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G_2–instantons over twisted connected sums: an example

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Using earlier work of Sá Earp and the author [SEW15] we construct an irreducible unobstructed G_2–instanton on an SO(3)–bundle over a twisted connected sum G_2–manifold recently discovered by Crowley and Nordström [CN14].

1. Introduction

In order to put this note into context and help the reader appreciate its significance, we (very) briefly recall some ideas from the study of gauge theory on G_2–manifolds.

Definition 1.1. A connection $A \in \mathscr{A}(E)$ on a G–bundle E over a G_2–manifold Y is called a G_2*–instanton* if its curvature satisfies

$$F_A \wedge \psi = 0 \tag{1.2}$$

with $\psi := *\phi$ and $\phi \in \Omega^3(Y)$ denoting the G_2–structure on Y.

In their visionary article [DT98] Donaldson and Thomas speculated that "counting" G_2–instantons might lead to an interesting enumerative invariant. Although almost two decades have passed, it is still not understood what the precise definition of this invariant ought to be; however, see Donaldson and Segal [DS11], the author [Wal12, Wal13a], and Haydys and the author [HW15] for some recent progress. What is clear, nonetheless, is that irreducible unobstructed G_2–instantons should contribute with ± 1 (depending on orientations).

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Definition 1.3. A G_2–instanton $A \in \mathscr{A}(E)$ is called *irreducible* (*unobstructed*) if the elliptic complex

$$\Omega^0(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(Y, \mathfrak{g}_E) \xrightarrow{\psi \wedge d_A} \Omega^6(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^7(Y, \mathfrak{g}_E)$$

has vanishing cohomology in degree zero (one).

In [SEW15] Sá Earp and the author developed a method for constructing irreducible unobstructed G_2–instantons over twisted connected sums. So far, however, we were unable to find a single instance of the input required for this construction. This brief note is meant to ameliorate this disgraceful situation by showing that our method can be used to produce at least one example.

Let us briefly recall the twisted connected sum construction, a rich source of G_2–manifolds, which was suggested by Donaldson, pioneered by Kovalev [Kov03] and later extended and improved by Kovalev and Lee [KL11], and Corti, Haskins, Nordström and Pacini [CHNP15].

Definition 1.4. A *building block* is a non-singular algebraic 3–fold Z together with a projective morphism $f\colon Z \to \mathbf{P}^1$ such that:

- the anticanonical class $-K_Z \in H^2(Z)$ is primitive,
- $\Sigma := f^{-1}(\infty)$ is a smooth $K3$ surface and $\Sigma \sim -K_Z$.

A *framing* of a building block (Z, Σ) consists of a hyperkähler structure $\boldsymbol{\omega} = (\omega_I, \omega_J, \omega_K)$ on Σ such that $\omega_J + i\omega_K$ is of type $(2,0)$ as well as a Kähler class on Z whose restriction to Σ is $[\omega_I]$.[1]

For the purpose of this article we are mostly interested in the following class of building blocks introduced by Kovalev [Kov03].

Definition 1.5. A building block is said to be of *Fano type* if it is obtained by blowing-up a Fano 3–fold W along the base locus of general pencil $|\Sigma_0, \Sigma_\infty| \subset |-K_W|$. (See Section 2.1 for more details on this construction.)

Given a framed building block $(Z, \Sigma, \boldsymbol{\omega})$, using the work of Haskins, Hein and Nordström [HHN12], we can make $V := Z \setminus \Sigma$ into an asymptotically cylindrical (ACyl) Calabi–Yau 3–fold with asymptotic cross-section $S^1 \times \Sigma$; hence, $Y := S^1 \times V$ is an ACyl G_2–manifold with asymptotic cross-section $T^2 \times \Sigma$.

[1] The existence of such a class is not guaranteed a priori.

Definition 1.6. A *matching* of pair of framed building blocks $(Z_\pm, \Sigma_\pm, \boldsymbol{\omega}_\pm)$ is a hyperkähler rotation $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$, i.e., a diffeomorphism such that

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}.$$

Given a matched pair of framed building blocks $(Z_\pm, \Sigma_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$, the twisted connected sum construction produces a simply-connected compact 7–manifold Y together with a family of torsion-free G_2–structures $\{\phi_T : T \gg 1\}$ by gluing truncations of $Y_\pm$ along their boundaries via interchanging the circle factors and $\mathfrak{r}$.

