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Eigenvarieties for classical groups and complex conjugations in Galois representations

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The goal of this paper is to remove the irreducibility hypothesis in a theorem of Richard Taylor describing the image of complex conjugations by p-adic Galois representations associated with regular, algebraic, essentially self-dual, cuspidal automorphic representations of GL_{2n+1} over a totally real number field F. We also extend it to the case of representations of GL_{2n}/F whose multiplicative character is "odd". We use a p-adic deformation argument, more precisely we prove that on the eigenvarieties for symplectic and even orthogonal groups, there are "many" points corresponding to (quasi-)irreducible Galois representations. Recent work of James Arthur describing the automorphic spectrum for these groups is used to define these Galois representations, and also to transfer self-dual automorphic representations of the general linear group to these classical groups.

1. Introduction

Let p be a prime. Let us choose once and for all algebraic closures $\overline{\mathbb{Q}}, \overline{\mathbb{Q}_p}, \mathbb{C}$ and embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let F be a totally real number field. A regular, L-algebraic, essentially self-dual, cuspidal (RLAESDC) representation of $\mathbf{GL}_n(\mathbb{A}_F)$ is a cuspidal automorphic representation π together with an algebraic character $\chi = \eta \,||\cdot||^q$ of $\mathbb{A}_F^\times / F^\times$ (η being an Artin character, and q an integer) such that

- $\pi^\vee \simeq (\chi \circ \det) \otimes \pi$,
- For any real place v of F, $\mathcal{LL}(\pi_v)|_{\mathrm{W}_\mathbb{C}} \simeq \bigoplus_i \left(z \mapsto z^{a_{v,i}} \bar{z}^{b_{v,i}}\right)$ where $\mathcal{LL}$ denotes the local Langlands correspondence, $\mathrm{W}_\mathbb{C} \simeq \mathbb{C}^\times$ is the Weil group of $\mathbb{C}$, and $a_{v,i}$, $b_{v,i}$ are integers satisfying $a_{v,i} \neq a_{v,j}$ for $i \neq j$.

By definition, π is regular, L-algebraic (in the sense of [10]), essentially self-dual, cuspidal (RLAESDC) if and only if $\pi \otimes ||\det||^{(n-1)/2}$ is regular, *algebraic* (in the sense of Clozel), essentially self-dual, cuspidal (RAESDC). The latter is the notion of "algebraic" usually found in the literature, and is called "C-algebraic" in [10]. For example, any cuspidal eigenform of weight $k \geq 2$, level $N \geq 1$ and Nebentypus $\alpha : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ gives rise to an RLAESDC representation π of $\mathbf{GL}_2(\mathbb{A}_\mathbb{Q})$ which is essentially self-dual with respect to $\alpha^{-1}||\cdot||^{k-1}$ and such that $\mathcal{LL}(\pi_\infty)|_{\mathrm{W}_\mathbb{C}} \simeq \left(z \mapsto \mathrm{diag}(z^{1-k}, \bar{z}^{1-k})\right)$, where ∞ denotes the real place of $\mathbb{Q}$. Up to twisting by a character, any regular L-algebraic cuspidal automorphic representation of $\mathbf{GL}_2(\mathbb{A}_\mathbb{Q})$ arises in this way.

Let Gal_F denote the absolute Galois group of F. Given a RLAESDC representation π of $\mathbf{GL}_n(\mathbb{A}_F)$, there is (Theorem 3.1.2) a unique continuous, semisimple Galois representation $\rho_{\iota_p,\iota_\infty}(\pi) : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ such that $\rho_{\iota_p,\iota_\infty}(\pi)$ is unramified at any finite place v of F not lying above p and for which π_v is unramified, and $\iota_\infty \iota_p^{-1} \mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(\mathrm{Frob}_v)\right)$ is equal to the trace of the Satake parameter of π_v (implicit in this assertion is the fact that this trace is algebraic over $\mathbb{Q}$). It is natural to ask if for other places v of F, the restriction of $\rho_{\iota_p,\iota_\infty}(\pi)$ to a decomposition group at v is also determined by π_v via the local Langlands correspondence. This problem is usually called "local-global compatibility". At any finite place v of F not dividing p, local-global compatibility is known (see [20], [12]), up to semisimplification in some cases. At p-adic places the Galois representation $\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}}$ is known to be de Rham and its associated Weil-Deligne representation coincides with the local Langlands parameter of π_v, up to semisimplification in some cases ([2], [13]). Moreover the Hodge-Tate weights of the representations $(\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}})_{v|p}$ are the integers $(a_{i,v})_{v|\infty}$ defined above (see Theorem 3.1.2 for a more precise statement). Thus the only remaining case of local-global compatibility is that of a real place v of F, where the only non-trivial element of the decomposition group Gal_{F_v} is a complex conjugation c_v. It is conjectured that the conjugacy class of $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ is determined by $\mathcal{LL}(\pi_v)$ (see [10][Lemma 2.3.2] for the case of an arbitrary reductive group). In the present case, by Clozel's purity lemma and by regularity, $\mathcal{LL}(\pi_v)$ is determined by its restriction to $\mathrm{W}_{\mathbb{C}}$, and since $\det\left(\rho_{\iota_p,\iota_\infty}(\pi)\right)$ is known, the determination of $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ amounts to the following

Conjecture. *Under the above hypotheses,* $\left|\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(c_v)\right)\right| \leq 1.$

There are several cases for which this is known. For example, if π is a RLAESDC automorphic representation of $\mathbf{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to a cuspidal eigenform of weight $k \geq 2$, level $N \geq 1$ and Nebentypus $\alpha : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, then necessarily $\alpha(-1)(-1)^k = 1$ and so $\det \rho_{\iota_p,\iota_\infty}(\pi)(c_\infty) = -1$ ("$\rho_{\iota_p,\iota_\infty}(\pi)$ is odd"), which implies the conjecture because the dimension is 2. More generally, according to [34], for v an infinite place of F the value of $\eta_v(-1) \in \{\pm 1\}$ does not depend on v, and we denote the common value by $\eta_\infty(-1)$. When $\eta_\infty(-1)(-1)^q = -1$ (this happens only if n is even, and by [4] this means that $\rho_{\iota_p,\iota_\infty}(\pi)$ together with the character $\rho_{\iota_p,\iota_\infty}(\eta\,||\cdot||^q) = (\eta \circ \mathrm{rec})\mathrm{cyclo}^q$, is "symplectic"), $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ is conjugate to $-\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$, so the trace is obviously zero. In all other cases, essential self-duality of $\rho_{\iota_p,\iota_\infty}(\pi)$ does not yield information at the real places and the conjecture is non-trivial.

Solving this conjecture is important to formulate, and probably also to prove, generalisations of Serre's modularity conjecture [38] stating that any *odd* irreducible continuous Galois representation $\mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ is modular, i.e. comes from a cuspidal eigenform.

In [42], Richard Taylor proves the following.

Theorem (Taylor). *Let F be a totally real number field, $n \geq 1$ an integer. Let π be a regular, L-algebraic, essentially self-dual, cuspidal automorphic representation of $\mathbf{GL}_{2n+1}/F$. Assume that the attached Galois representation $\rho_{\iota_p,\iota_\infty}(\pi) : \mathrm{Gal}_F \to \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}_p})$ is irreducible. Then for any real place v of F,*

$$\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\pi)(c_v)\right) = \pm 1.$$

Although one expects $\rho_{\iota_p,\iota_\infty}(\pi)$ to be always irreducible, this is not known in general. Nevertheless it is known when $n \leq 2$ by [11], and for arbitrary n but only for p in a set of positive Dirichlet density by [35].

In this paper, the following cases are proved:

Theorem A (Theorem 5.3.4). Let $n \geq 2$, F a totally real number field, π a regular, L-algebraic, essentially self-dual, cuspidal representation of $\mathbf{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq (\chi \circ \det) \otimes \pi$, where $\chi = \eta || \cdot ||^q$ for an Artin character η and an integer q. Suppose that one of the following conditions holds

1) n is odd.

2) n is even, q is even, and $\eta_\infty(-1) = 1$.

Then for any complex conjugation $c \in \mathrm{Gal}_F$, $|\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c))| \leq 1$.

This is achieved thanks to the result of Taylor, Arthur's endoscopic transfer between twisted general linear groups and symplectic or orthogonal groups, and using eigenvarieties for these groups. Let us describe the natural strategy that one might consider to prove the odd-dimensional case using these tools, to explain why it fails and how a detour through the even-dimensional case allows us to conclude.

Let π be a RLAESDC representation of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$. Up to a twist by an algebraic character π is self-dual and has trivial central character. Conjecturally, there should be an associated self-dual Langlands parameter $\phi_\pi : \mathrm{L}_F \to \mathrm{GL}_{2n+1}(\mathbb{C})$ where L_F is the conjectural Langlands group. Up to conjugation, ϕ_π takes values in $\mathrm{SO}_{2n+1}(\mathbb{C})$, and by functoriality there should be a discrete automorphic representation Π of $\mathbf{Sp}_{2n}(\mathbb{A}_F)$ such that $\mathcal{LL}(\Pi_v)$ is equal to $\mathcal{LL}(\pi_v)$ via the inclusion $\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$ for any place of F which is either archimedean or such that π_v is unramified. Arthur's results [1] imply that this (in fact, much more) holds. To construct p-adic families of automorphic representations, that is eigenvarieties, containing Π, it is preferable to work with a group which is *compact* at the real places of F, and work with representations having Iwahori-invariants at the p-adic places. A suitable solvable base change allows us to assume that $[F : \mathbb{Q}]$ is even and that π_v has Iwahori-invariants for $v|p$. Using [40] we can "transfer" π to an automorphic representation Π of $\mathbf{G}$, the inner form of the split reductive group $\mathbf{Sp}_{2n}/F$ which is split at the finite places and compact at the real places of F. By [32], generalizing [18], there is an eigenvariety $\mathscr{X}$ for $\mathbf{G}$. Using [1] and [40] again, one can associate p-adic Galois representations $\rho_{\iota_p,\iota_\infty}(\Pi)$ to automorphic representations Π of $\mathbf{G}$, yielding a family of Galois representations on $\mathscr{X}$, that is to say a continuous map $T : \mathrm{Gal}_F \to \mathcal{O}(\mathscr{X})$ which specializes to $\mathrm{Tr}\left(\rho_{\iota_p,\iota_\infty}(\cdot)\right)$ at the points of $\mathscr{X}$ corresponding to automorphic representations of $\mathbf{G}(\mathbb{A}_F)$. One can then hope to prove a result similar to [4, Lemma 3.3], i.e. show that one can "p-adically deform" Π to reach a point on $\mathscr{X}$ corresponding to an automorphic representation Π' whose Galois representation is irreducible (even when restricted to the decomposition group of a p-adic place of F). Since $\rho_{\iota_p,\iota_\infty}(\Pi')$ comes from an automorphic representation π' of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$, π' is necessarily cuspidal and satisfies the hypotheses of Taylor's theorem. Since $T(c_v)$ is locally constant on $\mathscr{X}$, we would be done.

Unfortunately, it does not appear to be possible to reach a representation Π' whose Galois representation is irreducible by using local arguments on the eigenvariety. However we will prove the following, which includes the case of some even-dimensional special orthogonal groups as it will be needed later:

Theorem B (Theorem 3.2.2, Theorem 4.0.1). Let $\mathbf{G}$ be an inner form of $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ over a totally real number field, compact at the real places and split at the p-adic ones. Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let $\rho_{\iota_p,\iota_\infty}(\Pi)$ denote the p-adic representation of the absolute Galois group Gal_F of F associated with Π and embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let N be an integer. There exists an automorphic representation Π' of $\mathbf{G}(\mathbb{A}_F)$ such that:

- Π' is unramified at the places above p, and has invariants under U;
- The restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to the decomposition group at any place above p is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$ (the latter occurring only in the symplectic case);
- For all g in Gal_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.

The possible presence of an Artin character (in the case of inner forms of $\mathbf{Sp}_{2n}$) comes from the fact that the standard representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$ is not minuscule: the set of characters of a torus $T(\mathbb{C})$ of $\mathrm{SO}_{2n+1}(\mathbb{C})$ in this representation has two orbits under the Weyl group, one of which contains only the trivial character. The key fact allowing us to prove the above theorem is that classical points on the eigenvariety for $\mathbf{G}$ correspond to automorphic representations Π of $\mathbf{G}(\mathbb{A}_F)$ (say, unramified at the p-adic places) *and* a refinement of each Π_v, $v|p$, that is to say a particular element in $T(\mathbb{C})$ in the conjugacy class of the Satake parameter of Π_v. The variation of the crystalline Frobenius of $\rho_{\iota_p,\iota_\infty}(\cdot)$ on the eigenvariety with respect to the weight and the freedom to change the refinement (by the action of the Weyl group) are at the heart of the proof of Theorem B. The proof is more delicate than that of [4, Lemma 3.3], essentially because the dimension of the Galois representations is greater than the dimension of the eigenvariety $\mathscr{X}$. In fact we will be able to strengthen Theorem B, to show that for any p-adic place v the Lie algebra of $\rho_{\iota_p,\iota_\infty}(\Pi')(\mathrm{Gal}_{F_v})$ can be assumed to be as large as one can expect: Corollaries 3.2.3 and 4.0.2. This could be useful in future applications.

Although the strategy outlined above fails because of the possible presence of an Artin character in Theorem B, Theorem A can still be deduced from Theorem B. Indeed, [1] and [40] imply that certain formal sums of distinct cuspidal self-dual representations of general linear groups "contribute" to the automorphic spectrum of inner forms of $\mathbf{Sp}_{2n}$ or $\mathbf{SO}_{4n}$ as above. The even-dimensional case in Theorem A will be proved by transferring $\pi \boxplus \pi_0$, where π, π_0 are regular, L-algebraic, self-dual, cuspidal representations of $\mathbf{GL}_{2n}(\mathbb{A}_F)$ (resp. $\mathbf{GL}_3(\mathbb{A}_F)$) with distinct weights at any real place of F, to an automorphic representation Π of an inner form $\mathbf{G}$ of $\mathbf{Sp}_{2n+2}/F$. Since $\rho_{\iota_p,\iota_\infty}(\pi) \oplus \rho_{\iota_p,\iota_\infty}(\pi_0)$ does not contain any Artin character (the zero Hodge-Tate weights come from $\rho_{\iota_p,\iota_\infty}(\pi_0)$, which is known to be irreducible), for big enough N any representation Π' as in B has an irreducible Galois representation.

To treat the original case of a regular, L-algebraic, self-dual, cuspidal representation of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$ having trivial central character, we appeal to Theorem B for special orthogonal groups. For example, if n is odd, $\pi \boxplus \pi_0$, where π_0 is the trivial character of $\mathbb{A}_F^\times/F^\times$, contributes to the automorphic spectrum of $\mathbf{G}$, which is now the special orthogonal group of a quadratic form on F^{2n+2} which is definite at the real places and split at the finite places of F. Note that $\pi \boxplus \pi_0$ is not regular: the zero weight appears twice at each real place of F. However the Langlands parameters of representations of the compact group $\mathbf{SO}_{2n+2}(\mathbb{R})$ are of the form

$$\bigoplus_{i=1}^{n+1} \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} \left(z \mapsto (z/\bar{z})^{k_i} \right)$$

when composed with $\mathrm{SO}_{2n+2}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$, with $k_1 > \cdots > k_{n+1} \geq 0$; and $\mathcal{LL}\left((\pi \boxplus \pi_0)_v\right)$ is of this form, with $k_{n+1} = 0$. The rest of the proof is identical to the even-dimensional case.

After the first version of this paper was written, Harris-Lan-Taylor-Thorne [27] and Scholze [36] have attached Galois representations to (not necessarily essentially self-dual) L-algebraic regular automorphic cuspidal representations of general linear groups over totally real number fields. Ana Caraiani and Bao Viet Le Hung [14], following Scholze's construction and using Theorem A, have proved that the above Conjecture also holds for these Galois representations.

We now fix some notations for the rest of the article. The valuation v_p of $\overline{\mathbb{Q}_p}$ is the one sending p to 1, and $|\cdot|$ will denote the norm $p^{-v_p(\cdot)}$. All the number fields in the paper will sit inside $\overline{\mathbb{Q}}$. We have chosen arbitrary embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. In fact, the constructions will only depend on the identification between the algebraic closures of $\mathbb{Q}$ in $\overline{\mathbb{Q}_p}$ and $\mathbb{C}$ (informally, $\iota_p \iota_\infty^{-1}$). Observe that the choice of a p-adic place v of a number field F and of an embedding $F_v \hookrightarrow \overline{\mathbb{Q}_p}$ is equivalent, via ι_p, to the choice of an embedding $F \hookrightarrow \overline{\mathbb{Q}}$. The same holds for the infinite places and ι_∞. Thus if F is totally real, $\iota_p \iota_\infty^{-1}$ defines a bijection between the set of infinite places of F and the set of p-adic places v of F together with an embedding $F_v \hookrightarrow \overline{\mathbb{Q}_p}$. The eigenvarieties will be rigid analytic spaces (in the sense of Tate). If $\mathscr{X}$ is a rigid analytic space over a finite extension E of $\mathbb{Q}_p$, $|\mathscr{X}|$ will denote its points.

2. The eigenvariety for definite symplectic groups

In this section we recall the main result of [32] in our particular case (existence of the eigenvariety for symplectic groups), and show that the points corresponding to unramified, "completely refinable" automorphic forms, with weight far from the walls, are "dense" in this eigenvariety.

2.1. The eigenvariety

2.1.1. Inner forms of symplectic groups compact at the archimedean places. Let F be a totally real number field of even degree over $\mathbb{Q}$, and let D be a quaternion algebra over F, unramified at all the finite places of F ($F_v \otimes_F D \simeq \mathrm{M}_2(F_v)$), and definite at all the real places of F. Such a D exists thanks to the exact sequence relating the Brauer groups of F and the F_v. Let n be a positive integer, and let $\mathbf{G}$ be the algebraic group over F defined by the equation $M^* M = \mathrm{I}_n$ for $M \in \mathrm{M}_n(D)$, where $(M^*)_{i,j} = M^*_{j,i}$, and $\cdot^*$ denotes conjugation in D.

Then $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ is a compact Lie group, and for all finite places v of F, $\mathbf{G} \times_F F_v \simeq \mathbf{Sp}_{2n}/F_v$. The connected reductive group $\mathbf{G}$ is an inner form of the split group $\mathbf{G}^* := \mathbf{Sp}_{2n}/F$.

Fix a prime p. We will apply the results of [32] to the group $\mathbf{G}' = \mathrm{Res}_{\mathbb{Q}}^{F}(\mathbf{G})$. Let E be a finite and Galois extension of $\mathbb{Q}_p$, containing all the F_v (v over p).

2.1.2. The Atkin-Lehner algebra. The algebraic group $\mathbf{G}' \times_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} \mathbf{G} \times_{\mathbb{Q}} F_v$ (where v runs over the places of F) is isomorphic to $\prod_{v|p} \mathrm{Res}_{\mathbb{Q}_p}^{F_v} \mathbf{Sp}_{2n}/F_v$, which is quasi-split but not split in general. The algebraic group $\mathbf{Sp}_{2n}$ is defined over $\mathbb{Z}$ by the equation ${}^tMJM = J$ for M in $\mathbf{M}_{2n}$, where $J = \left(\begin{smallmatrix} 0 & J_n \\ -J_n & 0 \end{smallmatrix}\right)$ and $J_n = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$. We define its algebraic subgroups $\mathbf{T}_v$, $\mathbf{B}_v$, $\bar{\mathbf{B}}_v$, $\mathbf{N}_v$, $\bar{\mathbf{N}}_v$ of diagonal, upper triangular, lower triangular, unipotent upper triangular, and unipotent lower triangular matrices of $\mathrm{Res}_{\mathbb{Q}_p}^{F_v}(\mathbf{Sp}_{2n}/F_v)$, and let $\mathbf{T} = \prod_{v|p} \mathbf{T}_v$, $\mathbf{B} = \prod_{v|p} \mathbf{B}_v$, and so on. In [32, 2.4], only the action of the maximal split torus of $\mathbf{G}' \times_{\mathbb{Q}} \mathbb{Q}_p$ is considered. For our purpose, we will need to extend this and consider the action of a maximal (non-split in general) torus, that is $\mathbf{T}$, instead of a maximal split torus $\mathbf{S} \subset \mathbf{T}$. The results in [32] are easily extended to this bigger torus, essentially because $\mathbf{T}(\mathbb{Q}_p)/\mathbf{S}(\mathbb{Q}_p)$ is compact. Moreover, we let I_v be the compact subgroup of $\mathbf{Sp}_{2n}(\mathcal{O}_v)$ consisting of matrices with invertible diagonal elements and elements of positive valuation below the diagonal. Finally, following Loeffler's notation, we let $G_0 = \prod_{v|p} I_v$. It is an Iwahori sugroup of $\mathbf{G}'(\mathbb{Q}_p)$ having an Iwahori decomposition: $G_0 \simeq \bar{N}_0 T_0 N_0$ where $*_0 = *(\mathbb{Q}_p) \cap G_0$.

For each place v of F above p, let us choose a uniformizer ϖ_v of F_v. Let Σ_v be the subgroup of $\mathbf{Sp}_{2n}(F_v)$ consisting of diagonal matrices whose diagonal elements are powers of ϖ_v, i.e. matrices of the form $\mathrm{Diag}(\varpi_v^{r_1}, \ldots, \varpi_v^{r_n}, \varpi_v^{-r_n}, \ldots, \varpi_v^{-r_1})$. Let Σ_v^+ be the submonoid of Σ_v whose elements satisfy $r_1 \leq \cdots \leq r_n \leq 0$, and Σ_v^{++} the one whose elements satisfy $r_1 < \cdots < r_n < 0$. Naturally, we set $\Sigma = \prod_{v|p} \Sigma_v$, and similarly for Σ^+ and Σ^{++}.

