Math. Res. Lett.
Volume 23, Number 3, 593–619, 2016

Rectifiers and the local Langlands correspondence: the unramified case

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Bushnell and Henniart define rectifiers, which provide a correction term in the local Langlands correspondence for $\mathrm{GL}_n(K)$. They also give a natural bijection between essentially tame supercuspidal Langlands parameters and characters of minisotropic tori, and a second bijection between characters of minisotropic tori and supercuspidal representations of $\mathrm{GL}_n(K)$. Rectifiers bridge the gap, adding an intermediate step so that the composition agrees with the local Langlands correspondence. In this paper, we begin the process of generalizing rectifiers to other connected reductive groups, focusing on the case of unramified minisotropic tori that satisfy a certain cohomology condition.

1. Introduction

Let G be a connected reductive group defined over a p-adic field K. The local Langlands conjecture predicts the existence of a finite to one map from the set of isomorphism classes of irreducible admissible representations of $G(K)$ to the set of Langlands parameters for $G(K)$.

There has been a significant amount of progress in recent years focusing on supercuspidal representations of $G(K)$. Bushnell–Henniart [12], DeBacker–Reeder [15], Kaletha [19] and Reeder [24] approach the task of constructing L-packets by first attaching a character of an elliptic torus to a Langlands parameter, and then associating a collection of supercuspidal representations to this character. Their constructions all use the local Langlands correspondence for tori in some way. However, since the image of the Langlands parameter $\mathcal{W}_K \to {}^LG$ is not necessarily contained within the L-group of a maximal torus, they need to make certain adjustments in order to produce a character of an elliptic torus. The different authors remedy this situation in various ways.

The second author was supported by the Pacific Institute for the Mathematical Sciences.

For example, consider the group $G = \mathrm{PGL}_2(K)$, and suppose that $\varphi : \mathcal{W}_K \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation. Then φ is a Langlands parameter corresponding to a supercuspidal representation of $\mathrm{PGL}_2(K)$. Moreover, the image of φ is contained in the normalizer of the dual torus $\hat{T}$, a non-split extension of $\mathrm{Gal}(L/K)$ by $\hat{T}$, but not in the L-group of any torus.

If $p \neq 2$, then there is a tamely ramified quadratic extension L/K and a character χ of $L^\times$ that is trivial on the norms $\mathrm{Nm}_{L/K}(L^\times)$ and nontrivial on $K^\times$, so that $\varphi = \mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\chi)$ [11, §34]. In particular, χ is a character of the group $L^\times/\,\mathrm{Nm}_{L/K}(L^\times)$. By Hilbert's theorem 90, the group $L^\times/\,\mathrm{Nm}_{L/K}(L^\times)$ appears as a covering group of the elliptic torus L^1 of norm 1 elements in L:

$$\begin{aligned} 1 \to \mathbb{Z}/2\mathbb{Z} \to L^\times/\,\mathrm{Nm}_{L/K}(L^\times) &\to L^1 \to 1 \\ x\,\mathrm{Nm}_{L/K}(L^\times) &\mapsto x/\sigma(x); \end{aligned}$$

here σ generates $\mathrm{Gal}(L/K)$. In particular, the Langlands parameter φ naturally provides a character χ, not of the elliptic torus $L^1 \subset \mathrm{PGL}_2(K)$ (since χ is nontrivial on $K^\times$), but of the two-fold cover $L^\times/\,\mathrm{Nm}_{L/K}(L^\times)$. We can obtain a character χ' of L^1 by tensoring χ by another character of $L^\times/\,\mathrm{Nm}_{L/K}(L^\times)$ that is nontrivial on $K^\times$. One can then attach a supercuspidal representation of $\mathrm{PGL}_2(K)$ to χ' via the construction of Bushnell and Kutzko [13]. The tensoring character that gives the correct supercuspidal representation of $\mathrm{PGL}_2(K)$ is precisely what appears in Bushnell and Henniart's rectifier. In [12], Bushnell and Henniart compute this rectifier in the *essentially tame* setting for $\mathrm{GL}_n(K)$.

Bushnell and Henniart motivate their rectifier as follows. Suppose that φ is an essentially tame supercuspidal Langlands parameter for $\mathrm{GL}_n(K)$. The local Langlands correspondence for tori then yields a degree n extension L/K and a character ξ of $L^\times$. We now fix a construction $\chi \mapsto \pi_\chi$ of supercuspidal representations of $\mathrm{GL}_n(K)$ from *admissible* characters of $L^\times$. Then the rectifier of ξ is a character μ_ξ of $L^\times$ such that $\varphi \mapsto \pi_{\xi\cdot\mu_\xi}$ is the local Langlands correspondence for $\mathrm{GL}_n(K)$. In the specific case of supercuspidal representations of $\mathrm{GL}_2(K)$ with trivial central character (i.e. supercuspidal representations of $\mathrm{PGL}_2(K)$), one computes that the character μ_ξ is, as discussed above, a character of $L^\times/\,\mathrm{Nm}_{L/K}(L^\times)$ that is nontrivial on $K^\times$.

In this paper, we initiate a program to generalize Bushnell and Henniart's rectifier to groups other than $\mathrm{GL}_n(K)$. Suppose that G is a connected reductive group defined over K. Let $\varphi : \mathcal{W}_K \to {}^L G$ be a Langlands parameter for $G(K)$, and suppose that φ factors through the normalizer of a maximal torus in the dual group. Benedict Gross has recently used his theory of groups of type L to show that one may attach to each such parameter a character of a group that covers a subgroup of a maximal torus in $G(K)$. We briefly describe this construction, since it is needed in our definition of rectifier; for more details see §5. To φ, one can associate a maximal K-torus T in G, unique up to stable conjugacy. Let L be the splitting field of T and set $\Gamma = \mathrm{Gal}(L/K)$. Restricting φ to $\mathcal{W}_L$, the local Langlands correspondence for tori associates to $\varphi|_{\mathcal{W}_L}$ a character ξ of $T(L)$. In fact, ξ factors through the coinvariants $T(L)_\Gamma$. Moreover, invariants and coinvariants are related by the norm map

$$\begin{aligned} N : T(L) &\to T(K), \\ t &\mapsto \prod_{\sigma\in\Gamma} \sigma(t), \end{aligned}$$

in the cohomology sequence

$$(1.1)\quad 1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \xrightarrow{N} T(K) = T(L)^\Gamma \to \hat{\mathrm{H}}^0(\Gamma, T(L)) \to 1,$$

where $\hat{\mathrm{H}}$ denotes Tate cohomology. We have attached to each φ a character ξ of $T(L)_\Gamma$, which is in general a covering group of a finite-index subgroup of $T(K)$. In order to use existing constructions of representations of $G(K)$, we need a character of $T(K)$ itself. The outer terms in (1.1) provide an obstruction in shifting to a character of $T(K)$.

In this paper, we sidestep half of the problem by assuming that $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$. This condition holds for unramified minisotropic tori in both GL_n and in semisimple groups. Such tori arise for discrete parameters, where φ does not factor through a proper Levi subgroup. In the case of GL_n, we also have $\hat{\mathrm{H}}^{-1}(\Gamma, T(L)) = 0$ for any unramified minisotropic torus, so $T(L)_\Gamma \cong T(K)$ can be identified with $L^\times$. To construct the local Langlands correspondence for $\mathrm{GL}_n(K)$, one would then proceed to attach the supercuspidal representation $\pi_{\xi\cdot\mu_\xi}$ to ξ, via the construction of Bushnell and Henniart.

For many G, there are constructions of supercuspidal L-packets $\Pi(\chi)$ associated to characters χ of elliptic tori $T(K) \subset G(K)$. However, a Langlands parameter φ does not naturally provide a character of $T(K)$, but rather a character ξ of $T(L)_\Gamma$. After fixing an association $\chi \mapsto \Pi(\chi)$ of supercuspidal L-packets of G to *admissible* characters of $T(K)$ (for example, the association of [15], [19], or [24]), the *rectifier* will be a character μ_ξ of the covering group $T(L)_\Gamma$ such that $\varphi \mapsto \Pi(\xi \cdot \mu_\xi)$ agrees with existing constructions of the local Langlands correspondence for $G(K)$, just as in the case of $\mathrm{GL}_n(K)$. We note that if $G(K) = \mathrm{PGL}_2(K)$ and if φ is supercuspidal, then the Tate cohomology sequence above is exactly the 2-fold cover $L^\times / \mathrm{Nm}_{L/K}(L^\times)$ of L^1 described earlier.

