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Local Fano-Mori contractions of high nef-value

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Let X be a variety with terminal singularities of dimension n. We study local contractions $f : X \to Z$ supported by a $\mathbb{Q}$-Cartier divisor of the type $K_X + \tau L$, where L is an f-ample Cartier divisor and $\tau > 0$ is a rational number. Equivalently, f is a Fano-Mori contraction associated to an extremal face in $\overline{NE(X)}_{K_X+\tau L=0}$. We prove that, if $\tau > (n-3) > 0$, the general element $X' \in |L|$ is a variety with at most terminal singularities. We apply this to characterize, via an inductive argument, some birational contractions as above with $\tau > (n-3) \geq 0$.

1. Introduction

Let X be a variety with at most log terminal singularities of dimension n; let $f : X \to Z$ be a local contraction on X (see Section 2). Assume that f is an adjoint contraction supported by a $\mathbb{Q}$-Cartier divisor of the type $K_X + \tau L$, where L is an f-ample Cartier divisor and τ is a positive rational number (Definition 2.2). Equivalently, f is a Fano-Mori contraction associated to an extremal face in $\overline{NE(X)}_{K_X+\tau L=0}$ (Definition 2.1 and Remark 2.3). These maps naturally arise in the context of the minimal model program.

The description and the classification of such contractions $f : X \to Z$ are often obtained by an inductive procedure, the so-called Apollonius method: it consists in finding a "good" element $X' \in |L|$ (that is an element of the linear system $|L|$ with good singularities), studying by induction the properties of $f_{|X'} : X' \to Z'$ and then lifting them to $f : X \to Z$. The first step, i.e. the proof of the existence of good elements in $|L|$, is a long lasting and delicate problem; the following is a result in this direction.

Theorem 1.1. *Let $f : X \to Z$, L and τ be as above; assume that X has terminal singularities and $\tau > (n-3) > 0$. Let $X' \in |L|$ be a general divisor. Then X' is a variety with at most terminal singularities and $f_{|X'} : X' \to f(X') =: Z'$ is a local contraction supported by $K_{X'} + (\tau - 1)L'$, where $L' := L_{|X'}$ (i.e. f' is again a Fano-Mori contraction).*

The next two results are proved by induction, applying Theorem 1.1. If $n = 3$, then part A of the following Theorem is the main result of [12].

Theorem 1.2. *Let $f : X \to Z$, L and τ be as above; assume also that X is terminal and $\mathbb{Q}$-factorial and that $\tau > (n-3) \geq 0$.*

A) *Assume that f is birational and contracts a prime divisor to a point. For $i = 1, \ldots, n-3$, let $H_i \in |L|$ be a general divisor and set $X'' = \cap H_i$. Then X'' is a threefold with terminal singularities and $f'' : X'' \to Z''$ is a divisorial contraction of an irreducible $\mathbb{Q}$-Cartier divisor $E'' \subset X''$ to a point $p \in Z''$. Assume that p is smooth in Z''. Then f is a weighted blow-up of a smooth point with weight $(1, a, b, c, \ldots, c)$, where a, b are positive integers, $(a, b) = 1$, c is the positive integer such that $L = f^* f_* L - cE$ and $ab|c$.*

B) *Let E be the exceptional locus of f. Assume that X has only points of index 1 and 2 and that each component of E has dimension $(n-2)$ (in particular f is a birational small contraction). Then $\tau = \frac{2n-5}{2}$, E is irreducible, it is contracted to a point and $(E, L_{|E}) = (\mathbb{P}^{n-2}, \mathcal{O}(1))$.*

Fano-Mori contractions of nef-value $\tau > (n-2)$ are classified, see [3] and [4]. In [4] we also describe divisorial contractions of nef-value $\tau > (n-3)$ such that the exceptional locus is not contracted to a point . The above Theorem is a further step towards a classification in the case $(n-2) \geq \tau > (n-3)$.

2. Notation

We use notations and definitions which are standard in the Minimal Model Program, they are compatible with the ones in the books [18] and [19].

In particular a *log pair* (X, D) consists of a normal variety X together with an effective Weil $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on X such that $K_X + D$ is $\mathbb{Q}$-Cartier.

Let $\mu : Y \to X$ be a log resolution of (X, D), then we can write

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + \sum_{E_i \text{ exceptional}} a(E_i, X, D) E_i.$$

We define the *discrepancy* of (X, D) as

$$\mathrm{discrep(X, D)} := \inf_{\mathrm{E}} \{\mathrm{a(E, X, D) : E \text{ is an exceptional divisor over } X}\}.$$

We say that (X, D) is terminal, resp.canonical, klt (or Kawamata log terminal), plt, lc (or log canonical) if discrep(X, D) is > 0, resp. ≥ 0, > -1 and $\lfloor D \rfloor = 0$, > -1, ≥ -1.

