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BLM realization for Frobenius–Lusztig Kernels of type A

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The infinitesimal quantum $\mathfrak{gl}_n$ was realized in [1, §6]. We will realize Frobenius–Lusztig kernels of type A in this paper.

1. Introduction

In 1990, Ringel discovered the Hall algebra realization [21] of the positive part of the quantum enveloping algebras of finite type. Almost at the same time, the entire quantum $\mathfrak{gl}_n$ was realized by A. A. Beilinson, G. Lusztig and R. MacPherson in [1]. They first used q-Schur algebras to construct a $\mathbb{Q}(v)$-algebra $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$, and then proved that the quantum enveloping algebra of $\mathfrak{gl}_n$ over $\mathbb{Q}(v)$ can be realized as a subalgebra of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$.

Let $U_{\Bbbk}(n)$ be the the quantum enveloping algebra of $\mathfrak{gl}_n$ over $\Bbbk$ with standard generators $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm 1}$ and $\left[{K_i;0 \atop t} \right]$, where $\Bbbk$ is a commutative ring containing a primitive l'th root ε of 1. Let $p = \mathrm{char}\Bbbk$. For $h \geqslant 1$, let $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$ be the $\Bbbk$-subalgebra of $U_{\Bbbk}(n)$ generated by $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\left[{K_j;0 \atop t} \right]$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$ and $0 \leqslant m, t < lp^{h-1}$, where $l = l'$ if l' is odd, and $l = l'/2$ otherwise. Then we have $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_1 \subseteq \widetilde{\mathfrak{u}}_{\Bbbk}(n)_2 \subseteq \cdots \subseteq U_{\Bbbk}(n)$. In the case where l' is an odd number, let $\mathfrak{u}_{\Bbbk}(n)_h = \widetilde{\mathfrak{u}}_{\Bbbk}(n)_h / \langle K_1^l - 1, \ldots, K_n^l - 1 \rangle$. The algebra $\mathfrak{u}_{\Bbbk}(n)_1$ is called the infinitesimal quantum $\mathfrak{gl}_n$ and the algebra $\mathfrak{u}_{\Bbbk}(n)_h$ is called Frobenius–Lusztig kernels of $U_{\Bbbk}(n)$ (cf. [7]). The algebra $\mathfrak{u}_{\Bbbk}(n)_1$ was realized in [1, §6]. In this paper, we will realize the algebra $\mathfrak{u}_{\Bbbk}(n)_h$ for all $h \geqslant 1$. More precisely, we will first construct the $\Bbbk$-algebra $\mathscr{K}'(n)_h$ in §4. Then we will prove in 5.5 that $\mathfrak{u}_{\Bbbk}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is odd, and that $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is even and $\Bbbk$ is a field.

Let $\mathcal{S}_{\Bbbk}(n,r)$ be the q-Schur algebra over $\Bbbk$. A certain subalgebra, denoted by $\widetilde{\mathfrak{u}}_{\Bbbk}(n,r)_h$, of $\mathcal{S}_{\Bbbk}(n,r)$ was constructed in [12, §4]. It is proved in [13] that $\widetilde{\mathfrak{u}}_{\Bbbk}(n,r)_1$ is isomorphic to the little q-Schur algebra introduced in [11, 14]. We will prove in 6.1 that the algebra $\widetilde{\mathfrak{u}}_{\Bbbk}(n,r)_h$ is a homomorphic image of $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$.

Infinitesimal q-Schur algebras are certain important subalgebras of q-Schur algebras (cf. [2, 3, 6]). For $h \geqslant 1$ let $\mathfrak{s}_{k}(n)_h$ be the k-subalgebra of $U_{k}(n)$ generated by the algebra $\widetilde{\mathfrak{u}}_{k}(n)_h$ and $\left[\begin{smallmatrix} K_j;0 \\ t \end{smallmatrix}\right]$ $(1 \leqslant j \leqslant n, t \in \mathbb{N})$. We will prove in 6.4 that the infinitesimal q-Schur algebra $\mathfrak{s}_{k}(n,r)_h$ is a homomorphic image of $\mathfrak{s}_{k}(n)_h$.

Throughout this paper, let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate and let $\mathcal{Q} = \mathbb{Q}(v)$ be the fraction field of $\mathcal{Z}$. For $i \in \mathbb{Z}$ let $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$. For integers N, t with $t \geqslant 0$, let

$$\begin{bmatrix} N \\ t \end{bmatrix} = \frac{[N][N-1]\cdots[N-t+1]}{[t]^!} \in \mathcal{Z}$$

where $[t]^! = [1][2]\cdots[t]$. For $\mu \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let $\left[\begin{smallmatrix} \mu \\ \lambda \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mu_1 \\ \lambda_1 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} \mu_n \\ \lambda_n \end{smallmatrix}\right]$. For $\lambda, \mu \in \mathbb{Z}^n$, write $\lambda \leqslant \mu \Leftrightarrow \lambda_i \leqslant \mu_i$ for $1 \leqslant i \leqslant n$. We say that $\lambda < \mu$ if $\lambda \leqslant \mu$ and $\lambda \neq \mu$.

Let k be a commutative ring containing a primitive l'th root ε of 1 with $l' \geqslant 1$. Let $l \geqslant 1$ be defined by

$$l = \begin{cases} l' & \text{if } l' \text{ is odd,} \\ l'/2 & \text{if } l' \text{ is even.} \end{cases}$$

Let p be the characteristic of k. We will regard k as a $\mathcal{Z}$-module by specializing v to ε. When v is specialized to ε, $\left[\begin{smallmatrix} c \\ t \end{smallmatrix}\right]$ specialize to the element $\left[\begin{smallmatrix} c \\ t \end{smallmatrix}\right]_\varepsilon$ in k.

2. The BLM construction of quantum $\mathfrak{gl}_n$

Following [17] we define the quantum enveloping algebra $U_{\mathcal{Q}}(n)$ of $\mathfrak{gl}_n$ to be the $\mathbb{Q}(v)$-algebra with generators

$$E_i,\ F_i \quad (1 \leqslant i \leqslant n-1),\ K_j,\ K_j^{-1} \quad (1 \leqslant j \leqslant n)$$

and relations

(a) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1$;
(b) $K_iE_j = v^{\delta_{i,j} - \delta_{i,j+1}} E_jK_i$;
(c) $K_iF_j = v^{\delta_{i,j+1} - \delta_{i,j}} F_jK_i$;
(d) $E_iE_j = E_jE_i,\ F_iF_j = F_jF_i$ *when* $|i-j| > 1$;
(e) $E_iF_j - F_jE_i = \delta_{i,j}\frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{v - v^{-1}}$, *where* $\widetilde{K}_i = K_iK_{i+1}^{-1}$;
(f) $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ *when* $|i-j| = 1$;
(g) $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ *when* $|i-j| = 1$.

Following [19], let $U_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-subalgebra of $U_{\mathcal{Q}}(n)$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm 1}$ and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$, where for $m, t \in \mathbb{N}$,

$$E_i^{(m)} = \frac{E_i^m}{[m]^!}, \quad F_i^{(m)} = \frac{F_i^m}{[m]^!}, \quad \text{and} \quad \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

Let $\Theta(n)$ be the set of all $n \times n$ matrices over $\mathbb{N}$. Let $\Theta^{\pm}(n)$ be the set of all $A \in \Theta(n)$ whose diagonal entries are zero. Let $\Theta^+(n)$ (resp. $\Theta^-(n)$) be the subset of $\Theta(n)$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geqslant j$ (resp. $i \leqslant j$). For $A \in \Theta^{\pm}(n)$, write $A = A^+ + A^-$ with $A^+ \in \Theta^+(n)$ and $A^- \in \Theta^-(n)$. For $A \in \Theta^{\pm}(n)$ let

$$E^{(A^+)} = \prod_{\substack{i \leqslant s < j \\ 1 \leqslant i,j \leqslant n}} E_s^{(a_{ij})}, \quad F^{(A^-)} = \prod_{\substack{j \leqslant s < i \\ 1 \leqslant i,j \leqslant n}} F_s^{(a_{i,j})}$$

where the ordering of the products is the same as in [1, 3.9]. According to [19, 4.5] and [20, 7.8] we have the following result.

Proposition 2.1. *The set*

$$\left\{ E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i;0 \\ \lambda_i \end{bmatrix} F^{(A^-)} \;\middle|\; A \in \Theta^{\pm}(n),\, \delta, \lambda \in \mathbb{N}^n,\, \delta_i \in \{0,1\},\, \forall i \right\}$$

forms a $\mathcal{Z}$-basis of $U_{\mathcal{Z}}(n)$.

Using the stabilization property of the multiplication of q-Schur algebras, an important algebra $\mathcal{K}_{\mathcal{Z}}(n)$ over $\mathcal{Z}$ (without 1), with basis $\{[A] \mid A \in \widetilde{\Theta}(n)\}$ was constructed in [1, 4.5], where $\widetilde{\Theta}(n) = \{(a_{ij}) \in M_n(\mathbb{Z}) \mid a_{ij} \geqslant 0 \ \forall 1 \leqslant i \neq j \leqslant n\}$. It should be noted that the algebra $\mathcal{K}_{\mathcal{Z}}(n)$ is isomorphic to the Lusztig integral form of the modified quantum $\mathfrak{gl}_n$ (cf. [10, 6.3]).