The associations $\chi \mapsto \Pi(\chi)$ that we use in our definition of rectifiers are those of DeBacker–Reeder and Reeder. We define rectifiers for unramified minisotropic tori T whose zeroth Tate cohomology is trivial. We show in Theorem 8.4 that rectifiers for semisimple G exist and are unique up to equivalence. In fact, the rectifier that we construct is canonical: it is a character of $T(L)_\Gamma$ that is constructed from a canonical Langlands parameter (see Definition 6.3). We show that our rectifier agrees with that of Bushnell–Henniart in the setting of depth zero supercuspidal representations of $\mathrm{GL}_n(K)$ (see Theorem 9.5).

Note that there is an obstruction to proving the compatibility of our rectifier with Bushnell–Henniart's in the positive depth case. In the depth zero case, Deligne–Lusztig representations provide a canonical way of attaching supercuspidal representations to characters. However, in positive depth there are many: Adler [3], Howe [17], Bushnell–Henniart [12], Bushnell–Kutzko [13], and Yu [30]. In the positive depth setting, the rectifier will depend on the methods used to construct representations from the character of $T(K)$, and our rectifier indeed differs from that of Bushnell and Henniart in positive depth.

We would like to remark that the notion of a covering group of a torus occurring in the local Langlands correspondence is an old one, dating back at least to work of Adams and Vogan [1] in the setting of real groups. In the theory of real groups, an admissible homomorphism $W_{\mathbb{R}} \to {}^L G$ automatically has image inside the normalizer of a torus in ${}^L G$. As such, it naturally produces a genuine character χ of the *ρ-cover of* $T(\mathbb{R})$, denoted $T(\mathbb{R})_\rho$, a certain double cover of $T(\mathbb{R})$. To χ, one can naturally attach a collection of representations of $G(\mathbb{R})$ to construct an L-packet.

We emphasize that our results do not extend the scope of DeBacker and Reeder's constructions to a broader class of parameters. Instead, we aim to translate Bushnell and Henniart's notion of rectifier in such a way that it can be applied to more general groups than GL_n. In this paper we only handle the case where we may attach a torus T to the parameter that is both unramified and satisfies $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$. We believe that these restrictions can be removed in future work.

We now present an outline of the paper. In §3 we recall the notion of rectifier due to Bushnell and Henniart and describe the rectifier in the setting that we will need. In §4 we present some results about Tate cohomology of

p-adic tori that will be used in the rest of the paper. In §5 we review the theory of groups of type L. In §6 we describe the relationship between the construction of Gross, via groups of type L, and the constructions $\chi \mapsto \Pi(\chi)$ of [15] and [24]. In §7 we study how translation by a character affects the map $\chi \mapsto \Pi(\chi)$. In §8 we introduce our notion of rectifier and prove our main result, Theorem 8.4. Finally, in §9 we show that our rectifier is compatible with the rectifier of Bushnell and Henniart in the setting of depth zero supercuspidal representations of $\mathrm{GL}_n(K)$.

2. Notation and preliminaries

Throughout, K will denote a nonarchimedean local field of characteristic zero, $\mathcal{O}_K$ its ring of integers, k its residue field of cardinality q, $\mathcal{P}_K$ the maximal ideal in $\mathcal{O}_K$ and ϖ a fixed uniformizer. Write K_n for the unramified extension of K of degree n, k_n for the degree n extension of k, and set $\Gamma_n = \mathrm{Gal}(K_n/K) = \mathrm{Gal}(k_n/k)$. Let $\bar{K}$ and $\bar{k}$ be algebraic closures of K, k, respectively.

A geometric Frobenius is an element of $\mathrm{Gal}(\bar{K}/K)$ inducing the automorphism $x \mapsto x^{1/p}$ of $\bar{k}$. Under the Artin reciprocity map of local class field theory the choice of ϖ determines a geometric Frobenius Fr [27, §2].

If $\chi : K^\times \to \mathbb{C}^\times$ is a character, we define the *depth* of χ to be the smallest integer r such that $\chi|_{1+\mathcal{P}_K^{r+1}} \equiv 1$ and $\chi|_{1+\mathcal{P}_K^r} \not\equiv 1$.

If T is a torus defined over K we write $X^*(T)$ for the character lattice $\mathrm{Hom}_{\bar{K}}(T, \mathbb{G}_m)$ and $X_*(T)$ for the cocharacter lattice $\mathrm{Hom}_{\bar{K}}(\mathbb{G}_m, T)$ [18, §16.2]. T will split over an extension L of K if and only if $\mathrm{Gal}(\bar{K}/L)$ acts trivially on $X^*(T)$. We may thus define *the* splitting field L of T as the minimal extension of K splitting T; note that L is necessarily Galois over K. Write Γ for $\mathrm{Gal}(L/K)$. Then $X_*(T)$, $X^*(T)$ and $T(L)$ are all Γ-modules.

Suppose now that $T \subset G$ for a connected reductive group G over K. We will write $\hat{T} \subset \hat{G}$ for the dual torus in the complex dual group of G [8, §I.2]. Let N be the normalizer $\mathrm{N}_G(T)$ of T in G and define $W = N/T$; set $\hat{N} = \mathrm{N}_{\hat{G}}(\hat{T})$ and $\hat{W} = \hat{N}/\hat{T}$. The identification of $X^*(T)$ and $X_*(\hat{T})$ yields a canonical anti-isomorphism between W and $\hat{W}$. Note that W is a scheme over K; in general $W(K) \neq N(K)/T(K)$.

Write Nm for the norm map

$$\begin{aligned} T(L) &\to T(K) \\ t &\mapsto \prod_{\sigma \in \Gamma} \sigma(t) \end{aligned}$$

and for its restriction to $X_*(T)$.

The following theorem, due to Lang [20], underpins the facts in §4 on tori over p-adic fields. Let H be a commutative connected algebraic group over a finite field k, and suppose H splits over k_n. Denote by $\hat{\mathrm{H}}^i$ the i^{th} Tate cohomology group.

Theorem 2.1. *$\hat{\mathrm{H}}^i(\Gamma_n, H(k_n)) = 0$ for all i.*

Proof. Since Γ_n is cyclic, $\hat{\mathrm{H}}^i(\Gamma_n, H(k_n)) \cong \hat{\mathrm{H}}^{i+2}(\Gamma_n, H(k_n))$ [6, Thm. 5], so it suffices to prove the result for $i = 1$ and $i = 2$, which is done by Serre [28, §VI.6]. □

3. Rectifier for $\mathrm{GL}_n(K)$

In this section we recall the rectifier of Bushnell and Henniart and their construction of the essentially tame local Langlands correspondence for $\mathrm{GL}_n(K)$. An irreducible smooth representation of the Weil group $\mathcal{W}_K$ of K is called *essentially tame* if its restriction to wild inertia is a sum of characters.

Definition 3.1. Let L/K be an extension of degree n, with n coprime to p. A character ξ of $L^\times$ is *admissible* if

1) ξ doesn't factors through the norm from a subfield of L containing K,
2) If $\xi|_{1+\mathcal{P}_L}$ factors through the norm from a proper subfield $L \supseteq M \supseteq K$, then L/M is unramified.

There is a natural bijection $\varphi_\xi \leftrightarrow (L/K, \xi)$ between irreducible smooth essentially tame $\varphi_\xi : \mathcal{W}_K \to \mathrm{GL}_n(\mathbb{C})$ and *admissible pairs* $(L/K, \xi)$. Bushnell and Henniart construct a map (see [12])

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{admissible pairs} \end{array} \right\} \to \left\{ \begin{array}{c} \text{supercuspidal representations} \\ \text{of } \mathrm{GL}_n(K) \end{array} \right\}$$

$$(L/K, \xi) \mapsto \pi_\xi$$

However, the map

$$\varphi_\xi \mapsto \pi_\xi$$

is not the local Langlands correspondence because π_ξ has the wrong central character. Instead, the local Langlands correspondence is given by

$$(\star) \qquad \varphi_\xi \mapsto \pi_{\xi \cdot_K \mu_\xi}$$

for some subtle finite order character ${}_K\mu_\xi$ of $L^\times$. Since we will not be changing K in this paper we will write μ_ξ for ${}_K\mu_\xi$.