If $D = 0$, then the notions klt and plt coincide and X is called log terminal (lt).

The *log canonical threshold* of a log pair (X, D) is defined as

$$\mathrm{lct}(X, D) := \sup\{t \in \mathbb{Q} : (X, tD) \text{ is log canonical}\}.$$

A subvariety $W \subset X$ is called a *lc centre* for (X, D) if there is a log resolution $\mu : Y \to X$ and an irreducible exceptional divisor E on Y such that $a(E, X, D) = -1$ and $\mu(E) = W$. The set of all the lc centres is denoted by $CLC(X, D)$. Note that if $W_1, W_2 \in CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$; in particular, there exist minimal elements in $CLC(X, D)$. An lc centre W is called *isolated* if for any log resolution $\mu : Y \to X$ and any exceptional divisor E on Y such that $a(E, X, D) = -1$, we have $\mu(E) = W$.

Let T be a normal projective variety over $\mathbb{C}$ and $n = \dim T$. A *contraction* is a surjective morphism $\varphi : T \to S$ with connected fibres onto a normal variety S. We take a contraction $\varphi : T \to S$ and we fix a non trivial fibre F of f; take an open affine set $Z \subset S$ such that $f(F) \in Z$.

Let $X := f^{-1}(Z)$; then $f : X \to Z$ will be called a *local contraction around* F, or simply a local contraction; eventually shrinking Z, we can assume that $\dim F \geq \dim F'$ for every fibre F' of f.

We assume that f is *projective*, that is we assume the existence of f-ample Cartier divisors L. We will also assume that X has log terminal, or milder type, singularities.

Definition 2.1. We will say that a local projective contraction $f : X \to Z$ is *Fano-Mori* (F-M) if $-K_X$ is f-ample.

Fano-Mori contractions are associated to extremal faces of the polyhedral part of the Mori-Kleiman cone $\overline{NE(X)}_{K_X<0} = \{[C] \in \overline{NE(X)} : K_X.C < 0\}$ in the vector space $N_1(X)$ generated by 1-cycles modulo numerical equivalence. In particular the contraction contracts exactly all the curves contained in the associated face. If the associated face has dimension 1 (a ray) the contraction is called *elementary*.

Definition 2.2. We will say that a local projective contraction $f : X \to Z$ is an *adjoint contraction supported by* $K_X + \tau L$ if there is a $\tau \in \mathbb{Q}$ such that $K_X + \tau L \sim_f \mathcal{O}_X$, where L is an f-ample Cartier divisor ($\sim_f$ stays for numerical equivalence over f).

Remark 2.3. Any F-M contraction $f : X \to Z$, once we fix a f-ample Cartier divisors L, is an adjoint contraction. To see this we define the *nef-value* of the pair $(f : X \to Z, L)$ as $\tau_f(X, L) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is } f\text{-nef}\}$. By the rationality theorem of Kawamata (Theorem 3.5 in [18]), $\tau(X, L)$ is a rational non-negative number and therefore f is an adjoint contraction supported by $K_X + \tau L$. Viceversa any adjoint contraction with positive τ is clearly a F-M contraction.

All through the paper, although not further specified, we will be in the following set up:

($\star$) X is a variety with at most log terminal singularities, $f : X \to Z$ is an adjoint contraction (Definition 2.2), local around a (non trivial) fibre F and supported by $K_X + \tau L$, where L is an f-ample Cartier divisor and τ is a rational number.

We will denote by E the exceptional locus of f and by $Bs|L|$ the relative base locus of L, i.e. the support of the cokernel of the natural map $f^*f_*L \to L$. Clearly $Bs|L| \subset E$.

Weighted projective spaces and weighted blow-up, under some conditions on the weights, are special Fano-Mori contractions. For a detailed treatment of weighted blow-ups we refer to Section 10 of [18] or Section 3 of [4]; here we just fix our notation.

Let $\sigma = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_n) = 1$.

We denote by $\mathbb{P}(a_1, \ldots, a_n)$ the *weighted projective space* with weight $(a_1, \ldots, a_k)$.