For $A \in \widetilde{\Theta}(n)$ let $\mathrm{ro}(A) = (\sum_{1 \leqslant j \leqslant n} a_{1,j}, \ldots, \sum_{1 \leqslant j \leqslant n} a_{n,j})$ and $\mathrm{co}(A) = (\sum_{1 \leqslant i \leqslant n} a_{i,1}, \ldots, \sum_{1 \leqslant i \leqslant n} a_{i,n})$. Note that if $A, B \in \widetilde{\Theta}(n)$ is such that $\mathrm{co}(B) \neq \mathrm{ro}(A)$ then $[B] \cdot [A] = 0$ in $\mathcal{K}_{\mathcal{Z}}(n)$. For $1 \leqslant i, j \leqslant n$ let $E_{i,j}$ be the $n \times n$ matrix whose i, j entry is 1 and all other entries are zero. For $r \in \mathbb{N}$ let $\Lambda(n, r) = \{\lambda \in \mathbb{N}^n \mid \sum_{1 \leqslant i \leqslant n} \lambda_i = r\}$. From [1, 4.6(c)] we see that the algebra $\mathcal{K}_{\mathcal{Z}}(n)$ is generated by the elements $[mE_{i,i+1} + \mathrm{diag}(\lambda)]$, $[mE_{i+1,i} + \mathrm{diag}(\lambda)]$ for $m \in \mathbb{N}$, $1 \leqslant i \leqslant n-1$ and $\lambda \in \mathbb{Z}^n$. Furthermore the following multiplication formulas in $\mathcal{K}_{\mathcal{Z}}(n)$ is given in [1, 4.6(a),(b)].

Proposition 2.2. *Assume that $1 \leqslant i \leqslant n-1$, $m \in \mathbb{N}$ and $A \in \widetilde{\Theta}(n)$.*

(1) *If* $B \in \widetilde{\Theta}(n)$ *is such that* $\mathrm{co}(B) = \mathrm{ro}(A)$ *and* $B - mE_{i,i+1}$ *is a diagonal matrix, then*

$$[B] \cdot [A] = \sum_{\substack{\mathbf{t} \in \Lambda(n,m) \\ \forall u \neq i+1, t_u \leqslant a_{i+1,u}}} v^{\beta(\mathbf{t},A)} \prod_{1 \leqslant u \leqslant n} \begin{bmatrix} a_{i,u} + t_u \\ t_u \end{bmatrix} \Big[A + \sum_{1 \leqslant u \leqslant n} t_u (E^{\Delta}_{i,u} - E^{\Delta}_{i+1,u}) \Big],$$

where $\beta(\mathbf{t}, A) = \sum_{j>u} (a_{i,j} - a_{i+1,j}) t_u + \sum_{u<u'} t_u t_{u'}$.

(2) *If* $C \in \widetilde{\Theta}(n)$ *is such that* $\mathrm{co}(C) = \mathrm{ro}(A)$ *and* $C - mE_{i+1,i}$ *is a diagonal matrix, then*

$$[C] \cdot [A] = \sum_{\substack{\mathbf{t} \in \Lambda(n,m) \\ \forall u \neq i, t_u \leqslant a_{i,u}}} v^{\gamma(\mathbf{t},A)} \prod_{1 \leqslant u \leqslant n} \begin{bmatrix} a_{i+1,u} + t_u \\ t_u \end{bmatrix} \Big[A - \sum_{1 \leqslant u \leqslant n} t_u (E^{\Delta}_{i,u} - E^{\Delta}_{i+1,u}) \Big],$$

where $\gamma(\mathbf{t}, A) = \sum_{j<u} (a_{i+1,j} - a_{i,j}) t_u + \sum_{u<u'} t_u t_{u'}$.

Following [1, 5.1], let $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ be the vector space of all formal $\mathbb{Q}(v)$-linear combinations $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]$ satisfying the following property: for any $\mathbf{x} \in \mathbb{Z}^n$,

$$\text{the sets } \begin{matrix} \{A \in \widetilde{\Theta}(n) \mid \beta_A \neq 0,\ \mathrm{ro}(A) = \mathbf{x}\} \\ \{A \in \widetilde{\Theta}(n) \mid \beta_A \neq 0,\ \mathrm{co}(A) = \mathbf{x}\} \end{matrix} \text{ are finite.} \tag{2.2.1}$$

The product of two elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]$, $\sum_{B \in \widetilde{\Theta}(n)} \gamma_B [B]$ in $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ is defined to be $\sum_{A,B} \beta_A \gamma_B [A] \cdot [B]$ where $[A] \cdot [B]$ is the product in $\mathcal{K}_{\mathcal{Z}}(n)$. Then $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ becomes an associative algebra over $\mathbb{Q}(v)$.

For $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let

$$A(\delta, \lambda) = \sum_{\mu \in \mathbb{Z}^n} v^{\mu \centerdot \delta} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} [A + \mathrm{diag}(\mu)] \in \widehat{\mathcal{K}}_{\mathcal{Q}}(n);$$
$$A(\delta) = \sum_{\mu \in \mathbb{Z}^n} v^{\mu \centerdot \delta} [A + \mathrm{diag}(\mu)] \in \widehat{\mathcal{K}}_{\mathcal{Q}}(n),$$

where $\mu \centerdot \delta = \sum_{1 \leqslant i \leqslant n} \mu_i \delta_i$.

The next result is proved in [1, 5.5,5.7].

Theorem 2.3. *There is an injective algebra homomorphism* $\varphi : U_{\mathcal{Q}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ *satisfying*

$$E_i \mapsto E_{i,i+1}(\mathbf{0}),\ K_1^{j_1}K_2^{j_2}\cdots K_n^{j_n} \mapsto 0(\mathbf{j}),\ F_i \mapsto E_{i+1,i}(\mathbf{0}).$$

Furthermore the set $\{A(\mathbf{j}) \mid A \in \Theta^{\pm}(n),\ \mathbf{j} \in \mathbb{Z}^n\}$ *forms a* $\mathbb{Q}(v)$*-basis for* $\varphi(U_{\mathcal{Q}}(n))$.

We shall identify $U_{\mathcal{Q}}(n)$ with $\varphi(U_{\mathcal{Q}}(n))$. According to [16, 4.2,4.3,4.4], we have the following result.

Proposition 2.4. *The algebra* $U_{\mathcal{Z}}(n)$ *is generated as a* $\mathcal{Z}$*-module by the elements* $A(\delta,\lambda)$ *for* $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ *and* $\lambda \in \mathbb{N}^n$. *Furthermore, each of the following set forms a* $\mathcal{Z}$*-basis for* $U_{\mathcal{Z}}(n)$:

(1) $\{A(\mathbf{0})0(\delta,\lambda) \mid A \in \Theta^{\pm}(n),\, \delta,\lambda \in \mathbb{N}^n,\, \delta_i \in \{0,1\}, \forall i\}$;

(2) $\{A(\delta,\lambda) \mid A \in \Theta^{\pm}(n),\, \delta,\lambda \in \mathbb{N}^n,\, \delta_i \in \{0,1\}, \forall i\}$.

We end this section by recalling an important triangular relation in $\mathcal{K}_{\mathcal{Z}}(n)$. For $A = (a_{s,t}) \in \widetilde{\Theta}(n)$ let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s\leqslant i;t\geqslant j} a_{s,t} & \text{if } i<j \\ \sum_{s\geqslant i;t\leqslant j} a_{s,t} & \text{if } i>j. \end{cases}$$

Following [1], for $A, B \in \widetilde{\Theta}(n)$, define $B \preccurlyeq A$ if and only if $\sigma_{i,j}(B) \leqslant \sigma_{i,j}(A)$ for all $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and $\sigma_{i,j}(B) < \sigma_{i,j}(A)$ for some $i \neq j$.

According to [1, 5.5(c)], for $A \in \Theta^{\pm}(n)$ and $\lambda \in \mathbb{Z}^n$ the following triangular relation holds in $\mathcal{K}_{\mathcal{Z}}(n)$:

$$E^{(A^+)}[\mathrm{diag}(\lambda)]F^{(A^-)} = [A + \mathrm{diag}(\lambda - \boldsymbol{\sigma}(A))] + f \tag{2.4.1}$$

where $\boldsymbol{\sigma}(A) = (\sigma_1(A), \ldots, \sigma_n(A))$ with $\sigma_i(A) = \sum_{j<i}(a_{i,j} + a_{j,i})$ and f is a finite $\mathcal{Z}$-linear combination of $[B]$ with $B \in \widetilde{\Theta}(n)$ such that $B \prec A$.

3. The algebra $\widetilde{\mathrm{u}}_{k}(n)_h$

Let $U_{k}(n) = U_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} k$. We shall denote the images of $E_i^{(m)}$, $F_i^{(m)}$, $A(\delta,\lambda)$, etc. in $U_{k}(n)$ by the same letters. For $h \geqslant 1$ let $\widetilde{\mathrm{u}}_{k}(n)_h$ be the k-subalgebra of $U_{k}(n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\left[{K_j;0 \atop t}\right]$ for $1 \leqslant i \leqslant n-1$,

$1 \leqslant j \leqslant n$ and $0 \leqslant m, t < lp^{h-1}$. If l' is an odd number, we let

$$\mathrm{u}_{\Bbbk}(n)_h = \widetilde{\mathrm{u}}_{\Bbbk}(n)_h / \langle K_1^l - 1, \ldots, K_n^l - 1 \rangle. \tag{3.0.1}$$

The algebra $\mathrm{u}_{\Bbbk}(n)_h$ is called Frobenius–Lusztig kernels of $U_{\Bbbk}(n)$. We will construct several $\Bbbk$-bases for $\widetilde{\mathrm{u}}_{\Bbbk}(n)_h$ in 3.7.