Sá Earp [SE15] proved that given a holomorphic vector bundle $\mathscr{E}$ on a building block with $\mathscr{E}|_\Sigma$ μ–stable,[2] the smooth vector bundle underlying $\mathscr{E}|_V$ can be equipped with a Hermitian Yang–Mills connection which is asymptotic at infinity to the anti-self-dual connection A_∞ inducing the holomorphic structure on $\mathscr{E}|_\Sigma$ [Don85]. Building on this, Sá Earp and the author [SEW15] developed a method for constructing G_2–instantons over twisted connected sums provided a pair $\mathscr{E}_\pm$ of such bundles and a lift $\bar{\mathfrak{r}}\colon \mathscr{E}_+|_{\Sigma_+} \to \mathscr{E}_-|_{\Sigma_-}$ of the hyperkähler rotation $\mathfrak{r}$, which pulls back $A_{\infty,-}$ to $A_{\infty,+}$ (and assuming certain transversality conditions). The following is a *very special* case of the main result of [SEW15].

Theorem 1.7. *Let $(Z_\pm, \Sigma_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ be a matched pair of framed building blocks. Denote by Y the compact 7–manifold and by $\{\phi_T : T \gg 1\}$ the family of torsion-free G_2–structures obtained from the twisted connected sum construction. Let $\mathscr{E}_\pm \to Z_\pm$ be a pair of rank r holomorphic vector bundles such that the following hold:*

- $c_1(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_1(\mathscr{E}_-|_{\Sigma_-})$ *and* $c_2(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_2(\mathscr{E}_-|_{\Sigma_-})$.
- $\mathscr{E}_\pm|_{\Sigma_\pm}$ *is μ–stable with respect to $\omega_{I,\pm}$ and spherical, i.e.,*

$$H^*(\Sigma_\pm, \mathscr{E}nd_0(\mathscr{E}_\pm|_{\Sigma_\pm})) = 0.$$

- $\mathscr{E}_\pm$ *is infinitesimally rigid:*

$$H^1(Z_\pm, \mathscr{E}nd_0(\mathscr{E}_\pm)) = 0. \tag{1.8}$$

[2]Recall that a holomorphic bundle $\mathscr{E}$ on a compact Kähler n–fold (X, ω) is μ–*(semi)stable* if for each torsion-free coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rk} \mathscr{F} < \operatorname{rk} \mathscr{E}$ we have $(\mu(\mathscr{F}) \leq \mu(\mathscr{E}))$ $\mu(\mathscr{F}) < \mu(\mathscr{E})$. Here $\mu(\mathscr{E}) := \langle c_1(\mathscr{E}) \cup [\omega]^{n-1}, [X]\rangle / \operatorname{rk} \mathscr{E}$ is the *slope* of $\mathscr{E}$ (and similarly for $\mathscr{F}$).

Then there exists a U(r)*–bundle* E *over* Y *with*

$$c_1(E) = \Upsilon(c_1(\mathscr{E}_+), c_1(\mathscr{E}_-)) \quad \text{and} \quad c_2(E) = \Upsilon(c_2(\mathscr{E}_+), c_2(\mathscr{E}_-)) \tag{1.9}$$

and a family of connections $\{A_T : T \gg 1\}$ *on the associated* **P**U(r)*–bundle with* A_T *being an irreducible unobstructed* G$_2$*–instanton over* (Y, ϕ_T)*.*

Remark 1.10. The map

$$\Upsilon\colon \left\{([\alpha_+], [\alpha_-]) \in H^{\mathrm{ev}}(Z_+) \times H^{\mathrm{ev}}(Z_-) : [\alpha_+]|_{\Sigma_+} = \mathfrak{r}^*([\alpha_-]|_{\Sigma_-})\right\} \to H^{\mathrm{ev}}(Y)$$

is the natural patching map denoted by Y in [CHNP15, Definition 4.15].

Let $\mathrm{res}_\pm\colon H^2(Z_\pm) \to H^2(\Sigma_\pm)$ denote the restriction maps associated with the inclusions $\Sigma_\pm \subset Z_\pm$ and set

$$N_\pm := \operatorname{im} \mathrm{res}_\pm.$$

If $\mathscr{E}_\pm|_{\Sigma_\pm}$ is spherical, then $c_1(\mathscr{E}_\pm|_{\Sigma_\pm})$ must be non-zero; hence, Theorem 1.7 cannot be applied in situations where $N_+ \cap \mathfrak{r}^* N_- = 0$. In particular, this rules out all the examples in [Kov03, KL11] as well as the mass-produced examples in [CHNP15]. This means that the list of currently known G$_2$–manifolds to which Theorem 1.7 could potentially be applied is relatively short. Moreover, it has proved rather difficult to find suitable $\mathscr{E}_\pm$.