Let $X = \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ and $p = (0, \ldots, 0) \subset X$. Consider the rational map $\varphi : \mathbb{A}^n \to \mathbb{P}(a_1, \ldots, a_n)$ given by $(x_1, \ldots, x_n) \mapsto (x_1^{a_1} : \cdots : x_n^{a_n})$. The *weighted blow-up* of $p \in X$ of weight σ is defined as the closure $\overline{X}$ in $\mathbb{A}^n \times \mathbb{P}(a_1, \ldots, a_n)$ of the graph of φ, together with the morphism $\pi : \overline{X} \to X$ given by the projection on the first factor. The map π is birational and contracts an exceptional irreducible divisor $E \cong \mathbb{P}(a_1, \ldots, a_n)$ to p. For any $d \in \mathbb{N}$ we define the σ-weighted ideal of degree d as $I_{\sigma,d} := (x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^n s_j a_j \geq d)$.
We have the following characterization: $\overline{X} = \operatorname{Proj}(\bigoplus_{d \geq 0} I_{\sigma,d})$ (see [4]).

A criterium to check that the singularities of $\overline{X}$ are terminal can be find in [23, Theorem 4.11]: for instance if $\sigma = (1, a, b, c, \ldots, c)$, where $(a, b) = 1$ and $ab|c$, then $\overline{X}$ has terminal singularities.

3. Existence of good sections

In this section we prove Theorem 1.1 and we provide a collection of technical results which could be useful by themselves (see Proposition 3.3).

We start with a non-vanishing lemma.

Lemma 3.1. *Let* $f : X \to Z$ *be as in Section 2* $(\star)$*. Let* $D \sim_f \beta L$ *be a* $\mathbb{Q}$*-divisor such that* (X, D) *is lc and let* $W \in CLC(X, D)$ *be a minimal centre. Assume that* $\tau - \beta > -1$*, or that* $\tau - \beta \geq -1$ *if* f *is birational; assume also that one of the following conditions is satisfied:*

(i) $\dim W \leq 2$,

(ii) $\dim W \geq 3$ *and* $\tau - \beta > \dim W - 3$.

Then $H^0(W, L_{|W}) \neq 0$.

Proof. By subadjunction formula (see Theorem 1.2 of [10]), there is an effective $\mathbb{Q}$-divisor D_W such that (W, D_W) is klt and

$$K_W + D_W \sim (K_X + D)_{|W} \sim -(\tau - \beta)L_{|W}.$$

If $\dim W \leq 2$, then we conclude by Theorem 3.1 of [14].

If $\dim W \geq 3$, then (W, D_W) is a log Fano variety of index $i(W, D_W) > \dim W - 3$ and the result follows by the main Theorem of [1]. □

The next is the first step to prove the existence of a good element in the linear system $|L|$.

Corollary 3.2. *Let* $f : X \to Z$ *be a as in Section 2* $(\star)$*. Let* $D \sim_f \beta L$ *be a* $\mathbb{Q}$*-divisor such that* (X, D) *is lc and let* $W \in CLC(X, D)$ *be a minimal centre. Assume that* $\tau - \beta > -1$ *or that* $\tau - \beta \geq -1$ *if* f *is birational; assume also that one of the following conditions is satisfied:*

(i) $\dim W \leq 2$,

(ii) $\dim W \geq 3$ *and* $\tau - \beta > \dim W - 3$.

Then there exists a section of $|L|$ *not vanishing identically on* W*.*

Proof. By a tie-breaking technique (see the discussion 1.15 in [22]), we may assume that W is an isolated lc centre and hence $I_W = \mathcal{J}(D)$, where I_W is the ideal sheaf of W and $J(D)$ is the multiplier ideal of D (see Lemma 2.19

of [8]). Consider the exact sequence

$$0 \to \mathcal{O}_X(L) \otimes \mathcal{I}_W \to \mathcal{O}_X(L) \to \mathcal{O}_W(L_{|W}) \to 0.$$

Since $L - (K_X + D) \sim_f (1 + \tau - \beta)L$ is f-nef and big, we can apply Nadel vanishing [19, Thm. 9.4.17] to obtain that

$$H^0(X, L) \to H^0(W, L_{|W})$$

is surjective. The result follows now by Lemma 3.1. □

The next proposition collects a series of useful technical results.

Proposition 3.3. *Let $f : X \to Z$ be as in Section 2 ($\star$).*

1 *([5, Theorem 5.1]) Assume that either* $\dim F < \tau + 1$, *if f is of fibre type, or* $\dim F \leq \tau + 1$, *if f is birational. Then L is relatively base-point free (i.e. $Bs|L| = \emptyset$).*

2 *If $\tau > -1$ and* $\dim F < \tau + 3$, *then there exists a section of $|L|$ not vanishing identically along F.*

3 *Assume that* $\dim F < \tau + 3$, *F is irreducible, and that either $\tau > 0$, if f is of fibre type, or $\tau \geq 0$, if f is birational. Then the general element of $|L|$ is a variety with lt singularities. If* $\dim F < \tau + 2$, *then the same holds without the assumption that F is irreducible.*

4 *Assume $\tau > 0$ and $n - 3 < \tau$. Then* $\dim Bs|L| \leq 1$.