We need some preparation before proving 3.7. The following result is due to [18, 3.2] (see also [15, 4.1]).

Lemma 3.1. *Let $m = m_0 + lm_1$, $0 \leqslant m_0 \leqslant l-1$, $m_1 \in \mathbb{N}$. Then*

$$\begin{bmatrix} m \\ t \end{bmatrix}_\varepsilon = \varepsilon^{l(t_1 l - t_1 m_0 - t m_1)} \begin{bmatrix} m_0 \\ t_0 \end{bmatrix}_\varepsilon \binom{m_1}{t_1}$$

for $0 \leqslant t \leqslant m$, where $t = t_0 + lt_1$ with $0 \leqslant t_0 \leqslant l-1$ and $t_1 \in \mathbb{N}$.

Lemma 3.2. *The following identity hold in $\Bbbk$: $\binom{m+p^{h-1}}{s} = \binom{m}{s}$ for $m \in \mathbb{Z}$ and $0 \leqslant s < p^{h-1}$.*

Proof. We consider the polynomial ring $\Bbbk[x, y]$. Since the characteristic of $\Bbbk$ is p we see that

$$\sum_{0 \leqslant j \leqslant p^{h-1}} \binom{p^{h-1}}{j} x^j y^{p^{h-1}-j} = (x+y)^{p^{h-1}} = x^{p^{h-1}} + y^{p^{h-1}}.$$

It follows that $\binom{p^{h-1}}{j} = 0$ for $0 < j < p^{h-1}$. This implies that

$$\binom{m+p^{h-1}}{s} = \sum_{0 \leqslant j \leqslant s} \binom{p^{h-1}}{j} \binom{m}{s-j} = \binom{m}{s}$$

for $m \in \mathbb{Z}$ and $0 \leqslant s < p^{h-1}$. □

We now generalize 3.2 to the quantum case.

Lemma 3.3. *Assume $0 \leqslant a < lp^{h-1}$ and $b \in \mathbb{Z}$. Then we have $\begin{bmatrix} b+lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon$. In particular, we have $\begin{bmatrix} b+l'p^{h-1} \\ a \end{bmatrix}_\varepsilon = \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon$*

Proof. We write $a = a_0 + a_1 l$ and $b = b_0 + b_1 l$ with $0 \leqslant a_0, b_0 < l$, $a_1 \in \mathbb{N}$ and $b_1 \in \mathbb{Z}$. If $b \in \mathbb{N}$, then by 3.1 and 3.2 we conclude that

$$\begin{aligned}\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon &= \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - a_1 b_1 l - a_0 b_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} b_1 + p^{h-1} \\ a_1 \end{pmatrix} \\ &= \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - a_1 b_1 l - a_0 b_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} b_1 \\ a_1 \end{pmatrix} \\ &= \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon .\end{aligned}$$

Furthermore if $b + lp^{h-1} < 0$, then $-b + a - 1 - lp^{h-1} \geqslant 0$ and hence

$$\begin{aligned}\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon &= (-1)^a \begin{bmatrix} -b + a - 1 - lp^{h-1} \\ a \end{bmatrix}_\varepsilon \\ &= (-1)^a \varepsilon^{alp^{h-1}} \begin{bmatrix} -b + a - 1 \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon .\end{aligned}$$

Now we assume $-lp^{h-1} \leqslant b < 0$. According to 3.1 we have

$$\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - a b_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} b_1 \\ a_1 \end{pmatrix}. \tag{3.3.1}$$

If $a_0 - b_0 - 1 \geqslant 0$ then $\left[\begin{smallmatrix} b_0 \\ a_0 \end{smallmatrix}\right]_\varepsilon = (-1)^{a_0} \left[\begin{smallmatrix} a_0 - b_0 - 1 \\ a_0 \end{smallmatrix}\right]_\varepsilon = 0$ and hence, by 3.1 and (3.3.1), we have

$$\begin{aligned}\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon &= (-1)^a \begin{bmatrix} l(a_1 - b_1) + (a_0 - b_0 - 1) \\ a \end{bmatrix}_\varepsilon \\ &= (-1)^a \varepsilon^{l(a_1 l - a_1 (a_0 - b_0 - 1) - a(a_1 - b_1))} \begin{bmatrix} a_0 - b_0 - 1 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} a_1 - b_1 \\ a_1 \end{pmatrix} \\ &= 0 \\ &= \varepsilon^{alp^{h-1}} \begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon .\end{aligned}$$

Now we assume $-lp^{h-1} \leqslant b < 0$ and $a_0 - b_0 - 1 < 0$. Then $a_1 - b_1 - 1 \geqslant 0$ and $0 \leqslant l + a_0 - b_0 - 1 < l$. According to 3.1 we have

$$\begin{aligned}
\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon &= (-1)^a \begin{bmatrix} -b+a-1 \\ a \end{bmatrix}_\varepsilon \\
&= (-1)^a \begin{bmatrix} l(a_1-b_1-1)+(l+a_0-b_0-1) \\ a \end{bmatrix}_\varepsilon \\
&= (-1)^a \varepsilon^{l(-a_1(a_0-b_0-1)-a(a_1-b_1-1))} \begin{bmatrix} l+a_0-b_0-1 \\ a_0 \end{bmatrix}_\varepsilon \binom{a_1-b_1-1}{a_1} \\
&= (-1)^{a_1 l + a_1} \varepsilon^{l(-a_1(a_0-b_0-1)-a(a_1-b_1-1))} \begin{bmatrix} b_0 - l \\ a_0 \end{bmatrix}_\varepsilon \binom{b_1}{a_1}.
\end{aligned}$$

Since $0 \leqslant a_0 < l$ and $[m+l]_\varepsilon = \varepsilon^{-l}[m]_\varepsilon$ we see that $\left[\begin{smallmatrix} b_0-l \\ a_0 \end{smallmatrix}\right] = \varepsilon^{a_0 l}\left[\begin{smallmatrix} b_0 \\ a_0 \end{smallmatrix}\right]_\varepsilon$. This implies that

$$\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon = (-1)^{a_1 l + a_1} \varepsilon^{l(a_0 - a_1(a_0-b_0-1)-a(a_1-b_1-1))} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \binom{b_1}{a_1}. \tag{3.3.2}$$

Furthermore since $\varepsilon^{2l} = 1$ and $(a_1^2 l - a_1) - (a_1 l + a_1) = -2a_1 + l a_1(a_1 - 1)$ is even, we see that

$$\begin{aligned}
\frac{\varepsilon^{l(a_1 l - a_1 b_0 - a b_1)}}{\varepsilon^{l(a_0 - a_1(a_0-b_0-1)-a(a_1-b_1-1))}} &= \varepsilon^{l(-2ab_1 - 2a_1 b_0 - 2a_0 + 2a_0 a_1)} \varepsilon^{l(a_1^2 l - a_1)} \\
&= \varepsilon^{l(a_1^2 l - a_1)} = \varepsilon^{l(a_1 l + a_1)} = (-1)^{a_1(l+1)}.
\end{aligned}$$

Thus by (3.3.1) and (3.3.2) we conclude that $\left[\begin{smallmatrix} b+lp^{h-1} \\ a \end{smallmatrix}\right]_\varepsilon = \varepsilon^{-alp^{h-1}}\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]_\varepsilon$. The proof is completed. □

Corollary 3.4. *Assume* $0 \leqslant a, b < lp^{h-1}$ *and* $a + b \geqslant lp^{h-1}$. *Then* $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_\varepsilon = 0$.

Proof. According to 3.3 we have $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right]_\varepsilon = \varepsilon^{-alp^{h-1}}\left[\begin{smallmatrix} a+b-lp^{h-1} \\ a \end{smallmatrix}\right]_\varepsilon$. Since $0 \leqslant a + b - lp^{h-1} < a$, we see that $\left[\begin{smallmatrix} a+b-lp^{h-1} \\ a \end{smallmatrix}\right]_\varepsilon = 0$. The assertion follows. □

Let $\widetilde{\mathfrak{u}}^0_{\not{k}}(n)_h$ be the $\not{k}$-subalgebra of $\widetilde{\mathfrak{u}}_{\not{k}}(n)_h$ generated by $K_j^{\pm 1}$, $\left[\begin{smallmatrix} K_j;0 \\ t \end{smallmatrix}\right]$ for $1 \leqslant j \leqslant n$ and $0 \leqslant t < lp^{h-1}$. For $h \geqslant 1$ let

$$\mathbb{N}^n_{lp^{h-1}} = \{\lambda \in \mathbb{N}^n \mid 0 \leqslant \lambda_i < lp^{h-1}, \forall i\}.$$

Lemma 3.5. *The set* $\mathfrak{M}^0 = \left\{ \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \left[\begin{smallmatrix} K_i;0 \\ \lambda_i \end{smallmatrix}\right] \mid \delta \in \mathbb{N}^n, \delta_i \in \{0,1\}, \lambda \in \mathbb{N}^n_{lp^{h-1}} \right\}$ *forms a* $\not{k}$-*basis for* $\widetilde{\mathfrak{u}}^0_{\not{k}}(n)_h$.