Crowley and Nordström [CN14] systematically studied twisted connected sums of building blocks arising from Fano 3–folds with Picard number two; in particular, those that arise from matchings with $N_+ \cap \mathfrak{r}^* N_- \neq 0$. This note shows that for one such twisted connected sum the hypotheses of Theorem 1.7 can be satisfied.

Theorem 1.11. *There exists a twisted connected sum* Y *of a pair of Fano type building blocks* $(Z_\pm, \Sigma_\pm)$*, arising from #13 and #14 in Mori and Mukai's classification of Fano* 3*–folds with Picard number two [MM81, Table 2], admitting a pair of rank* 2 *holomorphic vector bundles* $\mathscr{E}_\pm$ *as required by Theorem 1.7. In particular, each of the resulting twisted connected sums* (Y, ϕ_T) *with* $T \gg 1$ *carries an irreducible unobstructed* G$_2$*–instanton on an* SO(3)*–bundle.*

Remark 1.12. In earlier work [Wal13b] the author constructed examples of irreducible unobstructed G$_2$–instantons over G$_2$–manifolds arising from Joyce's generalised Kummer construction [Joy96a, Joy96b]. To the author's best knowledge, Theorem 1.11 provides the first example of an irreducible unobstructed G$_2$–instanton over a twisted connected sum.

The method of proof relies mostly on certain arithmetic properties enjoyed by the twisted connected sum listed as [CN14, Table 4, Line 16] by Crowley and Nordström. A more abstract existence theorem is stated as Theorem 3.14. It is an interesting question to ask whether there are any further twisted connected sums to which this result can be applied.

Finally, it should be pointed out that there is a very recent preprint by Menet, Nordström and Sá Earp [MNSE15] in which they use the more general main result of [SEW15] to construct one G_2–instanton.

2. The twisted connected sum

In this section we provide further details on Fano type building blocks, explain how to construct matching pairs of framed building blocks and describe the twisted connected sum mentioned in Theorem 1.11.

2.1. Building blocks of Fano type

If W is a Fano 3–fold, then according to Shokurov [Šok79] a general divisor $\Sigma \in |-K_W|$ is a smooth K3 surface. Given a general pencil $|\Sigma_0, \Sigma_\infty| \subset |-K_W|$, blowing-up its base locus yields a smooth 3–fold Z together with a base-point free anti-canonical pencil spanned by the proper transforms of Σ_0 and Σ_∞. The resulting projective morphism $f\colon\ Z \to \mathbf{P}^1$ makes (Z, Σ_∞) into a building block with

$$N := \operatorname{im}\bigl(\operatorname{res}\colon\ H^2(Z) \to H^2(\Sigma)\bigr) \cong \operatorname{Pic}(W), \tag{2.1}$$

see [Kov03, Proposition 6.42] and [CHNP15, Proposition 3.15].

Moishezon [Moĭ67] showed that for a very general[3] $\Sigma \in |-K_W|$ we have $\operatorname{Pic}(\Sigma) = \operatorname{Pic}(W)$. Moreover, according to Kovalev [Kov09, Proposition 2.14] we can assume that $f\colon\ Z \to \mathbf{P}^1$ is a *rational double point (RDP)* $K3$ *fibration*, by which we mean that it has at only finitely many singular fibres and the singular fibres have only RDP singularities. (In fact, Kovalev asserts that generically the singular fibres have only ordinary double points.)

[3]Here *very general* means that the set of $\Sigma \in |-K_W|$ not satisfying the asserted condition is a countable union of complex analytic submanifolds of positive codimension in $\mathbf{P}H^0(-K_W)$.

2.2. Matching building blocks

Fix a lattice L which is isomorphic to $(H^2(\Sigma), \cup)$ for Σ a $K3$ surface. Using the Torelli theorem and Yau's solution to the Calabi conjecture, Corti, Haskins, Nordström and Pacini [CHNP15, Section 6] showed that a set of framings of a pair of building blocks $Z_\pm$ together with a matching is equivalent (up to the action of $\mathrm{O}(L)$) to lattice isomorphisms $h_\pm\colon L \to H^2(\Sigma_\pm)$ and an orthonormal triple (k_+, k_-, k_0) of positive classes in $L_{\mathbf{R}} := L \otimes_{\mathbf{Z}} \mathbf{R}$ with $h_\pm(k_\pm)$ the restriction of a Kähler class on $Z_\pm$ and $\langle k_\mp, \pm k_0\rangle$ the period point of $(\Sigma_\pm, h_\pm)$. (The corresponding framings have $[\omega_{I,\pm}] = h_\pm(k_\pm)$ and the matching is such that $\mathfrak{r}^* = h_+ \circ h_-^{-1}$.) The following definition is useful to further simplify the matching problem.