The *Atkin-Lehner algebra* $\mathcal{H}_p^+$ is defined as the subalgebra of the Hecke-Iwahori algebra $\mathcal{H}(G_0 \backslash \mathbf{G}'(\mathbb{Q}_p)/G_0)$ (over $\mathbb{Q}$) generated by the characteristic functions $[G_0 u G_0]$, for $u \in \Sigma^+$. Let $\mathcal{H}_p$ be the subalgebra of $\mathcal{H}(G_0 \backslash \mathbf{G}'(\mathbb{Q}_p)/G_0)$ generated by the characteristic functions $[G_0 u G_0]$ and their inverses, for $u \in \Sigma^+$ (in [28], a presentation of the Hecke-Iwahori algebra is given, which shows that $[G_0 u G_0]$ is invertible if p is invertible in the ring of coefficients).

If S^p is a finite set of finite places of F not containing those over p, let $\mathcal{H}^S$ be the Hecke algebra (over $\mathbb{Q}$)

$$\bigotimes_{w \notin S^p \cup S_p \cup S_\infty}{}' \mathcal{H}(\mathbf{G}(\mathcal{O}_{F_w}) \backslash \mathbf{G}(F_w) / \mathbf{G}(\mathcal{O}_{F_w}))$$

where S_* denotes the set of places above $*$. This Hecke algebra has unit e^S. Let $\mathcal{H}_S^p$ be a commutative subalgebra of $\bigotimes_{w\in S^p}\mathcal{H}(\mathbf{G}(F_w))$, with unit e_{S^p}.

Finally, we let $\mathcal{H}^+ = \mathcal{H}_p^+ \otimes \mathcal{H}_{S^p} \otimes \mathcal{H}^S$, $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_{S^p} \otimes \mathcal{H}^S$ and $e = e_{G_0} \otimes e_{S^p} \otimes e^S$.

2.1.3. *p*-adic automorphic forms. The construction in [32] depends on the choice of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}'$ and a representation V of a compact subgroup of the Levi quotient $\mathbf{M}$ of $\mathbf{P}$. The parabolic subgroup we consider here is the Borel subgroup $\mathbf{B}$, and thus, using Loeffler's notation, $\mathbf{T} = \mathbf{M}$ is a maximal (non-split in general) torus contained in $\mathbf{B}$. The representation V is taken to be trivial.

The weight space $\mathscr{W}$ is the rigid space (over E, but it is well-defined over $\mathbb{Q}_p$) parametrizing locally $\mathbb{Q}_p$-analytic (equivalently, continuous) characters of the compact group T_0, isomorphic to $\left(\prod_{v|p}\mathcal{O}_v^\times\right)^n$. As $1+\varpi_v\mathcal{O}_v$ is isomorphic to $(\mu_{p^\infty}\cap F_v^\times)\times\mathbb{Z}_p^{[F_v:\mathbb{Q}_p]}$, $\mathscr{W}$ is the product of an open polydisc of dimension $n[F:\mathbb{Q}]$ and a rigid space finite over E.

The construction in [32] defines the k-analytic ($(G_k)_{k\geq 0}$ being a filtration of G_0) parabolic induction from T_0 to G_0 of the "universal character" $\chi : T_0 \to \mathcal{O}(\mathscr{W})^\times$, denoted by $\mathcal{C}(\mathscr{U},k)$ (k big enough such that χ is k-analytic on the open affinoid $\mathscr{U}$), which interpolates p-adically the restriction to $\mathbf{G}'(\mathbb{Q}_p)$ of algebraic representations of $\mathbf{G}'(\overline{\mathbb{Q}_p})$. From there one can define the spaces $M(e,\mathscr{U},k)$ ([32, Definition 3.7.1]) of p-adic automorphic forms (or overconvergent automorphic forms, by analogy with the rigid-geometric case of modular forms) above an open affinoid or a point $\mathscr{U}$ of $\mathscr{W}$ which are k-analytic and fixed by the idempotent e. This space has an action of $\mathcal{H}^+$. By [32, Corollary 3.7.3], when considering p-adic automorphic forms which are eigenvectors for $[G_0uG_0]$ for some $u\in\Sigma^{++}$ and for a non-zero eigenvalue ("finite slope" p-adic eigenforms), one can forget about k, and we will do so in the sequel.

2.1.4. Existence and properties of the eigenvariety. We choose the element

$$\eta = \left(\mathrm{Diag}(\varpi_v^{-n},\ldots,\varpi_v^{-1},\varpi_v,\ldots,\varpi_v^n)\right)_v \in \Sigma^{++}.$$

Theorem 2.1.1. *There exists a reduced rigid space $\mathscr{X}$ over E, together with an E-algebra morphism $\Psi : \mathcal{H}^+ \to \mathcal{O}(\mathscr{X})^\times$ and a morphism of rigid spaces $w : \mathscr{X} \to \mathscr{W}$ such that:*

1) *The morphism $\left(w, \Psi([G_0\eta G_0])^{-1}\right) : \mathscr{X} \to \mathscr{W}\times\mathbb{G}_m$ is finite*
2) *For each point x of $\mathscr{X}$, $\Psi\otimes w^\natural : \mathcal{H}^+\otimes_E \mathcal{O}_{w(x)} \to \mathcal{O}_x$ is surjective*

3) *For every finite extension E'/E, $\mathscr{X}(E')$ is in bijection with the finite slope systems of eigenvalues of $\mathcal{H}^+$ acting on the space of "overconvergent" automorphic forms, via evaluation of the image of Ψ at a given point.*

Moreover, for any point $x \in |\mathscr{X}|$, there is an arbitrarily small open affinoid $\mathscr{V}$ containing x and an open affinoid $\mathscr{U}$ of $\mathscr{W}$ such that $\mathscr{V} \subset w^{-1}(\mathscr{U})$, the morphism $w|_{\mathscr{V}} : \mathscr{V} \to \mathscr{U}$ is finite, and surjective when restricted to any irreducible component of $\mathscr{V}$.

Proof. This is [32, Theorems 3.11.2 and 3.12.3], except for the last assertion. To prove it, we need to go back to the construction of the eigenvariety in [9]. Buzzard begins by constructing the Fredholm hypersurface $\mathscr{Z}$ (encoding only the value of $\Psi([G_0\eta G_0])$), together with a flat morphism $\mathscr{Z} \to \mathscr{W}$, before defining the finite morphism $\mathscr{X} \to \mathscr{Z}$. By [9, Theorem 4.6], $\mathscr{Z}$ can be admissibly covered by its open affinoids $\mathscr{V}_0$ such that w restricted to $\mathscr{V}_0$ induces a finite, surjective morphism to an open affinoid $\mathscr{U}$ of $\mathscr{W}$, and $\mathscr{V}_0$ is a connected component of the pullback of $\mathscr{U}$. We can assume that $\mathscr{U}$ is connected, and hence irreducible, since $\mathscr{W}$ is normal. The morphism $\mathscr{V}_0 \to \mathscr{U}$ is both open (since it is flat: [7, Corollary 7.2]) and closed (since it is finite), so that any irreducible component of $\mathscr{V}_0$ is mapped onto $\mathscr{U}$. This can be seen more naturally by observing that the irreducible components of $\mathscr{V}_0$ are also Fredholm hypersurfaces, by [24, Theorem 4.3.2].

By [18, Proposition 6.4.2], if $\mathscr{V}$ denotes the pullback to $\mathscr{X}$ of $\mathscr{V}_0$, each irreducible component of $\mathscr{V}$ is mapped onto an irreducible component of $\mathscr{V}_0$ (more precisely, this is a consequence of [18, Lemme 6.2.10]). To conclude, we only need to show that if $x \in \mathscr{V}$, up to restricting $\mathscr{U}$, the connected component of $\mathscr{V}$ containing x can be arbitrarily small. This is a consequence of the following lemma. □

Lemma 2.1.2. *Let $f : \mathscr{X}_1 \to \mathscr{X}_2$ be a finite morphism of rigid analytic spaces. Then the connected components of $f^{-1}(U)$, for U admissible open of $\mathscr{X}_2$, form a basis for the canonical topology on $\mathscr{X}_1$.*

Proof. It is enough to consider the case $\mathscr{X}_1 = \mathrm{Sp} A_1$, $\mathscr{X}_2 = \mathrm{Sp} A_2$. Let x_1 be a maximal ideal of A_1. Then $f^{-1}(\{f(x_1)\}) = \{x_1, \dots, x_m\}$. We choose generators $t_1, \dots, t_n$ of $f(x_1)$, and $r_1^{(i)}, \dots, r_{k_i}^{(i)}$ of x_i. Using the maximum modulus principle, it is easily seen that $\Omega_{j,N} := \left\{ y \in \mathscr{X}_2 \mid |t_j(y)| \geq p^{-N} \right\}_{j,N}$ is an admissible covering of the admissible open $\mathscr{X}_2 \setminus \{f(x)\}$ of $\mathscr{X}_2$. Let V_M be the admissible open $\left\{ x \in \mathscr{X}_1 \mid \forall i, \exists k,\ |r_k^{(i)}(x)| \geq p^{-M} \right\}$, which is a finite union of open affinoids, hence quasi-compact. Consequently, the admissible open sets

$$\begin{aligned} U_{j,N} &:= V_M \cap f^{-1}(\Omega_{j,N}) \\ &= \left\{ x \in \mathscr{X}_1 \;\middle|\; \forall i, \exists k,\ |r_k^{(i)}(x)| \geq p^{-M} \text{ and } |f^\natural(t_j)(x)| \geq p^{-N} \right\}_{j,N} \end{aligned}$$

form an admissible covering of V_M. Therefore there is an N big enough so that

$$V_M = \bigcup_{j=1}^{r} U_{j,N}$$

which implies that

$$f^{-1}\left(\left\{y \in \mathscr{X}_2 \mid |t_j(y)| \leq p^{-N-1}\right\}\right) \subset \bigcup_i \left\{x \in \mathscr{X}_1 \mid \forall k,\ |r_k^{(i)}(x)| \leq p^{-M}\right\}$$

and when M goes to infinity, the right hand side is the disjoint union of arbitrarily small affinoid neighbourhoods of the x_i. □

We define the algebraic points of $\mathscr{W}(E)$ to be the ones of the form

$$(x_{v,i})_{v,i} \mapsto \prod_{v,\sigma} \sigma\left(\prod_{i=1}^{n} x_{v,i}^{k_{v,\sigma,i}}\right)$$

where $k_{v,\sigma,i}$ are integers, and such a point is called dominant if $k_{v,\sigma,1} \geq k_{v,\sigma,2} \geq \cdots \geq k_{v,\sigma,n} \geq 0$.

Recall that a set $S \subset |\mathscr{X}|$ is said to *accumulate* at a point $x \in |\mathscr{X}|$ if x has a basis of affinoid neighbourhoods in which S is Zariski dense.

Proposition 2.1.3. *Let $(\phi_r)_r$ be a finite family of linear forms on $\mathbb{R}^A$ where A is the set of triples (v,σ,i) for v a place of F above p, $\sigma : F_v \to E$ and $1 \leq i \leq n$, and let $(c_r)_r$ be a family of elements in $\mathbb{R}_{\geq 0}$. Assume that the open affine cone $C = \left\{y \in \mathbb{R}^A \mid \forall r,\ \phi_r(y) > c_r\right\}$ is nonempty. Then the set of algebraic characters in C yields a Zariski dense set in the weight space $\mathscr{W}$, which accumulates at all the algebraic points.*

Proof. [19, Lemma 2.7]. □

In particular the property of being dominant or "very regular" can be expressed in this way.

By finiteness of $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_{F,f})/U$ for any open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f})$, if Π is an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, the representation Π_f

is defined over $\iota_\infty(\bar{\mathbb{Q}})$. Loeffler defines ([32, Definition 3.9.1]) the classical subspace of the space of p-adic automorphic forms above an algebraic and dominant point w of the weight space. This subspace is isomorphic to $\iota_p \iota_\infty^{-1}\left(e\left(\mathcal{C}^\infty(\mathbf{G}(F)\backslash \mathbf{G}(\mathbb{A}_F)) \otimes W^*\right)^{\mathbf{G}(F\otimes_{\mathbb{Q}}\mathbb{R})}\right)$ as $\mathcal{H}^+$-module, with W the representation of $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ which is the restriction of the algebraic representation of $\mathbf{G}' \times_{\mathbb{Q}} \mathbb{C}$ having highest weight $\iota_\infty^{-1}\iota_p(w)$. The classical points of the eigenvariety are the ones having eigenvectors in the classical subspace.

We need to give an interpretation of classical points on the eigenvariety $\mathscr{X}$, in terms of automorphic representations of $\mathbf{G}(\mathbb{A}_F)$. Namely, there is a classical point $x \in \mathscr{X}(E')$ defining a character $\Psi_x : \mathcal{H} \to E'$ (here $E \subset E' \subset \overline{\mathbb{Q}_p}$) if and only if there is an automorphic representation $\Pi = \otimes'_v \Pi_v = \Pi_\infty \otimes \Pi_p \otimes \Pi_f^{(p)}$ of $\mathbf{G}(\mathbb{A}_F)$ such that:

- $\iota_p \iota_\infty^{-1}\left(\otimes_{v|\infty}\Pi_v\right)$ is the algebraic representation having highest weight $w(x)$;
- $\iota_p\left((e^S \otimes e_S)\Pi_f^{(p)}\right)$ contains a non-zero vector on which $\mathcal{H}^S \otimes \mathcal{H}_S$ acts according to Ψ_x;
- $\iota_p(e_{G_0}\Pi_p)$ contains a non-zero vector on which $\mathcal{H}_p$ acts according to $\mu_{w(x)}\Psi_x$, where $\mu_{w(x)}([G_0 \xi G_0]) = w(x)(\xi)$ if $\xi \in \Sigma^+$.

The twist by the character $\mu_{w(x)}$ is explained by the fact that the classical overconvergent automorphic forms are constructed by induction of characters of the torus extended from T_0 (on which they are defined by w) to T trivially on Σ.

2.2. Unramified and "completely refinable" points

2.2.1. Small slope *p*-adic eigenforms are classical. The algebraic and dominant points of $\mathscr{W}$ are the ones of the form

$$(x_{v,i})_{v,i} \mapsto \prod_{v,\sigma} \sigma\left(\prod_{i=1}^n x_{v,i}^{k_{v,\sigma,i}}\right)$$

where $k_{v,\sigma,1} \geq k_{v,\sigma,2} \geq \cdots \geq k_{v,\sigma,n} \geq 0$ are integers. The proof of the criterion given in [32, Theorem 3.9.6] contains a minor error, because it "sees" only the restriction of these characters to the maximal split torus $\mathbf{S}$ (over $\mathbb{Q}_p$), and the BGG resolution has to be applied to *split* semi-simple Lie algebras.

We correct it in the case of quasi-split reductive groups (in particular the restriction to a subfield of a quasi-split group remains quasi-split), and give a stronger criterion. This criterion could be used on an eigenvariety for which only the weights corresponding to a given p-adic place of F vary. For this purpose we use the "dual BGG resolution" given in [29]. We follow closely the proof of [32, Propositions 2.6.3-2.6.4]. In the following $\mathbf{G}'$ could be any quasi-split reductive group over $\mathbb{Q}_p$, and we could replace $E/\mathbb{Q}_p$ by any extension splitting $\mathbf{G}'$.

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}'$, $\mathbf{S}$ a maximal split torus in $\mathbf{B}$, $\mathbf{T}$ the centralizer of $\mathbf{S}$, a maximal torus. This determines an opposite Borel subgroup $\bar{\mathbf{B}}$ such that $\bar{\mathbf{B}} \cap \mathbf{B} = \mathbf{T}$. Let Φ^+ (resp. Δ) be the set of positive (resp. simple) roots of $\mathbf{G}' \times_{\mathbb{Q}_p} E$, with respect to the maximal torus $\mathbf{T}$ of the Borel subgroup $\mathbf{B}$. One can split $\Delta = \sqcup_i \Delta_i$ where α, β belong to the same Δ_i if and only if $\alpha|_{\mathbf{S}} = \beta|_{\mathbf{S}}$ (equivalently, the Δ_i are the Galois orbits of Δ). Let Σ be a subgroup of $\mathbf{T}(\mathbb{Q}_p)$ supplementary to its maximal compact subgroup, and Σ^+ the submonoid consisting of the $z \in \mathbf{T}(\mathbb{Q}_p)$ such that $|\alpha(z)| \geq 1$ for all $\alpha \in \Delta$. For each i, define η_i to be the element of $\Sigma^+ / (Z(\mathbf{G}')(\mathbb{Q}_p) \cap \Sigma)$ generating $\cap_{j \neq i} \ker |\alpha_j(\cdot)|$ (here α_j denotes any element of Δ_j, and $|\alpha_j(\cdot)|$ does not depend on this choice).

Assume that G_0 is a compact open subgroup of $\mathbf{G}'(\mathbb{Q}_p)$ having an Iwahori factorization $\bar{N}_0 T_0 N_0$. Using a lattice in the Lie algebra of N and the exponential map, it is easily seen that N_0 admits a decreasing, exhaustive filtration by open subgroups $(N_k)_{k \geq 1}$ having a canonical rigid-analytic structure. Moreover any ordering of Φ^+ endows the Banach space of $\mathbb{Q}_p$-analytic functions on N_k taking values in E with an orthonormal basis consisting of monomials on the weight spaces.

Let λ be an algebraic and dominant weight of $\mathbf{T} \times_{\mathbb{Q}_p} E$. By [29], there is an exact sequence of $E[\mathbb{I}]$-modules, where $\mathbb{I} = G_0 \Sigma^+ G_0 = \bar{B}_0 \Sigma^+ N_0$ is the monoid generated by G_0 and Σ^+:

$$\begin{aligned} (2.2.1) \qquad & 0 \to \text{alg-Ind}_{\bar{\mathbf{B}}}^{\mathbf{G}}(\lambda) \otimes \text{sm-Ind}_{\bar{B}_0}^{\bar{B}_0 N_0} 1 \to \text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(\lambda) \\ & \to \bigoplus_{\alpha \in \Delta} \text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(s_\alpha(\lambda + \rho) - \rho) \end{aligned}$$

where $2\rho = \sum_{\alpha \in \Phi^+} \alpha$, "sm" stands for "smooth" and "la" for "locally analytic". The relation with Loeffler's $\text{Ind}(V)_k$ is $\text{la-Ind}_{\bar{B}}^{\bar{B} N_0}(\lambda) \otimes \lambda_{\text{sm}}^{-1} = \varinjlim_k \text{Ind}(E_\lambda)_k$, where λ_{sm} is the character on T which is trivial on its maximal compact subgroup and agrees with λ on Σ. Naturally $\text{Ind}_{\bar{\mathbf{B}}}^{\mathbf{G}}(\lambda) \otimes \text{sm-Ind}_{\bar{B}_0}^{\bar{B}_0 N_0} 1 \otimes \lambda_{\text{sm}}^{-1} = \varinjlim_k \text{Ind}(E_\lambda)_k^{\text{cl}}$.

To prove a classicity criterion, we need to bound the action of η_i on the factors of the RHS of (2.2.1) twisted by $\lambda_{\mathrm{sm}}^{-1}$. Let $n_\alpha = \alpha^\vee(\lambda) \in \mathbb{N}$ for $\alpha \in \Delta$, then $s_\alpha(\lambda+\rho) - \lambda - \rho = -(1+n_\alpha)\alpha$. The Banach space of k-analytic functions on N_0 is the direct sum of the spaces of analytic functions on xN_k, $x \in N_0/N_k$, and each of these spaces has an orthonormal (with respect to the supremum norm) basis $(v_{j,x})_{j\in J}$ where $J = \mathbb{N}^{\Phi^+}$ (monomials on the weights spaces). This basis depends on the choice of a representative x, but if we fix i and $x_0 \in N_0$, we can choose $\eta_i^{-1}x_0\eta_i$ as a representative of its class. Then if $\phi = \sum_j a_j v_{j,\eta_i^{-1}x_0\eta_i}$ (with $a_j \to 0$) is an element of $\text{la-Ind}_{\bar{B}}^{\bar{B}N_0}(s_\alpha(\lambda+\rho) - \rho) \otimes \lambda_{\mathrm{sm}}^{-1}$, and $\xi \in N_k$,

$$\begin{aligned}(\eta_i \cdot \phi)(x_0\xi) &= \eta_i^{-(1+n_\alpha)\alpha} \sum_{j\in J} a_j v_{j,\eta_i^{-1}x_0\eta_i}(\eta_i^{-1}x_0\xi\eta_i) \\ &= \sum_{j\in J} a_j \eta_i^{-(1+n_\alpha)\alpha - s(j)} v_{j,x_0}(x_0\xi)\end{aligned}$$

where $s(j) = \sum_{\beta\in\Phi^+} j(\beta)\beta$. This shows that $|\eta_i \cdot \phi| \leq |\alpha(\eta_i)|^{-(1+n_\alpha)}|\phi|$, and so the operator η_i has norm less than or equal to $|\alpha(\eta_i)|^{-(1+n_\alpha)}$ on $\text{la-Ind}_{\bar{B}}^{\bar{B}N_0}(s_\alpha(\lambda+\rho) - \rho) \otimes \lambda_{\mathrm{sm}}^{-1}$.