The relation $(\star)$ does not determine μ_ξ uniquely. As pointed out in [12], the obstruction to uniqueness revolves around the group $\mathrm{GL}_2(\mathbb{F}_3)$. Bushnell and Henniart therefore make the following definition [12, Def. 1].

Definition 3.2. Let L/K be a finite, tamely ramified field extension of degree n. A *rectifier* for L/K is a function

$$\boldsymbol{\mu} : (L/K, \xi) \mapsto \mu_\xi$$

which attaches to each admissible pair $(L/K, \xi)$ a character μ_ξ of $L^\times$ satisfying the following conditions:

1) The character μ_ξ is tamely ramified.
2) Writing $\xi' = \xi \cdot \mu_\xi$, the pair $(L/K, \xi')$ is admissible and $\varphi_\xi \mapsto \pi_{\xi\cdot\mu_\xi}$ is the local Langlands correspondence for $\mathrm{GL}_n(K)$.
3) If $(L/K, \xi_i), i = 1, 2$, are admissible pairs such that $\xi_1^{-1}\xi_2$ is tamely ramified, then $\mu_{\xi_1} = \mu_{\xi_2}$.

Bushnell and Henniart then prove [12, Thm. A]:

Theorem 3.3. *Any finite, tamely ramified, field extension L/K admits a unique rectifier $\boldsymbol{\mu} : (L/K, \xi) \mapsto \mu_\xi$.*

Both the description of and the intuition behind the rectifiers $\boldsymbol{\mu}$ have been studied (see [12], [29], [4]). In order to generalize rectifiers to groups other than $\mathrm{GL}_n(K)$ we will will need a description of the characters μ_ξ in certain cases. The following result comes immediately from [12, Prop. 21].

Proposition 3.4. *Suppose that $(L/K, \xi)$ is an admissible pair, where ξ has depth 0. Then μ_ξ is unramified and $\mu_\xi(\varpi) = (-1)^{n-1}$.*

4. Tori over p-adic fields

We first recall the definition of a minisotropic torus.

Definition 4.1 ([24, §3.1]). If T is an unramified maximal K-torus in G, we say that T is *K-minisotropic* if $T(K)/Z(K)$ is compact.

Now let $T \subset G$ be a torus defined over K with splitting field L, let K_n be the maximal unramified subextension L/K, set $\Gamma = \mathrm{Gal}(L/K)$ and $I = \mathrm{Gal}(L/K_n)$. In studying rectifiers for groups other than GL_n, the following exact sequence in Tate cohomology plays a crucial role:

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \xrightarrow{\mathrm{Nm}} T(K) = T(L)^\Gamma \to \hat{\mathrm{H}}^0(\Gamma, T(L)) \to 1. \tag{4.1}$$

We make the following definition as a matter of notational convenience, since it will serve as a running hypothesis on T for most of the rest of the paper.

Definition 4.2. We say that T is *coverable* if it is K-minisotropic and $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$.

We require that T be minisotropic in order to be able to use the local Langlands correspondence given by Reeder [24] and Debacker–Reeder [15]; the condition on $\hat{\mathrm{H}}^0(\Gamma, T(L))$ will allow us to define characters of $T(K)$ from characters of $T(L)_\Gamma$. In the remainder of this section recall some tools for computing Tate cohomology groups over tori and give examples of coverable and non-coverable tori.

Let $\mathcal{T}$ be the Néron model of T, a canonical model of T over $\mathcal{O}_K$ [9, Ch. 10]. As a consequence of the Néron mapping property, we may identify $\mathcal{T}(\mathcal{O}_K)$ with $T(K)$. The connected component of the identity, $\mathcal{T}^\circ$, cuts out a subgroup $T(K)_0 = \mathcal{T}^\circ(\mathcal{O}_K)$ of $T(K)$; we also write $T(K_n)_0$ for $\mathcal{T}^\circ(\mathcal{O}_{K_n})$.

In fact, this subgroup of $T(K)$ is the first in a decreasing filtration. Moy and Prasad [22] define one such filtration by embedding T into an induced torus and defining the filtration of $\mathrm{Res}_{L/K}\,\mathbb{G}_m$ in terms of the valuation on L. Yu [30, §5] describes a different filtration, agreeing with that of Moy and Prasad in the case of tame tori but with nicer features in the presence of wild ramification. Let $\{\mathcal{T}_r\}_{r\geq 0}$ be the integral models of T defined in Yu's minimal congruent filtration and let $\{T(K)_r\}_{r\geq 0}$ and $\{T(K_n)_r\}_{r\geq 0}$ be the corresponding filtrations of $T(K)$ and $T(K_n)$.

Let $\mathcal{C}$ be the scheme of connected components of $\mathcal{T}$, which we may identify with the components of $\mathcal{T} \times \mathrm{Spec}(k)$ since $T = \mathcal{T} \times \mathrm{Spec}(K)$ is connected. The structure of $\mathcal{C}$ is described by Bertapelle and González-Avilés:

Proposition 4.3 ([7, Thm. 1.1]). *As* $\mathrm{Gal}(\bar{k}/k)$*-modules,*

$$\mathcal{C} \cong X_*(T)_I.$$

Using our filtration of $T(K_n)$, we may relate the cohomology of $T(K_n)$ with that of $\mathcal{C}$.

Proposition 4.4. $\hat{\mathrm{H}}^i(\Gamma_n, T(K_n)_0) = 0$ *for all* i.

Proof. Note that

$$T(K_n)_0 = \varprojlim_r T(K_n)_0/T(K_n)_r.$$

So by a result of Serre [27, Lem. 3], it suffices to prove that $\hat{\mathrm{H}}^i(\Gamma_n, T(K_n)_r/T(K_n)_{r+}) = 0$ for all i. But $T(K_n)_r/T(K_n)_{r+}$ is connected [30, Prop. 5.2] and thus has trivial cohomology by Theorem 2.1. □

Corollary 4.5. $\hat{\mathrm{H}}^i(\Gamma_n, T(K_n)) \cong \hat{\mathrm{H}}^i(\Gamma_n, X_*(T)_I)$.

Proof. This follows from the long exact sequence in cohomology associated to the sequence

$$0 \to \mathcal{T}^0 \to \mathcal{T} \to \mathcal{C} \to 0.$$

□

Using this corollary, we can give examples of coverable and non-coverable tori.

Example 4.6.

1) *Each minisotropic torus in* GL_m *is of the form* $T = \mathrm{Res}_{N/K}\,\mathbb{G}_m$ *for a degree* m *extension* N *of* K. *Write* L *for the Galois closure of* N *and* $H = \mathrm{Gal}(L/N)$. *As usual, set* $\Gamma = \mathrm{Gal}(L/K)$, *let* K_n *be the maximal unramified subextension of* N *and choose a Frobenius lift* $\mathrm{Fr} \in \Gamma$. *We may take a basis* $\{e_C\}$ *for* $X_*(T)$ *indexed by left cosets* C *of* H *in* Γ, *with* Γ *permuting the basis through left multiplication on the index. Then* $X_*(T)_I$ *is spanned by the classes of* $e_{\mathrm{Fr}^i H}$ *for* $0 \leq i < m$, *permuted cyclically. An easy computation using Corollary 4.5 now shows that*
$$\hat{\mathrm{H}}^{-1}(\Gamma_n, T(K_n)) = \hat{\mathrm{H}}^0(\Gamma_n, T(K_n)) = 0.$$
2) *For an example of an unramified torus with nontrivial cohomology, let* T *be a torus in* GSp_4 *[21, Prop. 1.3], split over a quartic unramified extension* L/K *with* $\Gamma = \mathrm{Gal}(L/K) = \langle\tau\rangle$, *with cocharacter lattice*
$$\begin{aligned} X_*(T) = &\{(a,b,c,d) \in \mathbb{Z}^4 \;:\; a+d = b+c\} \\ &\tau(a,b,c,d) = (c,a,d,b). \end{aligned}$$
Then T *is minisotropic but has* $\hat{\mathrm{H}}^0(\Gamma, T(L)) \cong \mathbb{Z}/2\mathbb{Z}$ *by Corollary 4.5:* $X_*(T)^\Gamma$ *is spanned by* $(1,1,1,1)$ *while the image of the norm map*

$(a,b,c,d) \mapsto (a+b+c+d,\dots,a+b+c+d)$ is spanned by $(2,2,2,2)$. *A similar computation shows that* $\hat{\mathrm{H}}^{-1}(\Gamma, T(L)) = 0$.