Definition 2.2. Let $\mathscr{Z}$ be a family of building blocks with constant N and a fixed primitive isometric embedding $N \subset L$. Let Amp be an open subcone of the positive cone in $N_{\mathbf{R}}$. $\mathscr{Z}$ is called (N, Amp)*–generic* if there exists a subset $U_{\mathscr{Z}} \subset D_N := \left\{\Pi \in \mathbf{P}(N^\perp_{\mathbf{C}}) : \Pi\bar{\Pi} > 0\right\}$ with complement a countable union of complex analytic submanifolds of positive codimension and with the property that for any $\Pi \in U_{\mathscr{Z}}$ and $k \in \mathrm{Amp}$ there exists a $(Z, \Sigma) \in \mathscr{Z}$ and a marking $h\colon L \to H^2(\Sigma)$ such that Π is the period point of (Σ, h) and $h(k)$ is the restriction to Σ of a Kähler class on Z.

This definition slightly deviates from [CHNP15, Definition 6.17]. There it is required that the complement of $U_{\mathscr{Z}}$ is a *locally finite* union of complex analytic submanifolds of positive codimension. The above slightly weaker condition still suffices for the proof of the next proposition to carry over verbatim.

Proposition 2.3 ([CHNP15, Proposition 6.18]). *Let $N_\pm \subset L$ be a pair of primitive sublattices of signature $(1, r_\pm - 1)$ and let $\mathscr{Z}_\pm$ be a pair of $(N_\pm, \mathrm{Amp}_\pm)$–generic families of building blocks. Suppose that $W := N_+ + N_-$ is an orthogonal pushout.*[4] *Set $T_\pm := N_\pm^\perp$ and $W_\pm := N_\pm \cap T_\mp$. If*

$$\mathrm{Amp}_\pm \cap W_\pm \neq \emptyset,$$

then there exist $(Z_\pm, \Sigma_\pm) \in \mathscr{Z}_\pm$, markings $h_\pm\colon L \to H^2(\Sigma_\pm)$ compatible with the given embeddings $N_\pm \subset L$ and an orthonormal triple (k_+, k_-, k_0) of positive classes in $L_{\mathbf{R}}$ with:

- $k_\pm \in \mathrm{Amp}_\pm \cap W_{\pm,\mathbf{R}}$ *and* $k_0 \in W^\perp$,

[4]This means that $W_{\mathbf{R}} = W_{+,\mathbf{R}} \oplus W_{-,\mathbf{R}} \oplus (N_{+,\mathbf{R}} \cap N_{-,\mathbf{R}})$.

- *$h_\pm(k_\pm)$ the restriction of a Kähler class on $Z_\pm$, and*
- *$\langle k_\mp, \pm k_0\rangle$ the period point of $(\Sigma_\pm, h_\pm)$.*

If $\mathcal{Z}$ is a family of building blocks arising from a full deformation type of Fano 3–folds, then we can always find an open subcone Amp of the positive cone such that $\mathcal{Z}$ is (N, Amp)–generic [CHNP13, Proposition 6.9]. (Also Definition 2.2 allows to slightly shrink $\mathcal{Z}$ from a full deformation type by imposing very general conditions in the sense of Footnote 3.) This reduces finding a matching of a pair such families of building blocks to the arithmetic problem of embedding $N_\pm$ into L compatible with Proposition 2.3.

2.3. An example due to Crowley and Nordström

We will now describe the twisted connected sum found by Crowley and Nordström [CN14, Table 4, Line 16] which we referred to in Theorem 1.11.

Consider the following pair of Fano 3–folds:

- Denote by $Q \subset \mathbf{P}^4$ a smooth quadric. Let $W_+ \to Q$ denote the blow-up of Q in a degree 6 genus 2 curve [MM81, Table 2, #13].
- Denote by V_5 a section of the Plücker-embedded Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$ by a subspace of codimension 3. Let $W_- \to V_5$ denote the blow-up of V_5 in a elliptic curve that is the intersection of two hyperplane sections [MM81, Table 2, #14].

$W_\pm$ both have Picard number 2 with $\mathrm{Pic}(W_\pm)$ generated by $H_\pm$, the pullback of a generator of $\mathrm{Pic}(Q)$ and $\mathrm{Pic}(V_5)$ respectively, and the exceptional divisor $E_\pm$. With respect to the bases $(H_\pm, E_\pm)$ the intersection forms on $N_\pm = \mathrm{Pic}(W_\pm)$, see (2.1), can be written as

$$\begin{pmatrix} 6 & 6 \\ 6 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 10 & 5 \\ 5 & 0 \end{pmatrix}$$

respectively.