We can then apply the exact functor which to an $E[\mathbb{I}]$-module W associates the automorphic forms taking values in W, and take the invariants under the idempotent e (this functor is left exact). We obtain that $M(e, E_\lambda)/M(e, E_\lambda)_{\mathrm{cl}}$ (the space of p-adic automorphic forms modulo the classical automorphic forms) embeds in $\bigoplus_{\alpha\in\Delta} M_\alpha$ where each M_α is a Banach space on which the operator $[G_0\eta_i G_0]$ has norm $\leq |\alpha(\eta_i)|^{-(1+n_\alpha)}$. The following criterion follows:

Lemma 2.2.1. *If an overconvergent eigenform $f \in M(e, E_\lambda)$ satisfies $[G_0\eta_i G_0]\, f = \mu_i f$ with $\mu_i \neq 0$ and*

$$v_p(\mu_i) < \inf_{\alpha\in\Delta_i} -(1+n_\alpha)v_p(\alpha(\eta_i))$$

for all i, then f is classical.

In the case of the symplectic group $\mathbf{G}'$, the family $(\eta_i)_i$ can be indexed by the couples (v, i) where v is a place of F above p and $1 \leq i \leq n$, and $\Delta_{v,i}$ is indexed by the embeddings $F_v \hookrightarrow E$. Specifically, $\eta_{v,i}$ is trivial at all the places except for v, where it equals

$$\mathrm{Diag}(x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1})$$

with $x_j = \begin{cases} \varpi_v^{-1} & \text{if } j \leq i \\ 1 & \text{if } j > i \end{cases}$.

The conditions in the previous lemma can be written

$$\begin{cases} v_p(\mu_{v,i}) < \frac{1}{e_v} \inf_\sigma (1 + k_{v,\sigma,i} - k_{v,\sigma,i+1}) & \text{for } i < n \\ v_p(\mu_{v,n}) < \frac{1}{e_v} \inf_\sigma (2 + 2k_{v,\sigma,n}) . \end{cases}$$

2.2.2. Representations having Iwahori-invariants and unramified principal series. We recall results of Casselman showing that irreducible representations having invariants under an Iwahori subgroup appear in unramified principal series, and giving the Atkin-Lehner eigenvalues in terms of the unramified character being induced.

In this subsection, we fix a place v of F above p. Recall I_v has an Iwahori decomposition $I_v = N_{v,0} T_{v,0} \bar{N}_{v,0}$. As in [15], if (Π, V) is a smooth representation of $\mathbf{G}(F_v)$, $V(\bar{N}_v)$ is the subspace of V spanned by the $\Pi(\bar{n})(x) - x$, $\bar{n} \in \bar{N}_v$, $V_{\bar{N}_v} = V/V(\bar{N}_v)$ and if $\bar{N}_{v,i}$ is a compact subgroup of $\bar{N}_v$, $V(\bar{N}_{v,i}) = \left\{ v \in V \mid \int_{\bar{N}_{v,i}} \Pi(\bar{n})(v) d\bar{n} = 0 \right\}$.

Lemma 2.2.2. *Let (Π, V) be an admissible representation of $\mathbf{G}(F_v)$ over $\mathbb{C}$. Then the natural (vector space) morphism from V^{I_v} to $\left(V_{\bar{N}_v}\right)^{T_{v,0}}$ is an isomorphism, inducing a Σ_v^+-equivariant isomorphism*

$$\Pi^{I_v} \xrightarrow{\sim} \left(\Pi_{\bar{N}_v}\right)^{T_{v,0}} \otimes \delta_{\bar{B}_v}^{-1}$$

where $\delta_{\bar{B}_v}$ denotes the modulus morphism of $\bar{B}_v$, and $u \in \Sigma_v^+$ acts on Π^{I_v} by $[I_v u I_v]$.

Proof. Let $\bar{N}_{v,1}$ be a compact subgroup of $\bar{N}_v$ such that $V^{I_v} \cap V(\bar{N}_v) \subset V(\bar{N}_{v,1})$. There is a $u \in \Sigma_v^+$ such that $u \bar{N}_{v,1} u^{-1} \subset \bar{N}_{v,0}$. By [15, Prop. 4.1.4], and using the fact that $[I_v u I_v]$ is invertible in the Hecke-Iwahori algebra, the natural morphism from V^{I_v} to $V_{\bar{N}}^{T_{v,0}}$ is an isomorphism (of vector spaces).

Lemmas 4.1.1 and 1.5.1 in [15] allow one to compute the action of Σ_v^+. □

Corollary 2.2.3. *Any smooth irreducible representation of $\mathbf{G}(F_v)$ over $\mathbb{C}$ having Iwahori-invariants is a subquotient of the parabolic induction (from $\bar{B}_v$) of a character of the torus T_v, which is unique up to the action of $W(T_v, \mathbf{G}(F_v))$, and unramified.*

Proof. Π is a subquotient of the parabolic induction of a character of the torus T_v if and only if $\Pi_{\bar{N}_v} \neq 0$, which is true by the previous lemma. The geometrical lemma [5, 2.12] shows that if χ is a smooth character of T_v,

$$\left(\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)}\chi\right)^{\mathrm{ss}}_{\bar{N}_v} \simeq \bigoplus_{w\in W(T_v,\mathbf{G}(F_v))} \chi^w \delta_{\bar{B}_v}^{1/2}.$$

□

Since $*_{\bar{N}}$ is left adjoint to *non-normalized* induction, the first argument in the proof shows that Π is actually a subrepresentation of $\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)}$ for at least one χ in the orbit under $W(T_v, \mathbf{G}(F_v))$. In that case we will say that (Π, χ) is a *refinement* of Π. Note that up to the action of $W(T_v, \mathbf{G}(F_v))$, there is a unique χ such that Π is a subquotient of $\mathrm{Ind}_{\bar{B}_v}^{\mathbf{G}(F_v)}$.

2.2.3. Most points of the eigenvariety arise from unramified, completely refinable representations. We will need a result of Tadić, characterizing the *irreducible* principal series. If $\chi_1, \dots, \chi_n$ are characters of $F_v^\times$, we denote simply by $\chi = (\chi_1, \dots, \chi_n)$ the character of T_v which maps

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ & & & & & x_1^{-1} \end{pmatrix}$$

to $\prod_{i=1}^n \chi_i(x_i)$. Let ν be the unramified character of $F_v^\times$ such that $\nu(\varpi_v) = |\mathbb{F}_v|^{-1}$.

The following Theorem is [39, Theorem 7.1].

Theorem 2.2.4 (Tadić). *Let $\chi = (\chi_1, \dots, \chi_n)$ be a character of T_v. The representation $\mathrm{Ind}_{\bar{B}_v}^{\mathrm{Sp}_{2n}(F_v)}(\chi)$ is irreducible if and only if the following conditions are satisfied*

1) *For all i, χ_i is not of order 2.*
2) *For all i, $\chi_i \neq \nu^{\pm 1}$.*
3) *For all distinct i, j, $\chi_i \chi_j^{-1} \neq \nu^{\pm 1}$ and $\chi_i \chi_j \neq \nu^{\pm 1}$.*

Definition 2.2.5. An irreducible representation Π_v of $\mathbf{G}(F_v)$ is *completely refinable* if it is isomorphic to $\mathrm{Ind}_{\bar{B}_v}^{\mathrm{Sp}_{2n}(F_v)}\chi$ for some unramified character χ.

An automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ is completely refinable if Π_v is completely refinable for any $v|p$.

Note that completely refinable representations are unramified (for any choice of hyperspecial subgroup). A representation Π_v is completely refinable if and only if $(\Pi_v)^{\mathrm{ss}}_{\bar{N}_v}$ is the sum of $|W(T_v, \mathbf{G}(F_v))|$ unramified characters.

Recall that classical points on the eigenvariety are determined by an automorphic representation Π together with a refinement of each Π_v, $v|p$. Completely refinable automorphic representations are the ones giving the greatest number of points on the eigenvariety. When one can associate Galois representations to automorphic representations, each refinement of Π comes with a "p-adic family" of Galois representations going through the same one.

Proposition 2.2.6. *Let $f_1, \ldots, f_r \in \mathcal{O}(\mathscr{X})^\times$, and let $(\Lambda_j)_{j\in J}$ be a finite family of non-constant affine functions on $(\mathbb{Q}^n)^{\mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{R})}$. The set $\mathcal{S}$ of points of $\mathscr{X}$ corresponding to classical, unramified and completely refinable points at which*

$$\begin{aligned}(2.2.2)\qquad &\min_{v,\sigma} \min\{k_{v,\sigma,1} - k_{v,\sigma,2}, \ldots, k_{v,\sigma,n-1} - k_{v,\sigma,n}, k_{v,\sigma,n}\}\\ &\geq \max\{v_p(f_1), \ldots, v_p(f_n)\}\end{aligned}$$

and $\Lambda_j((k_{v,\sigma,1}, \ldots, k_{v,\sigma,n})_{v,\sigma}) \neq 0$ for all v, σ, j, is Zariski dense and accumulates at all the algebraic points.

Compare [18, Proposition 6.4.7], [32, Corollary 3.13.3]. To evaluate a linear form Λ_j at $(k_{v,\sigma,1}, \ldots, k_{v,\sigma,n})_{v,\sigma}$ we have used the bijection between $\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$ and $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}_p})$ given by ι_p, ι_∞.

Proof. The hypotheses in the classicality criterion 2.2.1 and the ones in Theorem 2.2.4 are implied by inequalities of the form 2.2.2. First we prove the accumulation property. We can restrict to open affinoids $\mathscr{V}$ of the eigenvariety, and hence assume that the right hand side of 2.2.2 is replaced by a constant. By Theorem 2.1.1, $\mathscr{V}$ can be an arbitrarily small open affinoid containing an algebraic point x of $\mathscr{X}$, such that there is open affinoid $\mathscr{U}$ of $\mathscr{W}$ such that $\mathscr{V} \subset w^{-1}(\mathscr{U})$, the morphism $w|_{\mathscr{V}} : \mathscr{V} \to \mathscr{U}$ is finite, and surjective when restricted to any irreducible component of $\mathscr{V}$. By Proposition 2.1.3, the algebraic weights satisfying 2.2.2 and such that $\Lambda_j(k_{v,\sigma,1}, \ldots, k_{v,\sigma,n}) \neq 0$ for all v, σ, j are Zariski dense in the weight space $\mathscr{W}$ and accumulate at all the algebraic points of $\mathscr{W}$. [18, Lemme 6.2.8] shows that $\mathcal{S} \cap \mathscr{V}$ is Zariski-dense in $\mathscr{V}$.

Each irreducible component $\mathscr{X}'$ of $\mathscr{X}$ is mapped onto a Zariski-open subset of a connected component of $\mathscr{W}$, by [18, Corollaire 6.4.4] (which is a consequence of the decomposition of a Fredholm series into a product of irreducible Fredholm series, [24, Corollary 4.2.3]), so $\mathscr{X}'$ contains at least one algebraic point (the algebraic weights intersect all the connected components of $\mathscr{W}$), and hence the Zariski closure of $\mathcal{S} \cap \mathscr{X}'$ contains an open affinoid of $\mathscr{X}'$, which is Zariski dense in $\mathscr{X}'$. □

3. Galois representations associated with automorphic representations of symplectic groups

3.1. Existence of Galois representations

3.1.1. Automorphic self-dual representations of $\mathbf{GL_{2n+1}}$ of orthogonal type. If $\Pi = \otimes_v \Pi_v$ is an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, then for any Archimedean place v of F, the local Langlands parameter of Π_v (composed with $\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$) is of the form:

$$\mathcal{LL}(\Pi_v) \simeq \epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where ϵ is the only non-trivial character of $W_\mathbb{C}/W_\mathbb{R}$, and the r_i are integers, with $r_n > r_{n-1} > \cdots > r_1 > 0$. We define $A_{\mathbf{G}}^{\mathrm{vr}}$ to be the set of automorphic representations such that for each infinite place v of F, $r_1 \geq 2$ and $r_{i+1} \geq r_i + 2$. The equivalence above is meant as representations of $W_\mathbb{R}$ (i.e. morphisms $W_\mathbb{R} \to \mathrm{GL}_{2n+1}(\mathbb{C})$), although $\mathcal{LL}(\Pi_v)$ is a parameter taking values in $\mathrm{SO}_{2n+1}(\mathbb{C})$. These two notions of conjugacy actually coincide.

Similarly, let $A_{\mathbf{GL}_{2n+1}}$ be the set of formal sums of self-dual cuspidal representations $\pi = \boxplus_i \pi_i$ of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$ such that for each infinite place v of F, the local Langlands parameter $\mathcal{LL}(\pi_v) := \bigoplus_i \mathcal{LL}(\pi_{i,v})$ is isomorphic to

$$\epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

where the r_i's are integers such that $r_1 \geq 2$, $r_{i+1} \geq r_i + 2$, and such that the product of the central characters of the π_i's is trivial.

These inequalities between the r_i's are imposed to ensure that the corresponding global parameters are trivial on Arthur's $\mathrm{SL}_2(\mathbb{C})$, to simplify the statement of Proposition 3.1.1 below. That is why we take formal sums of *cuspidal* (not discrete) representations.

Note that there is no non-zero alternate bilinear form preserved by such a parameter (one could say that the parameter is "completely orthogonal").

Proposition 3.1.1. *For any* $\Pi \in A^{\mathrm{vr}}_{\mathbf{G}}$*, there is a* $\pi \in A_{\mathbf{GL}_{2n+1}}$*, such that the local Langlands parameters match at the infinite places, and for any finite place* v *of* F*,* π_v *is unramified if* Π_v *is unramified, and in that case the local parameters match, by means of the inclusion* $\mathrm{SO}_{2n+1}(\mathbb{C}) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$.

Here "Π_v is unramified" means that there exists a hyperspecial maximal compact subgroup K_v of $\mathbf{G}(F_v)$ such that $\Pi_v^{K_v} \neq 0$.

Proof. This follows from [40]: to any $\Pi \in A^{\mathrm{vr}}_{\mathbf{G}}$ one can associate a formal sum $\boxplus_i \pi_i[d_i]$ where the π_i's are self-dual cuspidal representations of $\mathbf{GL}_{n_i}/F$ and $d_i \geq 1$ are integers, and by compatibility with infinitesimal characters and the assumption on the r_i's associated to Π at any Archimedean place, all d_i's are equal to 1. □

3.1.2. Galois representations associated with RLASDC representations of $\mathbf{GL}_N$. An automorphic cuspidal representation π of $\mathbf{GL}_N(\mathbb{A}_F)$ is said to be *L-algebraic* if for any infinite place v of F, the restriction of the Langlands parameter $\mathcal{LL}(\pi_v)$ to $W_{\mathbb{C}} \simeq \mathbb{C}^\times$ is of the form

$$z \mapsto \mathrm{Diag}\left(\left(z^{a_{v,i}}\,\bar{z}^{b_{v,i}}\right)_i\right)$$

where $a_i, b_i \in \mathbb{Z}$. By the "purity lemma" [23, Lemme 4.9], $a_{v,i} + b_{v,i}$ does not depend on v, i. We will say that π is L-algebraic *regular* if for any v as above, the $a_{v,i}$ are distinct. By purity, this implies that if v is real,

$$\mathcal{LL}(\pi_v)|\cdot|^{-s} = \begin{cases} \epsilon^e \oplus_i \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}\left(z \mapsto (z/\bar{z})^{a'_{v,i}}\right) & \text{if } N \text{ is odd, with } e = 0, 1 \\ \oplus_i \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}\left(z \mapsto (z/\bar{z})^{a'_{v,i}}\right) & \text{if } N \text{ is even} \end{cases}$$

for some integer s, and integers $0 < a'_{v,1} < \cdots < a'_{v,\lfloor N/2 \rfloor}$.

As a special case of [20, Theorem 4.2] (which builds on previous work of Clozel, Harris, Kottwitz, Labesse, Shin, Taylor), we have the following theorem.

Theorem 3.1.2. *Let* π *be a regular L-algebraic, self-dual, cuspidal (RLASDC) representation of* $\mathbf{GL}_N(\mathbb{A}_F)$*. Then* π *is L-arithmetic, and there is a continuous Galois representation*

$$\rho_{\iota_p,\iota_\infty}(\pi) : \mathrm{Gal}_F \longrightarrow \mathrm{GL}_N(\overline{\mathbb{Q}_p})$$

such that if v *is a finite place of* F *and* π_v *is unramified,*

1) *if v is coprime to p, then $\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}}$ is unramified, and*

$$\det\left(T\mathrm{Id} - \rho_{\iota_p,\iota_\infty}(\pi)(\mathrm{Frob}_v)\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)$$

where A is the semisimple conjugacy class in $\mathrm{GL}_N(\mathbb{C})$ associated with π_v via the Satake isomorphism.

2) *if v lies above p, $\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}}$ is crystalline. The associated filtered φ-module (over $F_{v,0}\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}_p}$) is such that*

$$\det_{\overline{\mathbb{Q}_p}}\left(T\mathrm{Id} - \varphi^{f_v}\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)^{f_v}$$

where A is the semisimple conjugacy class in $\mathrm{GL}_N(\mathbb{C})$ associated with π_v via the Satake isomorphism. For any $\sigma : F_v \to \overline{\mathbb{Q}_p}$, the σ-Hodge-Tate weights are the $a_{w,i}$, where w is the real place of F defined by σ, ι_p and ι_∞.

The power f_v appearing at places above p may seem more natural to the reader after reading Section 3.2.1, where we give an equivalent formulation which does not involve this power.

Combining this theorem with Proposition 3.1.1, we obtain

Corollary 3.1.3. *Let Π be an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$, whose highest weights $k_{w,1} \geq k_{w,2} \geq \cdots \geq k_{w,n} \geq 0$ at the real places w are far from the walls ($\Pi \in A_{\mathbf{G}^{\mathrm{vr}}}$ is enough), and unramified at all the p-adic places of F. There exists a continuous semisimple Galois representation*

$$\rho_{\iota_p,\iota_\infty}(\Pi) : \mathrm{Gal}_F \longrightarrow \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}_p})$$

such that for any finite place v of F such that Π_v is unramified

1) *if v is coprime to p, then $\rho_{\iota_p,\iota_\infty}(\Pi)|_{\mathrm{Gal}_{F_v}}$ is unramified, and*

$$\det\left(T\mathrm{Id} - \rho_{\iota_p,\iota_\infty}(\Pi)(\mathrm{Frob}_v)\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)$$

where $A \in \mathrm{GL}_N(\mathbb{C})$ is associated with Π_v via the Satake isomorphism.

2) *if v lies above p, $\rho_{\iota_p,\iota_\infty}(\Pi)|_{\mathrm{Gal}_{F_v}}$ is crystalline. The associated filtered φ-module is such that*

$$\det_{\overline{\mathbb{Q}_p}}\left(T\mathrm{Id} - \varphi^{f_v}\right) = \iota_p\iota_\infty^{-1}\det\left(T\mathrm{Id} - A\right)^{f_v}$$

where $A \in \mathrm{SO}_{2n+1}(\mathbb{C}) \subset \mathrm{GL}_{2n+1}(\mathbb{C})$ is associated with Π_v via the Satake isomorphism. For any $\sigma : F_v \to \overline{\mathbb{Q}_p}$, the σ-Hodge-Tate weights are

$k_{w,1}+n > k_{w,2}+n-1 > \cdots > k_{w,1}+1 > 0 > -k_{w,1}-1 > \cdots > -k_{w,1}-n$, where w is the real place of F defined by σ, ι_p and ι_∞.

Proof. There is a formal sum of self-dual cuspidal automorphic representations of general linear groups $\pi = \boxplus_i \pi_i$ corresponding to Π by Proposition 3.1.1. Let $\rho_{\iota_p,\iota_\infty}(\Pi) = \bigoplus_i \rho_{\iota_p,\iota_\infty}(\pi_i)$. □

Note that in that case, since Π_∞ is C-algebraic, Π is obviously C-arithmetic (which is equivalent to L-arithmetic in the case of Sp_{2n}), and thus the coefficients of the polynomials appearing in the corollary lie in a finite extension of $\mathbb{Q}$.

Using [4, Corollary 1.3] one sees that $\rho_{\iota_p,\iota_\infty}(\Pi)$ is orthogonal, i.e. factors as the composition of a continuous morphism $\mathrm{Gal}_F \to \mathrm{SO}_{2n+1}(\overline{\mathbb{Q}_p})$ and the standard representations $\mathrm{SO}_{2n+1}(\overline{\mathbb{Q}_p}) \to \mathrm{GL}_{2n+1}(\overline{\mathbb{Q}_p})$. We will not need this fact in order to reach our goal of determining the image of complex conjugations in certain Galois representations, but we will use it in the proof of Corollaries 3.2.3 and 4.0.2.

3.1.3. The Galois pseudocharacter on the eigenvariety. To study families of representations, it is convenient to use *pseudorepresentations* (or *pseudocharacters*), which are simply the traces of semi-simple representations when the coefficient ring is an algebraically closed field of characteristic zero. We refer to [41] for the definition, and [41, Theorem 1] is the "converse theorem" we will need.

On $\mathcal{O}(\mathscr{X})$, we put the topology of uniform convergence on open affinoids.

The Zariski-density of the classical points at which we can define an attached Galois representation implies the following

Proposition 3.1.4. *There is a continuous pseudocharacter $T : \mathrm{Gal}_F \to \mathcal{O}(\mathscr{X})$, such that at every classical unramified point of the eigenvariety having weight far from the walls, T specializes to the character of the Galois representation associated with the automorphic representation by Corollary 3.1.3.*

Proof. This is identical to the unitary case, and thus is a consequence of [18, Proposition 7.1.1], by Proposition 2.2.6. □

Thus at any (classical or not) point of the eigenvariety, there is an attached Galois representation.