On the other hand, if G is semisimple, examples as in (2) above do not occur, as seen from the following proposition.

Proposition 4.7. *If T is unramified and anisotropic, then* $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$.

Proof. Since T is anisotropic, $X_*(T)^\Gamma = 0$, giving $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$ by Corollary 4.5. □

For unramified T the jumps in the filtration on $T(K)$ and $T(L)$ occur at integers, and we write

$$\begin{aligned} T(\mathcal{O}_K) &= T(K)_0, \\ T(\mathcal{O}_L) &= T(L)_0, \\ T(\mathcal{P}_K^r) &= T(K)_r \qquad \text{for } r > 0, \\ T(\mathcal{P}_L^r) &= T(L)_r \qquad \text{for } r > 0. \end{aligned}$$

5. Groups of type L

We now review the theory of groups of type L [26, §7]. For a torus T over K recall that the dual torus $\hat{T}$ is equipped with an action of Γ.

Definition 5.1. A *group of type L* is a group extension of Γ by $\hat{T}$.

For such a group D we have by definition an exact sequence

$$1 \to \hat{T} \to D \to \Gamma \to 1.$$

We now describe how we can naturally attach a character of the coinvariants $T(L)_\Gamma$ to a Langlands parameter

$$\varphi : \mathcal{W}_K \to D$$

with values in a group of type L. Restricting φ to $\mathcal{W}_L$ we get a homomorphism

$$\varphi|_{\mathcal{W}_L} : \mathcal{W}_L \to \hat{T},$$

and by the Langlands correspondence for tori a character $\xi_\varphi : T(L) \to \mathbb{C}^\times$. Since $\varphi|_{\mathcal{W}_L}$ extends to φ we have that

$$\xi_\varphi(\sigma(t)) = \xi_\varphi(t) \text{ for all } \sigma \in \Gamma.$$

Thus ξ_φ is trivial on the augmentation ideal $I_\Gamma(T(L))$ and descends to

$$\xi_\varphi : T(L)_\Gamma \to \mathbb{C}^\times.$$

Invariants and coinvariants are related by the norm map in the Tate cohomology sequence

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \xrightarrow{\mathrm{Nm}} T(K) = T(L)^\Gamma \to \hat{\mathrm{H}}^0(\Gamma, T(L)) \to 1.$$

We will assume in §8 that $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$, in which case ξ_φ is a character of a cover of $T(K)$.

We will need the following structural result about Langlands parameters mapping to groups of type L for the proof of Proposition 6.5. Suppose now that L/K is unramified of degree n and that φ and φ' are two Langlands parameters with $\varphi'(\mathrm{Fr})\varphi(\mathrm{Fr})^{-1} \in \hat{T}$. Let ξ and ξ' be the associated characters of $T(L)_\Gamma$.

Lemma 5.2. *ξ and ξ' have the same restriction to $\hat{\mathrm{H}}^{-1}(\Gamma, T(L))$.*

Proof. Set $y = \varphi(\mathrm{Fr}) \in D$. By Corollary 4.5, $\hat{\mathrm{H}}^{-1}(\Gamma, T(L)) = \hat{\mathrm{H}}^{-1}(\Gamma, X_*(T))$. If suffices to show that $\xi(\lambda) = \xi'(\lambda)$ for any $\lambda \in \ker(\mathrm{Nm} : X_*(T) \to X_*(T))$.

Via the canonical identification $X_*(T) \cong X^*(\hat{T})$, we regard λ as an element of $X^*(\hat{T})$. Note that $y^n \in \hat{T} \cong X_*(\hat{T}) \otimes \mathbb{C}^\times$. We assume that y^n is a simple tensor $y^n = \mu \otimes z$, for some $\mu \in X_*(\hat{T})$ and $z \in \mathbb{C}^\times$; for general y^n, the ensuing computations are analogous. Write $\langle , \rangle$ for the canonical pairing $X_*(\hat{T}) \times X^*(\hat{T}) \to \mathbb{Z}$. The local Langlands correspondence for tori implies that $\xi(\lambda) = z^{\langle \mu, \lambda \rangle}$.

Let $t = \varphi'(\mathrm{Fr})\varphi(\mathrm{Fr})^{-1} \in \hat{T}$, and let w be the image of y in the Weyl group. Again we assume that t is a simple tensor $t = \mu' \otimes z'$ for $\mu' \in X_*(\hat{T})$ and $z' \in \mathbb{C}^\times$. Then

$$\begin{aligned}(ty)^n &= tyty^{-1}y^2ty^{-2} \cdots y^{n-1}ty^{1-n}y^n \\ &= tw(t)w^2(t) \cdots w^{n-1}(t)y^n \\ &= (\mathrm{Nm}(\mu') \otimes z') \cdot (\mu \otimes z),\end{aligned}$$

where here Nm denotes norm map on $\hat{T}$ corresponding to the given Galois action on $\hat{T}$.

We conclude again by the local Langlands correspondence for tori that

$$\xi'(\lambda) = z^{\langle \mu,\lambda\rangle}(z')^{\langle \mathrm{Nm}(\mu'),\lambda\rangle}.$$

Since $\langle , \rangle$ is Galois equivariant, $\langle \mathrm{Nm}(\mu'), \lambda\rangle = \langle \mu', \mathrm{Nm}(\lambda)\rangle$. But $\lambda \in \ker(\mathrm{Nm} : X_*(T) \to X_*(T))$, so $\langle \mu', \mathrm{Nm}(\lambda)\rangle = 0$. Therefore $\xi'(\lambda) = z^{\langle \mu,\lambda\rangle} = \xi(\lambda)$. □

We will also need the following lemma in order to define our notion of admissible pair in §8.

Lemma 5.3. *Let G be a connected reductive K-group and let T be a maximal K-torus of G that has splitting field L.*

1) $\mathrm{N}_{G(L)}(T(K))/T(L) \cong W(K)$.
2) *The conjugation action of $\mathrm{N}_{G(L)}(T(L))/T(L)$ on $T(L)$ determines actions of $\mathrm{N}_{G(L)}(T(L))^\Gamma/T(K)$ and $W(K)$ on $T(L)$ which factor to actions on $T(L)_\Gamma$.*

Proof. See [5, Lem. 9.1]. □

6. The relationship between the Gross construction and the DeBacker–Reeder and Reeder construction

Let $\varphi : \mathcal{W}_K \to {}^L G$ be a regular semisimple elliptic Langlands parameter for an unramified connected reductive group G (see [15] and [24]). Here, ${}^L G = \langle\hat{\theta}\rangle \ltimes \hat{G}$, where $\hat{\theta}$ is the dual Frobenius automorphism on $\hat{G}$ (see [15, §3]). Since φ is elliptic, its image is contained in a group of type L for an unramified K-minisotropic torus T. Let L be the splitting field of T and let $\Gamma = \mathrm{Gal}(L/K)$ and ξ_φ be as in Section 5. Then $\varphi(I_K) \subset \hat{T}$ and $\varphi(\mathrm{Fr}) = \hat{\theta} f$ for some $f \in \hat{N}$. Let $\hat{w}$ be the image of f in $\hat{W}$. DeBacker–Reeder [15] and Reeder [24] associate a character χ_φ of $T(K)$ to φ.