$N_\pm$ can be thought of as the overlattices $\mathbf{Z}^2 + \frac{1}{5}(3,-1)\mathbf{Z}$ and $\mathbf{Z}^2 + \frac{1}{6}(1,1)\mathbf{Z}$ of $\mathbf{Z}^2$, generated by

$$\begin{aligned} A_+ &= 3H_+ - E_+ \quad \text{and} \quad B_+ = 4H_+ - 3E_+, \\ \text{and} \quad A_- &= 3H_- - 2E_- \quad \text{and} \quad B_- = 3H_- - 4E_-, \end{aligned} \tag{2.4}$$

with intersection forms

$$\begin{pmatrix} 20 & 0 \\ 0 & -30 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 30 & 0 \\ 0 & -30 \end{pmatrix}$$

respectively. The overlattice $W := \mathbf{Z}^3 + \frac{1}{5}(3,0,-1)\mathbf{Z} + \frac{1}{6}(0,1,1)\mathbf{Z}$ of $\mathbf{Z}^3$ with intersection form

$$\begin{pmatrix} 20 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & -30 \end{pmatrix}$$

is an orthogonal pushout of $N_\pm$ along $R = N_+ \cap N_- = (-30)$.

By Nikulin [Nik79, Theorem 1.12.4 and Corollary 1.12.3] the lattice W (and thus also $N_\pm$) can be embedded primitively into L. Since we can choose $\mathrm{Amp}_\pm$ such that $\mathrm{Amp}_\pm \cap W_\pm$ is spanned by $A_\pm$, Proposition 2.3 yields matching data with $k_\pm$ a multiple of $A_\pm$ for a pair of building blocks $(Z_\pm, \Sigma_\pm)$ of Fano type arising from $W_\pm$. Moreover, the resulting matching $\mathfrak{r}$ is such that $B_+ = \mathfrak{r}^* B_-$ (which generates $N_+ \cap \mathfrak{r}^* N_-$). By the discussion at the end of Section 2.1 we may assume that for all but countably many $b \in \mathbf{P}^1$ the fibre $\Sigma_{\pm,b} := f_\pm^{-1}(b)$ satisfies $\mathrm{Pic}(\Sigma_{\pm,b}) = N_\pm$; in particular, we may assume that this holds for $\Sigma_\pm = \Sigma_{\pm,\infty}$. Moreover, we may assume that $f_\pm \colon Z_\pm \to \mathbf{P}^1$ is an RDP $K3$ fibration.

3. Bundles on the building blocks

We will now construct holomorphic vector bundles $\mathscr{E}_\pm$ over the building blocks $Z_\pm$ such that the hypotheses of Theorem 1.7 are satisfied.

The following theorem provides a spherical μ–semistable vector bundle $\mathscr{E}_{\pm,b}$ with

$$\operatorname{rk} \mathscr{E}_{\pm,b} = 2, \quad c_1(\mathscr{E}_{\pm,b}) = B_\pm \quad \text{and} \quad c_2(\mathscr{E}_{\pm,b}) = -6 \tag{3.1}$$

with $B_\pm$ as in (2.4) on each non-singular fibre $\Sigma_{\pm,b} := f_\pm^{-1}(b)$.

Theorem 3.2 (Kuleshov [Kul90, Theorem 2.1]). *Let (Σ, A) be a polarised smooth $K3$ surface. If $(r, c_1, c_2) \in \mathbf{N} \times H^{1,1}(\Sigma, \mathbf{Z}) \times \mathbf{Z}$ are such that*

$$2rc_2 - (r-1)c_1^2 - 2(r^2-1) = 0,^{5} \tag{3.3}$$

then there exists a spherical μ–semistable vector bundle $\mathscr{E}$ on Σ with

$$\operatorname{rk}\mathscr{E} = r, \quad c_1(\mathscr{E}) = c_1 \quad \textit{and} \quad c_2(\mathscr{E}) = c_2.$$

Remark 3.4. By Hirzebruch–Riemann–Roch, (3.3) is equivalent to $\chi(\mathscr{E}nd_0(\mathscr{E})) = 0$, a necessary condition for $\mathscr{E}$ to be spherical.

Set

$$U_\pm := \left\{ b \in \mathbf{P}^1 : \Sigma_{\pm,b} \text{ is non-singular and } \operatorname{Pic}(\Sigma_{\pm,b}) \cong N_\pm \right\}.$$

Since $A_\pm^\perp \subset N_\pm$ is generated by $B_\pm$ and $B_\pm^2 = -30 < -6$, for $b \in U_\pm$ the following guarantees that $\mathscr{E}_{\pm,b}$ is indeed μ–stable (and thus stable[6]).

