$$

on the other hand,

$$\#\{A_{\omega m + \gamma}\} \leq \#\{A_{(\omega + 1)m}\} \leq H_{[m]}((\omega + 1)m) = H_{[m]}(\omega m) + \omega + 1.$$

In this way,

$$\#\{A_{\omega m + \gamma}\} \leq H_{[m]}(\omega m) + \min\{\gamma, \omega + 1\} = H_{[m]}(\omega m + \gamma)$$

as desired.

Now assume that  $\omega + 1 < \gamma < m$ , then we have  $H_{[m]}(\omega m + \gamma) = H_{[m]}((\omega + 1)m) \geq \#\{A_{(\omega + 1)m, l}\} \geq \#\{A_{\omega m + \gamma, l}\}$  (here we have used (5.2.5) for  $(\omega + 1)m$  instead of  $\omega m$ ).

Next we show that for any  $l$  there exists a  $j$  for which equality holds. From the above, it is clear which conditions do we want to be satisfied. We will choose a  $j$  such that  $0 \leq j \leq l$ ,  $j \leq (m - 1)m$ ,  $B_{j, l} = \emptyset$  and  $\#\{r \in \mathbb{Z}_{\geq 0} : l - j \leq a_i + rm < l - im\} = \#\{A_{j, l}^{(i)}\} = \max\{\lceil \frac{j}{m} \rceil - i, 0\}$  for all  $i = 0, 1, \dots, m - 1$ .

For any  $l$ , let  $i_0$  be the smallest index  $i$  among  $0, 1, \dots, m - 1$  for which  $l - im \leq a_i$  is already valid, if such index exists. If not, take  $i_0 = m - 1$ .

It is not hard to see that  $j = \min\{i_0 m, l\}$  is a good choice, i.e. for  $j = \min\{i_0 m, l\}$  we will have equality in the upper bound. For if  $j = i_0 m$ , then by the choice of  $i_0$  we have  $B_{i_0 m, l} = \emptyset$  and  $A_{i_0 m, l}^{(i)} = \max\{i_0 - i, 0\}$  for all possible  $i$ , and these two conditions (via (5.2.5)) are enough to guarantee the equality  $\#\{A_{i_0 m, l}\} - \#\{B_{i_0 m, l}\} = H_{[m]}(i_0 m)$ . If  $j = l$  happens to be the case (i.e. if  $l < i_0 m$ ), then by a similar argument as above, we have  $H_{[m]}(i_0 m) = A_{i_0 m, l}$ , hence  $A_{l, l} \leq H_{[m]}(l) \leq H_{[m]}(i_0 m) = A_{i_0 m, l} = A_{l, l}$ , which implies  $\#\{A_{l, l}\} - \#\{B_{l, l}\} = H_{[m]}(l)$  again, as  $B_{l, l} = \emptyset$ .  $\square$

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## References

- [1] R. Apéry, *Sur les branches superlinéaires des courbes algébriques*, C. R. Acad. Sci. Paris, **222** (1946), 1198–1200.
- [2] V. Barucci, M. D’Anna, and R. Fröberg, *On plane algebroid curves*, in Commutative ring theory and applications (Fez, 2001), Vol. 231 of *Lecture Notes in Pure and Appl. Math.*, 37–50, Dekker, New York (2003).
- [3] M. Borodzik and C. Livingston, *Heegaard Floer homology and rational cuspidal curves*, Forum Math. Sigma, **2** (2014), e28, 23.
- [4] G. Braun and A. Némethi, *Surgery formula for Seiberg-Witten invariants of negative definite plumbed 3-manifolds*, J. Reine Angew. Math., **638** (2010), 189–208.
- [5] E. Brieskorn and H. Knörrer, *Plane algebraic curves*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel (1986), ISBN 978-3-0348-0492-9. Translated from the German original by John Stillwell, [2012] reprint of the 1986 edition.
- [6] D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Vol. 110 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ (1985), ISBN 0-691-08380-0; 0-691-08381-9.
- [7] T. Fenske, *Rational cuspidal plane curves of type  $(d, d-4)$  with  $\chi(\Theta_V\langle D \rangle) \leq 0$* , Manuscripta Math., **98** (1999), no. 4, 511–527.
- [8] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, and A. Némethi, *On rational cuspidal plane curves, open surfaces and local singularities*, in Singularity theory, 411–442, World Sci. Publ., Hackensack, NJ (2007).
- [9] J. Fernández de Bobadilla, I. Luengo-Velasco, A. Melle-Hernández, and A. Némethi, *On rational cuspidal projective plane curves*, Proc. London Math. Soc. (3), **92** (2006), no. 1, 99–138.