Proof. Let $V_1 = \mathrm{span}_{\mathcal{k}} \mathfrak{M}^0$. From 2.1, we see that the set $\mathfrak{M}^0$ is linearly independent. Thus it is enough to prove that $\widetilde{\mathfrak{u}}^0_{\mathcal{k}}(n)_h = V_1$. Let V_2 be the $\mathcal{k}$-submodule of $\widetilde{\mathfrak{u}}^0_{\mathcal{k}}(n)_h$ spanned by the elements $\prod_{1\leqslant i\leqslant n} K_i^{\delta_i} \left[{K_i;0 \atop \lambda_i} \right]$ ($\delta \in \mathbb{Z}^n$, $\lambda \in \mathbb{N}^n$, $0 \leqslant \lambda_i < lp^{h-1}$, for all i). According to [19, 2.3(g8)], for $0 \leqslant t, t' < lp^{h-1}$ we have

$$\varepsilon^{t't} \begin{bmatrix} K_i;0 \\ t' \end{bmatrix} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = \begin{bmatrix} t+t' \\ t \end{bmatrix}_\varepsilon \begin{bmatrix} K_i;0 \\ t+t' \end{bmatrix} - \sum_{0<j\leqslant t'} (-1)^j \varepsilon^{t(t'-j)} \begin{bmatrix} t+j-1 \\ j \end{bmatrix}_\varepsilon K_i^j \begin{bmatrix} K_i;0 \\ t'-j \end{bmatrix} \begin{bmatrix} K_i;0 \\ t \end{bmatrix}.$$

Note that by 3.4 we have $\left[{t+t' \atop t} \right]_\varepsilon \left[{K_i;0 \atop t+t'} \right] = 0$ for $0 \leqslant t, t' < lp^{h-1}$ with $t + t' \geqslant lp^{h-1}$. Thus, by induction on t' we see that $\left[{K_i;0 \atop t'} \right]\left[{K_i;0 \atop t} \right] \in V_2$ for $0 \leqslant t, t' < lp^{h-1}$. It follows that $\widetilde{\mathfrak{u}}^0_{\mathcal{k}}(n)_h = V_2$. Furthermore, by the proof of [19, 2.14], for $m \geqslant 0$ and $0 \leqslant t < lp^{h-1}$ we have

$$K_i^{m+2} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = \varepsilon^t(\varepsilon^{t+1} - \varepsilon^{-t-1}) K_i^{m+1} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} + \varepsilon^{2t} K_i^m \begin{bmatrix} K_i;0 \\ t \end{bmatrix},$$
$$K_i^{-m-1} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = -\varepsilon^{-t}(\varepsilon^{t+1} - \varepsilon^{-t-1}) K_i^{-m} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} + \varepsilon^{-2t} K_i^{-m+1} \begin{bmatrix} K_i;0 \\ t \end{bmatrix}.$$

If $t + 1 = lp^{h-1}$, then

$$\varepsilon^t(\varepsilon^{t+1} - \varepsilon^{-t-1}) K_i^{m+1} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} = -\varepsilon^{-t}(\varepsilon^{t+1} - \varepsilon^{-t-1}) K_i^{-m} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} = 0.$$

Thus by induction on $m \geqslant 0$ we see that $K_i^{\pm m} \left[{K_i;0 \atop t} \right] \in V_1$ for $0 \leqslant t < lp^{h-1}$. This implies that $V_1 = V_2$. The assertion follows. □

We are now ready to prove 3.7. Let $\Theta^\pm(n)_h = \{A \in \Theta^\pm(n) \mid 0 \leqslant a_{s,t} < lp^{h-1}, \forall s \neq t\}$.

Lemma 3.6. *The algebra $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h$ is generated as a $\mathcal{k}$-module by the elements $A(\delta, \lambda)$ for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$.*

Proof. Let V_h be the $\mathcal{k}$-submodule of $U_{\mathcal{k}}(n)$ spanned by $A(\delta, \lambda)$ for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$. According to [16, 3.5(1)] for $A \in \Theta^\pm(n)_h$,

$0 \leqslant m < lp^{h-1}$, $1 \leqslant i \leqslant n-1$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$, we have

$$\begin{aligned}&(mE_{i,i+1})(\mathbf{0})A(\delta,\lambda)\\ &= \sum_{\substack{\mathbf{t}\in\Lambda(n,m),\,0\leqslant j\leqslant\lambda_i\\ t_u\leqslant a_{i+1,u},\,\forall u\neq i+1\\ 0\leqslant k\leqslant\lambda_{i+1}\\ 0\leqslant c\leqslant\min\{t_i,j\}}} f^{\mathbf{t}}_{j,c,k}\Big(A+\sum_{u\neq i}t_uE_{i,u}-\sum_{u\neq i+1}t_uE_{i+1,u}\Big)(\delta+\alpha^{\mathbf{t}}_{j,c,k},\lambda+\beta^{\mathbf{t}}_{j,c,k}).\end{aligned}$$

where $\alpha^{\mathbf{t}}_{j,c,k} = \big(\sum_{i>u}t_u+\lambda_i-j-c\big)\boldsymbol{e}_i+\big(\lambda_{i+1}-k-\sum_{i+1>u}t_u\big)\boldsymbol{e}_{i+1}$, $\beta^{\mathbf{t}}_{j,c,k} = (t_i+j-c-\lambda_i)\boldsymbol{e}_i+(k-\lambda_{i+1})\boldsymbol{e}_{i+1}$ with $\boldsymbol{e}_i = (0,\ldots,0,\underset{i}{1},0,\ldots,0)\in\mathbb{N}^n$, and

$$f^{\mathbf{t}}_{j,c,k} = \varepsilon^{g^{\mathbf{t}}_{j,k}}\prod_{u\neq i}\begin{bmatrix}a_{i,u}+t_u\\ t_u\end{bmatrix}_\varepsilon\begin{bmatrix}-t_i\\ \lambda_i-j\end{bmatrix}_\varepsilon\begin{bmatrix}t_i+j-c\\ t_i\end{bmatrix}_\varepsilon\begin{bmatrix}t_i\\ c\end{bmatrix}_\varepsilon\begin{bmatrix}t_{i+1}\\ \lambda_{i+1}-k\end{bmatrix}_\varepsilon$$

with $g^{\mathbf{t}}_{j,k}=\sum_{j>u,\,j\neq i}a_{i,j}t_u-\sum_{j>u,\,j\neq i+1}a_{i+1,j}t_u+\sum_{u'\neq i,i+1,\,u<u'}t_ut_{u'}-t_i\delta_i+t_{i+1}\delta_{i+1}+2jt_i-kt_{i+1}$. If $A+\sum_{u\neq i}t_uE_{i,u}-\sum_{u\neq i+1}t_uE_{i+1,u}\notin\Theta^\pm(n)_h$ then $a_{i,u}+t_u\geqslant lp^{h-1}$ for some $u\neq i$. From 3.4 we see that $\left[\begin{smallmatrix}a_{i,u}+t_u\\ t_u\end{smallmatrix}\right]_\varepsilon=0$ and hence $f^{\mathbf{t}}_{j,c,k}=0$. Furthermore, if $\lambda+\beta^{\mathbf{t}}_{j,c,k}\notin\mathbb{N}^n_{lp^{h-1}}$ then $(\lambda+\beta^{t}_{j,c,k})_i=t_i+j-c\geqslant lp^{h-1}$. From 3.4 we see that $\left[\begin{smallmatrix}t_i+j-c\\ t_i\end{smallmatrix}\right]_\varepsilon=0$ and hence $f^{\mathbf{t}}_{j,c,k}=0$. Thus we conclude that

$$(mE_{i,i+1})(\mathbf{0})V_h\subseteq V_h, \tag{3.6.1}$$

for $0\leqslant m<lp^{h-1}$ and $1\leqslant i\leqslant n-1$. Similarly, using [16, 3.4,3.5(2)] we see that

$$(mE_{i+1,i})(\mathbf{0})V_h\subseteq V_h \quad\text{and}\quad 0(\gamma,\mu)V_h\subseteq V_h \tag{3.6.2}$$

for $0\leqslant m<lp^{h-1}$, $1\leqslant i\leqslant n-1$, $\gamma\in\mathbb{Z}^n$ and $\mu\in\mathbb{N}^n_{lp^{h-1}}$. Combining (3.6.1) with (3.6.2) implies that

$$\widetilde{\mathfrak{u}}_\Bbbk(n)_h\subseteq\widetilde{\mathfrak{u}}_\Bbbk(n)_hV_h\subseteq V_h. \tag{3.6.3}$$

On the other hand, from [16, 3.4] we see that for $A\in\Theta^\pm(n)_h$, $\delta\in\mathbb{Z}^n$ and $\lambda\in\mathbb{N}^n_{lp^{h-1}}$,

$$\begin{aligned}A(\mathbf{0})0(\delta,\lambda) &= \varepsilon^{\mathrm{co}(A)\centerdot(\delta+\lambda)}A(\delta,\lambda)\\ &\quad+\sum_{\mathbf{j}\in\mathbb{N}^n,\,\mathbf{0}<\mathbf{j}\leqslant\lambda}\varepsilon^{\mathrm{co}(A)\centerdot(\delta+\lambda-\mathbf{j})}\begin{bmatrix}\mathrm{co}(A)\\ \mathbf{j}\end{bmatrix}A(\delta-\mathbf{j},\lambda-\mathbf{j}).\end{aligned} \tag{3.6.4}$$