We now recall the definition of the Tits group and some of its properties. Choose a set $\{X_\alpha\}$ of root vectors indexed by the set of simple roots of $\hat{T}$ in $\hat{B}$. Then $(\hat{T}, \hat{B}, \{X_\alpha\})$ is a pinning as in [25, §3.1]. For each simple root α, define $\phi_\alpha : \mathrm{SL}_2 \to \hat{G}$ by

$$\phi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \alpha^\vee(z)$$
$$d\phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha.$$

Let $\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Definition 6.1. The Tits group $\widetilde{W}$ is the subgroup of $\hat{N}$ generated by $\{\sigma_\alpha\}$ for simple roots α.

For each simple root α, let $m_\alpha = \sigma_\alpha^2 = \alpha^\vee(-1)$ and let $\hat{T}_2$ be the subgroup of $\hat{T}$ generated by the m_α.

Theorem 6.2. *[2, Prop. 5.2]*

1) *The kernel of the natural map $\widetilde{W} \to \hat{W}$ is $\hat{T}_2$,*
2) *The elements σ_α satisfy the braid relations,*
3) *There is a canonical lifting of $\hat{W}$ to a subset of $\widetilde{W}$: take a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$, and let $\tilde{w} = \sigma_{\alpha_1} ... \sigma_{\alpha_n}$.*

We remark that the lifting $\hat{W} \to \widetilde{W}$ is not necessarily a homomorphism, as shown by the example of SL_2.

Definition 6.3. Given $\hat{u} \in \hat{W}$, let $\tilde{u}$ be its canonical lift to $\widetilde{W}$. We define a homomorphism $\varphi_{\hat{u}} : \mathcal{W}_K \to {}^L G$ by

1) $\varphi_{\hat{u}}|_{I_K} \equiv 1$,
2) $\varphi_{\hat{u}}(\mathrm{Fr}) = \hat{\theta}\tilde{u}$.

By §5, φ and $\varphi_{\hat{w}}$ give rise to characters ξ_φ and $\xi_{\varphi_{\hat{w}}}$ of $T(L)_\Gamma$ respectively.

The following proposition relates the character ξ_φ defined through groups of type L to the character χ_φ constructed by DeBacker–Reeder and Reeder.

Lemma 6.4. *ξ_φ and $\chi_\varphi \circ \mathrm{Nm}$ have the same restriction to $T(\mathcal{O}_L)_\Gamma$.*

Proof. We have the exact sequence

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \to T(K) \to \hat{\mathrm{H}}^0(\Gamma, T(L)) \to 1.$$

Recall that the character ξ_φ is associated to φ by the local Langlands correspondence for tori (see §5). Note that the above exact sequence restricts to an exact sequence

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(\mathcal{O}_L)) \to T(\mathcal{O}_L)_\Gamma \to T(\mathcal{O}_K) \to \hat{\mathrm{H}}^0(\Gamma, T(\mathcal{O}_L)) \to 1.$$

Moreover, by Proposition 4.4, we have $\hat{\mathrm{H}}^{-1}(\Gamma, T(\mathcal{O}_L)) = \hat{\mathrm{H}}^0(\Gamma, T(\mathcal{O}_L)) = 1$. Therefore, the map

$$T(\mathcal{O}_L)_\Gamma \xrightarrow{\mathrm{Nm}} T(\mathcal{O}_K)$$

is an isomorphism, so $\xi_\varphi|_{T(\mathcal{O}_L)_\Gamma}$ factors to a character of $T(\mathcal{O}_K)$ via this isomorphism. But this is exactly how the character $\chi_\varphi|_{T(\mathcal{O}_K)}$ is constructed in [15] and [24]. $\square$

In the case that G is semisimple, we can say even more.

Proposition 6.5. *If G is semisimple, then $\chi_\varphi \circ \mathrm{Nm} = \xi_\varphi \otimes \xi_{\varphi_{\hat{w}}}^{-1}$.*

Proof. Since G is semisimple, $T(K)$ is compact. In particular, $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$ by Proposition 4.7, so we have the following exact sequence:

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \to T(K) \to 1.$$

Note that $T(K) = T(\mathcal{O}_K)$ and thus $T(\mathcal{O}_L)_\Gamma$ surjects onto $T(K)$ via the norm map Nm. Therefore $\hat{\mathrm{H}}^{-1}(\Gamma, T(L))$ and $T(\mathcal{O}_L)_\Gamma$ together generate $T(L)_\Gamma$. It thus suffices to check that $\xi_\varphi \otimes \xi_{\varphi_{\hat{w}}}^{-1} = \chi_\varphi \circ \mathrm{Nm}$ on each of these two subgroups.

Since $\varphi_{\hat{w}}|_{I_K} \equiv 1$, $\xi_{\varphi_{\hat{w}}}$ is trivial on $T(\mathcal{O}_L)_\Gamma$ so Lemma 6.4 implies equality on $T(\mathcal{O}_L)_\Gamma$. Equality on $\hat{\mathrm{H}}^{-1}(\Gamma, T(L))$ is Lemma 5.2. $\square$

The character $\xi_{\varphi_{\hat{w}}}^{-1}$ is the key ingredient in the rectifier that we construct (see the proof of Theorem 8.4). It is canonical in the sense that the Langlands parameter $\varphi_{\hat{w}}$ is a canonical Langlands parameter.

We note that for semisimple G we may replace $\tilde{w}$ by another lift w' of $\hat{w}$ to $\hat{N}$ in the definition of $\varphi_{\hat{w}}$. In fact, if we define φ' by

$$\begin{aligned} \varphi'|_{I_K} &\equiv 1 \\ \varphi'(\mathrm{Fr}) &= w' \end{aligned}$$

then Lemma 5.2 implies $\xi_{\varphi_{\hat{w}}} = \xi_{\varphi'}$. We will justify the Tits group lift $\tilde{w}$ in §9 for $\mathrm{GL}_n(K)$.

7. L-packets fixed under translation by a character

The general definition of rectifier is complicated by the fact that different characters of a torus can yield the same L-packet. Consider the following archetypical example. Let $K = \mathbb{Q}_3$, $G = \mathrm{SL}_2$ and T be an unramified anisotropic torus in G. There are four depth zero characters: two admissible and two inadmissible, notions defined below. Since the two admissible characters are interchanged by the action of the Weyl group, the corresponding L-packets are isomorphic [23, §10]. In this section we investigate nontrivial depth zero characters α of $T(K)$ that leave the association $\chi \mapsto \Pi(\chi)$ of [15] invariant upon translation:

$$\Pi(\chi) = \Pi(\alpha \cdot \chi) \text{ for all depth zero admissible } \chi.$$

Definition 7.1. Let T be a K-minisotropic torus, that splits over an unramified extension L (see [24, §3]). Suppose ξ is a character of $T(L)_\Gamma$.

1) The pair (T, ξ) is called *admissible* if ξ is not fixed by any nontrivial element of $W(K)$ (c.f. Lemma 5.3); we denote by $P_G(K)$ the set of admissible pairs in G.
2) We call two admissible pairs (T, ξ) and (T', ξ') *isomorphic* if there exists a $g \in G(K)$ such that $gT(K)g^{-1} = T'(K)$ and $\xi(t) = \xi'(gtg^{-1})$ for all $t \in T(K)$.

Similarly, we will call a character of $T(K)$ *admissible* if it is not fixed by any nontrivial element of $W(K)$ (c.f. [15, p. 802] and [24, §3])

Note that this definition of admissible pair generalizes Bushnell–Henniart's notion of admissible pair [12] in the case of unramified tori. Indeed, if $G = \mathrm{GL}_n$, and T is an elliptic torus in G splitting over an unramified extension L/K, then one can show that $W(K) = \Gamma$. In this case, the following are equivalent conditions on a character ξ of $T(K) = L^\times$:

1) ξ is fixed by a nontrivial element of $W(K)$,
2) ξ is fixed by a nontrivial subgroup of Γ,
3) ξ factors through the norm map $\mathrm{Nm}_{L/M}$ for some intermediate field $K \subseteq M \subset L$.

Note that for non-adjoint groups it is not sufficient to consider only reflections. For example, the depth zero character of the split torus in $\mathrm{SL}_3(\mathbb{Q}_7)$

inflated from

$$\begin{pmatrix} 3^x & & \\ & 3^y & \\ & & 3^{-x-y} \end{pmatrix} \mapsto \zeta_3^{x+y}$$

is fixed by a 3-cycle in the Weyl group and thus not admissible.