3.2. Galois representations stemming from symplectic forms are generically almost irreducible

3.2.1. Crystalline representations over $\overline{\mathbb{Q}_p}$. We fix a finite extension K of $\mathbb{Q}_p$, and denote by K_0 the maximal unramified subextension, $e = [K : K_0]$, $f = [K_0 : \mathbb{Q}_p]$. Let $\rho : \mathrm{Gal}_K \to \mathrm{GL}(V)$ be a continuous representation of the absolute Galois group of K, where V is a finite dimensional vector space over L, a finite Galois extension of $\mathbb{Q}_p$. We will take L to be big enough so as to be able to assume in many situations that $L = \overline{\mathbb{Q}_p}$. For example, we can assume that L is an extension of K, and that ρ has a composition series $0 = V_1 \subset \cdots \subset V_r = V$ such that each quotient V_{i+1}/V_i is absolutely irreducible.

For any such ρ, let $D_{\mathrm{cris}}(V) = \left(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V\right)^{\mathrm{Gal}_K}$. From now on we assume that ρ is a crystalline representation, which means that $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$. It is well-known that $D_{\mathrm{cris}}(V)$ is a filtered φ-module over K, and since V is a vector space over L, $D_{\mathrm{cris}}(V)$ is a φ-module over $K_0 \otimes_{\mathbb{Q}_p} L$, and $D_{\mathrm{dR}}(V) = K \otimes_{K_0} D_{\mathrm{cris}}(V)$ is a module over $K \otimes_{\mathbb{Q}_p} L$ with a filtration by projective submodules.

We have a natural decomposition $K_0 \otimes_{\mathbb{Q}_p} L \simeq \prod_{\sigma_0 \in \Upsilon_0} L_{\sigma_0}$ where Υ_0 is defined as $\mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K_0, L)$ and $L_{\sigma_0} \simeq L$, given by the morphisms $\sigma_0 \otimes \mathrm{Id}_L$. Similarly, $K \otimes_{\mathbb{Q}_p} L \simeq \prod_{\sigma \in \Upsilon} L_\sigma$ with $\Upsilon = \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K, L)$.

Hence we have decompositions

$$D_{\mathrm{cris}}(V) = \prod_{\sigma_0 \in \Upsilon_0} D_{\mathrm{cris}}(V)_{\sigma_0}, \quad D_{\mathrm{dR}}(V) = \prod_{\sigma \in \Upsilon} D_{\mathrm{dR}}(V)_\sigma.$$

The operator φ restricts as linear isomorphisms from $D_{\mathrm{cris}}(V)_{\sigma_0}$ to $D_{\mathrm{cris}}(V)_{\sigma_0 \circ \varphi^{-1}}$, and so φ^f is a L_{σ_0}-linear automorphism on each $D_{\mathrm{cris}}(V)_{\sigma_0}$, which are isomorphic as vector spaces over L equipped with the linear automorphism φ^f.

Each $D_{\mathrm{dR}}(V)_\sigma$ comes with a filtration, and hence defines $\dim_L V = N$ Hodge-Tate weights $k_{\sigma,1} \leq \cdots \leq k_{\sigma,N}$ (the jumps of the filtration).

Although we will not use it, it should be noted that by [8, Proposition 3.1.1.5], to verify the weak admissibility of a filtered φ-module D over K with an action of L commuting with φ and leaving the filtration stable, it is enough to check the inequality $t_N(D') \geq t_H(D')$ for sub-$K_0 \otimes L$-modules stable under φ.

If φ^f has eigenvalues $\varphi_1, \ldots, \varphi_N$, with $v_p(\varphi_1) \leq \cdots \leq v_p(\varphi_n)$, we can in particular choose $D' = \oplus_{i \leq j} \ker(\varphi^f - \varphi_i)$ (if the eigenvalues are distinct, but even if they are not, we can choose D' such that $\varphi^f|_{D'}$ has eigenvalues

$\varphi_1, \ldots, \varphi_j$, counted with multiplicities). The worst case for the filtration yields the inequalities

$$\begin{aligned} v_p(\varphi_1) &\geq \frac{1}{e}\sum_\sigma k_{\sigma,1} \\ v_p(\varphi_1\varphi_2) &\geq \frac{1}{e}\sum_\sigma k_{\sigma,1} + k_{\sigma,2} \\ &\vdots \end{aligned}$$

In the sequel, we will only use these inequalities, and we will not be concerned with the subtleties of the filtrations.

It will be useful to know how these objects behave when ρ is restricted to an open subgroup of Gal_K. Let K'/K be a finite extension and denote by K'_0 the maximal unramified subextension of $K'/\mathbb{Q}_p$, $e' = [K' : K'_0]$, $f' = [K'_0 : \mathbb{Q}_p]$. Denote by V' the L-vector space V considered only with its $\mathrm{Gal}_{K'}$-action. Then $D_{\mathrm{cris}}(V') = K'_0 \otimes_{K_0} D_{\mathrm{cris}}(V)$ as φ-modules, and so for $\sigma'_0 \in \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K'_0, L)$ the eigenvalues of $\varphi^{f'}$ on $D_{\mathrm{cris}}(V')_{\sigma'_0}$ are the f'/f-powers of the eigenvalues of φ^f on $D_{\mathrm{cris}}(V)_{\sigma_0}$, where σ_0 is the restriction of σ'_0 to K_0. Similarly, $D_{\mathrm{dR}}(V') = K' \otimes_K D_{\mathrm{dR}}(V)$ and thus for any $\sigma' \in \mathrm{Hom}_{\mathbb{Q}_p-\mathrm{alg.}}(K', L)$, the σ'-Hodge-Tate weights of V' are simply the σ-Hodge-Tate weights of V, where $\sigma = \sigma'|_K$. Note that the inequalities above remain unchanged when replacing ρ with $\rho|_{\mathrm{Gal}_{K'}}$.

3.2.2. Variation of the crystalline Frobenius on the eigenvariety. In this section we make explicit the formulae relating the eigenvalues of the crystalline Frobenius at classical, unramified points of the eigenvariety and the eigenvalues of the Hecke-Iwahori operators acting on p-adic automorphic forms. Let x be a classical point on the eigenvariety. There is an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that $\iota_p\iota_\infty^{-1}(\Pi_\infty)$ is the representation having highest weight $w(x)$. Assume that Π_p is unramified. The point x defines a refinement of Π_p, that is an unramified character $\chi_x : T_0 \to \mathbb{C}^\times$ such that $\Pi_p \hookrightarrow \mathrm{Ind}_{\bar{B}}^{\mathbf{G}'(\mathbb{Q}_p)}\chi_x$, or equivalently the character $\delta_{\bar{B}}^{1/2}\chi_x$ appearing in $(\Pi_p)_{\bar{N}}$. By 2.2.2, for any $u \in \Sigma^+$, $\mu_{w(x)}\Psi_x|_{\mathcal{H}_p} = (\iota_p \circ \iota_\infty^{-1} \circ \chi_x)\delta_B^{1/2}$.

The diagonal torus in $\mathrm{SO}_{2n+1}(\mathbb{C})$ and the identification of it with the dual of the diagonal torus of $\mathbf{Sp}_{2n}/F_v$ being fixed, the character χ_x is mapped by the unramified Langlands correspondence for tori to $y = \left(\mathrm{Diag}(y_{v,1}, \ldots, y_{v,n}, 1, y_{v,n}^{-1}, \ldots, y_{v,1}^{-1})\right)_{v|p}$ where $y_{v,i} = \chi_x(\mathrm{Diag}(1, \ldots, \varpi_v, \ldots, 1, 1, \ldots, \varpi_v^{-1}, \ldots, 1))$ (ϖ_v being the i-th element). Thus for any choice of $\sigma_0 : F_v \to E$ in $\Upsilon_{0,v}$, the linearization of the crystalline Frobenius φ^{f_v} on

$D_{\mathrm{cris}}(\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}})_{\sigma_0}$ has eigenvalues

$$\iota_p\iota_\infty^{-1}(y_{v,i}) = q_v^{n+1-i}\phi_{v,n+1-i}(x)\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,i}}$$

and their inverses, together with the eigenvalue 1. Here $\phi_{v,n+1-i}\in\mathcal{O}(\mathscr{X})$ is defined by

$$\phi_{v,n+1-i} = \frac{\Psi\left([G_0u_{i-1}G_0]\right)}{\Psi\left([G_0u_iG_0]\right)}$$

with $u_i = \mathrm{Diag}(\varpi_{\mathrm{v}}^{-1},\dots,\varpi_{\mathrm{v}}^{-1},1,\dots,1,\varpi_{\mathrm{v}},\dots,\varpi_{\mathrm{v}})$ (the last ϖ_v^{-1} is the i-th element), and $k_{v,\sigma,i}$ the integers defining the weight $w(x)$.

Assume furthermore that Π_p admits another refinement $\chi_{x'} = \chi_x^a$ for some $a=(a_v)_{v|p}$ in the Weyl group $W(\mathbf{G}'(\mathbb{Q}_p),\mathbf{T}(\mathbb{Q}_p)) = \prod_v W(\mathbf{G}(F_v),T_v)$. Each factor $W(\mathbf{G}(F_v),T_v)$ can be identified with the group of permutations $a_v:\{-n,\dots,n\}\to\{-n,\dots,n\}$ such that $a_v(-i)=-a_v(i)$ for all i, acting by

$$\begin{aligned}&a_v(\mathrm{Diag}(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}))\\&=\mathrm{Diag}(x_{a_v^{-1}(1)},\dots,x_{a_v^{-1}(n)},x_{a_v^{-1}(-n)},\dots,x_{a_v^{-1}(1)})\end{aligned}$$

on T_v, where by convention $x_{-i}=x_i^{-1}$ for $i<0$. Similarly we define $k_{v,\sigma,-i} = -k_{v,\sigma,i}$ and $\phi_{v,-i}=\phi_{v,i}^{-1}$. We also set $k_{v,\sigma,0}=0$, $\phi_{v,0}=1$. The equality $\chi_{x'} = \chi_x^a$ can also be written

$$\begin{aligned}&q_v^{(n+1)\mathrm{sign}(w(i))-w(i)}\phi_{v,n+1-w(i)}(x)\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,w(i)}}\\&=q_v^{(n+1)\mathrm{sign}(i)-i}\phi_{v,n+1-i}(x')\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,i}}\end{aligned}$$

for all $-n\le i\le n$, where $\mathrm{sign}(j)=+1$ (resp. 0, -1) if $j>0$ (resp. $j=0$, $j<0$). Equivalently,

$$\begin{aligned}&\phi_{v,n+1-i}(x')\\&=\phi_{v,n+1-w(i)}(x)q_v^{i-w(i)+(n+1)(\mathrm{sign}(i)-\mathrm{sign}(w(i)))}\prod_{\sigma\in\Upsilon_v}\sigma(\varpi_v)^{k_{v,\sigma,w(i)}-k_{v,\sigma,i}}.\end{aligned}$$

This last formula will be useful in the proof of the main result.

3.2.3. Main result.

Lemma 3.2.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\rho : \mathrm{Gal}_K \to \mathrm{GL}_N(\overline{\mathbb{Q}_p})$ be a continuous and crystalline representation. Let $(D, \varphi, \mathrm{Fil}^i D \otimes_{K_0} K)$ be the associated filtered φ-module. Denote by $\kappa_{\sigma,1} \leq \cdots \leq \kappa_{\sigma,N}$ the Hodge-Tate weights associated with the embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. Let $\varphi_1, \ldots, \varphi_N$ be the eigenvalues of the linear operator φ^f (on any of the D_{σ_0}, $\sigma_0 \in \Upsilon_0$), and suppose they are distinct. Finally, assume that for some $\tau \in \Upsilon$, for all i,*

$$\left| v_p(\varphi_i) - \frac{1}{e} \sum_{\sigma \in \Upsilon} \kappa_{\sigma,i} \right| \leq \frac{1}{eN} \min_{1 \leq j \leq N-1} \kappa_{\tau,j+1} - \kappa_{\tau,j}.$$

Then if $D' \subset D$ is an admissible sub-φ-module over $K_0 \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ (corresponding to a subrepresentation of ρ), there is a subset I of $\{1, \ldots, N\}$ such that D' has φ^f-eigenvalues $(\varphi_i)_{i \in I}$ and τ-Hodge-Tate weights $(\kappa_{\sigma,i})_{i \in I}$.

Proof. Since the eigenvalues of φ^f are distinct, and D' is stable under φ, there is a subset I of $\{1, \ldots, N\}$ such that $D' = \ker \prod_{i \in I} \left(\varphi^f - \varphi_i\right)$. There are unique increasing functions $\theta_{1,\sigma} : I \to \{1, \ldots, N\}$ such that the σ-weights of D' are the $\kappa_{\sigma,\theta_{1,\sigma}(i)}$, for $i \in I$. By ordering similarly the weights of D/D', we define increasing functions $\theta_{2,\sigma} : \{1, \ldots, N\} \setminus I \to \{1, \ldots, N\}$, and we can glue the $\theta_{\cdot,\sigma}$ to get bijective maps $\theta_\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$. We will show that $\theta_\tau = \mathrm{Id}$.

We now write the admissibility condition for D' and D/D'. Let i_1 be the smallest element of I. Then $\ker \left(\varphi^f - \varphi_{i_1}\right)$ is a sub-φ-module of D'. Its induced σ-weight is one of the $\kappa_{\sigma,\theta_\sigma(i)}$ for $i \in I$, thus it is greater than or equal to $\kappa_{\sigma,\theta_\sigma(i_1)}$. This implies that $v_p(\varphi_{i_1}) \geq 1/e \sum_{\sigma \in \Upsilon} \kappa_{\sigma,\theta_\sigma(i_1)}$. We can proceed similarly for the submodules

$$\ker \left(\left(\varphi^f - \varphi_{i_1}\right) \cdots \left(\varphi^f - \varphi_{i_r}\right) \right)$$

(where the $i_\cdot$ are the ordered elements of I), to get the inequality

$$\sum_{1 \leq x \leq r} v_p(\varphi_{i_x}) \geq \frac{1}{e} \sum_{1 \leq x \leq r} \sum_{\sigma \in \Upsilon} \kappa_{\sigma,\theta_\sigma(i_x)}$$

The same applies to D/D', and by adding both inequalities, we finally get

$$\sum_{1 \leq i \leq s} v_p(\varphi_i) \geq \frac{1}{e} \sum_{1 \leq i \leq s} \sum_{\sigma \in \Upsilon} \kappa_{\sigma,\theta_\sigma(i)}$$

We now isolate τ, using the fact that $\sum_{1\leq i\leq s}\kappa_{\sigma,\theta_\sigma(i)} \geq \sum_{1\leq i\leq s}\kappa_{\sigma,i}$ for $\sigma\neq\tau$, and obtain the inequality

$$\sum_{1\leq i\leq s} v_p(\varphi_i) - \frac{1}{e}\sum_{1\leq i\leq s}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i} \geq \frac{1}{e}\sum_{1\leq i\leq s}\kappa_{\tau,\theta_\tau(i)} - \kappa_{\tau,i}$$

Let r be minimal such that $\theta_\tau(s)\neq s$ (if no such s exists, we are done). In that case, we necessarily have $\theta_\tau(s)\geq s+1$, and the previous inequality yields

$$\sum_{1\leq i\leq s} v_p(\varphi_i) - \frac{1}{e}\sum_{1\leq i\leq s}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i} \geq \frac{\kappa_{\tau,s+1}-\kappa_{\tau,s}}{e}$$

but the hypothesis implies that the left hand side is less than $\min_j(\kappa_{\tau,j+1} - \kappa_{\tau,j})/e$, and we get a contradiction. □

Theorem 3.2.2. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. For each p-adic place v of F, let $K(v)/F_v$ be a finite extension. Let $(\Lambda_j)_{j\in J}$ be a finite family of non-constant affine functions on $(\mathbb{Q}^n)^{\mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{R})}$. Let N be an integer.*

There exists an automorphic representation $\Pi'=\otimes'_w\Pi'_w$ of $\mathbf{G}(\mathbb{A}_F)$ such that:

- *Π' is unramified at the places above p, and has invariants under U.*
- *For any p-adic place v of F, the restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to $\mathrm{Gal}_{K(v)}$ is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$.*
- *For any $j\in J$, $\Lambda_j((k'_{w,1},\dots,k'_{w,n})_w)\neq 0$ where for w a real place of F $k'_{w,1}\geq\cdots\geq k'_{w,n}\geq 0$ denotes the highest weight of Π'_w.*
- *For all g in Gal_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

Proof. We will write $\Pi'\equiv\Pi \mod p^N$ for the last property. For simplicity we only give the proof in the case where $K(v)=F_v$ for all v, the proof in the general case being identical up to notational changes.

Recall that for v a place of F above p, there are elements $\phi_{v,1},\dots,\phi_{v,n}\in\mathcal{O}(\mathscr{X})^\times$ such that for any unramified classical point $x\in\mathscr{X}(\overline{\mathbb{Q}_p})$ refining an automorphic representation Π, the filtered φ-module associated with the crystalline representation $\rho_{\iota_p,\iota_\infty}(\Pi)|_{\mathrm{Gal}_{F_v}}$ has φ^{f_v}-eigenvalues

$$\Big(\phi_{v,-n}(x)q_v^{-n}\prod_\sigma \sigma(\varpi_v)^{k_{v,\sigma,-1}},\ldots,\phi_{v,-1}(x)q_v^{-1}\prod_\sigma \sigma(\varpi_v)^{k_{v,\sigma,-n}},1,$$
$$\phi_{v,1}(x)q_v\prod_\sigma \sigma(\varpi_v)^{k_{v,\sigma,n}},\ldots,\phi_{v,n}(x)q_v^{n}\prod_\sigma \sigma(\varpi_v)^{k_{v,\sigma,1}}\Big)$$

and σ-Hodge-Tate weights

$$k_{v,\sigma,-1}-n,\ldots,k_{v,\sigma,-n}-1,0,k_{v,\sigma,n}+1,\ldots,k_{v,\sigma,1}+n$$

In the following if x_b or x'_b is a classical point, $k^{(b)}_{v,\sigma,i}$ will be the weights defining $w(x_b)$. The representation Π corresponds to at least one point x of the eigenvariety $\mathscr{X}$ for $\mathbf{G}'$ and the idempotent $e_U\otimes e_{G_0}$. By Proposition 2.2.6, and since Gal_F is compact, there exists a point $x_1\in\mathscr{X}(E')$ (near x, and for some finite extension E' of E) corresponding to an unramified, completely refinable automorphic representation Π_1 and a refinement χ, such that for any v,

$$\frac{2}{e_v}\sum_{i=1}^n\sum_\sigma k^{(1)}_{v,\sigma,i} > -v_p\left(\phi_{v,1}(x_1)\cdots\phi_{v,n}(x_1)\right)+3n(n+1)f_v$$

and $\Pi_1\equiv\Pi \mod p^N$. Since Π_1 is completely refinable, there is a point $x'_1\in\mathscr{X}(E')$ associated with the representation Π_1 and the character χ^a, where a is the element of the Weyl group acting as $-\mathrm{Id}$ on the roots. Specifically, $\Psi_{x_1}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}=\Psi_{x'_1}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}$, but

$$\phi_{v,n+1-i}(x'_1)=\phi_{v,-n-1+i}(x_1)q_v^{2i+(2n+2)}\prod_\sigma\sigma(\varpi_v)^{-2k^{(1)}_{v,\sigma,i}}$$

for $i=1,\ldots,n$, and all places v. There exists a point $x_2\in\mathscr{X}(E')$ (near x'_1, and up to enlarging E') corresponding to an unramified, completely refinable automorphic representation Π_2 and a refinement, such that for any v and any $j<0$,

$$\frac{1}{e_v}\sum_\sigma\left(k^{(2)}_{v,\sigma,n+j}-k^{(2)}_{v,\sigma,n+j+1}\right) > -v_p(\phi_{v,-j+1}(x_2))-f_v$$

and $\Pi_2\equiv\Pi_1\equiv\Pi \mod p^N$. Like before, since Π_2 is completely refinable, there is a point $x'_2\in\mathscr{X}(E')$ such that $\Psi_{x_2}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}=\Psi_{x'_2}|_{\mathcal{H}^S\otimes\mathcal{H}_S\otimes e_{G_0}}$, and

$$\phi_{v,n}(x_2') = \phi_{v,1}(x_2) q_v^{1-n} \prod_\sigma \sigma(\varpi_v)^{k^{(2)}_{v,\sigma,n} - k^{(2)}_{v,\sigma,1}}$$
$$\phi_{v,i}(x_2') = \phi_{v,i+1}(x_2) q_v \prod_\sigma \sigma(\varpi_v)^{k^{(2)}_{v,\sigma,n-i} - k^{(2)}_{v,\sigma,n-i+1}} \text{ for } i = 1, \dots, n-1.$$

Here we used the element of the Weyl group corresponding (at each v) to the permutation

$$\begin{pmatrix} -n & -n+1 & \cdots & -2 & -1 & 1 & \cdots & n \\ -n+1 & -n+2 & \cdots & -1 & -n & n & \cdots & n-1 \end{pmatrix}.$$