This implies that

$$V_h = \mathrm{span}_{\mathscr{k}}\{A(\mathbf{0})0(\delta,\lambda) \mid A \in \Theta^{\pm}(n)_h,\ \delta \in \mathbb{Z}^n,\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}. \tag{3.6.5}$$

Furthermore, combining (2.4.1) with 2.4 shows that for $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$E^{(A^+)}F^{(A^-)}\prod_{1\leqslant i\leqslant n} K_i^{\delta_i}\begin{bmatrix} K_i;0 \\ \lambda_i \end{bmatrix} = E^{(A^+)}F^{(A^-)}0(\delta,\lambda) = A(\mathbf{0})0(\delta,\lambda) + f$$

where f is a $\mathscr{k}$-linear combination of $B(\mathbf{0})0(\gamma,\mu)$ with $B \in \Theta^{\pm}(n)$, $B \prec A$, $\gamma \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}^n$. From (3.6.3) and (3.6.5) we see that f must be a $\mathscr{k}$-linear combination of $B(\mathbf{0})0(\gamma,\mu)$ with $B \in \Theta^{\pm}(n)_h$, $B \prec A$, $\gamma \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}^n_{lp^{h-1}}$. Thus we conclude that

$$V_h = \mathrm{span}_{\mathscr{k}}\left\{ E^{(A^+)}F^{(A^-)}\prod_{1\leqslant i\leqslant n} K_i^{\delta_i}\begin{bmatrix} K_i;0 \\ \lambda_i \end{bmatrix} \,\middle|\, A \in \Theta^{\pm}(n)_h,\ \delta \in \mathbb{Z}^n,\ \lambda \in \mathbb{N}^n_{lp^{h-1}} \right\} \subseteq \widetilde{\mathfrak{u}}_{\mathscr{k}}(n)_h. \tag{3.6.6}$$

The assertion follows. □

Proposition 3.7. *Each of the following set forms a $\mathscr{k}$-basis for $\widetilde{\mathfrak{u}}_{\mathscr{k}}(n)_h$:*

1. $\mathfrak{M} := \{E^{(A^+)}\prod_{1\leqslant i\leqslant n} K_i^{\delta_i}\left[\begin{smallmatrix} K_i;0 \\ \lambda_i \end{smallmatrix}\right]F^{(A^-)} \mid A\in\Theta^{\pm}(n)_h,\ \delta\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\ \forall i,\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$;
2. $\mathfrak{B} := \{A(\delta,\lambda) \mid A \in \Theta^{\pm}(n)_h,\ \delta \in \mathbb{N}^n,\ \delta_i \in \{0,1\},\ \forall i,\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$;
3. $\mathfrak{B}' := \{A(\mathbf{0})0(\delta,\lambda) \mid A \in \Theta^{\pm}(n)_h,\ \delta \in \mathbb{N}^n,\ \delta_i \in \{0,1\},\ \forall i,\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$.

Proof. According to 2.1 and 2.4, it is enough to prove that $\widetilde{\mathfrak{u}}_{\mathscr{k}}(n)_h = \mathrm{span}_{\mathscr{k}}\,\mathfrak{M} = \mathrm{span}_{\mathscr{k}}\,\mathfrak{B} = \mathrm{span}_{\mathscr{k}}\,\mathfrak{B}'$. From 3.5, 3.6, (3.6.5) and (3.6.6) we see that $\widetilde{\mathfrak{u}}_{\mathscr{k}}(n)_h = \mathrm{span}_{\mathscr{k}}\,\mathfrak{M} = \mathrm{span}_{\mathscr{k}}\,\mathfrak{B}'$. For $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ we have

$$\begin{aligned} A(\delta,\lambda) &= \varepsilon^{\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1})A(\delta - \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) + \varepsilon^{2\lambda_i}A(\delta - 2\boldsymbol{e}_i, \lambda) \\ &= -\varepsilon^{-\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1})A(\delta + \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) + \varepsilon^{-2\lambda_i}A(\delta + 2\boldsymbol{e}_i, \lambda) \end{aligned}$$

Note that if $\lambda_i + 1 = lp^{h-1}$ then $\varepsilon^{\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1})A(\delta - \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) = -\varepsilon^{-\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1})A(\delta + \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) = 0$. This together with 3.6 shows that $\widetilde{\mathfrak{u}}_{\mathscr{k}}(n)_h = \mathrm{span}_{\mathscr{k}}\,\mathfrak{B}$. □

4. The algebra $\mathscr{K}'(n)_h$

We will construct the algebra $\mathscr{K}'(n)_h$ in this section. We will prove in 5.5 the algebra $\mathscr{K}'(n)_h$ is the realization of $\widetilde{\mathfrak{u}}_k(n)_h$.

Let $\mathcal{K}_k(n) = \mathcal{K}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} k$, where k is regarded as a $\mathcal{Z}$-module by specializing v to ε. For $A \in \widetilde{\Theta}(n)$ let

$$[A]_\varepsilon = [A] \otimes 1 \in \mathcal{K}_k(n).$$

Let $\widetilde{\Theta}(n)_h$ be the set of all $A = (a_{i,j}) \in \widetilde{\Theta}(n)$ such that $a_{i,j} < lp^{h-1}$ for all $i \neq j$. We will denote by $\mathscr{K}(n)_h$ the k-submodule of $\mathcal{K}_k(n)$ spanned by the elements $[A]_\varepsilon$ with $A \in \widetilde{\Theta}(n)_h$.

To construct the algebra $\mathscr{K}'(n)_h$ we need the following lemma (cf. [1, 6.2] and [14, 5.1]).

Lemma 4.1. (1) *$\mathscr{K}(n)_h$ is a subalgebra of $\mathcal{K}_k(n)$. It is generated by $[mE_{i,i+1} + \mathrm{diag}(\lambda)]_\varepsilon$ and $[mE_{i+1,i} + \mathrm{diag}(\lambda)]_\varepsilon$ for $0 \leqslant m < lp^{h-1}$, $1 \leqslant i \leqslant n-1$ and $\lambda \in \mathbb{Z}^n$.*

(2) *Let D be any diagonal matrix in $\widetilde{\Theta}(n)$. The map $\tau_D : \mathscr{K}(n)_h \to \mathscr{K}(n)_h$ given by $[A]_\varepsilon \to [A + l'p^{h-1}D]_\varepsilon$ is an algebra homomorphism.*

Proof. Let $A = (a_{s,t}) \in \widetilde{\Theta}(n)_h$ and $0 \leqslant m < lp^{h-1}$. Assume that $B = (b_{s,t}) \in \widetilde{\Theta}(n)_h$ is such that $B - mE_{i,i+1}$ is a diagonal matrix such that $\mathrm{co}(B) = \mathrm{ro}(A)$. By 2.2 we have

$$[B]_\varepsilon \cdot [A]_\varepsilon = \sum_{\substack{\mathbf{t}\in\Lambda(n,m) \\ \forall u\neq i+1, t_u \leqslant a_{i+1,u}}} \varepsilon^{\beta(\mathbf{t},A)} \times \prod_{1\leqslant u\leqslant n} \begin{bmatrix} a_{i,u}+t_u \\ t_u \end{bmatrix}_\varepsilon \Big[A + \sum_{1\leqslant u\leqslant n} t_u(E_{i,u} - E_{i+1,u})\Big]_\varepsilon$$

where $\beta(\mathbf{t}, A) = \sum_{j>u}(a_{i,j} - a_{i+1,j})t_u + \sum_{u<u'} t_u t_{u'}$. Assume that $A + \sum_u t_u(E_{i,u} - E_{i+1,u}) \notin \widetilde{\Theta}(n)_h$ for some $\mathbf{t}$; then $a_{i,u} + t_u \geqslant lp^{h-1}$ for some $u \neq i$. Since $0 \leqslant a_{i,u}, t_u < lp^{h-1}$, by 3.4, we conclude that $\left[\begin{smallmatrix} a_{i,u}+t_u \\ t_u \end{smallmatrix}\right]_\varepsilon = 0$ and hence $[B]_\varepsilon \cdot [A]_\varepsilon \in \mathscr{K}(n)_h$. Similarly, we have $[C]_\varepsilon \cdot [A]_\varepsilon \in \mathscr{K}(n)_h$, where C is such that $C - mE_{i+1,i}$ is a diagonal matrix such that $\mathrm{co}(C) = \mathrm{ro}(A)$. Now using [1, 4.6(c)], (1) can be proved in a way similar to the proof of [1, 6.2].

By 2.2 and 3.3 we conclude that $\tau_D([A']_\varepsilon[A]_\varepsilon) = \tau_D([A']_\varepsilon)\tau_D([A]_\varepsilon)$ for any A' of the form B, C as above. Since $\mathscr{K}(n)_h$ is generated by elements like $[B]_\varepsilon$, $[C]_\varepsilon$ above, we conclude that τ_D is an algebra homomorphism. □

Let $\widetilde{\Theta}'(n)_h$ be the set of all $n \times n$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N}$, $a_{i,j} < lp^{h-1}$ for all $i \neq j$ and $a_{i,i} \in \mathbb{Z}/l'p^{h-1}\mathbb{Z}$ for all i. We have an obvious map $pr : \widetilde{\Theta}(n)_h \to \widetilde{\Theta}'(n)_h$ defined by reducing the diagonal entries modulo $l'p^{h-1}\mathbb{Z}$.