Proposition 3.5. *In the situation of Theorem 3.2, if the divisibilities of r and c_1 are coprime and for all non-zero $x \in H^{1,1}(\Sigma, \mathbf{Z})$ perpendicular to $c_1(A)$ we have*

$$x^2 < -\frac{r^2(r^2-1)}{2}, \tag{3.6}$$

then $\mathscr{E}$ is μ–stable.

Proof. Suppose $\mathscr{F}$ were a destabilising sheaf, i.e., a torsion-free subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}$ and $\mu(\mathscr{F}) = \mu(\mathscr{E})$. Since $c_1(\mathscr{E})c_1(A) = \operatorname{rk}\mathscr{E} \cdot \mu(\mathscr{E})$ (and similarly for $\mathscr{F}$), $x := \operatorname{rk}\mathscr{E} \cdot c_1(\mathscr{F}) - \operatorname{rk}\mathscr{F} \cdot c_1(\mathscr{E}) \in c_1(A)^\perp$. The

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[5] Here and in the following, for $x \in H^2(\Sigma)$, we write $x^2 \in \mathbf{Z}$ to denote $x \cup x \in H^4(\Sigma) \cong \mathbf{Z}$.

[6] Recall that a torsion-free coherent sheaf $\mathscr{E}$ on a projective variety $(X, \mathscr{O}(1))$ is called *(semi)stable* if for each torsion-free coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}$ we have $(p_\mathscr{F} \leq p_\mathscr{E})$ $p_\mathscr{F} < p_\mathscr{E}$. Here $p_\mathscr{E}$ denotes the *reduced Hilbert polynomial* of $\mathscr{E}$, the unique polynomial satisfying $p_\mathscr{E}(m) = \chi(\mathscr{E} \otimes \mathscr{O}(m))/\operatorname{rk}\mathscr{E}$ for all $m \in \mathbf{Z}$, and we compare polynomials using the lexicographical order of their coefficients.

The notions of μ–stability and stability are closely related in case $(X, \mathscr{O}(1))$ is smooth (and thus Kähler): because $p_\mathscr{E}(m) = \deg \mathscr{O}(1)/n! \cdot m^n + (\mu(\mathscr{E}) + \frac{1}{2}\deg(K_X))/(n-1)! \cdot m^{n-1} + \cdots$, μ–stable implies stable (and semistable implies μ–semistable).

discriminant of $\mathscr{E}$ is

$$\Delta(\mathscr{E}) := 2\,\mathrm{rk}\,\mathscr{E} \cdot c_2(\mathscr{E}) - (\mathrm{rk}\,\mathscr{E} - 1)c_1(\mathscr{E})^2 = 2(r^2 - 1)$$

by (3.3). According to [HL10, Theorem 4.C.3] we must have either

$$-\frac{(\mathrm{rk}\,\mathscr{E})^2}{4}\Delta(\mathscr{E}) \le x^2,$$

which violates (3.6), or $x = 0$.

The latter, however, implies

$$\mathrm{rk}\,\mathscr{E} \cdot c_1(\mathscr{F}) = \mathrm{rk}\,\mathscr{F} \cdot c_1(\mathscr{E}),$$

which is impossible because the divisibilities of $\mathrm{rk}\,\mathscr{E}$ and $c_1(\mathscr{E})$ are coprime. □

As a consequence of this and the following, for $b \in U_\pm$ the moduli space of semistable bundles on $\Sigma_{\pm,b}$ satisfying (3.1) is a reduced point.

Theorem 3.7 (Mukai [HL10, Theorem 6.1.6]). *Let (Σ, A) be a polarised smooth $K3$ surface. Suppose that $\mathscr{E}$ is a stable sheaf satisfying (3.3) with $r = \mathrm{rk}\,\mathscr{E}$, $c_1 = c_1(\mathscr{E})$ and $c_2 = c_2(\mathscr{E})$. Then $\mathscr{E}$ is locally free and any other semistable sheaf satisfying the same condition must be isomorphic to $\mathscr{E}$.*

If we were able construct holomorphic vector bundles $\mathscr{E}_\pm$ on $Z_\pm$ whose restrictions to the fibres $\Sigma_{\pm,b}$ with $b \in U_\pm$ agree with $\mathscr{E}_{\pm,b}$ and which satisfy (1.8), then we could apply Theorem 1.7 and the proof of Theorem 1.11 would be complete. To see this, note that $\infty \in U_\pm$ and thus $\mathscr{E}_\pm|_{\Sigma_{\pm,\infty}}$ have the same rank, their characteristic classes are identified by $\mathfrak{r}^*$ (since $\mathfrak{r}^* B_- = B_+$ by construction) and both are μ–stable. The construction of $\mathscr{E}_\pm$ is achieved using the following tool. (Note that $\frac{1}{2}B_\pm^2 + 6 = -9$; hence, (3.9) holds in our situation in view of (3.1).)