Again, we can choose a point $x_3 \in \mathscr{X}(E')$ (near x_1', and up to enlarging E') corresponding to an unramified automorphic representation Π_3 and a refinement, such that for any v and any $\tau \in \Upsilon$,

$$\frac{1}{e_v(2n+1)} \min\left\{k^{(3)}_{v,\tau,1} - k^{(3)}_{v,\tau,2}, \dots, k^{(3)}_{v,\tau,n-1} - k^{(3)}_{v,\tau,n}, k^{(3)}_{v,\tau,n}\right\}$$
$$> \max\left\{0, |v_p(\phi_{v,\tau,1}(x_3))|, \dots, |v_p(\phi_{v,\tau,n}(x_3))|\right\}$$

and $\Pi_3 \equiv \Pi \mod p^N$. By Proposition 2.2.6 we can also assume that none of the affine functions Λ_j vanishes at the highest weight of $(\Pi_{3,w})_{w|\infty}$. Let us show that Π_3 has the desired properties. First we apply the previous lemma to the crystalline representation $\rho_{\iota_p,\iota_\infty}(\Pi_3)|_{\mathrm{Gal}_{F_v}}$ where v is any p-adic place of F. Since the differences $v_p(\varphi_i) - \frac{1}{e}\sum_{\sigma\in\Upsilon}\kappa_{\sigma,i}$ in the hypotheses of the lemma are equal in our case to

$$-v_p(\phi_{v,n}(x_3)), \dots, -v_p(\phi_{v,1}(x_3)), 0, v_p(\phi_{v,1}(x_3)), \dots, v_p(\phi_{v,n}(x_3)),$$

the hypotheses of the lemma are satisfied for all $\tau \in \Upsilon$. Thus if $\rho_{\iota_p,\iota_\infty}(\pi_3)|_{\mathrm{Gal}_{F_v}}$ is not irreducible, there is a subset $\emptyset \subsetneq I \subsetneq \{-n, \dots, n\}$ such that if $i_1 < \cdots < i_r$ are the elements of I and $j_1 < \cdots < j_{2n+1-r}$ those of $J = \{-n, \dots, n\} \setminus I$,

$$\begin{aligned} v_p(\phi_{v,i_1}(x_3)) &\geq 0 \\ v_p(\phi_{v,i_1}(x_3)) + v_p(\phi_{v,i_2}(x_3)) &\geq 0 \\ &\vdots \\ v_p(\phi_{v,i_1}(x_3)) + \cdots + v_p(\phi_{v,i_r}(x_3)) &= 0 \\ v_p(\phi_{v,j_1}(x_3)) &\geq 0 \\ v_p(\phi_{v,j_1}(x_3)) + v_p(\phi_{v,j_2}(x_3)) &\geq 0 \\ &\vdots \\ v_p(\phi_{v,j_1}(x_3)) + \cdots + v_p(\phi_{v,j_{2n+1-r}}(x_3)) &= 0 \end{aligned}$$

by the admissibility of the corresponding filtered φ-modules. For all i, $v_p(\phi_{v,i}(x'_2)) = v_p(\phi_{v,i}(x_3))$, so all these conditions hold also at x'_2. Up to exchanging I and J, we can assume that $i_1 = -n$. If $j_1 < 0$,

$$\begin{aligned} v_p(\phi_{v,j_1}(x'_2)) &= -v_p(\phi_{v,-j_1}(x'_2)) \\ &= -v_p(\phi_{v,-j_1+1}(x_2)) - f_v - \frac{1}{e_v}\sum_{\sigma} k^{(2)}_{v,\sigma,n+j_1} - k^{(2)}_{v,\sigma,n+j_1+1} \end{aligned}$$

and x_2 was chosen to ensure that this quantity is negative, so we are facing a contradiction. Thus J has only nonnegative elements, and $\{-n, \ldots, -1\} \subset I$. If we do not assume that $i_1 = -n$, we have in general that $\{-n, \ldots, -1\}$ is contained in I or J. Similarly, suppose $i_r = n$. If $j_{2n+1-r} > 0$, denoting $j = j_{2n+1-r}$ for simplicity, we have that

$$\begin{aligned} v_p(\phi_{v,j}(x'_2)) &= v_p(\phi_{v,j}(x'_2)) \\ &= v_p(\phi_{v,j}(x_2)) + f_v + \frac{1}{e_v}\sum_{\sigma} k^{(2)}_{v,\sigma,n-j} - k^{(2)}_{v,\sigma,n-j+1} \end{aligned}$$

is positive, another contradiction. Therefore $\{1, \ldots, n\}$ is contained in I or J.

Assume for example that $\{-n, \ldots, -1\} \subset I$ and $\{1, \ldots, n\} \subset J$. In that case

$$\begin{aligned} & v_p(\phi_{v,j_1}(x_3) \cdots \phi_{v,j_{2n+1-r}}(x_3)) \\ &= v_p(\phi_{v,1}(x_2) \cdots \phi_{v,n}(x_2)) \\ &= v_p(\phi_{v,1}(x'_1) \cdots \phi_{v,n}(x'_1)) \\ &= -v_p(\phi_{v,1}(x_1) \cdots \phi_{v,n}(x_1)) + 3n(n+1)f_v - \frac{2}{e_v}\sum_{i=1}^{n}\sum_{\sigma} k^{(1)}_{v,\sigma,i} \end{aligned}$$

is negative, which is yet another contradiction.

As a consequence, we can conclude that I or J is equal to $\{0\}$, and this shows that at each place v of F above p, the semisimplification of $\rho_{\iota_p,\iota_\infty}(\Pi_3)|_{\mathrm{Gal}_{F_v}}$ is either irreducible or the sum of an Artin character and an irreducible representation of dimension $2n$. Consequently Π_3 has the required properties. □

3.2.4. From irreducibility to large image. Theorem 3.2.2 can be strengthened as follows, although we will not need this stronger result in this paper.

Corollary 3.2.3. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let N be an integer.*

There exists an automorphic representation $\Pi' = \otimes'_w \Pi'_w$ of $\mathbf{G}(\mathbb{A}_F)$ such that:

- *Π' is unramified at the places above p, and has invariants under U.*
- *For any p-adic place v of F, there exists a finite extension E_0 of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ and a Lie algebra $\mathfrak{h}_0$ over E_0 such that either*
  - *$\overline{\mathbb{Q}_p} \otimes_{E_0} \mathfrak{h}_0 \simeq \mathfrak{so}_{2n+1}$ and the Lie algebra of $\rho_{\iota_p,\iota_\infty}(\Pi')(\mathrm{Gal}_{F_v})$ is conjugate under $\mathrm{GL}_{2n+1}(\overline{\mathbb{Q}_p})$ to the image of $\mathfrak{h}_0$ in the standard representation of $\mathfrak{so}_{2n+1}$, or*
  - *$\overline{\mathbb{Q}_p} \otimes_{E_0} \mathfrak{h}_0 \simeq \mathfrak{so}_{2n}$ and the Lie algebra of $\rho_{\iota_p,\iota_\infty}(\Pi')(\mathrm{Gal}_{F_v})$ is conjugate under $\mathrm{GL}_{2n+1}(\overline{\mathbb{Q}_p})$ to the image of $\mathfrak{h}_0$ in the direct sum of the standard representation of $\mathfrak{so}_{2n}$ and the one-dimensional trivial representation,*
- *For all g in Gal_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

Proof. For any p-adic place v of F, let $K(v)$ be the compositum of all finite extensions of F_v of degree dividing $2n+1$ or $2n$, so that $K(v)/F_v$ is a finite Galois extension. We will apply Theorem 3.2.2 using these extensions and for a finite family $(\Lambda_j)_{j\in J}$ that we will determine below.

Let $\Pi' \in A_{\mathbf{G}}^{\mathrm{vr}}$. As we observed after Corollary 3.1.3, [4, Corollary 1.3] implies that the Galois representation $\rho := \rho_{\iota_p,\iota_\infty}(\Pi')$ is orthogonal, i.e. there exists a non-degenerate quadratic form Q on $\overline{\mathbb{Q}_p}^{2n+1}$ preserved by $\rho(\mathrm{Gal}_F)$. We abusively still denote by ρ the resulting continuous morphism $\mathrm{Gal}_F \to \mathrm{SO}(\overline{\mathbb{Q}_p}^{2n+1}, Q)$. Assume that for any p-adic place v, $\rho|_{\mathrm{Gal}_{K(v)}}$ is either an irreducible representation of dimension $2n+1$ or the direct sum of an irreducible $2n$-dimensional representation and a character. Consequently if

$\rho_v := \rho|_{\mathrm{Gal}_{F_v}}$ is not irreducible, it is isomorphic to the direct sum of an irreducible representation $\rho_{v,0}$ and $\det \rho_{v,0} : \mathrm{Gal}_{F_v} \to \{\pm 1\}$. If ρ_v is irreducible then $\rho|_{\mathrm{Gal}_{K(v)}}$ is the direct sum of irreducible representations in the same $\mathrm{Gal}(K(v)/F_v)$-orbit, in particular having same dimension, and so $\rho|_{\mathrm{Gal}_{K(v)}}$ is irreducible. If ρ_v is irreducible (resp. $\rho_v = \rho_{v,0} \oplus \det(\rho_{v,0})$), we claim that for any finite extension L/F_v, the orthogonal representation $\rho_v|_{\mathrm{Gal}_L}$ (resp. $\rho_{v,0}|_{\mathrm{Gal}_L}$) remains irreducible. This follows from [11][Lemma 4.3] and the definition of $K(v)$. Let $E \subset \overline{\mathbb{Q}_p}$ be a finite extension of $\mathbb{Q}_p$ such that Q takes values in E on E^{2n+1} and ρ takes values in $\mathrm{SO}(E^{2n+1}, Q)$. In the case where $\rho_v = \rho_{v,0} \oplus \det(\rho_{v,0})$ we can also assume that the line in $\overline{\mathbb{Q}_p}^{2n+1}$ corresponding to $\det(\rho_{v,0})$ is defined over E, i.e.

$$\bigcap_{\gamma \in \mathrm{Gal}_{F_v}} \ker(\rho_v(\gamma) - \det(\rho_{v,0}(\gamma))) = \overline{\mathbb{Q}_p} \otimes_E D$$

for a well-defined line $D \subset E^{2n+1}$. Let V be E^{2n+1} if ρ_v is irreducible, and $V = D^\perp$ if $\rho_v = \rho_{v,0} \oplus \det(\rho_{v,0})$. The Lie algebra $\mathfrak{h} = \mathrm{Lie}(\rho_v(\mathrm{Gal}_{F_v}))$ is a sub-$\mathbb{Q}_p$-Lie algebra of the E-Lie algebra $\mathfrak{so}(V, Q)$. Let $\mathfrak{g}$ be the E-span of $\mathfrak{h}$. The representation V of $\mathfrak{g}$ is faithful, traceless, and *absolutely irreducible* by the above. Therefore $\mathfrak{g}$ is a semi-simple E-Lie algebra, and the rank of $\overline{\mathbb{Q}_p} \otimes_E \mathfrak{g}$ is at most n. Note that $\dim_E V = 2$ is impossible.

We use Sen theory to relate $\mathfrak{g}$ to the weights of Π', in order to define a family $(\Lambda_j)_j$ forcing this rank to be n. Recall that $D_{\mathrm{Sen}}(V)$ is a free $F_{v,\infty} \otimes_{\mathbb{Q}_p} E$-module of rank $\dim_E V$, where $F_{v,\infty} = F_v(\mu_{p^\infty})$. It can be defined as the subspace of $\mathbb{Q}_p[\mathrm{Gal}(F_{v,\infty}/F_v)]$-finite vectors in $\left(\widehat{\overline{F_v}} \otimes_{\mathbb{Q}_p} V\right)^{\mathrm{Gal}(\overline{F_v}/F_{v,\infty})}$ where $\widehat{\overline{F_v}}$ is the completion of an algebraic closure $\overline{F_v}$ of $F_{v,\infty}$. In particular $D_{\mathrm{Sen}}(V)$ is equipped with a continuous semi-linear action of $\mathrm{Gal}(F_{v,\infty}/F_v)$, which the cyclotomic character identifies with a finite index subgroup of $\mathbb{Z}_p^\times$. This action is locally described by the linear operator $\Theta \in \mathrm{End}_{F_{v,\infty} \otimes_{\mathbb{Q}_p} E}(D_{\mathrm{Sen}}(V))$. There is a natural decomposition

$$F_{v,\infty} \otimes_{\mathbb{Q}_p} E = \prod_{\sigma : F_v \hookrightarrow E} F_{v,\infty} \otimes_{F_v} E_\sigma$$

inducing a similar decomposition of $D_{\mathrm{Sen}}(V)$ and Θ, and the characteristic polynomial of Θ_σ has coefficients in $E_\sigma \subset F_{v,\infty} \otimes_{F_v} E_\sigma$. Since V is de Rham, the roots of this polynomial are the σ-Hodge-Tate weights of V. By [37, Theorem 1], the endomorphism $1 \otimes \Theta$ of

$$\widehat{\overline{F_v}} \otimes_{F_{v,\infty}} D_{\mathrm{Sen}}(V) = \widehat{\overline{F_v}} \otimes_{\mathbb{Q}_p} V$$

belongs to $\widehat{\overline{F_v}} \otimes_{\mathbb{Q}_p} \mathfrak{h}$. In particular, for any $\mathbb{Q}_p$-embedding $\sigma : F_v \hookrightarrow E$ we have that $1 \otimes \Theta_\sigma$ belongs to $\widehat{\overline{F_v}} \otimes_{F_v,\sigma} \mathfrak{g}$, and so $\overline{\mathbb{Q}_p} \otimes_E \mathfrak{g}$ contains an element whose eigenvalues on $\overline{\mathbb{Q}_p} \otimes_E V$ are the σ-Hodge-Tate weights of V. Recall that the σ-Hodge-Tate weights of ρ_v are $\pm(k_{v,\sigma,1} + n), \ldots, \pm(k_{v,\sigma,n} + 1)$ and 0, where $k_{v,\sigma,1} \geq \cdots \geq k_{v,\sigma,n} \geq 0$ is the weight of Π'_w. Here w denotes the real place of F corresponding to (v, σ) via ι_p, ι_∞.

Let $\mathcal{R}$ be the set of isomorphism classes of pairs $(\mathfrak{k}, u)$ where $\mathfrak{k}$ is a semi-simple $\overline{\mathbb{Q}_p}$-Lie algebra of rank less than n and u is a faithful irreducible orthogonal representation of $\mathfrak{k} \to \overline{\mathbb{Q}_p}$ of dimension $2n$ or $2n+1$. From the classification of simple Lie algebras and the Weyl dimension formula one can deduce that $\mathcal{R}$ is finite. Consider such a pair $(\mathfrak{k}, u)$, and fix a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{k}$. There are finitely many n-tuples $(f_1, \ldots, f_n)$ of linear forms on $\mathfrak{c}$ such that for any $x \in \mathfrak{c}$, the eigenvalues of $u(x)$ (counted with multiplicity) are $f_1(x), -f_1(x), \ldots, f_n(x), -f_n(x)$, along with 0 if $\dim u = 2n+1$. Since $\dim \mathfrak{c} < n$, there is a non-zero linear form F on $\overline{\mathbb{Q}_p}^n$ killing $(f_1(x), \ldots, f_n(x))$ for all $x \in \mathfrak{c}$, and from the classification of representations of $\mathfrak{k}$ we see that we can take F to be (induced from) a linear form on $\mathbb{Q}^n$. Then $\Lambda(x_1, \ldots, x_n) = F(x_1 + n, \ldots, x_n + 1)$ defines a non-constant affine form Λ on $\mathbb{Q}^n$, and considering all elements of $\mathcal{R}$ and all n-tuples of linear forms on a Cartan subalgebra, we get a finite family $(\Lambda_j)_j$. If we apply Theorem 3.2.2 with $(K(v))_{v|p}$ as above and this family $(\Lambda_j)_j$, we see that for any p-adic place v of F, the Lie algebra $\mathfrak{g}$ defined above has rank n over $\overline{\mathbb{Q}_p}$. We will see that this family $(\Lambda_j)_j$ is enough to conclude, except perhaps in the case where $\dim_E V = 4$.

By examining [25, Table 9 on p.147], which classifies sub-Lie algebras containing a Cartan subalgebra of any given simple Lie algebra, and using the fact that the representation V of $\mathfrak{g}$ is absolutely irreducible, we see that $\mathfrak{g} = \mathfrak{so}(V, Q)$. We now distinguish two cases:

- If $\dim_E V \neq 4$, then $\mathfrak{so}(V, Q)$ is absolutely simple and by Lemma 3.2.4 below, this implies that $\mathfrak{h} \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}_p}(\mathfrak{g})$ is isomorphic to $\mathrm{Res}_{E_0/K}(\mathfrak{h}_0) \hookrightarrow \mathrm{Res}_{E/K}(E \otimes_{E_0} \mathfrak{h}_0)$ where E_0/K is the subextension of E/K defined by $E_0 = \mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{h}})$.
- If $\dim_E V = 4$, then $\mathfrak{so}(V, Q) \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2$ as Lie algebras over E. The proof of Lemma 3.2.4 below still shows that $\mathfrak{h}$ is semi-simple. If either projection $\mathfrak{h} \to \mathrm{Res}_{E/\mathbb{Q}_p}(\mathfrak{sl}_2)$ is not injective, it is easy to conclude that $\mathfrak{h} \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}_p}(\mathfrak{g})$ is canonically isomorphic to $\mathrm{Res}_{E_1/\mathbb{Q}_p}(\mathfrak{sl}_2) \times \mathrm{Res}_{E_2/\mathbb{Q}_p}(\mathfrak{sl}_2) \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}_p}(\mathfrak{g})$ for canonical E_1, E_2 as before. Otherwise we can apply the lemma to the images of $\mathfrak{h}$ in both projections, and

conclude that $\mathfrak{h} \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}_p}(\mathfrak{g})$ is isomorphic to $\mathrm{Res}_{E_0/\mathbb{Q}_p}(\mathfrak{h}_0) \hookrightarrow \mathrm{Res}_{E/\mathbb{Q}_p}((E_{\tau_1} \otimes_{E_0} \mathfrak{h}_0) \times (E_{\tau_2} \otimes_{E_0} \mathfrak{h}_0))$ given by two *distinct* $\mathbb{Q}_p$-embeddings $\tau_1, \tau_2 : E_0 \to E$. Here E_τ denotes E seen as an extension of E_0 using τ. For $\sigma : F_v \hookrightarrow E$ denote by κ_σ the non-negative eigenvalue of Θ_σ in the 2-dimensional representation of $\mathfrak{h}_0$ defined by $\mathfrak{h}_0 \hookrightarrow E_{\tau_1} \otimes_{E_0} \mathfrak{h}_0 \simeq \mathfrak{sl}_2$ (over E). Let $\gamma \in \mathrm{Gal}(E/\mathbb{Q}_p)$ be such that $\tau_1 = \gamma \circ \tau_2$. Then the eigenvalues of Θ_σ for the representation $\rho_{v,0}$ on V are $\pm\kappa_\sigma \pm \kappa_{\gamma\circ\sigma}$. This yields a relation between the σ-Hodge-Tate weights of $\rho_{v,0}$, of the form

$$\sum_{\sigma : F_v \hookrightarrow E} \epsilon_\sigma (k_{v,\sigma,2} + 1) = 0$$

for signs $\epsilon_\sigma \in \{\pm 1\}$. By adding these relations (for all choices of signs ϵ_σ) to the family $(\Lambda_j)_j$ we prevent this special case from occurring. □

Lemma 3.2.4. *Let $\mathfrak{g}$ be an absolutely simple Lie algebra over a field E of characteristic 0. Let K be a subfield of E over which E has finite dimension, and let $\mathfrak{h}$ be a sub-K-Lie algebra of $\mathrm{Res}_{E/K}(\mathfrak{g})$ such that $\mathfrak{g}$, considered as a vector space over E, is generated by $\mathfrak{h}$. Then there exists a canonical subextension E_0/K of E/K such that the K-Lie algebra embedding $\mathfrak{h} \hookrightarrow \mathrm{Res}_{E/K}(\mathfrak{g})$ is isomorphic to $\mathrm{Res}_{E_0/K}(\mathfrak{h}_0) \hookrightarrow \mathrm{Res}_{E/K}(E \otimes_{E_0} \mathfrak{h}_0)$ for an absolutely simple Lie algebra $\mathfrak{h}_0$ over E_0.*

Proof. First we observe that $\mathfrak{h}$ is semi-simple: let $\mathfrak{r}$ be the radical of $\mathfrak{h}$, then the E-span of $\mathfrak{r}$ is a solvable ideal in the E-span of $\mathfrak{h}$, that is $\mathfrak{g}$, and thus $\mathfrak{r} = 0$.

Furthermore, $\mathfrak{h}$ is simple. Otherwise $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero ideals of $\mathfrak{h}$, and so their E-spans $\mathfrak{a}'$ and $\mathfrak{b}'$ are non-zero ideals of $\mathfrak{g}$ such that $[\mathfrak{a}', \mathfrak{b}'] = 0$, in contradiction with the assumption that $\mathfrak{g}$ is simple.