Let $\mathscr{K}'(n)_h$ be the free $\Bbbk$-module with basis $\{[A]_\varepsilon \mid A \in \widetilde{\Theta}'(n)_h\}$. We shall define an algebra structure on $\mathscr{K}'(n)_h$ as follows. If the column sums of A are not equal to the row sums of A' (as integers modulo $l'p^{h-1}$), then the product $[A]_\varepsilon \cdot [A']_\varepsilon$ for $A, A' \in \widetilde{\Theta}'(n)_h$ is zero. Assume now that the column sums of A are equal to the row sums of A' (as integers modulo $l'p^{h-1}$). We can then represent A, A' by elements $\widetilde{A}$, $\widetilde{A}' \in \widetilde{\Theta}(n)_h$ such that the column sums of $\widetilde{A}$ are equal to the row sums of $\widetilde{A}'$ (as integers). According to 4.1(1), we can write $[\widetilde{A}]_\varepsilon \cdot [\widetilde{A}']_\varepsilon = \sum_{\widetilde{A}'' \in I} \rho_{\widetilde{A}''}[\widetilde{A}'']_\varepsilon$ (product in $\mathscr{K}(n)_h$) where $I = \{\widetilde{A}'' \in \widetilde{\Theta}(n)_h \mid \mathrm{ro}(\widetilde{A}'') = \mathrm{ro}(\widetilde{A}),\ \mathrm{co}(\widetilde{A}'') = \mathrm{co}(\widetilde{A}')\}$ (a finite set) and $\rho_{\widetilde{A}''} \in \Bbbk$. Then the product $[A]_\varepsilon \cdot [A']_\varepsilon$ is defined to be $\sum_{\widetilde{A}'' \in I} \rho_{\widetilde{A}''}[pr(\widetilde{A}'')]_\varepsilon$. From 4.1(2) we see that the product is well defined and $\mathscr{K}'(n)_h$ becomes an associative algebra over $\Bbbk$.

In the case where l' is odd, the algebra $\mathscr{K}'(n)_1$ is the algebra $\mathscr{K}'$ constructed in [1, 6.3]. Furthermore, it was remarked at the end of [1] that $\mathscr{K}'$ is "essentially" the algebra defined in [19, §5] for type A. We will prove in 5.5 that $\mathscr{K}'(n)_h$ is isomorphic to the algebra $\mathtt{u}_\Bbbk(n)_h$ in the case where l' is odd.

Mimicking the construction of $\widehat{\mathcal{K}}_\mathcal{Q}(n)$, we define $\widehat{\mathcal{K}}_\Bbbk(n)$ to be the $\Bbbk$-module of all formal $\Bbbk$-linear combinations $\sum_{A \in \widetilde{\Theta}(n)} \beta_A[A]_\varepsilon$ satisfying the property (2.2.1). The product of two elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A[A]_\varepsilon$, $\sum_{B \in \widetilde{\Theta}(n)} \gamma_B[B]_\varepsilon$ in $\widehat{\mathcal{K}}_\Bbbk(n)$ is defined to be $\sum_{A,B} \beta_A\gamma_B[A]_\varepsilon \cdot [B]_\varepsilon$ where $[A]_\varepsilon \cdot [B]_\varepsilon$ is the product in $\mathcal{K}_\Bbbk(n)$. Then $\widehat{\mathcal{K}}_\Bbbk(n)$ becomes an associative algebra over $\Bbbk$.

We end this section by interpreting $\mathscr{K}'(n)_h$ as a $\Bbbk$-subalgebra of $\widehat{\mathcal{K}}_\Bbbk(n)$. For $h \geqslant 1$ let $\mathbb{Z}_{l'p^{h-1}} = \mathbb{Z}/l'p^{h-1}\mathbb{Z}$ and let $^- : \mathbb{Z}^n \to (\mathbb{Z}_{l'p^{h-1}})^n$ be the map defined by $\overline{(j_1, j_2, \ldots, j_n)} = (\overline{j_1}, \overline{j_2}, \ldots, \overline{j_n})$. For $A \in \Theta^\pm(n)_h$ and $\bar{\mu} \in (\mathbb{Z}_{l'p^{h-1}})^n$ let

$$(4.1.1) \qquad [\![A + \mathrm{diag}(\bar{\mu})]\!]_h = \sum_{\substack{\nu \in \mathbb{Z}^n \\ \bar{\mu} = \bar{\nu}}} [A + \mathrm{diag}(\nu)]_\varepsilon.$$

Let $\mathcal{W}_\Bbbk(n)_h$ be the $\Bbbk$-submodule of $\widehat{\mathcal{K}}_\Bbbk(n)$ spanned by the set $\{[\![A + \mathrm{diag}(\bar{\lambda})]\!]_h \mid A \in \Theta^\pm(n)_h,\ \bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n\}$. From 4.1 we see that $\mathcal{W}_\Bbbk(n)_h$ is a $\Bbbk$-subalgebra of $\widehat{\mathcal{K}}_\Bbbk(n)$. Furthermore, it is easy to prove that there is an algebra isomorphism

$$(4.1.2) \qquad \mathcal{W}_\Bbbk(n)_h \xrightarrow{\sim} \mathscr{K}'(n)_h$$

defined by sending $[\![A]\!]_h$ to $[A]_\varepsilon$ for $A \in \widetilde{\Theta}'(n)_h$.

5. Realization of $\mathtt{u}_{\mathcal{k}}(n)_h$

For $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let

$$A(\delta, \lambda)_\varepsilon = \sum_{\mu \in \mathbb{Z}^n} \varepsilon^{\mu \centerdot \delta} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_\varepsilon [A + \mathrm{diag}(\mu)]_\varepsilon \in \widehat{\mathcal{K}}_{\mathcal{k}}(n).$$

Let $\mathcal{V}_{\mathcal{k}}(n)$ be the $\mathcal{k}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ spanned by the elements $A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. For $h \geqslant 1$ let $\mathcal{V}_{\mathcal{k}}(n)_h$ be the $\mathcal{k}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ spanned by the elements $A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$. We will prove in 5.5 that $\mathtt{u}_{\mathcal{k}}(n)_h \cong \mathcal{V}_{\mathcal{k}}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is odd, and that $\widetilde{\mathtt{u}}_{\mathcal{k}}(n)_h \cong \mathcal{V}_{\mathcal{k}}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is even and $\mathcal{k}$ is a field.

Let $\widehat{\mathcal{K}}_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ consisting of the elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]$ with $\beta_A \in \mathcal{Z}$. Then $\widehat{\mathcal{K}}_{\mathcal{Z}}(n)$ is a $\mathcal{Z}$-subalgebra of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$. There is a natural algebra homomorphism

$$\theta : \widehat{\mathcal{K}}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \mathcal{k} \to \widehat{\mathcal{K}}_{\mathcal{k}}(n)$$

defined by sending $(\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]) \otimes 1$ to $\sum_{A \in \widetilde{\Theta}(n)} (\beta_A \cdot 1)[A]_\varepsilon$, where 1 is the identity element in $\mathcal{k}$.

Recall the injective algebra homomorphism $\varphi : U_{\mathcal{Q}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ defined in 2.3. From 2.4 we see that $\varphi(U_{\mathcal{Z}}(n)) \subseteq \widehat{\mathcal{K}}_{\mathcal{Z}}(n)$. Thus, by restriction, we get a map $\varphi : U_{\mathcal{Z}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Z}}(n)$. It induces an algebra homomorphism $\varphi_{\mathcal{k}} : U_{\mathcal{k}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \mathcal{k}$. The map θ, composed with $\varphi_{\mathcal{k}}$ gives an algebra homomorphism

$$\xi := \theta \circ \varphi_{\mathcal{k}} : U_{\mathcal{k}}(n) \to \widehat{\mathcal{K}}_{\mathcal{k}}(n). \tag{5.0.1}$$

By definition we have $\xi(A(\delta, \lambda)) = A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. This together with 2.4 and 3.6 implies that

$$\xi(U_{\mathcal{k}}(n)) = \mathcal{V}_{\mathcal{k}}(n) \text{ and } \xi(\widetilde{\mathtt{u}}_{\mathcal{k}}(n)_h) = \mathcal{V}_{\mathcal{k}}(n)_h. \tag{5.0.2}$$

In particular, $\mathcal{V}_{\mathcal{k}}(n)$ and $\mathcal{V}_{\mathcal{k}}(n)_h$ are all $\mathcal{k}$-subalgebras of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$.

We will now construct several bases for $\mathcal{V}_{\mathcal{k}}(n)_h$ and $\mathcal{V}_{\mathcal{k}}(n)$ in 5.1 and 5.3. These results will be used to prove 5.5. According to 3.3 we see that $\left[\begin{smallmatrix} \nu \\ \lambda \end{smallmatrix}\right]_\varepsilon =$

$\left[{\nu+l'p^{h-1}\delta \atop \lambda}\right]_\varepsilon$ for $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\nu, \delta \in \mathbb{Z}^n$. This implies that

$$A(\delta,\lambda)_\varepsilon = \sum_{\bar{\mu}\in(\mathbb{Z}_{l'p^{h-1}})^n} \varepsilon^{\delta\centerdot\mu} \begin{bmatrix}\mu\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\bar{\mu})]\!]_h \tag{5.0.3}$$

for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$, where $[\![A+\mathrm{diag}(\bar{\mu})]\!]_h$ is defined in (4.1.1). For $\lambda, \mu \in \mathbb{N}^n$, we write $\lambda \leqslant \mu$ if and only if $\lambda_i \leqslant \mu_i$ for $1 \leqslant i \leqslant n$. If $\lambda \leqslant \mu$ and $\lambda_i < \mu_i$ for some $1 \leqslant i \leqslant n$ then we write $\lambda < \mu$.