5 *Assume* $\dim F < \tau + 3$, *F irreducible and $\tau \geq 1$. Let $S \in |L|$ be a general element. If X has canonical singularities, then S has canonical singularities. If X has terminal singularities, then S has terminal singularities, except possibly when $\tau = 1$ and f is of fibre-type.*
If $\dim F < \tau + 2$, *then the same holds without the assumption that F is irreducible.*

6 *Assume that* $\dim F < \tau + 3$, *F is irreducible and $\tau > 0$ if f is of fibre type or $\tau \geq 0$ if f is birational. If X has canonical Gorenstein singularities, then the general element of $|L|$ has canonical singularities.*

7 *Assume that* $\dim F = \tau + 3$, *F is irreducible and $\tau > 0$ if f is of fibre type or $\tau \geq 0$ if f is birational. If there exists a section of L not vanishing along F and X has canonical Gorenstein singularities, then the general element of $|L|$ has canonical singularities.*

Remark 3.4. Point 1 is the main result of [5]. Points 2 and 3 are generalisations of Proposition 2.4 and Proposition 3.3 in [22]. Points 4, 5 and 6 are generalizations of results in [21] and [22]. Point 7 is the analogous of [9, Thm. 1.1] in the relative set-up.

At the Points 3 and 6 of Proposition 3.3 the assumption $\tau > 0$ if f is of fiber type is necessary, as the following trivial example shows. Let E be a smooth elliptic curve and D an ample line bundle with a base point (i.e. $D = p$). Consider $X = E \times \mathbb{P}^m \to \mathbb{P}^m$ for $m \geq 0$ and $L = D \boxtimes (-2K_{\mathbb{P}^m})$. This is an adjoint contraction of fibre-type with $\tau = 0$ for which the conclusions of Points 3 and 6 do not hold. Similar examples can be constructed for point 7.

Counter-examples for the statement in the point 5 for $\tau = 1$ and f of fiber type were given by Mella; in [21] he actually classified all terminal Mukai 3-folds Y such that the general element of $|-K_Y|$ is not smooth. Taking $X := Y \times \mathbb{P}^m \to \mathbb{P}^m$ for $m \geq 0$ and $L = -(K_Y \boxtimes 2K_{\mathbb{P}^m})$, we get examples of fibre-type contractions (not necessarily to a point) with $\tau = 1$ which do not satisfy the conclusions of Point 5.

Proof of Proposition 3.3.2. Let $\{h_i\} \in H^0(Z, \mathcal{O}_Z)$ be general functions vanishing at $f(F)$ such that (X, D) is not lc, where $D = \sum f^*(h_i)$. Let $\gamma = \mathrm{lct}(X, D)$ and let $W \in CLC(X, \gamma D)$ be a minimal lc centre; by the general choice of h_i outside $f(F)$, we can assume that $W \subset F$. Note that $\gamma D \sim_f 0$ and that, by assumption, $\dim W \leq \dim F < \tau + 3$. Therefore by Corollary 3.2 there exists a section of $|L|$ not vanishing identically on W and thus on F. □

Proof of Proposition 3.3.3. We start proving that $Bs|L|$ has codimension at least two. Assume by contradiction that there exists an irreducible component $V \subset Bs|L|$ of dimension $n - 1$.

Suppose first that $V \subset F$. Let $H \in |L|$ be a general element and set $c = \mathrm{lct}(X, H)$. If $c < 1$, then $LCC(X, cH) \subset Bs|L|$; consider a minimal lc centre $W \in CLC(X, cH)$. By Proposition 3.3.2, $W \subsetneq F$. If F is irreducible, then $\dim W \leq \dim F - 1 < \tau + 2$. If F is not irreducible, then $\dim W \leq \dim F < \tau + 2$ by hypothesis. Therefore by Corollary 3.2 there exists a section of $|L|$ not vanishing identically on W, thus on $Bs|L|$, which is a contradiction. If $c = 1$, then $V \subset Bs|L|$ is an lc centre of (X, H) and, by Proposition 3.3.2, $V \subsetneq F$. Since $dim V = n - 1$, f is a contraction to a point. Therefore, by assumptions, we have $\tau > 0$. We can conclude again by Corollary 3.2.