Therefore $\mathfrak{h} = \mathrm{Res}_{E_0/K}(\mathfrak{h}_0)$, where the (commutative) field E_0 is the endomorphism K-algebra of the adjoint representation of $\mathfrak{h}$, and $\mathfrak{h}_0$ is absolutely simple. To lighten notation we will denote $\mathfrak{g}' = \mathrm{Res}_{E/K}(\mathfrak{g})$. As K-representations of $\mathfrak{h}$, $E \otimes_K \mathrm{ad}_{\mathfrak{h}} \twoheadrightarrow \mathrm{ad}_{\mathfrak{g}'}$ and thus $\mathrm{ad}_{\mathfrak{g}'} = V \otimes_{E_0} \mathrm{ad}_{\mathfrak{h}}$ where V is the space of $\mathfrak{h}$-invariants in $\mathrm{ad}_{\mathfrak{g}'} \otimes_K (\mathrm{ad}_{\mathfrak{h}})^*$ endowed with the natural action of E_0 on $(\mathrm{ad}_{\mathfrak{h}})^*$. This action is naturally a right action, but the distinction is irrelevant because E_0 is commutative. Thus $\mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{g}'}) = \mathrm{End}_{E_0}(V)$ and E_0 is canonically identified with the center of $\mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{g}'})$. Note that $E = \mathrm{End}_{\mathfrak{g}'}(\mathrm{ad}_{\mathfrak{g}'})$ is contained in $\mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{g}'})$, and thus any element of the center of $\mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{g}'})$ is E-linear. Since $\mathfrak{h}$ spans $\mathfrak{g}$ over E,

$\mathrm{End}_{\mathfrak{h}}(\mathrm{ad}_{\mathfrak{g}'}) \cap \mathrm{End}_E(\mathrm{ad}_{\mathfrak{g}'}) = \mathrm{End}_{\mathfrak{g}'}(\mathrm{ad}_{\mathfrak{g}'}) = E$, and thus E_0/K is a subextension of E/K.

The natural morphism $E \otimes_{E_0} \mathfrak{h}_0 \to \mathfrak{g}$ is surjective, and also injective because $E \otimes_{E_0} \mathfrak{h}_0$ is simple. □

4. Similar results for even orthogonal groups

In this section we explain (very) briefly how the same method as in the previous sections applies to orthogonal groups.

Let F be a totally real number field of even degree over $\mathbb{Q}$. Then F has an even number of 2-adic places of odd degree over $\mathbb{Q}_2$, and as these are the only finite places of F at which $(-1,-1)_v = -1$ (where $(\cdot,\cdot)_v$ denotes the Hilbert symbol), we have $\prod_v (-1,-1)_v = 1$ where the product ranges over the finite places of F. Consequently, there is a unique quadratic form on F^4 which is positive definite at the real places of F, and split (isomorphic to $(x,y,z,t) \mapsto xy + zt$) at the finite places. It has Hasse invariant $(-1,-1)_v$ at each finite place v of F, and its discriminant is 1. As a consequence, for any integer $n \geq 1$, there is a connected reductive group $\mathbf{G}$ over F which is compact (and connected) at the real places (isomorphic to $\mathbf{SO}_{4n}/\mathbb{R}$) and split at all the finite places (isomorphic to the split reductive group $\mathbf{SO}_{4n}$). As before, we let $\mathbf{G}' = \mathrm{Res}_{\mathbb{Q}}^F(\mathbf{G})$. The proofs of the existence and properties of the attached eigenvariety $\mathscr{X} \to \mathscr{W}$ are identical to the symplectic case. We could not find a result as precise as Theorem 2.2.4 in the literature, however by [16, Proposition 3.5] unramified principal series are irreducible on an explicit Zariski-open subset of the unramified characters. Specifically, if $\mathbf{SO}_{4n}(F_v) = \left\{ M \in \mathrm{M}_{4n}(F_v) \mid {}^t M J_{4n} M = J_{4n} \right\}$,

$$T = \left\{ \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_{2n} & & & \\ & & & x_{2n}^{-1} & & \\ & & & & \ddots & \\ & & & & & x_1^{-1} \end{pmatrix} \middle| \; x_i \in F_v^\times \right\}$$

and P is any parabolic subgroup containing T, then for an unramified character $\chi = (\chi_1, \ldots, \chi_n)$ of T (χ_i is a character of the variable x_i), $\mathrm{Ind}_P^{\mathbf{SO}_{4n}(F_v)}(\chi)$ is irreducible if $\chi_i(\varpi_v)^2 \neq 1$ for all i and $\chi_i(\varpi_v)\chi_j(\varpi_v)^{\pm 1} \neq 1, q_v, q_v^{-1}$ for all $i < j$. Note that this is not an equivalence.

The existence of Galois representations $\rho_{\iota_p,\iota_\infty}(\Pi)$ attached to automorphic representations Π of $\mathbf{G}(\mathbb{A}_F)$ is identical to Proposition 3.1.1. We now state the main result for orthogonal groups.

Theorem 4.0.1. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. For each p-adic place v of F, let $K(v)/F_v$ be a finite extension. Let $(\Lambda_j)_{j\in J}$ be a finite family of non-constant affine functions on $(\mathbb{Q}^{2n})^{\mathrm{Hom}_{\mathbb{Q}}(F,\mathbb{R})}$. Let N be an integer. There exists an automorphic representation Π' of $\mathbf{G}(\mathbb{A}_F)$ such that:*

- *Π' is unramified at the places above p, and has invariants under U.*
- *For any p-adic place v of F, the restriction of $\rho_{\iota_p,\iota_\infty}(\Pi')$ to $\mathrm{Gal}_{K(v)}$ is irreducible.*
- *For any $j \in J$, $\Lambda_j((k'_{w,1},\dots,k'_{w,n})_w) \neq 0$ where for w a real place of F, $k'_{w,1} \geq \cdots \geq k'_{w,2n-1} \geq |k'_{w,2n}| \geq 0$ denotes the highest weight of Π'_w.*
- *For all g in Gal_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.*

Proof. The proof is nearly identical to that of Theorem 3.2.2. In the orthogonal case the Weyl group is a bit smaller: it is the semi-direct product of S_{2n} and a hyperplane of $(\mathbb{Z}/2\mathbb{Z})^{2n}$. Alternatively, it is the group of permutations w of $\{-2n,\dots,-1,1,\dots,2n\}$ such that $w(-i) = -w(i)$ for all i and $\prod_{i=1}^{2n} w(i) > 0$. The two elements of the Weyl group used in the proof of Theorem 3.2.2 have natural counterparts in this Weyl group. The only difference lies in the fact that there is no Hodge-Tate weight equal to 0 in the present case, at least for generic weights, hence the simpler conclusion "$\rho_{\iota_p,\iota_\infty}(\Pi')|_{\mathrm{Gal}_{F_v}}$ is irreducible for $v|p$". □

As in the previous case, this result can be strengthened as follows.

Corollary 4.0.2. *Let Π be an irreducible automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ having Iwahori invariants at all the places of F above p, and having invariants under an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$. Let N be an integer.*

There exists an automorphic representation $\Pi' = \otimes'_w \Pi'_w$ of $\mathbf{G}(\mathbb{A}_F)$ such that:

- *Π' is unramified at the places above p, and has invariants under U.*
- *For any p-adic place v of F, there exists a finite extension E_0 of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ and a Lie algebra $\mathfrak{h}_0$ over E_0 such that $\overline{\mathbb{Q}_p} \otimes_{E_0} \mathfrak{h}_0 \simeq \mathfrak{so}_{4n}$ and*

the Lie algebra of $\rho_{\iota_p,\iota_\infty}(\Pi')(\mathrm{Gal}_{F_v})$ *is conjugate under* $\mathrm{GL}_{4n}(\overline{\mathbb{Q}_p})$ *to the image of* $\mathfrak{h}_0$ *in the standard representation of* $\mathfrak{so}_{4n}$.

- *For all* g *in* Gal_F, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(g)) \equiv \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(g)) \mod p^N$.

The proof is similar to that of Corollary 3.2.3, and even simpler.

Remark 4.0.3. For simplicity we chose to work with reductive groups $\mathbf{G}$ over F which are split at all finite places of F, but with nearly identical proofs one can show similar results at least for even special orthogonal groups $\mathbf{G}$ such that

- $\mathbf{G}(\mathbb{R} \otimes_{\mathbb{Q}} F)$ is compact,
- for any finite place v of F, $\mathbf{G}_{F_v}$ is quasi-split,
- for any p-adic place v of F, $\mathbf{G}_{F_v}$ is split.

In fact the second assumption, which is only required to apply [40] in order to show that one can attach Galois representations to automorphic representations for $\mathbf{G}$, is unnecessary since we do not need the precise multiplicity formula [40, Theorem 4.0.1], but simply the analogue of Proposition 3.1.1. This can be proved directly using the stabilization of the trace formula, similarly to the proof of [40, Theorem 4.0.1].

5. The image of complex conjugation: relaxing hypotheses in Taylor's theorem

Let us apply the previous results to the determination of the image of the complex conjugations under the p-adic Galois representations associated with regular, algebraic, essentially self-dual, cuspidal automorphic representations of $\mathbf{GL}_n(\mathbb{A}_F)$, F totally real. Recall that these representations are constructed by "patching" representations of Galois groups of CM extensions of F, on Shimura varieties for unitary groups. The complex conjugations are lost when we restrict to CM fields. In [42], Taylor proves that the image of any complex conjugation is given by (the "discrete" part of) the local Langlands parameter at the corresponding real place, assuming n is odd and the Galois representation is irreducible, by constructing the complex conjugation on the Shimura datum. Of course the Galois representation associated with a cuspidal representation of $\mathbf{GL}_n$ is conjectured to be irreducible, but unfortunately this is (at the time of writing) still out of reach in the general case (however, see [11] for $n \leq 5$; [3, Theorem D] for a "density one" result for arbitrary n but under the assumption that F is CM and the automorphic representation is "extremely regular" at the archimedean places; and [35] for a "positive density" result for arbitrary n and without these assumptions).

The results of the first part of this paper allow us to remove the irreducibility hypothesis in Taylor's theorem, and to extend it to some ("half") cases of even n, using Arthur's endoscopic transfer. Unfortunately some even-dimensional cases are out of reach using this method, because odd-dimensional essentially self-dual cuspidal representations are (up to a twist) self-dual, whereas some even-dimensional ones are not.

Since the proof is not direct, let us outline the strategy. First we deduce the even-dimensional self-dual case from Taylor's theorem by adding a cuspidal self-dual (with appropriate weights) representation of $\mathbf{GL}_3$, we get an automorphic self-dual representation of $\mathbf{GL}_{2n+3}$ which (up to base change) can be "transferred" to a discrete representation of the symplectic group in dimension $2n+2$. Since the associated Galois representation contains no Artin character, it can be deformed irreducibly, and Taylor's theorem applies. Then the general odd-dimensional case is deduced from the even-dimensional one, by essentially the same method, using the eigenvariety for orthogonal groups.

5.1. Regular, L-algebraic, self-dual, cuspidal representations of $\mathbf{GL}_{2n}(\mathbb{A}_F)$ having Iwahori-invariants

In this subsection $\mathbf{G}$ will denote the symplectic group in dimension $2n+2$ defined in Section 2. The following is due to C. Mœglin and J.-L. Waldspurger.

Lemma 5.1.1. *Let K be a finite extension of $\mathbb{Q}_p$. Let $\phi : W_K \times \mathrm{SU}(2) \to \mathrm{SO}_{2n+3}(\mathbb{C})$ be a Langlands parameter (equivalently, a generic Arthur parameter). Assume that the subgroup $I \times \{1\}$ (I being the inertia subgroup of W_K) is contained in the kernel of ϕ.*

Then the Arthur packet associated with ϕ contains a representation having a non-zero vector fixed under the Iwahori subgroup of $\mathbf{Sp}_{2n+2}(K)$.

Proof. Denote by $\{\Pi_1, \dots, \Pi_k\}$ this Arthur packet. Since Arthur's construction of the Π_i's is inductive for parameters trivial on the supplementary $\mathrm{SL}_2(\mathbb{C})$, and subquotients of parabolic inductions of representations having Iwahori-invariants also have Iwahori-invariants, it is enough to prove the result in the case where ϕ is discrete. Let τ be the irreducible smooth representation of $\mathbf{GL}_{2n+3}(K)$ having parameter ϕ, then $\tau \simeq \mathrm{Ind}_{\mathbf{L}}^{\mathbf{GL}_{2n+3}}(\sigma)$, where σ is the tensor product of (square-integrable) Steinberg representations $\mathrm{St}(\chi_i, n_i)$

of $\mathbf{GL}_{n_i}(K)$ ($i \in \{1, \ldots, r\}$), χ_i are unramified, self-dual characters of $K^\times$ (thus $\chi_i = 1$ or $(-1)^{v(\cdot)}$), and the pairs (χ_i, n_i) are distinct. Here $\mathbf{L}$ denotes the standard parabolic associated with the decomposition $2n+3 = \sum_i n_i$. Since ϕ is self-dual, τ can be extended (not uniquely, but this will not matter for our purpose) to a representation of $\widetilde{\mathbf{GL}}^+_{2n+3} = \mathbf{GL}_{2n+3} \rtimes \{1, \theta\}$, where

$$\theta(g) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} {}^t g^{-1} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

Let also $\widetilde{\mathbf{GL}}_{2n+3} = \mathbf{GL}_{2n+3} \rtimes \theta$.

Let N_0 be the number of i such that n_i is odd, and for $j \geq 1$ let N_j be the number of i such that $n_i \geq 2j$. Then $N_0 + 2\sum_{j \geq 1} N_j = 2n+3$, and if s is maximal such that $N_s > 0$, we let

$$\mathbf{M} = \mathbf{GL}_{N_s} \times \cdots \times \mathbf{GL}_{N_1} \times \mathbf{GL}_{N_0} \times \mathbf{GL}_{N_1} \times \cdots \times \mathbf{GL}_{N_s}$$

which is a θ-stable Levi subgroup of $\mathbf{GL}_{2n+3}$, allowing us to define $\widetilde{\mathbf{M}}^+$ and $\widetilde{\mathbf{M}}$. Since the standard (block upper triangular) parabolic containing $\mathbf{M}$ is also stable under θ, the Jacquet module $\tau_{\mathbf{M}}$ is naturally a representation of $\widetilde{\mathbf{M}}^+$, denoted by $\tau_{\widetilde{\mathbf{M}}}$. The constituents of the semi-simplification of $\tau_{\widetilde{\mathbf{M}}}$ either stay irreducible when restricted to $\mathbf{M}$, in which case they are of the form $\sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$ where σ_1 is a representation of $\mathbf{GL}_{N_s}(K) \times \cdots \times \mathbf{GL}_{N_1}(K)$ and σ_0 is a representation of $\widetilde{\mathbf{GL}}_{N_0}(K)$; or they are induced from $\mathbf{M}(K)$ to $\widetilde{\mathbf{M}}^+(K)$, and the restriction of their character to $\widetilde{\mathbf{M}}(K)$ is zero. Since we are precisely interested in that character, we can forget about the second case. By the geometrical lemma,

$$\tau_{\mathbf{M}}^{\mathrm{ss}} \simeq \bigoplus_{w \in W^{\mathbf{L},\mathbf{M}}} \mathrm{Ind}_{\mathbf{M} \cap w(\mathbf{L})}^{\mathbf{M}} w\left(\sigma_{\mathbf{L} \cap w^{-1}\mathbf{M}}\right)$$

where $W^{\mathbf{L},\mathbf{M}}$ is the set of $w \in S_{2n+3}$ such that w is increasing on $I_1 = \{1, \ldots, n_1\}$, $I_2 = \{n_1+1, \ldots, n_1+n_2\}$, etc. and w^{-1} is increasing on $J_{-s} = \{1, \ldots, N_s\}$, $J_{-s+1} = \{N_s+1, \ldots, N_s+N_{s-1}\}$, etc. Fix the irreducible representation of $\mathbf{GL}_{N_s}(K) \times \cdots \times \mathbf{GL}_{N_1}(K)$

$$\sigma_1 = \bigotimes_{j=1}^{s} \mathrm{Ind}_{\mathbf{T}_j}^{\mathbf{GL}_{N_j}} \bigotimes_{i \mid n_i \geq 2j} \chi_i |\cdot|^{j-\nu_i}$$

where $\mathbf{T}_j$ is the standard maximal torus of $\mathbf{GL}_{N_j}$, $\nu_i = \begin{cases} 0 & n_i \text{ odd} \\ 1/2 & n_i \text{ even} \end{cases}$.

There is a unique w such that $\mathrm{Ind}_{\mathbf{M}\cap w(\mathbf{L})}^{\mathbf{M}} w\left(\sigma_{\mathbf{L}\cap w^{-1}(\mathbf{M})}\right)$ admits a subquotient of the form $\sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$ as above, moreover

$$\mathrm{Ind}_{\mathbf{M}\cap w(\mathbf{L})}^{\mathbf{M}} w\left(\sigma_{\mathbf{L}\cap w^{-1}(\mathbf{M})}\right)$$

is irreducible, and

$$\sigma_0 = \mathrm{Ind}_{\mathbf{T}_0}^{\mathbf{GL}_{N_0}} \bigotimes_{i \mid n_i \text{ odd}} \chi_i$$

Specifically, w maps the first element of I_i in $J_{-\lfloor (n_i+1)/2 \rfloor}$, the second in $J_{-\lfloor (n_i+1)/2 \rfloor} + 1$, ..., the central element (if n_i is odd) in J_0, etc.

Let $\mathbf{M}'$ be the parabolic subgroup of $\mathbf{Sp}_{2n+2}/K$ corresponding to $\mathbf{M}$, i.e.

$$\mathbf{M}' = \mathbf{GL}_{N_s} \times \cdots \times \mathbf{GL}_{N_1} \times \mathbf{Sp}_{N_0 - 1}$$

By [1, 2.2.6], $\sum_i \mathrm{Tr}\Pi_i$ is a stable transfer of $\mathrm{Tr}_{\mathbf{GL}^+_{2n+3}} \tau$. By [33, Lemme 4.2.1] (more accurately, the proof of the lemma),

$$\sum_i \mathrm{Tr}\left((\Pi_i)^{\mathrm{ss}}_{\mathbf{M}'}[\sigma_1]\right)$$

is a stable transfer of $\mathrm{Tr}\left(\tau^{\mathrm{ss}}_{\widetilde{\mathbf{M}}}[\sigma_1]\right)$ (where $\cdot[\cdot]$ denotes the isotypical component on the factor $\mathbf{GL}_{N_s} \times \cdots \times \mathbf{GL}_{N_1}$).

Since $\tau^{\mathrm{ss}}_{\widetilde{\mathbf{M}}}[\sigma_1] = \sigma_1 \otimes \sigma_0 \otimes \theta(\sigma_1)$, the stable transfer of $\mathrm{Tr}\left(\tau^{\mathrm{ss}}_{\widetilde{\mathbf{M}}}[\sigma_1]\right)$ is equal to the product of $\mathrm{Tr}(\sigma_1)$ and $\sum_l \mathrm{Tr}\Pi'_l$ where the Π'_l are the elements of the Arthur packet associated with the parameter

$$\bigoplus_{i \mid n_i \text{ odd}} \chi_i$$

At least one representation Π'_l is unramified for some hyperspecial compact subgroup of $\mathbf{Sp}_{N_0-1}(K)$, and so a Jacquet module of a Π_i contains a nonzero vector fixed by an Iwahori subgroup. This proves that at least one of the Π_i has Iwahori-invariant vectors. □

Proposition 5.1.2. *Let F_0 be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal (RLASDC) representation of $\mathbf{GL}_{2n}(\mathbb{A}_{F_0})$. Assume that for any place $v|p$ of F_0, π_v has vectors fixed under an Iwahori* *subgroup of* $\mathbf{GL}_{2n}(\mathbb{A}_{F_{0,v}})$*. Then there exists a RLASDC representation* π_0 *of* $\mathbf{GL}_3(\mathbb{A}_{F_0})$*, a totally real quadratic extension* F/F_0*, and an automorphic representation* Π *of* $\mathbf{G}(\mathbb{A}_F)$ *such that*

1) *for any place* $v|p$ *of* F_0*,* $\pi_{0,v}$ *is unramified,*
2) $\mathrm{BC}_{F/F_0}(\pi)$ *and* $\mathrm{BC}_{F/F_0}(\pi_0)$ *remain cuspidal,*
3) *for any place* v *of* F *above* p*,* Π_v *has invariants under the action of the Iwahori subgroup of* $\mathbf{G}(F_v)$*,*
4) *for any finite place* v *of* F *such that* $\mathrm{BC}_{F/F_0}(\pi)_v$ *and* $\mathrm{BC}_{F/F_0}(\pi_0)_v$ *are unramified,* Π_v *is unramified, and via the inclusion* $\mathrm{SO}_{2n+3}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+3}(\mathbb{C})$*, the Satake parameter of* Π_v *is equal to the direct sum of those of* $\mathrm{BC}_{F/F_0}(\pi)_v$ *and* $\mathrm{BC}_{F/F_0}(\pi_0)_v$*.*

Proof. First we construct π_0. Let δ be a cuspidal automorphic representation of $\mathbf{PGL}_2/F_0$ which is unramified at the p-adic places, Steinberg at some finite place not lying above p, and whose local langlands parameters at the real places are of the form $\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}(z \mapsto (z/\bar{z})^a)$ where a is a half-integer big enough with respects to the weights appearing in the local Langlands parameters of π. Such a representation exists thanks to [22, Theorem 1B]. Let π_0 be the automorphic representation of $\mathbf{GL}_3/F_0$ obtained by functoriality from δ through the adjoint representation of $\widehat{\mathbf{PGL}_2} = \mathrm{SL}_2(\mathbb{C})$ on its Lie algebra. The representation π_0 exists and is cuspidal by [26, Theorem 9.3]. The Steinberg condition ensures that no twist of δ by a non-trivial character (seen as a representation of $\mathbf{GL}_2/F_0$) is isomorphic to δ, and cuspidality of π_0 follows. We can twist π_0 by the central character of π to ensure that π and π_0 have the same central character. Clearly π_0 is a RLASDC representation of $\mathbf{GL}_3/F_0$.