Lemma 5.1. *Assume l' is odd. Then $\mathcal{V}_\Bbbk(n)_h = \mathcal{W}_\Bbbk(n)_h$ and the set $\mathcal{N}_h := \{A(\mathbf{0},\lambda)_\varepsilon \mid A \in \Theta^\pm(n)_h,\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ forms a $\Bbbk$-basis for $\mathcal{V}_\Bbbk(n)_h$. Furthermore, if $p > 0$, then the set $\mathcal{N} := \{A(\mathbf{0},\lambda) \mid A \in \Theta^\pm(n),\ \lambda \in \mathbb{N}^n\}$ forms a $\Bbbk$-basis for $\mathcal{V}_\Bbbk(n)$.*

Proof. From (5.0.3) we see that for $A \in \Theta^\pm(n)_h$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$A(\mathbf{0},\lambda)_\varepsilon = [\![A+\mathrm{diag}(\bar{\lambda})]\!]_h + \sum_{\mu\in\mathbb{N}^n_{lp^{h-1}},\,\lambda<\mu} \begin{bmatrix}\mu\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\bar{\mu})]\!]_h.$$

This, together with the fact that the set $\{[\![A+\mathrm{diag}(\bar{\lambda})]\!]_h \mid A \in \Theta^\pm(n)_h,\ \bar{\lambda} \in (\mathbb{Z}_{lp^{h-1}})^n\}$ forms a $\Bbbk$-basis for $\mathcal{W}_\Bbbk(n)_h$, shows that the set $\mathcal{N}_h$ forms a $\Bbbk$-basis for $\mathcal{W}_\Bbbk(n)_h$. It follows that $\mathcal{W}_\Bbbk(n)_h \subseteq \mathcal{V}_\Bbbk(n)_h$. Furthermore from (5.0.3) we see that $\mathcal{V}_\Bbbk(n)_h \subseteq \mathcal{W}_\Bbbk(n)_h$. Thus $\mathcal{V}_\Bbbk(n)_h = \mathcal{W}_\Bbbk(n)_h$. Now we assume $p = \mathrm{char}\Bbbk > 0$. Since $\mathcal{V}_\Bbbk(n) = \bigcup_{h\geqslant1}\mathcal{V}_\Bbbk(n)_h$, $\mathcal{N} = \bigcup_{h\geqslant1}\mathcal{N}_h$ and the set $\mathcal{N}_h$ forms a $\Bbbk$-basis for $\mathcal{V}_\Bbbk(n)_h$, we conclude that the set $\mathcal{N}$ forms a $\Bbbk$-basis for $\mathcal{V}_\Bbbk(n)$. □

Lemma 5.2. *For $m \geqslant 1$, let $X_m = ((-1)^{\delta\centerdot\beta})_{\delta,\beta\in\mathcal{I}_m}$, where $\mathcal{I}_m = \{\delta \in \mathbb{N}^m \mid \delta_i \in \{0,1\}\, for\, 1 \leqslant i \leqslant m\}$. If we order the set $\mathcal{I}_m$ lexicographically, then $\det(X_m) = (-2)^{2^{m-1}m}$ for all m.*

Proof. Since $\mathcal{I}_m = \{(0,\delta) \mid \delta \in \mathcal{I}_{m-1}\} \cup \{(1,\delta) \mid \delta \in \mathcal{I}_{m-1}\}$ we see that

$$X_m = \begin{pmatrix} X_{m-1} & X_{m-1}\\ X_{m-1} & -X_{m-1}\end{pmatrix}.$$

This, together with the fact that $\det(X_1) = -2$, implies that

$$\det(X_m) = \det\begin{pmatrix} X_{m-1} & X_{m-1}\\ 0 & -2X_{m-1}\end{pmatrix} = (-2)^{2^{m-1}}\det(X_{m-1})^2 = (-2)^{2^{m-1}m}$$

as required. □

Corollary 5.3. *Assume l' is even and k is a field. Then $\mathcal{V}_k(n)_h = \mathcal{W}_k(n)_h$ and the set $\mathcal{B}_h := \{A(\delta,\lambda)_\varepsilon \mid A \in \Theta^\pm(n)_h,\ \lambda \in \mathbb{N}^n_{lp^{h-1}},\ \delta \in \mathbb{N}^n,\ \delta_i \in \{0,1\},\ \forall i\}$ forms a k-basis for $\mathcal{V}_k(n)_h$. Furthermore, if $p > 0$, then the set $\mathcal{B} := \{A(\delta,\lambda) \mid A \in \Theta^\pm(n),\ \lambda,\delta \in \mathbb{N}^n,\ \delta_i \in \{0,1\},\ \forall i\}$ forms a k-basis for $\mathcal{V}_k(n)$.*

Proof. Note that there is a bijective map from $\{(\delta,\lambda) \mid \delta \in \mathbb{N}^n,\ \delta_i \in \{0,1\},\ \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ to $(\mathbb{Z}_{l'p^{h-1}})^n$ defined by sending (δ,λ) to $\overline{\lambda + lp^{h-1}\delta}$. Thus by (5.0.3) and 3.3 we conclude that for $A \in \Theta^\pm(n)_h$, $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\delta \in \mathbb{N}^n$

$$\begin{aligned} A(\delta,\lambda)_\varepsilon &= \sum_{\substack{\beta\in\mathbb{N}^n,\ \beta_i\in\{0,1\},\ \forall i\\ \alpha\in\mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta\centerdot(\alpha+lp^{h-1}\beta)} \begin{bmatrix}\alpha+lp^{h-1}\beta\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\overline{\alpha+lp^{h-1}\beta})]\!]_h \\ &= \sum_{\substack{\beta\in\mathbb{N}^n,\ \beta_i\in\{0,1\},\ \forall i\\ \alpha\in\mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta\centerdot\alpha}\varepsilon^{lp^{h-1}(\delta\centerdot\beta-\beta\centerdot\lambda)} \begin{bmatrix}\alpha\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\overline{\alpha+lp^{h-1}\beta})]\!]_h. \end{aligned}$$

Since l' is even and $(l',p)=1$ we see that p is an odd prime. This, together with the fact that $\varepsilon^l = -1$, implies that $\varepsilon^{lp^{h-1}} = (-1)^{p^{h-1}} = -1$. Thus for $A \in \Theta^\pm(n)_h$, $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\delta \in \mathbb{N}^n$ we have

$$\begin{aligned} (5.3.1)\quad A(\delta,\lambda)_\varepsilon &= \sum_{\substack{\beta\in\mathbb{N}^n,\ \beta_i\in\{0,1\},\ \forall i\\ \alpha\in\mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta\centerdot\alpha}(-1)^{\beta\centerdot(\delta-\lambda)} \begin{bmatrix}\alpha\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\overline{\alpha+lp^{h-1}\beta})]\!]_h \\ &= \sum_{\beta\in\mathbb{N}^n,\ \beta_i\in\{0,1\},\ \forall i} \varepsilon^{\delta\centerdot\lambda}(-1)^{\beta\centerdot(\delta-\lambda)}[\![A+\mathrm{diag}(\overline{\lambda+lp^{h-1}\beta})]\!]_h \\ &\quad+ \sum_{\substack{\beta\in\mathbb{N}^n,\ \beta_i\in\{0,1\},\ \forall i\\ \alpha\in\mathbb{N}^n_{lp^{h-1}},\ \lambda<\alpha}} \varepsilon^{\delta\centerdot\alpha}(-1)^{\beta\centerdot(\delta-\lambda)} \begin{bmatrix}\alpha\\ \lambda\end{bmatrix}_\varepsilon [\![A+\mathrm{diag}(\overline{\alpha+lp^{h-1}\beta})]\!]_h. \end{aligned}$$

From 5.2 we see that for $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$\begin{aligned} \det(\varepsilon^{\delta\centerdot\lambda}(-1)^{\beta\centerdot(\delta-\lambda)})_{\delta,\beta\in\mathcal{I}_n} &= (-\varepsilon)^{\sum_{\delta\in\mathcal{I}_n}\lambda\centerdot\delta}(-2)^{2^{n-1}n} \\ &= (-\varepsilon)^{\sum_{\delta\in\mathcal{I}_n}\lambda\centerdot\delta}(\varepsilon^l-1)^{2^{n-1}n} \\ &\neq 0, \end{aligned}$$

where $\mathcal{I}_n = \{\delta \in \mathbb{N}^n \mid \delta_i \in \{0,1\} \text{ for } 1 \leqslant i \leqslant n\}$. It follows that the martix $(\varepsilon^{\delta\centerdot\lambda}(-1)^{\beta\centerdot(\delta-\lambda)})_{\delta,\beta\in\mathcal{I}_n}$ is invertible since k is a field. Thus by (5.3.1) we conclude that the set $\mathcal{B}_h$ forms a k-basis for $\mathcal{W}_k(n)_h$ and $\mathcal{V}_k(n)_h = \mathcal{W}_k(n)_h$. Now we assume $p = \mathrm{char}\,k > 0$. Then $\mathcal{B} = \bigcup_{h\geqslant 1}\mathcal{B}_h$. Since the set $\mathcal{B}_h$ is linear independent for all h, we conclude that the set $\mathcal{B}$ is linear independent. Consequently, the set $\mathcal{B}$ forms a k-basis for $\mathcal{V}_k(n)$. □

We are now ready to prove the main result of this paper.