Assume now that V is not contained in any fibre of f and consider $h_1, \ldots, h_d$ general functions on Z, where $d := \dim f(V) > 0$. Set $X_{h_i} = f^*h_i$

and $X' = \cap X_{h_i}$. Note that $\dim X' = n - d$. By vertical slicing ([5, Lemma 2.5]), we get a local contraction $f' : X' \to Z'$, supported by $K_{X'} + \tau L'$ where $L' = L_{|X'}$ and there exists an irreducible component V' of $V \cap X' \subset Bs|L'|$ (actually, by Bertini, $V \cap X'$ is irreducible if it has positive dimension) such that $\dim V' = n - d - 1$ and $V' \subset F'$, where F' is a fibre of f'. Note that if f' is of fiber type also f is of fiber type, therefore in this case τ is positive by assumption. We are in the situation of the previous step and we can reach a contradiction.

We now prove that the general element of $|L|$ has lt singularities. Let $S \in |L|$ be general element; by Bertini Theorem (see [11, Thm. 6.3]) and the fact that $Bs|L|$ has codimension at least two, we see that S is irreducible and generically reduced. Assume by contradiction that S has singularities worse than log terminal. Then, by Proposition 7.5.1 of [16], (X, S) is not plt.

Assume first that $\tau > 0$. Set $\gamma = \mathrm{lct}(X, S) \leq 1$ and consider a minimal lc centre $W \in CLC(X, \gamma S)$ such that $W \subset Bs|L|$ (such a center exists by Bertini Theorem, see for instance [1, Lemma 5.1]). We want to show that there is a section of $|L|$ not vanishing identically on W, obtaining in this way a contradiction.

As above, via a vertical slicing argument, we may assume $W \subset F$. In fact, let $d = \dim f(W)$. Consider $h_1, \ldots, h_d$ general functions on Z. Set $X_{h_i} = f^* h_i$ and $X' = \cap X_{h_i}$. By vertical slicing ([5, Lemma 2.5]), we get a local contraction $f' : X' \to Z'$ around a fibre F', supported by $K_{X'} + \tau L'$ where $L' = L_{|X'}$. Let $S' \in |L'|$ be general. Since each X_{h_i} is general and intersects W, we have that $LLC(X', \gamma S') \subset W \cap X' \subset F'$ and the claim is proved.

By Proposition 3.3.2, $W \subsetneq F$. If F is irreducible, then $\dim W \leq \dim F - 1 < \tau + 2$. If F is not irreducible, then $\dim W \leq \dim F < \tau + 2$ by hypothesis. If $\dim W \geq 3$, then $\tau - \gamma > \dim W - 3 \geq 0$ and we can apply point (ii) of Corollary 3.2. If $\dim W \leq 2$, then the contradiction follows by point (i) of Corollary 3.2.

Assume now that $\tau = 0$ and f is not of fibre-type. Let $H = \varepsilon f^*(h)$, where h is a general function on Z vanishing at $f(F)$ and $0 < \varepsilon << 1$. Set $D = S + H$ and $\delta = \mathrm{lct}(X, D) < 1$. We can consider a minimal centre $W \in CLC(X, \delta D)$ and reason as before. □

Proof of Proposition 3.3.4. If $\dim F \leq (n - 2)$ then 3.3.4 follows from the main Theorem of [5], as quoted in 3.3.1. Assume that $F \geq (n - 1)$, then the result follows by the next Lemma. □

Lemma 3.5. *Assume that X has log terminal singularities, $\tau > 0$ and $\dim F = n - 1 < \tau + 2$. Then $\dim Bs|L| \leq 1$.*

Proof. The proof of the Lemma is by induction on $n \geq 3$. We have proved above that $|L|$ has not fixed components, therefore the lemma is true for $n \leq 3$.

Assume $n > 3$. Let $X' \in |L|$ general. Since $|L|$ has no fixed component, by Bertini we get that X' does not contain any irreducible component of F (and that it is irreducible and reduced). Moreover, by Proposition 3.3.3, we have that X' is log terminal. Hence, by horizontal slicing ([5, Lemma 2.6]), $f : X' \to Z'$ is a contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$ around a fibre $F' = F \cap X'$. It also follows that $\dim Bs|L| \leq \dim Bs|L'|$, because any section of L' lifts to a section of L by [5, Lemma 2.6.1]. By induction, we are done. □

Proof of Proposition 3.3.5. Let S be a general element of $|L|$; by Proposition 3.3.3, S has lt singularities. Let $\mu : Y \to X$ be a log resolution of the pair (X, S) and of the base locus of $|L|$. We can write

$$\mu^* S = \overline{S} + \sum_i r_i E_i$$

$$K_Y = \mu^* K_X + \sum_i a_i E_i$$

$$K_Y + \overline{S} = \mu^*(K_X + S) + \sum_i (a_i - r_i) E_i$$

where $\overline{S} = \mu_*^{-1} S$ is the strict transform of S and $|\overline{S}|$ is basepoint free. Moreover, $r_i \in \mathbb{N}$ and $r_i \neq 0$ if and only if $\mu(E_i) \subset Bs|L|$.