Note that if F/F_0 is a quadratic extension, for $\mathrm{BC}_{F/F_0}(\pi)$ and $\mathrm{BC}_{F/F_0}(\pi_0)$ to remain cuspidal it is enough for F/F_0 to be ramified above a finite place of F_0 at which π and π_0 are unramified. Fix a totally real quadratic extension F/F_0 satisfying this condition and which is split at any p-adic place of F_0. Since $[F:\mathbb{Q}]$ is even we can define the connected reductive group $\mathbf{G}$ over F as before, which is an inner form of $\mathbf{Sp}_{2n+2}$. We use [40] to check the existence of Π. For the rest of the proof we use notations from [40].

We claim that $\psi := \mathrm{BC}_{F/F_0}(\pi) \boxplus \mathrm{BC}_{F/F_0}(\pi_0)$ defines an element of $\Psi_{\mathrm{disc}}(\mathbf{G})$. The only non-obvious property that we have to check is that the quasi-split reductive group $\mathbf{H}$ over F associated to $\mathrm{BC}_{F/F_0}(\pi)$ by [1, Theorem 1.4.1] is such that $\widehat{\mathbf{H}} \simeq \mathrm{SO}_{2n}(\mathbb{C})$, not $\mathrm{Sp}_{2n}(\mathbb{C})$. By [1, Theorem 1.4.2], for any real place w of F the Langlands parameter of $\mathrm{BC}_{F/F_0}(\pi)_w$, that is the Langlands parameter of π_{w_0} where w_0 is the place of F_0 below w, factors through the dual group of $\mathbf{H}$ and the standard representation ${}^L\mathbf{H} \to \mathrm{GL}_{2n}(\mathbb{C})$. The Langlands parameter of π_{w_0} is the direct sum of distinct orthogonal 2-dimensional irreducible representations of W_{F_w}, and so $\widehat{\mathbf{H}}$ cannot be symplectic.

We now apply Arthur's multiplicity formula, proved in [40] for the group $\mathbf{G}$: we will choose an element $\Pi = \otimes'_w \Pi_w$ of the adélic packet $\Pi_\psi(\mathbf{G})$, and check that it is automorphic. Note that $C_\psi = \mathcal{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, and that ϵ_ψ is the trivial character of $\mathcal{S}_\psi$ (see [1, (1.5.6)]), and recall that C_ψ^+ denotes the preimage of $\mathcal{C}_\psi$ in $\widehat{\mathbf{G}}_{\mathrm{sc}} = \mathrm{Spin}_{2n+1}(\mathbb{C})$. Realise $\mathbf{G}$ as a rigid inner twist $(\mathbf{G}, \Xi, z)$ of $\mathbf{G}^* := \mathbf{Sp}_{2n+2}$. The multiplicity formula depends on the choice of an isomorphism class (for $\mathbf{G}^*(F)$-conjugacy) of global Whittaker data $\mathfrak{w}$ for $\mathbf{G}^* := \mathbf{Sp}_{2n+2}/F$ and on the choice of a realisation of $\mathbf{G}$ as a *rigid inner twist* $(\mathbf{G}, \Xi, z)$ of $\mathbf{G}^*$, a notion introduced in [30]. Let us fix the Whittaker datum $\mathfrak{w}$, and realise $\mathbf{G}$ as an inner twist $(\mathbf{G}, \Xi)$ of $\mathbf{G}^*$, i.e. choose an isomorphism $\Xi : \mathbf{G}^*_{\overline{F}} \to \mathbf{G}_{\overline{F}}$ such that for any $\sigma \in \mathrm{Gal}_F$, the automorphism $\Xi^{-1}\sigma(\Xi)$ of $\mathbf{G}^*_{\overline{F}}$ is inner. This defines $z_{\mathrm{ad}} \in Z^1(F, \mathbf{G}^*_{\mathrm{ad}})$, and below we will choose a lift z of z_{ad} in $Z^1(P_{F,\dot{V}} \to \mathcal{E}_{F,\dot{V}}, \mathbf{Z} \to \mathbf{G}^*)$, where $\mathbf{Z} = \mathbf{Z}(\mathbf{G}^*) \simeq \mu_2$. For any place v of F_0 splitting into v_1 and v_2 in F, $F_{v_1} = F_{v_2} = F_{0,v}$ and we claim that there exists an isomorphism $f_v : (\mathbf{G}^*_{F_{v_1}}, \mathfrak{w}_{v_1}) \simeq (\mathbf{G}^*_{F_{v_2}}, \mathfrak{w}_{v_2})$. Indeed, there exists an isomorphism $\mathbf{G}^*_{F_{v_1}} \simeq \mathbf{G}^*_{F_{v_2}}$, well-defined up to $\mathbf{G}^*_{\mathrm{ad}}(F_{v_2})$, and this group acts transitively on the set of isomorphism classes of Whittaker data for $\mathbf{G}^*_{F_{v_2}}$. This isomorphism f_v is well-defined up to composing with $\mathrm{Ad}(g)$ for some $g \in \mathbf{G}^*(F_{v_2})$, and f_v induces an isomorphism $H^1(u_{v_1} \to \mathcal{E}_{v_1}, \mathbf{Z} \to \mathbf{G}^*_{F_{v_1}}) \simeq H^1(u_{v_2} \to \mathcal{E}_{v_2}, \mathbf{Z} \to \mathbf{G}^*_{F_{v_2}})$. Recall [40, §3.1.1] that there are two elements of $H^1(u_{v_i} \to \mathcal{E}_{v_i}, \mathbf{Z} \to \mathbf{G}^*_{F_{v_i}})$ lifting $\mathrm{cl}(z_{\mathrm{ad},v_i}) \in H^1(F_{v_i}, \mathbf{G}^*_{F_{v_i}})$. By [40, Proposition 3.1.2] there exists $z \in Z^1(P_{F,\dot{V}} \to \mathcal{E}_{F,\dot{V}}, \mathbf{Z} \to \mathbf{G}^*)$ lifting z_{ad} and such that for any real place v of F_0, the classes of the localizations z_{v_i} in $H^1(u_{v_i} \to \mathcal{E}_{v_i}, \mathbf{Z} \to \mathbf{G}^*)$ *differ*. Recall also that for any finite place w of F the class of z_w in $H^1(u_w \to \mathcal{E}_w, \mathbf{Z} \to \mathbf{G}^*_{F_w})$ is trivial. We now choose Π_w at each place w of F.

- For any real place v of F_0, splitting into v_1, v_2 in F, the localizations $\psi_{v_i} : W_{F_{v_i}} \to \mathrm{SO}_{2n+1}(\mathbb{C})$ for $i = 1, 2$ are conjugated under $\mathrm{SO}_{2n+1}(\mathbb{C})$, giving rise to an isomorphism between the abelian groups $\mathcal{S}^+_{\psi_{v_i}} = C^+_{\psi_{v_i}}$ which is compatible with the morphisms $C^+_\psi \to \mathcal{S}^+_{\psi_{v_i}}$. These localizations ψ_{v_i} are discrete Langlands parameters, by the choice of π_0. As explained in [40, §3.1.1] each L-packet $\Pi_{\psi_{v_i}}(\mathbf{G}_{F_{v_i}})$ has one element

Π_{v_i}, and the associated characters $\langle \cdot, \Pi_{v_i} \rangle$ of $\mathcal{S}^+_{\psi_{v_i}}$ are opposite of each other, so that $\langle \cdot, \Pi_{v_1} \rangle|_{C^+_\psi} \times \langle \cdot, \Pi_{v_2} \rangle|_{C^+_\psi}$ is the trivial character of C^+_ψ.

- For any p-adic place v of F_0, splitting into v_1, v_2 in F, the localizations $\psi_{v_i} : W_{F_{v_i}} \times \mathrm{SU}(2) \to \mathrm{SO}_{2n+1}(\mathbb{C})$ for $i = 1, 2$ are conjugated under $\mathrm{SO}_{2n+1}(\mathbb{C})$, giving rise to an isomorphism between the abelian groups $\mathcal{S}_{\psi_{v_i}}$ which is compatible with the morphisms $\mathcal{S}_\psi \to \mathcal{S}_{\psi_{v_i}}$. Furthermore, since the class of z_{v_i} in $H^1(u_{v_i} \to \mathcal{E}_{v_i}, \mathbf{Z} \to \mathbf{G}^*_{F_{v_i}})$ is trivial we obtain an isomorphism $\mathbf{G}^*_{F_{v_i}} \simeq \mathbf{G}_{F_{v_i}}$ well-defined up to composing with $\mathrm{Ad}(g)$ for some $g \in \mathbf{G}(F_{v_i})$, and by definition ([40, §3.4]) this isomorphism is compatible with the internal parametrization of Arthur packets. Using these isomorphisms to conjugate f_v, we get an isomorphism $\mathbf{G}_{F_{v_1}} \simeq \mathbf{G}_{F_{v_2}}$ which is compatible with the internal parametrization of Arthur packets. Thanks to Lemma 5.1.1 we can choose $\Pi_{v_1} \in \Pi_{\psi_{v_1}}(\mathbf{G}_{F_{v_1}})$ having Iwahori-invariants, and we let Π_{v_2} be the corresponding element of $\Pi_{\psi_{v_2}}(\mathbf{G}_{F_{v_2}})$, so that the characters $\langle \cdot, \Pi_{v_i} \rangle$ of the abelian 2-torsion groups $\mathcal{S}_{\psi_{v_i}}$ coincide. Thus $\langle \cdot, \Pi_{v_1} \rangle|_{\mathcal{S}_\psi} \times \langle \cdot, \Pi_{v_2} \rangle|_{\mathcal{S}_\psi}$ is the trivial character of $\mathcal{S}_\psi$.
- For any place w of F which is neither real nor p-adic, the class of z_w in $H^1(u_w \to \mathcal{E}_w, \mathbf{Z} \to \mathbf{G}^*_{F_w})$ is trivial and again we obtain an isomorphism $\mathbf{G}^*_{F_w} \simeq \mathbf{G}_{F_w}$ which allows to identify $\mathfrak{w}_w$ with a Whittaker datum for $\mathbf{G}_{F_w}$. Let Π_w be the trivial element of $\Pi_{\psi_w}(\mathbf{G}_{F_w})$, i.e. the representation Π_w such that $\langle \cdot, \Pi_w \rangle = 1$. If ψ_w is unramified, then Π_w is unramified for the unique $\mathbf{G}(F_w)$-conjugacy class of hyperspecial maximal compact subgroups of $\mathbf{G}(F_w)$ determined by $\mathfrak{w}_v$ (see [17]).

The element $\Pi := \otimes'_w \Pi_w$ of $\Pi_\psi(\mathbf{G})$ is such that the associated character $\langle \cdot, \Pi \rangle$ of $\mathcal{S}_\psi$, that is $\prod_w \langle \cdot, \Pi_w \rangle|_{C^+_\psi}$, is trivial, therefore [40, Theorem 4.0.1] Π occurs in the automorphic spectrum of $\mathbf{G}$. □

Proposition 5.1.3. *Let F be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal representation of $\mathbf{GL}_{2n}(\mathbb{A}_F)$. Suppose that for any place v of F above p, π_v has invariant vectors under an Iwahori subgroup. Then for any complex conjugation $c \in \mathrm{Gal}_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = 0$.*

Proof. By the previous proposition, up to quadratic base change to a totally real extension (which only restricts the Galois representation to this totally real field, so that we get even more complex conjugations), we can choose a RLASDC representation π_0 of $\mathbf{GL}_3(\mathbb{A}_F)$ and transfer $\pi \boxplus \pi_0$ to an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$. The representation Π defines (at least) one point x of the eigenvariety $\mathscr{X}$ defined by $\mathbf{G}$ (and by an open subgroup U of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$). Of course, by the Čebotarev density theorem and compatibility of transfer at the unramified places, the representation associated with Π is equal to $\rho_{\iota_p,\iota_\infty}(\pi) \oplus \rho_{\iota_p,\iota_\infty}(\pi_0)$. Since the Hodge-Tate weights of $\rho_{\iota_p,\iota_\infty}(\pi)|_{\mathrm{Gal}_{F_v}}$ are non-zero for any place $v|p$, $\rho_{\iota_p,\iota_\infty}(\pi)$ does not contain an Artin character. By [6], $\rho_{\iota_p,\iota_\infty}(\pi_0)$ is irreducible and thus does not contain any character. There are only finitely many Artin characters taking values in $\{\pm 1\}$ and unramified at all the finite places at which Π is unramified. For any such character η, the pseudocharacter T on the eigenvariety is such that $T_x - \eta$ is not a pseudocharacter, hence we can find $g_{\eta,1}, \dots, g_{\eta,2n+3}$ such that

$$t_\eta := \sum_{\sigma \in S_{2n+3}} (T_x - \eta)_\sigma (g_{\eta,1}, \dots, g_{\eta,2n+3}) \neq 0$$

Let us choose N greater than all the $v_p(t_\eta)$ and such that $p^N > 2n+4$. Let Π' be an automorphic representation of $\mathbf{G}(\mathbb{A}_F)$ satisfying the requirements of Theorem 3.2.2 for this choice of N. Then the $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')) - \eta$ are not pseudocharacters, thus $\rho_{\iota_p,\iota_\infty}(\Pi')$ does not contain an Artin character and by Theorem 3.2.2 it is irreducible. This Galois representation is (by construction in the proof of Corollary 3.1.3) the direct sum of representations associated with cuspidal representations. Since it is irreducible, there is only one of them, and it has the property that its associated Galois representations is irreducible, so that the theorem of [42] can be applied: for any complex conjugation $c \in \mathrm{Gal}_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c)) = \pm 1$. Since $\det \rho_{\iota_p,\iota_\infty}(\Pi') = 1$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c)) = (-1)^{n+1}$.

As $p^N > 2n+4$ and $|\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(c)) - \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi')(c))| \leq 2n+4$, we can conclude that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\Pi)(c)) = (-1)^{n+1}$, and hence that

$$\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) + \mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = (-1)^{n+1}.$$

We also know that

$$\det \rho_{\iota_p,\iota_\infty}(\pi_0) = \det \rho_{\iota_p,\iota_\infty}(\pi)(c) = (-1)^n,$$

and that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = \pm 1$ by Taylor's theorem, from which we can conclude that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = (-1)^{n+1}$. Thus $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = 0$. □

5.2. Regular, L-algebraic, self-dual, cuspidal representations of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$ having Iwahori-invariants

In this subsection, $\mathbf{G}$ is the orthogonal reductive group defined in Section 4, in dimension $2n+2$ if n is odd, $2n+4$ if n is even.

Lemma 5.2.1. *Let K be a finite extension of $\mathbb{Q}_p$, and $m \geq 1$ an integer. Let $\phi : W_K \times \mathrm{SU}(2) \to \mathrm{SO}_{2m}(\mathbb{C})$ be a Langlands parameter. Assume that the subgroup $I \times \{1\}$ (I being the inertia subgroup of W_K) is contained in the kernel of ϕ.*

Then the packet of representations of the split group $\mathrm{SO}_{2m}(K)$ associated with ϕ by Arthur contains a representation having a non-zero vector fixed under the Iwahori subgroup.

Proof. Of course this result is very similar to Lemma 5.1.1. However, Mœglin and Waldspurger have not put their lemma in writing in this case, and the transfer factors are no longer trivial, so that one needs to modify the definition of “stable transfer”. For this one needs to use the transfer factors $\Delta_{\widetilde{\mathbf{GL}}_{2m},\mathbf{SO}_{2m}}(\cdot,\cdot)$ defined in [31]. They depend in general on the choice of an inner class of inner twistings [31, 1.2] (in our case an inner class of isomorphisms between $\mathbf{GL}_{2m}/K$ and its quasi-split inner form defined over $\overline{K}$, which we just take to be the identity), and a Whittaker datum of the quasi-split inner form. Arthur chooses the standard splitting of $\mathbf{GL}_{2m}$ and an arbitrary character $K \to \mathbb{C}^{\times}$, but this will not matter to us since both $\mathbf{GL}_{2m}$ and $\mathbf{SO}_{2m}$ are *split*, so that the factor $\langle z_J, s_J \rangle$ of [31, 4.2] (by which the transfer factors are multiplied when another splitting is chosen) is trivial. Indeed to compute this factor we can choose the split torus $\mathbf{T_H}$ of $\mathbf{SO}_{2m}/K$, which is a norm group (see [31, Lemma 3.3B]) for the split torus $\mathbf{T}$ of $\mathbf{GL}_{2m}/K$, and thus, using the notations of [31, 4.2], $\mathbf{T}^x$ is split and $H^1(K, \mathbf{T}^x)$ is trivial, so that $z' = 1$ (z_J is the image of z' in $H^1(K, \mathbf{J})$, thus it is trivial). Since both groups are split the ϵ-factor of [31, 5.3] is also trivial, so the transfer factors are canonical.

Let $\mathbf{H} = \mathbf{SO}_{2m}/K$, $\widetilde{\tau}$ the representation of $\widetilde{\mathbf{GL}}_{2m}^{+}$ associated with ϕ, and $\tau^{\mathbf{H}}$ the sum of the elements of the packet associated with ϕ by Arthur. Note that by construction, this packet is only a finite set of orbits under $\mathbf{O}_{2m}(K)/\mathbf{H}(K) \simeq \mathbb{Z}/2\mathbb{Z}$ of isomorphism classes of irreducible, square-integrable representations of $\mathbf{H}(K)$. Each orbit has either one or two elements. In the latter case where the orbit is (say) $\{\tau_1, \tau_2\}$ one can still define a "partial" character (in the sense of Harish-Chandra):

$$\Theta_{\tau_1}(h) + \Theta_{\tau_1}(h') = \Theta_{\tau_2}(h) + \Theta_{\tau_2}(h') := \Theta_{\tau_1}(h) + \Theta_{\tau_2}(h)$$

whenever h is regular semisimple conjugacy class in $\mathbf{H}(K)$ and h' is the complement of h in its conjugacy class under $\mathbf{O}_{2m}(K)$. Although the individual terms on the left cannot be distinguished, their sum does not depend on the choice of a particular element (e.g. τ_1) in the orbit. In that setting, Arthur shows ([1, 8.3]) that the following character identity holds:

$$\sum_h |D_{\mathbf{H}}(h)|^{1/2}\Theta_{\tau^{\mathbf{H}}}(h)\Delta(h,g) = |D_{\widetilde{\mathbf{GL}}_{2m}}(g)|^{1/2}\Theta_{\tilde{\tau}}(g) \tag{5.2.1}$$

where the sum on the left runs over the the stable conjugacy classes h in $\mathbf{H}(K)$ which are norms of the conjugacy class g in $\widetilde{\mathbf{GL}}_{2m}(K)$, both assumed to be strongly $\widetilde{\mathbf{GL}}_{2m}^+$-regular. There are two such stable conjugacy classes h, they are conjugate under $\mathbf{O}_{2m}(K)$ and the two transfer factors on the left are equal (this can be seen either by going back to the definition of Kottwitz and Shelstad, or by Waldspurger's formulas recalled below). This fact together with the stability of the "partial" distribution $\Theta_{\tau^{\mathbf{H}}}$ (which is part of Arthur's results) imply that the expression on the left is well-defined. Note that as in [33] and [1], the term Δ_{IV} is not included in the product defining the transfer factor Δ. Contrary to the case of symplectic and odd orthogonal groups treated in [33], the transfer factors are not trivial, and the terms $|D_{\mathbf{H}}(h)|^{1/2}$ and $|D_{\widetilde{\mathbf{GL}}_{2m}}(g)|^{1/2}$ are not equal. However the latter play no particular role in the proof. This character identity 5.2.1 is the natural generalization of the notion of "stable transfer" of [33].

Let

$$\mathbf{M} = \mathbf{GL}_{N_s} \times \cdots \times \mathbf{GL}_{N_1} \times \mathbf{GL}_{N_0} \times \mathbf{GL}_{N_1} \times \cdots \times \mathbf{GL}_{N_s}$$

be a θ-stable Levi subgroup of $\mathbf{GL}_{2m}$, and $\mathbf{M}' = \mathbf{GL}_{N_s} \times \cdots \times \mathbf{GL}_{N_1} \times \mathbf{SO}_{N_0}$ the corresponding parabolic subgroup of $\mathbf{H}$. To mimic the proof of Lemma 5.1.1, we only need to show that $\mathrm{Tr}\left(\tau^H_{\mathbf{M}'}\right)$ is a stable transfer of $\mathrm{Tr}_{\widetilde{\mathbf{M}}}\left(\tau_{\widetilde{M}}\right)$, where "stable transfer" has the above meaning, that is the character identity 5.2.1 involving transfer factors. Note that $\widetilde{\mathbf{M}}^+$ has a factor $\mathbf{GL}_{m-N_0/2} \times \mathbf{GL}_{m-N_0/2}$ together with the automorphism $\theta(a,b) = (\theta(b),\theta(a))$, for which the theory of endoscopy is trivial: θ-conjugacy classes are in bijection with conjugacy classes in $\mathbf{GL}_{m-N_0/2}$ (over K or $\bar{K}$) via $(a,b) \mapsto a\theta(b)$ and the θ-invariant irreducible representations are the ones of the form $\sigma \otimes \theta(\sigma)$.