In the next section we will be particularly interested in depth zero characters; write T^* for the set of depth zero characters of $T(\mathcal{O}_K)$, T^*_{adm} for the admissible ones and T^*_{in} for the inadmissible ones. Each of these sets is finite since they may be identified with characters of $T(k)$.

Definition 7.2. Write Q_T for the set of $\alpha \in T^*$ with the following property:

- For every $\chi \in T^*_{\mathrm{adm}}$ there is a $w \in W(K)$ with $\alpha = \frac{\chi}{w(\chi)}$.

The $\mathrm{SL}_2(\mathbb{Q}_3)$ example above has Q_T of order two, but Q_T is trivial for most tori. We spend the rest of this section giving criteria constraining Q_T.

Proposition 7.3. *The set Q_T is a subgroup of T^*, contained within T^*_{in} and stable under the action of $W(K)$.*

Proof. If $\alpha \in T^*_{\mathrm{adm}} \cap Q_T$ then there is some $w \in W(K)$ with $\frac{\alpha}{w(\alpha)} = \alpha$, so $\alpha = 1$ which is not admissible.

We now show that Q_T is a group. Certainly $1 \in Q_T$. Suppose $\alpha, \alpha' \in Q_T$ and $\chi \in T^*_{\mathrm{adm}}$. Then there are $w, w' \in W(K)$ with

$$\frac{\chi}{w(\chi)} = \alpha,$$
$$\frac{w(\chi)}{w'(w(\chi))} = \alpha'.$$

Multiplying the two relations yields $\frac{\chi}{w'w(\chi)} = \alpha\alpha'$, so $\alpha\alpha' \in Q_T$. We finish by noting that Q_T is finite and thus closure under multiplication implies closure under inversion.

Finally, suppose $\tau \in W(K)$. Given $\chi \in T^*_{\mathrm{adm}}$ with $\alpha = \frac{\chi}{w(\chi)}$ we have

$$\tau(\alpha) = \frac{\tau(\chi)}{\tau w(\chi)} = \frac{\tau(\chi)}{w'\tau(\chi)}$$

for some $w' \in W(K)$. Since τ permutes the admissible characters we get that $\tau(\alpha) \in Q_T$. $\square$

The condition on $\alpha \in Q_T$ is an extremely stringent one, and an abundance of admissible characters will preclude a nontrivial α. We can make this statement precise:

Proposition 7.4. *Suppose* $\#T^*_{\mathrm{adm}} > (\#W(K) - 1) \cdot \#T^*_{\mathrm{in}}$. *Then* $Q_T = \{1\}$.

Proof. Suppose $\alpha \in Q_T$. For $w \in W(K)$, set

$$S_w = \left\{ \chi \in T^*_{\mathrm{adm}} \;\middle|\; \frac{\chi}{w(\chi)} = \alpha \right\}.$$

Note that if S_1 is nonempty then we get $\alpha = 1$ immediately, so we may assume the contrary. Then by the pigeonhole principle, there is a $w \in W(K)$ with $\#S_w > \#T^*_{\mathrm{in}}$. Pick $\chi \in S_w$; since $\#S_w > \#T^*_{\mathrm{in}}$ there is some $\chi' \in S_w$ with $\frac{\chi}{\chi'}$ admissible. We now have

$$\frac{\chi}{w(\chi)} = \alpha = \frac{\chi'}{w(\chi')}$$

and therefore $\frac{\chi}{\chi'}$ is fixed by w. Since $\frac{\chi}{\chi'}$ is admissible, we must have $w = 1$ and thus

$$\alpha = \frac{\chi}{\chi} = 1.$$

□

Recall that Frobenius acts on $X^*(T)$ via an endomorphism $F = qF_0$, where F_0 is an automorphism of finite order [14, p. 82]. So it makes sense to vary q: we fix F_0 and consider the tori dual to the $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-modules with Frobenius acting through qF_0.

Corollary 7.5 (cf [14, Lem. 8.4.2]). *Consider the family of tori* T_q *with the same* F_0. *Then for sufficiently large* q, $Q_{T_q} = \{1\}$ *(regardless of the* G *in which* T_q *is embedded).*

Proof. We will write T for a general torus in the family and r for the common dimension. Note that T^* is the set of $\mathbb{F}_q$ points of a dual torus, also of rank r over $\mathbb{F}_q$. For $w \in W(K)$ with $w \neq 1$ the centralizer $\mathrm{Z}_{T^*}(w)$ is a proper F-stable subgroup of T^*, and thus $\dim(\mathrm{Z}_{T^*}(w)) \leq r - 1$. By [14, 3.3.5], $\#T^*$ is a polynomial in q of degree r and $\#\,\mathrm{Z}_{T^*}(w)$ is a polynomial in q of degree at most $r-1$. Thus the ratio

$$\frac{\#T^*_{\mathrm{adm}}}{\#T^*_{\mathrm{in}}} = \frac{\#T^* - \sum_{1\neq w\in W} \# \mathrm{Z}_{T^*}(w)}{\sum_{1\neq w\in W} \# \mathrm{Z}_{T^*}(w)}$$

grows without bound as q does. There are finitely many possibilities for the absolute Weyl group of T, so Proposition 7.4 gives the desired result. □

In computing Q_T for small q the following result is useful:

Proposition 7.6. *If $\alpha \in Q_T$ has order d and $\chi \in T^*_{\mathrm{adm}}$ has order m then d divides m.*

Proof. There is a $w \in W(K)$ with

$$\frac{\chi}{w(\chi)} = \alpha.$$

Since $w(\chi)$ also has order m, raising both sides to the mth power yields $\alpha^m = 1$. □

We would like to make a remark about associations $\chi \mapsto \Pi(\chi)$ of *positive depth* supercuspidal L-packets to characters of minisotropic tori. In particular, we would like to remark that there cannot generally be such an association that could be left invariant by a depth zero character α. To see this, suppose that

$$\Pi(\chi) = \Pi(\alpha\cdot\chi) \text{ for all admissible } \chi \text{ of positive depth } r.$$

Then for each such χ, there exists $w(\chi)$ such that

$${}^{w(\chi)}(\alpha\cdot\chi) = \chi.$$

Restricting this equation to $T(\mathcal{P}_K)$ yields $({}^{w(\chi)}\chi)|_{T(\mathcal{P}_K)} \equiv \chi|_{T(\mathcal{P}_K)}$, since α is depth zero. As long as one can find a χ of depth r that is *minimal* (see Definition 8.3), we get a contradiction.

Finally, we note that Lemma 8 of Bushnell–Henniart [12, p. 511] is equivalent to the statement that Q_T is trivial when T is a K-minisotropic torus in GL_n.

8. Rectifiers for general reductive groups

Suppose that G is a connected reductive group defined over a p-adic field K. Fix an unramified K-torus $T \subset G$ with splitting field L. Let $\varphi : \mathcal{W}_K \to {}^L G$

be a Langlands parameter for $G(K)$, and suppose that φ factors through a group of type L for T. Any Langlands parameter with image in the normalizer of a maximal torus will factor in this way for some T.

As in §5, one can canonically associate to φ a character ξ_φ of $T(L)_\Gamma$. Recall again the Tate cohomology sequence

$$1 \to \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) \to T(L)_\Gamma \xrightarrow{\mathrm{Nm}} T(K) = T(L)^\Gamma \to \hat{\mathrm{H}}^0(\Gamma, T(L)) \to 1.$$

Suppose that $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$, in which case $T(L)_\Gamma$ surjects onto $T(K)$. Let us also suppose that φ does not factor through a proper Levi subgroup, so that the representations in the L-packet associated to φ are conjecturally all supercuspidal (see [15, §3.5]). When $G = \mathrm{GL}_n$ we show in §9 that $\hat{\mathrm{H}}^0(\Gamma, T(L)) = \hat{\mathrm{H}}^{-1}(\Gamma, T(L)) = 0$ and thus $T(L)_\Gamma \cong T(K) \cong L^\times$. In this case $(L/K, \xi_\varphi)$ is an admissible pair; to construct the local Langlands correspondence one proceeds as in §3 by attaching the supercuspidal representation $\pi_{\xi_\varphi \cdot \mu_{\xi_\varphi}}$ to ξ_φ, via the construction of Bushnell and Henniart.