Assume that S has not canonical singularities (resp. terminal singularities); after reordering we can assume that $a_0 < r_0$ (resp. $a_0 \leq r_0$). Since S is generic, by Bertini we can assume that $\mu(E_i) \subset Bsl|L|$, for all i such that $r_i > 0$.

Let $D = S + S_1$, where S_1 is another generic section in $|L|$; note that μ is a log resolution also for the pair (X, D). Let $r_0^1 \geq 1$ be the multiplicity of S_1 at the centre of valuation associated to E_0. Then (X, D) is not LC since $a_0 + 1 < r_0 + r_0^1$ (resp. $a_0 + 1 \leq r_0 + r_0^1$). Let $\gamma = \mathrm{lct}(X, D) \leq 1$ and $W \in CLC(X, \gamma D)$ be a minimal lc centre. Now we can reason as in the proof of Proposition 3.3.3. □

Proof of Proposition 3.3.6. In the notation of the proof of Proposition 3.3, assume by contradiction that S is not canonical. Then $a_i - r_i < 0$ for some

i; since a_i and r_i are integers, we get $a_i - r_i \leq -1$ and hence (X, S) is not plt. Set $\gamma = \mathrm{lct}(X, S) \leq 1$ and let $W \in CLC(X, \gamma S)$ be minimal lc centre. Now, as in the proof above, we derive a contradiction. □

Proof of Proposition 3.3.7. If f is a contraction to a point, then the result is exactly [9, Thm. 1.1], so assume that f is not a contraction to a point. Let $S \in |L|$ be general and assume by contradiction that S is not canonical. Then (X, S) is not plt. Let $H = \varepsilon f^*(h)$, where h is a general function on Z vanishing at $f(F)$ and $0 < \varepsilon << 1$. Set $D = S + H$ and $\delta = \mathrm{lct}(X, D) < 1$. We can consider a minimal centre $W \in CLC(X, \delta D)$ and reason as in the proof above. □

Proof of Theorem 1.1. The fact that X' is terminal follows by Proposition 3.3.5. The fact that $f_{|X'} : X' \to Z'$ is a local contraction supported by $K_{X'} + (\tau - 1)L'$ follows by the so called horizontal slicing ([5, Lemma 2.6]). □

4. Lifting of contractions

Let X be a terminal variety of dimension $n \geq 4$ and let $f : X \to Z$ be a local contraction supported by $K_X + \tau L$ such that $\tau > n - 3$; assume that f contracts a prime $\mathbb{Q}$-Cartier divisor E to a smooth point $p \in Z$.

By Theorem 1.1 the general $X' \in |L|$ has terminal singularities and $f' = f_{|X} : X' \to Z'$ is a divisorial contraction to $p \in Z'$. Since f_*L is a Cartier divisor let c be a positive integer c such that $f^*f_*L = L + cE$.

Lemma 4.1. *In the situation above, assume that p is smooth in Z' and that f' is a weighted blow-up of type $(1, a, b, c, \dots, c)$, where c appears $(n-4)$ times. Then f is a also a weighted blow-up of type $(1, a, b, c, \dots, c)$, where c appears $(n-3)$ times.*

Proof. Let $x_1, \dots, x_n$ local coordinates for p; we may also assume that $f_*(X') = \{x_n = 0\}$.

Note that $\mathcal{O}_X(-cE)$ is f-ample and that the map f is proper; so we have that

$$X = \mathrm{Proj}(\oplus_{d \geq 0} f_*\mathcal{O}_X(-dcE)).$$

Using the notation of Section 2, we need to prove that

$$f_*\mathcal{O}_X(-dcE) = \left(x_1^{s_1} \cdots x_n^{s_n} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} c s_j \geq dc \right).$$

The proof is by induction on $d \geq 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(-L - dcE) \to \mathcal{O}_X(-dcE) \to \mathcal{O}_{X'}(-dcE) \to 0.$$

Note that

$$-L - dcE \sim_f -(d-1)cE \sim_f K_X + \left(n - 3 + d - 1 + \frac{a+b}{c}\right) L,$$

Hence, pushing down to Z the above exact sequence and applying the relative Kawamata-Viehweg Vanishing, we have

$$(4.1) \qquad 0 \to f_*\mathcal{O}_X(-(d-1)cE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dcE) \to f_*\mathcal{O}_{X'}(-dcE) \to 0.$$

Since by assumption f' is a weighted blow of type $(1, a, b, c, \ldots, c)$, we have

$$f_*\mathcal{O}_{X'}(-dcE) = \left(x_1^{s_1} \cdots x_{n-1}^{s_{n-1}} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} c s_j \geq dc\right),$$

where $s_j \in \mathbb{N}$. By induction on d, we can also assume that

$$f_*\mathcal{O}_X(-(d-1)cE) = \left(x_1^{s_1} \cdots x_n^{s_n} : s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} c s_j \geq (d-1)c\right),$$

the case $d = 0$ being trivial.