- [10] H. Flenner and M. Zaidenberg, *On a class of rational cuspidal plane curves*, Manuscripta Math., **89** (1996), no. 4, 439–459.
- [11] H. Flenner and M. Zaidenberg, *Rational cuspidal plane curves of type  $(d, d - 3)$* , Math. Nachr., **210** (2000), 93–110.
- [12] S. M. Guseĭn-Zade, F. Del’gado, and A. Kampil’o, *On the monodromy of a plane curve singularity and the Poincaré series of its ring of functions*, Funktsional. Anal. i Prilozhen., **33** (1999), no. 1, 66–68.
- [13] T. László and A. Némethi, *Reduction theorem for lattice cohomology*, Int. Math. Res. Not. IMRN (2015), no. 11, 2938–2985.
- [14] I. Luengo, *The  $\mu$ -constant stratum is not smooth*, Invent. Math., **90** (1987), no. 1, 139–152.
- [15] I. Luengo-Velasco, A. Melle-Hernández, and A. Némethi, *Links and analytic invariants of superisolated singularities*, J. Algebraic Geom., **14** (2005), no. 3, 543–565.
- [16] T. K. Moe, *Rational cuspidal curves*, Master’s thesis, University of Oslo (2008).
- [17] T. K. Moe, *Cuspidal curves on Hirzebruch surfaces*, Ph.D. thesis, University of Oslo (2013).
- [18] M. Namba, *Geometry of projective algebraic curves*, M. Dekker (1984).
- [19] P. Nayar and B. Pilat, *A note on the rational cuspidal curves*, Bull. Pol. Acad. Sci. Math., **62** (2014), no. 2, 117–124.
- [20] A. Némethi, *Lattice cohomology of normal surface singularities*, Publ. Res. Inst. Math. Sci., **44** (2008), no. 2, 507–543.
- [21] A. Némethi, *The Seiberg-Witten invariants of negative definite plumbed 3-manifolds*, J. Eur. Math. Soc. (JEMS), **13** (2011), no. 4, 959–974.
- [22] A. Némethi, *Two exact sequences for lattice cohomology*, in Noncommutative geometry and global analysis, Vol. 546 of *Contemp. Math.*, 249–269, Amer. Math. Soc., Providence, RI (2011).
- [23] A. Némethi and L. I. Nicolaescu, *Seiberg-Witten invariants and surface singularities*, Geom. Topol., **6** (2002), 269–328 (electronic).
- [24] A. Némethi and F. Román, *The lattice cohomology of  $S^3_{-d}(K)$* , in Zeta functions in algebra and geometry, Vol. 566 of *Contemp. Math.*, 261–292, Amer. Math. Soc., Providence, RI (2012).

- [25] A. Némethi and B. Sigurðsson, *The geometric genus of hypersurface singularities* (2013), to appear in Journal of the European Mathematical Society (JEMS), [arXiv:1310.1268](#).
- [26] W. D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc., **268** (1981), no. 2, 299–344.
- [27] S. Y. Orevkov, *On rational cuspidal curves. I. Sharp estimate for degree via multiplicities*, Math. Ann., **324** (2002), no. 4, 657–673.
- [28] P. Ozsváth, A. I. Stipsicz, and Z. Szabó, *A spectral sequence on lattice homology*, Quantum Topol., **5** (2014), no. 4, 487–521.
- [29] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), **159** (2004), no. 3, 1027–1158.
- [30] J. Piontkowski, *On the number of cusps of rational cuspidal plane curves*, Experiment. Math., **16** (2007), no. 2, 251–255.
- [31] J. Stevens, *Universal abelian covers of superisolated singularities*, Math. Nachr., **282** (2009), no. 8, 1195–1215.
- [32] K. Tono, *On the number of the cusps of cuspidal plane curves*, Math. Nachr., **278** (2005), no. 1-2, 216–221.

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