For other groups G there are some constructions of supercuspidal L-packets $\Pi(\chi)$ from characters χ of $T(K)$ [15, 19, 24]. These constructions depend on a particular construction of supercuspidal representations. In the case of [19, 24], the choice of construction is Adler's [3]. As we have seen, a Langlands parameter φ does not naturally provide a character of $T(K)$, but rather a character of $T(L)_\Gamma$. We consider here the maps $\chi \mapsto \Pi(\chi)$ of [15, 24].

Moreover, for general groups G there are some constructions of correspondences from Langlands parameters to L-packets of G [15, 19, 24]. More specifically, in the mentioned work, a map

$$\mathcal{L} : \Phi_{\mathcal{L}}(G, T) \to \Pi(G) \tag{8.1}$$

is constructed, where $\Phi_{\mathcal{L}}(G, T)$ is some subset of Langlands parameters that factor through a group of type L, and $\Pi(G)$ is the set of L-packets of G. We denote by $\mathcal{L}_{DR}$ the map coming from either [15] or [24].

Let $\mathcal{L}$ be any map of the form (8.1), and set $P_{G,\mathcal{L}}(K) = \{(T, \xi_\varphi) \in P_G(K) | \varphi \in \Phi_{\mathcal{L}}(G, T)\}$.

Definition 8.1. Let T be a coverable torus in G, with unramified splitting field L. An *$\mathcal{L}$-rectifier* for T is a function

$$\boldsymbol{\mu} : (T, \xi) \mapsto \mu_\xi$$

which attaches to each $(T,\xi) \in P_{G,\mathcal{L}}(K)$ a character μ_ξ of $T(L)_\Gamma$ satisfying the following conditions:

1) The character μ_ξ is tamely ramified (i.e. trivial on $T(\mathcal{P}_L)_\Gamma$),
2) The character $\xi \cdot \mu_\xi$ descends to $T(K)$, is admissible, and $\varphi \mapsto \Pi(\xi_\varphi \cdot \mu_{\xi_\varphi})$ agrees with the correspondence $\mathcal{L}$,
3) If (T,ξ_1) and (T,ξ_2) are admissible pairs such that $\xi_1^{-1}\xi_2$ is tamely ramified then $\mu_{\xi_1} = \mu_{\xi_2}$.

We say that two rectifiers $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ for T are *equivalent* if there is some $\alpha \in Q_T$ so that

$$\begin{aligned} \mu'_\xi &= \alpha\mu_\xi && \text{for depth zero } \xi, \\ \mu'_\xi &= \mu_\xi && \text{for positive depth } \xi. \end{aligned}$$

This notion of equivalence is made in order to prove uniqueness of a $\mathcal{L}_{DR}$-rectifier (see Theorem 8.4). More specifically, this notion is needed because of the very rare phenomenon that we described in §7 whereby all supercuspidal L-packets could be left invariant by a depth zero character α.

Since we have assumed $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$, the condition (in Definition 8.1) that $\xi \cdot \mu_\xi$ descends to $T(K)$ is equivalent to $\xi \cdot \mu_\xi$ vanishing on $\hat{\mathrm{H}}^{-1}(\Gamma, T(L))$.

Conjecture 8.2. *For T as in Definition 8.1 and $\mathcal{L}$ any map of the form* (8.1), *T admits a unique $\mathcal{L}$-rectifier up to equivalence.*

We note that, as the local Langlands correspondence is not known in general, we must restrict ourselves to cases where supercuspidal L-packets have been constructed. Since we are in the present paper considering the situation when T is unramified, we consider those L-packets constructed in [15] and [24]. We also note that an $\mathcal{L}$-rectifier implicitly depends on a construction of supercuspidal representations, as it does in the work of Bushnell and Henniart.

Definition 8.3. Suppose $(T,\xi) \in P_G(K)$.

1) The *depth* of (T,ξ) is the integer r so that ξ is trivial on $T(\mathcal{P}_L^{r+1})_\Gamma$ but nontrivial on $T(\mathcal{P}_L^r)_\Gamma$
2) An admissible pair of depth r is *minimal* if $\xi|_{T(\mathcal{P}_L^r)_\Gamma}$ is not fixed by any element of $W(K)$.

We note that this definition of minimal admissible pair generalizes the definition of minimal admissible pair of Bushnell and Henniart in the case of unramified tori (see [10, §2.2]). We also note that the character of a minisotropic torus that is produced from a Langlands parameter considered in [24] can be seen to automatically be minimal.

Theorem 8.4. *For G unramified and semisimple, and T as in Definition 8.1, T admits a unique $\mathcal{L}_{DR}$-rectifier up to equivalence.*

Proof. We first prove existence. We defined in §6 a Langlands parameter $\varphi_{\hat{w}} : \mathcal{W}_K \to {}^L G$ by sending Frobenius to the canonical lift $\tilde{w} \in \widetilde{W}$ of $\hat{w} \in \hat{W}$, and by setting $\varphi_{\hat{w}}$ to be trivial on I_K. For semisimple G we proved in Proposition 6.5 that the function

$$(T, \xi) \mapsto \xi_{\varphi_{\hat{w}}}^{-1}$$

satisfies condition (2) of Definition 8.1. Moreover, the function also satisfies condition (1): $\varphi_{\hat{w}}|_{I_K} \equiv 1$ and thus $\xi_{\varphi_{\hat{w}}}^{-1}$ is unramified. Finally, $\xi_{\varphi_{\hat{w}}}$ is independent of ξ and thus condition (3) is automatically satisfied. We may therefore set $\boldsymbol{\mu}(T, \xi) = \xi_{\varphi_{\hat{w}}}^{-1}$, proving existence.

We now prove uniqueness. Let ξ range over the set of characters of $T(L)_\Gamma$ such that $(T, \xi) \in P_{G,\mathcal{L}_{DR}}(K)$, and let $\boldsymbol{\mu}'$ and $\boldsymbol{\mu}''$ be rectifiers for $T \subset G$. By hypothesis, we have

$$\Pi(\mu'_\xi \cdot \xi) = \Pi(\mu''_\xi \cdot \xi).$$

By [23, Prop. 10.8], there exists $w_\xi \in W(K)$, depending on ξ, such that

$$^{w_\xi}(\mu'_\xi \cdot \xi) = \mu''_\xi \cdot \xi.$$

Suppose that ξ has positive depth. Restricting the equation $^{w_\xi}(\mu'_\xi \cdot \xi) = \mu''_\xi \cdot \xi$ to $T(\mathcal{P}_L)_\Gamma$, we get that $^{w_\xi}(\xi) = \xi$, by condition (1) of Definition 8.1. Since ξ is minimal, we get that $w_\xi = 1$, which implies that $\mu'_\xi = \mu''_\xi$.

Now suppose that ξ has depth zero, and suppose without loss of generality that $\boldsymbol{\mu}' = \boldsymbol{\mu}$, so that we will now prove that $\boldsymbol{\mu}''$ is equivalent to $\boldsymbol{\mu}$. Define λ on $T(\mathcal{O}_L)_\Gamma \cong T(\mathcal{O}_K)$ by $\lambda = ((^{w_\xi}(\mu_\xi))^{-1} \cdot \mu''_\xi)|_{T(\mathcal{O}_L)_\Gamma}$. We claim that λ is independent of ξ. To see this, let ξ_1, ξ_2 be depth zero characters. We wish to show that

$$((^{w_{\xi_1}}(\mu_{\xi_1}))^{-1} \cdot \mu''_{\xi_1})|_{T(\mathcal{O}_L)_\Gamma} = ((^{w_{\xi_2}}(\mu_{\xi_2}))^{-1} \cdot \mu''_{\xi_2})|_{T(\mathcal{O}_L)_\Gamma}. \tag{8.2}$$

But since ξ_1, ξ_2 are depth zero, we have that $\xi_1\xi_2^{-1}$ is also of depth zero, and so condition (3) implies that $\mu''_{\xi_1} = \mu''_{\xi_2}$. Therefore, (8.2) is equivalent to

$$({}^{w_{\xi_1}}(\mu_{\xi_1}))|_{T(\mathcal{O}_L)_\Gamma} = ({}^{w_{\xi_2}}(\mu_{\xi_2}))|_{T(\mathcal{O}_L)_\Gamma}. \tag{8.3}$$

But by definition of $\boldsymbol{\mu}$, we have that $\mu_{\xi_1}|_{T(\mathcal{O}_L)_\Gamma} \equiv \mu_{\xi_2}|_{T(\mathcal{O}_L)_\Gamma} \equiv 1$. Finally, since $W(K)$ preserves $T(\mathcal{O}_L)_\Gamma$, (8.3) is proven since both sides of the equality are the trivial character.