Let $g = x_1^{s_1} \cdots x_n^{s_n} \in f_*\mathcal{O}_X(-dcE)$ be a monomial. If $s_n \geq 1$ then g, looking at the sequence (4.1), comes from $f_*\mathcal{O}_X(-(d-1)cE)$ by the multiplication by x_n; therefore

$$s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} s_j c + s_n c \geq (d-1)c + s_n c \geq dc.$$

If $s_n = 0$, then $g \in f_*\mathcal{O}_{X'}(-dcE)$ and so

$$s_1 + s_2 a + s_3 b + \sum_{j=4}^{n} s_j c = s_1 + s_2 a + s_3 b + \sum_{j=4}^{n-1} s_j c \geq dc.$$

The non-monomial case follows immediately. □

Proof of Theorem 1.2.A. Let $H_i \in |L|$ be general divisors for $i = 1, \dots, n-3$. By Theorem 1.1, for any i, H_i is a variety with terminal singularities and the morphism $f_i = f_{|H_i} : H_i \to f(H_i) =: Z_i$ is a local contraction supported by $K_{H_i} + (\tau - 1)L_{|H_i}$. Since Z is terminal and $\mathbb{Q}$-factorial (see [18, Corollary 3.36] and [18, Corollary 3.43]), then the Z_i's are $\mathbb{Q}$-Cartier divisors on Z.

For any $t = 0, \dots, n-3$ define $Y_t = \cap_{i=1}^{n-3-t} H_i$ and $g_t = f_{|Y_t} : Y_t \to g_t(Y_t) =: W_t$; in particular $Y_{n-3} = X$, $g_{n-3} = f$ and $W_{n-3} = Z$. Let, as in the statement of the Theorem, $X'' = Y_0$ and $f'' = g_0$.

By induction on t, applying Theorem 1.1, one sees that, for any $t = 0, \dots, n-4$, Y_t is terminal and $g_t = f_{|Y_t} : Y_t \to W_t$ is a Fano Mori contraction. Therefore W_t is a terminal variety (by [18, Corollary 3.43]) and it is a $\mathbb{Q}$-Cartier divisor in W_{t+1}, because intersection of $\mathbb{Q}$-Cartier divisors (by construction $W_t = \cap_{i=1}^{n-3-t} Z_i$). Therefore by [20, Lemma 1.7], and by induction on t, it follows that p is a smooth point in W_t, for all t.

Set $L_t := L_{|W_t}$. Since $Bs|L_t|$ has dimension at most 1 by Proposition 3.3.4, by Bertini's theorem (see [11, Thm. 6.3]) $E_t := Y_t \cap E$ is a prime divisor. E_t is the intersection of $\mathbb{Q}$-Cartier divisors and hence it is $\mathbb{Q}$-Cartier.

Therefore $f'' : X'' \to Z''$ is a divisorial contraction from a 3-fold X'' with terminal singularities, which contracts a prime $\mathbb{Q}$-Cartier divisor $E'' := E_0$ to a point $p \in Z''$, which we assume to be smooth. By [12] we know then that f'' is a blow-up of type $(1, a, b)$ (note that in [12] the $\mathbb{Q}$-factoriality of the domain is not needed, see also [13, Thm. 1.9]).

We conclude by induction on t applying Lemma 4.1. □

Proof of Theorem 1.2.B. We first show that E is contracted to a point. By [2, Theorem 2.1] $dim f(E) \leq 1$. Since $\dim E = n-2$ and the non-Gorenstein locus of X has codimension 3, if $dim f(E) = 1$ then there is a fiber which is not contained in the non-Gorenstein locus; by [6, Lemma2.1] we get a contradiction. (See the following Remark 4.3 for a further analysis).

By the rationality theorem, [15, Theorem 4.1.1], we have $2\tau = \frac{u}{v}$ where $u, v \in \mathbb{N}$ and $u \leq 2(n-1)$. Therefore we have :

$$n - 3 < \tau = \frac{u}{2v} \leq \frac{n-1}{v}.$$

If $n = 4$ this gives $v = 1$ and $u = 3$ or $v = 2$ and $u = 5$. If $n > 4$ we can have only $v = 1$ and $u = 2n - 5$.