Proposition 3.8. *Let $f\colon Z \to B$ be RDP $K3$ fibration from a projective 3–fold Z to a smooth curve B and set $S := \{b \in B : \Sigma_b := f^{-1}(b)$ is singular$\}$. Let $(r, c_1, c_2) \in \mathbf{N} \times \mathrm{im}(\mathrm{res}\colon H^2(Z) \to H^2(\Sigma_b)) \times \mathbf{Z}$ for some $b \notin S$ be such* *that* (3.3) *holds and*

$$\gcd\left(r, \frac{1}{2}c_1^2 - c_2\right) = 1. \tag{3.9}$$

Suppose that there is a set $U \subset B \setminus S$ whose complement is countable and for each $b \in U$ the moduli space M_b of semistable bundles $\mathscr{E}_b$ on Σ_b with

$$\operatorname{rk} \mathscr{E}_b = r, \quad c_1(\mathscr{E}_b) = c_1 \quad \textit{and} \quad c_2(\mathscr{E}_b) = c_2 \tag{3.10}$$

consists of a single reduced point: $M_b = \{[\mathscr{E}_b]\}$. Then there exists a holomorphic vector bundle $\mathscr{E}$ over Z such that, for all $b \in U$, $\mathscr{E}|_{\Sigma_b} \cong \mathscr{E}_b$. $\mathscr{E}$ is spherical, i.e., $H^*(\mathscr{E}nd_0(\mathscr{E})) = 0$ and unique up to twisting by a line bundle pulled-back from B.

Remark 3.11. Note that by Hirzebruch–Riemann–Roch $\chi(\mathscr{E}_b) = \frac{1}{2}c_1^2 - c_2 + 2\operatorname{rk}\mathscr{E}_b$, so (3.9) is asking that $\operatorname{rk}\mathscr{E}_b$ and $\chi(\mathscr{E}_b)$ be coprime.

This result is essentially contained in Thomas' work on sheaves on $K3$ fibrations [Tho00, Theorem 4.5]. Its proof heavily relies on the following generalisation of Theorem 3.7.

Theorem 3.12 (Thomas [Tho00, Proof of Theorem 4.5]). *Let (Σ, A) be a polarised $K3$ surface with at worst RDP singularities. If $\mathscr{E}$ is a stable coherent sheaf on Σ with $\chi(\mathscr{E}nd_0(\mathscr{E})) = 0$, then $\mathscr{E}$ is locally free.*

We also use the following simple observation.

Proposition 3.13. *If $\mathscr{E}$ is a semistable sheaf with $\operatorname{rk}\mathscr{E}$ and $\chi(\mathscr{E})$ coprime, then $\mathscr{E}$ is stable.*

Proof. If $\mathscr{E}$ is destabilised by $\mathscr{F} \subset \mathscr{E}$ with $0 < \operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}$, then $p_{\mathscr{F}} = p_{\mathscr{E}}$. In particular, evaluating at $m = 0$ we have

$$\operatorname{rk}\mathscr{E} \cdot \chi(\mathscr{F}) = \operatorname{rk}\mathscr{F} \cdot \chi(\mathscr{E}).$$

This contradicts $\operatorname{rk}\mathscr{E}$ and $\chi(\mathscr{E})$ being coprime. □

Proof of Proposition 3.8. Consider the moduli functor $\underline{\mathbf{M}}\colon \mathbf{Sch}_B^{\mathrm{op}} \to \mathbf{Set}$ which assigns to a B–scheme U the set

$$\underline{\mathbf{M}}(B) := \{\mathscr{E} \text{ a coherent sheaf over } Z \times_B U \text{ satisfying } (\clubsuit)\} / \sim .$$

Here ($\clubsuit$) means that $\mathscr{E}$ is flat over U, for each $b \in U$, $\mathscr{E} \otimes_{\mathscr{O}_U} k(b)$ is semistable and its Hilbert polynomial P agrees with that of a sheaf on a smooth fibre with characteristic classes given by (3.10). We write $\mathscr{E} \sim \mathscr{F}$ if and only if there exists a line bundle $\mathscr{L}$ oover U such that $\mathscr{E}$ and $\mathscr{F} \otimes \mathscr{L}$ are S–equivalent; cf. Maruyaka [Mar78, p. 561] and Huybrechts and Lehn [HL10, Section 4.1].