Since λ is seen to be independent of ξ, the equation ${}^{w_\xi}(\mu_\xi \cdot \xi) = \mu''_\xi \cdot \xi$ implies that $\lambda \in Q_T$. Since $\mu_\xi \cdot \xi$ and $\mu''_\xi \cdot \xi$ descend to $T(K)$ by condition (2) of Definition 8.1, μ_ξ and μ''_ξ have the same restriction to $\hat{H}^{-1}(\Gamma, T(L))$. Since G is semisimple we may pull λ back to a character on $T(L)_\Gamma$, vanishing on $\hat{\mathrm{H}}^{-1}(\Gamma, T(L))$. Note also that

$$\mu''_\xi|_{T(\mathcal{O}_L)_\Gamma} \equiv (\lambda\mu_\xi)|_{T(\mathcal{O}_L)_\Gamma}.$$

by definition of λ, by the fact that $\mu_\xi|_{T(\mathcal{O}_L)_\Gamma} \equiv 1$, and since $W(K)$ preserves $T(\mathcal{O}_L)_\Gamma$. Therefore, we finally obtain that $\mu''_\xi = \lambda\mu_\xi$ on all of $T(L)_\Gamma$ and thus $\boldsymbol{\mu}$ is equivalent to $\boldsymbol{\mu}''$. □

Remark 8.5.

1) The condition $\hat{\mathrm{H}}^0(\Gamma, T(L)) = 0$ was necessary in order to obtain a character on $T(K)$ rather than the image of the norm map $T(L) \mapsto T(K)$. For non-semisimple groups where $\hat{\mathrm{H}}^0(\Gamma, T(L))$ is nontrivial we hope that the recipe for the central character in [16] will provide an extension to all of $T(K)$.

2) The rectifier in our setting is constant as a function of ξ. We expect a dependence on ξ for ramified tori.

3) The behavior of rectifiers under change of group is not yet clear to us. There may be a natural relationship between rectifiers when a torus is embedded into two different reductive groups with isomorphic Weyl groups. Similarly, when given an embedding $H \subset G$, a natural relationship between the rectifiers for tori in H and G would allow us to apply the results of [12] to rectifiers for general groups.

9. Compatibility with Bushnell–Henniart

In this section we show that our function $\boldsymbol{\mu}^{\min}$ agrees with the rectifier of Bushnell–Henniart in the depth zero setting: see Theorem 9.5. Let $L = K_n$

and set $T = \mathrm{Res}_{L/K}(\mathbb{G}_m)$. We begin by computing the Tate cohomology groups of T.

Proposition 9.1. $\hat{\mathrm{H}}^0(\Gamma, X_*(T)) = 0$.

Proof. Since Γ acts on $X_*(T)$ by permuting basis vectors, $X_*(T)^\Gamma$ is the copy of $\mathbb{Z}$ embedded diagonally in $X_*(T) = \mathbb{Z}^n$. Note that

$$\mathrm{Nm}(1, 0, 0, \ldots, 0) = (1, 1, \ldots, 1),$$

so $X_*(T)^\Gamma \subset \mathrm{Nm}(X_*(T))$. □

Proposition 9.2. $\hat{\mathrm{H}}^{-1}(\Gamma, X_*(T)) = 0$.

Proof. We note that $(a_1, a_2, \ldots, a_n) \in \ker(\mathrm{Nm})$ if and only if $\sum_{i=1}^n a_i = 0$. It is then easy to see that $\ker(\mathrm{Nm})$ is generated by $e_i - e_j$ for $i < j$, where e_i are the standard basis of $\mathbb{Z}^n$. But $e_i - e_j = (1 - \tau)e_i$ for some $\tau \in \Gamma$, since Γ acts by cyclic shift. Thus $\ker(\mathrm{Nm}) \subset I_\Gamma(X_*(T))$. □

The Tate cohomology exact sequence for T therefore reduces to

$$1 \to T(L)_\Gamma \xrightarrow{\sim} T(K) \to 1$$

by Corollary 4.5. We now need a basic result about powers of lifts of Coxeter elements in $\mathrm{GL}_n(\mathbb{C})$.

Proposition 9.3. *Let $\hat{w}$ be a Coxeter element of $\mathrm{GL}_n(\mathbb{C})$, and let $\tilde{w}$ be the canonical lift of $\hat{w}$ to $\widetilde{W}$. Then $\tilde{w}^n = (-1)^{n-1}$ as as scalar matrix in $\mathrm{GL}_n(\mathbb{C})$.*

Proof. See [31, §3.1]. □

We can now describe the image of $\boldsymbol{\mu}^{\min}$ in the setting of depth zero supercuspidal representations of $\mathrm{GL}_n(K)$. Write φ for $\varphi_{\hat{w}}$ (see Definition 6.3) and μ for ξ_φ^{-1} .

Proposition 9.4. *μ is unramified and $\mu(\varpi) = (-1)^{n-1}$.*

Proof. Let σ generate $\mathrm{Gal}(L/K)$. Then $T(L) \cong L^\times \times L^\times \times \cdots \times L^\times$ and

$$T(K) = \{(x, \sigma(x), \sigma^2(x), \ldots, \sigma^{n-1}(x)) : x \in L^\times\} \cong L^\times.$$

A uniformizer ϖ in $K^\times \subset L^\times$ therefore corresponds to $(\varpi, \varpi, \ldots, \varpi) \in T(K)$, whose preimage under Nm is the class of $(\varpi, 1, 1, \ldots, 1)$ in $T(L)_\Gamma$. By [27,

§2.4], ϖ corresponds to Fr^n under the Artin reciprocity map for L. Now by Proposition 9.3 and the local Langlands correspondence for tori we get $\mu(\varpi) = (-1)^{n-1}$. Finally, $\varphi|_{I_K} \equiv 1$ implies that μ is unramified. $\square$

Theorem 9.5. *If $G = \mathrm{GL}_n(K)$ and fixed T, the constant function $(T, \xi) \mapsto \mu$ agrees with the rectifier of Bushnell–Henniart for depth zero ξ.*

Proof. This result follows from Proposition 9.4 and Proposition 3.4. $\square$

We end this section by explaining why the Tits group lift $\tilde{w}$ is forced upon us. Suppose we define $\varphi' : \mathcal{W}_K \to \mathrm{GL}_n(\mathbb{C})$ by $\varphi'|_{I_K} \equiv 1$ and $\varphi'(\mathrm{Fr})$ to be a lift of an elliptic element $\hat{w}$ in $\hat{W}$. Then [15, p. 824] and [24, §6] imply that the characteristic polynomial of $\varphi'(\mathrm{Fr})$ is $X^n - a$, for some $a \in \mathbb{C}^\times$. One can see that, by arguments analogous to those in Proposition 9.4, $\xi_{\varphi'}(\varpi) = a$. By Proposition 3.4, we are forced to set $a = (-1)^{n-1}$. Finally, one can show by an inductive argument that the canonical lift $\tilde{w}$ of $\hat{w}$ to $\widetilde{W}$ has characteristic polynomial $X^n - (-1)^{n-1}$, so that $\varphi'(\mathrm{Fr})$ is indeed the canonical lift of $\hat{w}$ to $\widetilde{W}$ up to conjugacy.

Acknowledgements

This paper has benefited from conversations with Jeffrey Adams, Jeffrey Adler, Colin Bushnell, Andrew Fiori, Guy Henniart, Gordan Savin, and Geo Kam-Fai Tam. We thank them all.

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Received November 9, 2014