So we need to check that if $g = (g_1, g_0)$ is a strongly regular $\mathbf{GL}_{2m}(K)$-conjugacy class in $\widetilde{\mathbf{GL}}_{2m}(K)$ determined by a conjugacy class g_1 in $\mathbf{GL}_{m-N_0/2}(K)$ and a $\mathbf{GL}_{N_0}(K)$-conjugacy class g_0 in $\widetilde{\mathbf{GL}}_{N_0}(K)$, and if h_0 is the $\mathbf{O}_{2m}(\bar{K})$-conjugacy class in $\mathbf{H}(K)$ corresponding to g_0, then

$$\Delta_{\widetilde{\mathbf{GL}}_{N_0},\mathbf{SO}_{N_0}}(h_0, g_0) = \Delta_{\widetilde{\mathbf{GL}}_{2m},\mathbf{H}}((g_1, h_0), (g_1, g_0)).$$

Although this is most likely known by the experts (even in a general setting) we will check it. Fortunately the transfer factors have been computed by Waldspurger in [43]. We recall his notations and formulas. The conjugacy class g_1, being regular enough, is parametrized by a finite set I_1, a collection of finite extensions $K_{\pm i}$ of K for $i \in I_1$, and (regular enough, i.e. generating $K_{\pm i}$ over K) elements $x_{i,1} \in K_{\pm i}$. As in [43], g_0 is parametrized by a finite set I_0, finite extensions $K_{\pm i}$ of K, $K_{\pm i}$-algebras K_i, and $x_i \in K_i$. Each K_i is either a quadratic field extension of $K_{\pm i}$ or $K_{\pm i} \times K_{\pm i}$, and x_i is determined only modulo $N_{K_i/K_{\pm i}} K_i^\times$. Then g is parametrized by $I = I_1 \sqcup I_0$, with $K_i = K_{\pm i} \times K_{\pm i}$ and $x_i = (x_{i,1}, 1)$ for $i \in I_1$, and the same data for I_0. Let τ_i be the non-trivial $K_{\pm i}$-automorphism of K_i, and $y_i = -x_i/\tau_i(x_i)$. Let I^* be the set of $i \in I$ such that K_i is a field (so $I^* \subset I_0$). For any $i \in I$, let Φ_i be the set of K-morphisms $K_i \to \bar{K}$, and let $P_I(T) = \prod_{i \in I} \prod_{\phi \in \Phi_i} (T - \phi(y_i))$. Define P_{I_0} similarly. For $i \in I^*$ (resp. I_0^*), let $C_i = x_i^{-1} P_I'(y_i) P_I(-1) y_i^{1-m}(1 + y_i)$ (resp. $C_{i,0} = x_i^{-1} P_{I_0}'(y_i) P_{I_0}(-1) y_i^{1-m}(1 + y_i)$). We have dropped the factor η of [43, 1.10], because as remarked above, the transfer factors do not depend on the chosen splitting. Observe also that the factors computed by Waldspurger are really the factors $\Delta_0/\Delta_{\mathrm{IV}}$ of [31, 5.3], but the ϵ factor is trivial so they are complete.

Waldspurger shows that

$$\Delta_{\widetilde{\mathbf{GL}}_{2m},\mathbf{H}}((g_1, h_0), (g_1, g_0)) = \prod_{i \in I^*} \mathrm{sign}_{K_i/K_{\pm i}}(C_i)$$

where $\mathrm{sign}_{K_i/K_{\pm i}}$ is the nontrivial character of $K_{\pm i}^\times / N_{K_i/K_{\pm i}} K_i^\times$. We are left to show that $\prod_{i \in I^*} \mathrm{sign}_{K_i/K_{\pm i}}(C_i/C_{i,0}) = 1$.

$$\begin{aligned}
C_i/C_{i,0} &= y_i^{N_0/2-m} \prod_{j \in I_1} \prod_{\phi \in \Phi_j} (y_i - \phi(y_j))(-1 - \phi(y_j)) \\
&= \prod_{j \in I_1} \prod_{\phi \in \Phi_{\pm j}} y_i^{-1} \left(y_i + \phi(x_{j,1})\right) \left(y_i + \phi(x_{j,1})^{-1}\right) \left(\phi(x_{j,1}) - 1\right) \left(\phi(x_{j,1})^{-1} - 1\right) \\
&= (-1)^{m-N_0/2} N_{K_i/K_{\pm i}} \left(\prod_{j \in I_1} \prod_{\phi \in \Phi_{\pm j}} (y_i + \phi(x_{j,1}))(\phi(x_{j,1})^{-1} - 1) \right)
\end{aligned}$$

where $\Phi_{\pm j}$ is the set of K-morphisms $K_{\pm j} \to \bar{K}$. Thus

$$\prod_{i \in I^*} \mathrm{sign}_{K_i/K_{\pm i}}(C_i/C_{i,0}) = \prod_{i \in I^*} \mathrm{sign}_{K_i/K_{\pm i}}|_{K^\times}\left((-1)^{m-N_0/2}\right) = 1$$

since $\prod_{i\in I^*} \mathrm{sign}_{K_i/K_{\pm i}}|_{K^\times}$ is easily checked to be equal to the Hilbert symbol with the discriminant of our special orthogonal group, which is 1 (this is the condition for g_0 to have a norm in the special orthogonal group). □

Proposition 5.2.2. *Let F_0 be a totally real field, and let π be a regular, L-algebraic, self-dual, cuspidal representation of $\mathbf{GL}_{2n+1}(\mathbb{A}_{F_0})$. Assume that for any place $v|p$ of F_0, π_v has vectors fixed under the Iwahori. Then there exists a RLASDC representation π_0 of $\mathbf{GL}_1(\mathbb{A}_{F_0})$ if n is odd (resp. $\mathbf{GL}_3(\mathbb{A}_{F_0})$ if n is even), a totally real extension F/F_0 which is trivial or quadratic, and an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$ such that*

1) *For any place $v|p$ of F_0, $\pi_{0,v}$ is unramified.*
2) $\mathrm{BC}_{F/F_0}(\pi)$ *and* $\mathrm{BC}_{F/F_0}(\pi_0)$ *remain cuspidal.*
3) *For any place v of F above p, Π_v has invariants under the action of the Iwahori subgroup of $\mathbf{G}(F_v)$.*
4) *For any finite place v of F such that $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$ are unramified, Π_v is unramified, and via the inclusion $\mathrm{SO}_{2n+2}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$ (resp. $\mathrm{SO}_{2n+4}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+2}(\mathbb{C})$), the Satake parameter of Π_v is equal to the direct sum of those of $\mathrm{BC}_{F/F_0}(\pi)_v$ and $\mathrm{BC}_{F/F_0}(\pi_0)_v$.*

Proof. This is very similar to Proposition 5.1.2, and we only give details for the differences. Let π_0 be the central character of π if n is odd, or a self-dual L-algebraic cuspidal automorphic representation of $\mathbf{GL}_3(\mathbb{A}_{F_0})$ having central character equal to that of π and as in the proof of Proposition 5.1.2 if n is even (i.e. Steinberg at some finite non-p-adic place and with generic Hodge weights). Let F be any totally real quadratic extension of F_0 such that $\mathrm{BC}_{F/F_0}(\pi)$ remains cuspidal.

The crucial observation is that the direct sum of the local Langlands parameters of $\mathrm{BC}_{F/F_0}(\pi)$ and $\mathrm{BC}_{F/F_0}(\pi_0)$ at the infinite places correspond to parameters for the compact groups $\mathbf{SO}_{2n+2}/\mathbb{R}$ (resp. $\mathbf{SO}_{2n+4}/\mathbb{R}$). These parameters are of the form

$$\epsilon^n \oplus \bigoplus_{i=1}^{n} \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \left(z \mapsto (z/\bar{z})^{r_i}\right)$$

$(r_1 > \cdots > r_n > 0)$ for $\mathrm{BC}_{F/F_0}(\pi)$, and

$$\begin{cases} 1 & \text{if } n \text{ is odd} \\ \epsilon \oplus \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \left(z \mapsto (z/\bar{z})^r \right) & \text{if } n \text{ is even, with } r > r_1 \end{cases}$$

for $\mathrm{BC}_{F/F_0}(\pi_0)$, so that the direct sum of the two is always of the form

$$1 \oplus \epsilon \oplus \bigoplus_{i=1}^{k-1} \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \left(z \mapsto (z/\bar{z})^{r_i} \right)$$

for distinct, positive r_i. This is the composition with the standard representation of $\mathrm{SO}_{2m}(\mathbb{C})$ of the Langlands parameter corresponding to the representation of the compact group $\mathbf{SO}_{2m}(\mathbb{R})$ having highest weight $\sum_{i=1}^{m}(r_i - (m-i))e_i$ with $r_m = 0$, where the root system consists of the $\pm e_i \pm e_j$ $(i \neq j)$ and the simple roots are $e_1 - e_2, \ldots, e_{m-1} - e_m, e_{m-1} + e_m$.

In the present case the group $\mathcal{S}_\psi$ associated to $\psi = \mathrm{BC}_{F/F_0}(\pi) \boxplus \mathrm{BC}_{F/F_0}(\pi_0)$ is trivial, so that the multiplicity formula is trivial as well, and the quadratic extension of F_0 is only necessary in order to be able to define the group $\mathbf{G}$. □

Note that, contrary to the symplectic case, there is a non-trivial outer automorphism of the even orthogonal group, and so there may be two choices for the Satake parameters of Π_v, mapping to the same conjugacy class in the general linear group. Fortunately we only need the existence.

Proposition 5.2.3. *Let F be a totally real field, and let π be an L-algebraic, self-dual, cuspidal representation of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$. Suppose that for any place v of F above p, π_v has vectors invariant under an Iwahori subgroup. Then for any complex conjugation $c \in \mathrm{Gal}_F$, $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = \pm 1$.*

Proof. The proof is similar to that of Proposition 5.1.3. We use the previous proposition to be able to assume (after base change) that there is a representation π_0 (of $\mathbf{GL}_1(\mathbb{A}_F)$ if n is odd, $\mathbf{GL}_3(\mathbb{A}_F)$ if n is even) such that $\pi \boxplus \pi_0$ transfers to an automorphic representation Π of $\mathbf{G}(\mathbb{A}_F)$, with compatibility at the unramified places. The representation Π has Iwahori-invariants at the p-adic places of F, and thus it defines a point of the eigenvariety $\mathscr{X}$ associated with $\mathbf{G}$ (and an idempotent defined by an open subgroup of $\mathbf{G}(\mathbb{A}_{F,f}^{(p)})$). By Theorem 4.0.1, Π is congruent (modulo arbirarily big powers of p) to an automorphic representation Π' of $\mathbf{G}$ such that $\rho_{\iota_p,\iota_\infty}(\Pi')$ is irreducible. Hence $\rho_{\iota_p,\iota_\infty}(\Pi') = \rho_{\iota_p,\iota_\infty}(\pi')$ for some RLASDC π' of $\mathbf{GL}_{2m}(\mathbb{A}_F)$, which is unramified at all the p-adic places of F, and we can apply Proposition 5.1.3 to π'. This proves that $\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi)(c)) = -\mathrm{Tr}(\rho_{\iota_p,\iota_\infty}(\pi_0)(c)) = \pm 1$. □

5.3. Almost general case

We will now remove the hypothesis of being Iwahori-spherical at p, and allow more general similitude characters, using Arthur and Clozel's base change.

Lemma 5.3.1. *Let E be a number field, S a finite set of (possibly infinite) places of E, and for each $v \in S$, let $K^{(v)}$ be a finite abelian extension of E_v. There is an abelian extension F of E such that for any $v \in S$ and any place w of F above v, the extension F_v/E_v is isomorphic to $K^{(v)}/E_v$.*

Proof. After translation to local and global class field theory, this is a consequence of [21, Théorème 1]. □

Before proving the last theorem, we need to reformulate the statement, in order to make the induction argument more natural. Let π be a regular, L-algebraic, cuspidal representation of $\mathbf{GL}_{2n+1}(\mathbb{A}_F)$. At a real place v of F, the Langlands parameter of π_v is of the form

$$\epsilon^e \oplus \bigoplus_i \mathrm{Ind}_{W_\mathbb{R}}^{W_\mathbb{C}} z \mapsto (z/\bar{z})^{n_i}$$

and according to the recipe given in [10, Lemma 2.3.2], $\rho_{\iota_p,\iota_\infty}(\pi)(c_v)$ should be in the same conjugacy class as

$$\begin{pmatrix} (-1)^e & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

Since it is known that $\det \rho_{\iota_p,\iota_\infty}(\pi)(c_v) = (-1)^{e+n}$, $\rho_{\iota_p,\iota_\infty}(\pi)(c_v) \sim \mathcal{LL}(\pi_v)(j)$ if and only if $|\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c_v)| = 1$. Similarly, in the even-dimensional case, $\rho_{\iota_p,\iota_\infty}(\pi)(c_v) \sim \mathcal{LL}(\pi_v)(j)$ if and only if $\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c_v) = 0$.

Theorem 5.3.2. *Let $n \geq 2$, F a totally real number field, π a regular, L-algebraic, essentially self-dual, cuspidal representation of $\mathbf{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq (\chi \circ \det) \otimes \pi$, where $\chi = \eta || \cdot ||^q$ for an Artin character η and an integer q. Suppose that one of the following conditions holds*

1) *n is odd.*

2) *n is even, q is even, and $\eta_\infty(-1) = 1$.*

Then for any complex conjugation $c \in \mathrm{Gal}_F$, $|\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c)| \leq 1$.

Proof. We can twist π by an algebraic character, thus multiplying the similitude character $\eta\,||\cdot||^q$ by the square of an algebraic character. If n is odd, this allows us to assume $\eta = 1, q = 0$ (by comparing central characters, we see that $\eta\,||\cdot||^q$ is a square). If n is even, we can assume that $q = 0$ (we could also assume that the order of η is a power of 2, but this is not helpful). The Artin character η defines a cyclic, totally real extension F'/F. Since local Galois groups are pro-solvable, the preceding lemma shows that there is a totally real, solvable extension F''/F' such that $\mathrm{BC}_{F''/F}(\pi)$ has Iwahori invariants at all the places of F'' above p. In general $\mathrm{BC}_{F''/F}(\pi)$ is not cuspidal, but only induced by cuspidals: $\mathrm{BC}_{F''/F}(\pi) = \pi_1 \boxplus \cdots \boxplus \pi_k$. However it is self-dual, and the particular form of the Langlands parameters at the infinite places imposes that all π_i be self-dual. We can then apply Propositions 5.1.3 and 5.2.3 to the π_i, and conclude by induction that for any complex conjugation $c \in \mathrm{Gal}_F$, the conjugacy class of $\rho_{\iota_p,\iota_\infty}(\pi)(c)$ is given by the recipe found in [10, Lemma 2.3.2], that is to say $\left|\mathrm{Tr}\rho_{\iota_p,\iota_\infty}(\pi)(c)\right| \leq 1$. □

Remark 5.3.3. The case n even, $\eta_\infty(-1) = (-1)^{q+1}$ is trivial. The case n even, q odd and $\eta_\infty(-1) = -1$ is not addressed by the present article, but is proved (as a special case) by Caraiani and Le Hung in [14] using Scholze's construction of the Galois representation associated with π (ignoring the fact that π is essentially self-dual) and Theorem 5.3.2 for self-dual representations in dimension $2n + 1$.

For the sake of clarity, we state the theorem using the more common normalization of C-algebraic representations.

Theorem 5.3.4. *Let $n \geq 2$, F a totally real number field, π a regular, algebraic, essentially self-dual, cuspidal representation of $\mathbf{GL}_n(\mathbb{A}_F)$, such that $\pi^\vee \simeq (\chi \circ \det) \otimes \pi$, where $\chi = \eta\,||\cdot||^q$ for an Artin character η and an integer q. Suppose that one of the following conditions holds*

1) *n is odd.*

2) *n is even, q is odd, and $\eta_\infty(-1) = 1$.*

Then for any complex conjugation $c \in \mathrm{Gal}_F$, $|\mathrm{Tr}(r_{\iota_p,\iota_\infty}(\pi)(c))| \leq 1$.

Proof. Apply the previous theorem to $\pi \otimes ||\det||^{(n-1)/2}$. □

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References

[1] J. Arthur, *The Endoscopic Classification of Representations: Orthogonal and Symplectic groups*, Vol. 61 of American Mathematical Society Colloquium Publications, American Mathematical Society (2013).

[2] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Local-global compatibility for l=p. II*, Ann. Sci. de l'ENS **47** (2014), no. 1, 161–175.

[3] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609.

[4] J. Bellaïche and G. Chenevier, *The sign of Galois representations attached to automorphic forms for unitary groups*, Compositio Math. **147** (2011), 1337–1352.

[5] J. Bernstein and A. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. I*, Ann. Sci. École Norm. Sup. **10** (1977), no. 4, 441–472.

[6] D. Blasius and J. D. Rogawski, *Tate classes and arithmetic quotients of the two-ball*, The zeta functions of Picard modular surfaces. CRM Workshop, Montreal / Can. 1988, 421–444 (1992). (1992).

[7] S. Bosch, *Half a century of rigid analytic spaces*, Pure Appl. Math. Q. **5** (2009), no. 4, Special Issue: In honor of John Tate. Part 1, 1435–1467.

[8] C. Breuil and A. Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ *en* $\ell = p$, Duke Math. J. **115** (2002), no. 2, 205–310. With an appendix by Guy Henniart.

[9] K. Buzzard, *Eigenvarieties*, in L-functions and Galois representations, Vol. 320 of London Math. Soc. Lecture Notes, 59–120, Cambridge Univ. Press (2007). Durham, 2004.

[10] K. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, in Automorphic Forms and Galois Representations, Volume 1, Vol. 414 of *London Math. Soc. Lecture Note Ser.*, 135–187, Cambridge Univ. Press, Cambridge (2014).

[11] F. Calegari and T. Gee, *Irreducibility of automorphic Galois representations of $GL(n)$, n at most 5*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 5, 1881–1912.

[12] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413.

[13] A. Caraiani, *Monodromy and local-global compatibility for l=p*, Algebra Number Theory **8** (2014), no. 7, 1597–1646.

[14] A. Caraiani and B. V. Le Hung, *On the image of complex conjugation in certain Galois representations*, Compos. Math. **152** (2016), no. 7, 1476–1488.

[15] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*. Available at `http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`.

[16] W. Casselman, *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Mathematica **40** (1980), no. 3, 387–406.

[17] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II: The Whittaker function*, Compos. Math. **41** (1980), 207–231.

[18] G. Chenevier, *Familles p-adiques de formes automorphes pour* GL_n, J. Reine Angew. Math. **570** (2004), 143–217.

[19] G. Chenevier, *Une application des variétés de Hecke des groupes unitaires* (2009). Part of the Book Project under the direction of Michael Harris at Paris 7: `http://www.imj-prg.fr/fa/spip.php?article10`.

[20] G. Chenevier and M. Harris, *Construction of automorphic Galois representations. II.*, Camb. J. Math. **1** (2013), no. 1, 53–73.

[21] C. Chevalley, *Deux Théorèmes d'Arithmétique*, J. Math. Soc. Japan **3** (1951), no. 1, 36–44.

[22] L. Clozel, *On limit multiplicities of discrete series representations in spaces of automorphic forms*, Invent. Math. **83** (1986), no. 2, 265–284.

[23] L. Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*, in L. Clozel and J. S. Milne, editors, Automorphic

Forms, Shimura Varieties, and L-functions, Vol. 1 (1988). Proceedings of a Conference held at the University of Michigan, Ann Arbor.

[24] B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier **49** (1999), no. 2, 473–541.

[25] E. B. Dynkin, *Selected papers of E. B. Dynkin with commentary*, American Mathematical Society, Providence, RI; International Press, Cambridge, MA (2000). Edited by A. A. Yushkevich, G. M. Seitz and A. L. Onishchik.

[26] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Annales scientifiques de l'ENS **11** (1978), no. 4, 471–542.

[27] M. Harris, K.-W. Lan, R. Taylor, and J. Thorne, *On the rigid cohomology of certain Shimura varieties*. `http://www.math.ias.edu/~rtaylor/rigcoh.pdf`.

[28] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups*, Publications Mathématiques de l'IHÉS **25** (1965), 5–48.

[29] O. T. R. Jones, *An analogue of the BGG resolution for locally analytic principal series*, J. Number Theory **131** (2011), no. 9, 1616–1640.

[30] T. Kaletha, *Global rigid inner forms and multiplicities of discrete automorphic representations*, `arXiv:1501.01667`.

[31] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.

[32] D. Loeffler, *Overconvergent algebraic automorphic forms*, Proc. London Math. Soc. **102** (2011), no. 2, 193–228.

[33] C. Mœglin and J.-L. Waldspurger, *Sur le transfert des traces d'un groupe classique p-adique à un groupe linéaire tordu*, Selecta Math. **12** (2006), no. 3-4, 433–515.

[34] S. Patrikis, *On the sign of regular algebraic polarizable automorphic representations*, Math. Ann. **362** (2015), no. 1-2, 147–171.

[35] S. Patrikis and R. Taylor, *Automorphy and irreducibility of some l-adic representations*, Compos. Math. **151** (2015), no. 2, 207–229.

[36] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066.

[37] S. Sen, *Lie algebras of Galois groups arising from Hodge-Tate modules*, Ann. of Math. (2) **97** (1973) 160–170.

[38] J.-P. Serre, *Sur les représentations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[39] M. Tadić, *Representations of p-adic symplectic groups*, Compositio Mathematica **90** (1994), no. 2, 123–181.

[40] O. Taïbi, *Arthur's multiplicity formula for certain inner forms of special orthogonal and symplectic groups*. arXiv:1510.08395.

[41] R. Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332.

[42] R. Taylor, *The image of complex conjugation in l-adic representations associated to automorphic forms*, Algebra Number Theory **6** (2012), no. 3, 405–435.

[43] J.-L. Waldspurger, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), no. 1-2, 41–82.

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