$\underline{\mathbf{M}}$ is universally corepresented by a proper and separated B–scheme $\mathbf{M}$, i.e., the moduli problem has a proper and separated coarse moduli space, see Simpson [Sim94, Section 1]. The fibre of $\mathbf{M}$ over $b \in B$ is the coarse moduli space of semistable sheaves on Σ_b with Hilbert polynomial P.

Denote by M the component of $\mathbf{M}$ whose fibres over $B \setminus S$ are the coarse moduli space M_b of semistable sheaves $\mathscr{E}$ on Σ_b satisfying (3.10). By assumption, for each $b \in U$, M_b consists of a single reduced point $[\mathscr{E}_b]$. By (3.9) and Proposition 3.13, $\mathscr{E}_b$ is stable and, hence, spherical because $\chi(\mathscr{E}nd_0(\mathscr{E}_b)) = 0$. By Theorem 3.7 it is locally free. Using deformation theory, see, e.g., Hartshorne [Har10, Section 7], one can show that $M \to B$ is surjective onto a open neighbourhood of each $b \in U$ and thus to all of B, since it is proper. Using (3.9) and Proposition 3.13 as well as Theorem 3.7 again we see that for each $b \notin S$ the fibre M_b is a reduced point. Since M is separated, it follows that $M = B$.

By [HL10, Corollary 4.6.7], (3.9) guarantees the existence a universal sheaf $\mathscr{E}$ on $Z \times_B M = Z$.[7] By flatness, for each $b \in B$, $\chi(\mathscr{E}|_{\Sigma_b}) = \frac{1}{2}c_1^2 - c_2 + 2r$ and $\chi(\mathscr{E}nd_0(\mathscr{E}|_{\Sigma_b})) = 0$. From (3.9), Proposition 3.13 and Theorem 3.12 (resp. Theorem 3.7) it follows that $\mathscr{E}|_{\Sigma_b}$ is locally free and spherical for arbitrary $b \in B$. Therefore, $\mathscr{E}$ is also locally free by [Sim94, Lemma 1.27] and spherical by Grothendieck's spectral sequence.

The asserted uniqueness property follows from the fact that $\mathscr{E}$ is a universal sheaf and the definition of the moduli functor. □

This completes the construction of the bundles $\mathscr{E}_\pm$ and thus the proof of Theorem 1.11. Clearly, the above argument also proves the following more abstract result.

[7]Strictly speaking, the quoted result only provides the universal sheaf over $f^{-1}(B \setminus S)$; however, Langton's theorem, see, e.g., [HL10, Section 2.B] or [Har10, Proposition 28.7], shows that it extends to all of Z. Alternatively, the existence of the universal sheaf can be deduced from [Sim94, Theorem 1.21] and Tsen's theorem $H^2_{\mathrm{et}}(B, \mathscr{O}^*) = 0$.

Theorem 3.14. *Let* $(Z_\pm, \Sigma_\pm, \boldsymbol{\omega}_\pm; \mathfrak{r})$ *be a matched pair of framed building blocks. Suppose that* $f_\pm\colon Z_\pm \to \mathbf{P}^1$ *are RDP* $K3$ *fibrations and that for all but countably many* $b \in \mathbf{P}^1$ *we have*

$$\mathrm{Pic}(f_\pm^{-1}(b)) \cong N_\pm := \mathrm{im}\big(\mathrm{res}_\pm\colon\ H^2(Z_\pm) \to H^2(\Sigma_\pm)\big).$$

Suppose there exists a $(r, c_1, c_2) \in \mathbf{N} \times (N_+ \cap \mathfrak{r}^* N_-) \times \mathbf{Z}$ *such that*

$$2rc_2 - (r-1)c_1^2 - 2(r^2-1) = 0$$

and

$$\gcd\left(r, \frac{1}{2}c_1^2 - c_2\right) = 1.$$

If $[\omega_{I,\pm}] \in H^2(\Sigma_\pm, \mathbf{Q})$ *and for all non-zero* $x \in [\omega_{I,\pm}]^\perp \subset N_\pm$ *we have*

$$x^2 < -\frac{r^2(r^2-1)}{2},$$

then there exists rank r *holomorphic vector bundles* $\mathscr{E}_\pm$ *on* $Z_\pm$ *with*

$$c_1(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_1(\mathscr{E}_-|_{\Sigma_-}) = c_1 \quad \textit{and} \quad c_2(\mathscr{E}_+|_{\Sigma_+}) = \mathfrak{r}^* c_2(\mathscr{E}_-|_{\Sigma_-}) = c_2$$

satisfying the hypotheses of Theorem 1.7.

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