We want to exclude the case $n = 4$ and $\tau = 5/4$. Assume by contradiction that $4K_X + 5L$ is a supporting divisor for f and set $H = 2K_X + 3L$. Then H is an ample Cartier divisor such that

$$2K_X + 5H = 3(4K_X + 5L).$$

This implies that $2K_X + 5H$ is also a supporting divisor for f and that $5/2 = \tau(X, H)$, which is impossible because in dimension 4 birational contractions with nef-value greater than 2 are divisorial (see [4]).

By [5, Theorem 5.1] we can suppose that L is globally generated. Pick $(n-3)$ general members $H_i \in |L|$ $(1 \leq i \leq n-3)$ and let $X' = \cap H_i$ be the scheme intersection. By Theorem 1.1 X' is a 3-fold with terminal singularities and, by horizontal slicing ([5, Lemma 2.6]), the restricted morphism $f' := f_{|X'} : X' \to Z'$ is a small contraction supported by $K_{X'} + (\tau - n + 3)L_{|X'}$ with exceptional locus $C = (\cap H_i) \cap E$. Note also that X' has terminal singularities and has index at most 2, in fact $2K_{X'} = 2(K_X + (n-3)L)_{|X'}$ is Cartier.

Small contractions on a 3-fold with terminal 2-factorial singularities are classified in [17, Theorem 4.2]. In particular this gives that C is irreducible and isomorphic to $\mathbb{P}^1$ and $-K_{X'}.C = \frac{1}{2}$.

Therefore also E is irreducible. Moreover, $\tau = \frac{2n-5}{2}$ implies $L_{|X'}.C = 1$ and thus $L_{|E}^{n-2} = 1$.

By [2, Thm. 2.1] we have that E is normal and $\Delta(E, L) = 0$; by the classification of varieties with Δ-genus equal to zero, we get that $(E, L) = (\mathbb{P}^{n-2}, \mathcal{O}(1))$. □

Example 4.2. We construct a family of examples of small contractions as in Theorem 1.2.B. We follow a construction via GIT as explained in [24] and further in [7]. Our examples are just higher dimensional versions of the examples of point (4) of the main theorem in [7], to which we refer for more details.

Fix $n \geq 3$. Let $x_1, \dots, x_{n-1}, y_1, y_2, z$ be coordinates on $\mathbb{C}^{n+2}$ and consider the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^{n+2}$ with weights $(1, 2, \dots, 2, -1, -1, 0)$, that is for any $\lambda \in \mathbb{C}^*$ we have $x_1 \mapsto \lambda x_1$, $x_i \mapsto \lambda^2 x_i$ for $i = 2, \dots, n-1$, $y_j \mapsto \lambda^{-1} y_j$ for $j = 1, 2$ and $z \mapsto z$.

Let

$$f = x_1 y_1 + (x_2 + \cdots + x_{n-1}) y_2^2 + z^k$$

with $k \geq 0$ and consider the hypersurface $A : \{f = 0\} \subset \mathbb{C}^{n+2}$. In the notation of [7], we are considering an action of type $(1, 2, \dots, 2, -1, -1, 0; 0)$.

Setting $B^- = A \cap \{x_1 = \cdots = x_{n-1} = 0\}$ and $B^+ = A \cap \{y_1 = y_2 = 0\}$ we can define $X = A /\!\!/ \mathbb{C}^*$, $X^- = A^- /\!\!/ \mathbb{C}^*$ and $X^+ = A^+ /\!\!/ \mathbb{C}^*$ to obtain the diagram

$$\begin{array}{ccc} X^- & \dashrightarrow & X^+ \\ & {\scriptstyle f^-}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & X & \end{array}$$

It is not difficult to check that this construction gives a flip $X^- \dashrightarrow X^+$ with exceptional loci $E^- = \mathbb{P}(1, 2, \ldots, 2) \cong \mathbb{P}^{n-2}$ and $E^+ = \mathbb{P}^1$. Since $K_{X^-} \sim \mathcal{O}(2n-5)$ we obtain that the contraction f^- is supported by $2K_{X^-} + (2n-5)L$, where $L = \mathcal{O}(2)$. Finally, note that the singular locus of X^+ is of the form $\mathbb{C}^{n-3} \times P$ where

$$P = 0 \in (x_1y_1 + y_2^2 + z^k)/\mathbb{Z}_2(1, 1, 1, 0)$$

is a $cA/2$ singularity.

Remark 4.3. Let $f : X \to Z$, L and τ be as in Theorem 1.2. . Assume also that $\dim E \leq n-3$ (in particular f is small). It follows by [2, Theorem 2.1(II.ii)] and [6, Lemma 2.1] that E is irreducible, it is contained in the non-Gorenstein locus of X, is contracted to a point and $(E, L_{|E}) = (\mathbb{P}^{n-3}, \mathcal{O}(1))$.

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