Theorem 5.4. (1) *If l' is odd, then* $\ker(\xi) = \langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle$ *and hence* $U_{\Bbbk}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle \cong \mathcal{V}_{\Bbbk}(n)$.

(2) *If l' is even and $\Bbbk$ is a field with $p = \operatorname{char}\Bbbk > 0$, then ξ is injective and hence $U_{\Bbbk}(n) \cong \mathcal{V}_{\Bbbk}(n)$.*

Proof. The assertion (1) can be proved in a way similar to the proof of [16, 4.6]. The assertion (2) follows from 2.4, 5.3 and (5.0.2). □

Theorem 5.5. (1) *If l' is odd, then $\mathtt{u}_{\Bbbk}(n)_h \cong \mathcal{V}_{\Bbbk}(n)_h \cong \mathscr{K}'(n)_h$ for $h \geqslant 1$.*

(2) *If l' is even and $\Bbbk$ is a field, then $\widetilde{\mathtt{u}}_{\Bbbk}(n)_h \cong \mathcal{V}_{\Bbbk}(n)_h \cong \mathscr{K}'(n)_h$ for $h \geqslant 1$.*

Proof. If either l' is odd or both l' is even and $\Bbbk$ is a field, then by (4.1.2), 5.1 and 5.3, we deduce that $\mathcal{V}_{\Bbbk}(n)_h \cong \mathscr{K}'(n)_h$. If l' is odd, then $\xi(K_i^l - 1) = 0$ and hence the map $\xi : U_{\Bbbk}(n) \to \widehat{\mathcal{K}}_{\Bbbk}(n)$ induces an algebra homomorphism

$$\bar{\xi} : U_{\Bbbk}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle \to \widehat{\mathcal{K}}_{\Bbbk}(n).$$

One can prove that the set $\{E^{(A^+)} \prod_{1\leqslant i\leqslant n} K_i^{-\lambda_i} \left[\begin{smallmatrix} K_i;0 \\ \lambda_i \end{smallmatrix}\right] F^{(A^-)} \mid A \in \Theta^{\pm}(n)_h, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ forms a $\Bbbk$-basis of $\mathtt{u}_{\Bbbk}(n)_h$ in a way similar to the proof of [19, 6.5]. Thus we may regard $\mathtt{u}_{\Bbbk}(n)_h$ as a $\Bbbk$-subalgebra of $U_{\Bbbk}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n\rangle$. From (5.0.2) we see that $\bar{\xi}(\mathtt{u}_{\Bbbk}(n)_h) = \mathcal{V}_{\Bbbk}(n)_h$. Thus the restriction of $\bar{\xi}$ to $\mathtt{u}_{\Bbbk}(n)_h$ yields a surjective algebra homomorphism

$$\bar{\xi}' : \mathtt{u}_{\Bbbk}(n)_h \twoheadrightarrow \mathcal{V}_{\Bbbk}(n)_h.$$

This, together with 5.4(1), implies that $\mathtt{u}_{\Bbbk}(n)_h \cong \mathcal{V}_{\Bbbk}(n)_h$. Now we assume l' is even and $\Bbbk$ is a field. Since $\xi(\widetilde{\mathtt{u}}_{\Bbbk}(n)_h) = \mathcal{V}_{\Bbbk}(n)_h$ by (5.0.2), the restriction of ξ to $\widetilde{\mathtt{u}}_{\Bbbk}(n)_h$ yields a surjective algebra homomorphism

$$\xi' : \widetilde{\mathtt{u}}_{\Bbbk}(n)_h \twoheadrightarrow \mathcal{V}_{\Bbbk}(n)_h.$$

From 3.7 and 5.3 we see that ξ' is injective. Consequently, $\widetilde{\mathtt{u}}_{\Bbbk}(n)_h \cong \mathcal{V}_{\Bbbk}(n)_h$. □

6. The infinitesimal q-Schur algebras and little q-Schur algebras

Let $\mathcal{S}_{\mathcal{Z}}(n, r)$ be the algebra over $\mathcal{Z}$ introduced in [1, 1.2]. It has a $\mathcal{Z}$-basis $\{[A] \mid A \in \Theta(n, r)\}$ defined in [1], where $\Theta(n, r) = \{A \in \Theta(n) \mid \sigma(A) :=$

$\sum_{1\leqslant i,j\leqslant n} a_{i,j} = r\}$. It is proved in [8, A.1] that the algebra $\mathcal{S}_{\mathcal{Z}}(n,r)$ is isomorphic to the q-Schur algebra introduced in [4, 5]. Let $\mathcal{S}_{\mathcal{k}}(n,r) = \mathcal{S}_{\mathcal{Z}}(n,r) \otimes_{\mathcal{Z}} \mathcal{k}$. For $A \in \Theta(n,r)$ let

$$[A]_\varepsilon = [A] \otimes 1 \in \mathcal{S}_{\mathcal{k}}(n,r).$$

Recall that $\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \mid \sum_{1\leqslant i\leqslant n} \lambda_i = r\}$. Let $\overline{\Lambda(n,r)}_{l'p^{h-1}} = \{\bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n \mid \lambda \in \Lambda(n,r)\}$. For $A \in \Theta^\pm(n)_h$ and $\overline{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n$ we define the element $[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h \in \mathcal{S}_{\mathcal{k}}(n,r)$ as follows:

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h = \sum_{\substack{\mu\in\Lambda(n,r-\sigma(A))\\ \overline{\mu}=\overline{\lambda}}} [A + \mathrm{diag}(\mu)]_\varepsilon$$

Note that $[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h = 0$ if either $\sigma(A) > r$ or $\overline{\lambda} \notin \overline{\Lambda(n, r-\sigma(A))}_{l'p^{h-1}}$. Let $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n,r)_h$ be the $\mathcal{k}$-submodule of $\mathcal{S}_{\mathcal{k}}(n,r)$ spanned by the set $\{[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h \mid A \in \Theta^\pm(n)_h, \overline{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n\}$. According to [12, 4.8], $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n,r)_h$ is a $\mathcal{k}$-subalgebra of $\mathcal{S}_{\mathcal{k}}(n,r)$. Note that the algebra $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n,r)_1$ is the little q-Schur algebra introduced in [11, 14]. We will prove in 6.1 that the algebra $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n,r)_h$ is a homomorphic image of $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h$.

Let $\mathcal{S}_{\mathcal{Q}}(n,r) = \mathcal{S}_{\mathcal{Z}}(n,r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. For $A \in \Theta^\pm(n)$, $\delta \in \mathbb{Z}^n$ let

$$A(\delta, r) = \sum_{\mu\in\Lambda(n,r-\sigma(A))} v^{\mu\cdot\delta}[A + \mathrm{diag}(\mu)] \in \mathcal{S}_{\mathcal{Q}}(n,r).$$

According to [1], there is an algebra epimorphism

$$\zeta_r : U_{\mathcal{Q}}(n) \twoheadrightarrow \mathcal{S}_{\mathcal{Q}}(n,r)$$

satisfying $\zeta_r(E_i) = E_{i,i+1}(\mathbf{0}, r)$, $\zeta_r(K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n}) = 0(\mathbf{j}, r)$ and $\zeta_r(F_i) = E_{i+1,i}(\mathbf{0}, r)$, for $1 \leqslant i \leqslant n-1$ and $\mathbf{j} \in \mathbb{Z}^n$. It is proved in [9] that $\zeta_r(U_{\mathcal{Z}}(n)) = \mathcal{S}_{\mathcal{Z}}(n,r)$. By restriction, the map $\zeta_r : U_{\mathcal{Q}}(n) \to \mathcal{S}_{\mathcal{Q}}(n,r)$ induces a surjective algebra homomorphism $\zeta_r : U_{\mathcal{Z}}(n) \to \mathcal{S}_{\mathcal{Z}}(n,r)$. The map $\zeta_r : U_{\mathcal{Z}}(n) \to \mathcal{S}_{\mathcal{Z}}(n,r)$ induces an algebra homomorphism

$$\zeta_{r,\mathcal{k}} := \zeta_r \otimes id : U_{\mathcal{k}}(n) \to \mathcal{S}_{\mathcal{k}}(n,r).$$

Proposition 6.1. *If either l' is odd or both l' is even and $\mathcal{k}$ is a field then* $\zeta_{r,\mathcal{k}}(\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h) = \widetilde{\mathfrak{u}}_{\mathcal{k}}(n,r)_h$.

Proof. According to [10, 6.7], there is a surjective algebra homomorphism

$$\dot{\zeta}_r : \mathcal{K}_{\mathcal{Z}}(n) \to \mathcal{S}_{\mathcal{Z}}(n, r)$$

such that

$$\dot{\zeta}_r([A]) = \begin{cases} [A] & \text{if } A \in \Theta(n, r); \\ 0 & \text{otherwise.} \end{cases}$$

The map $\dot{\zeta}_r$ induces a surjective algebra homomorphism

$$\widehat{\dot{\zeta}}_{r,k} : \widehat{\mathcal{K}}_k(n) \to \mathcal{S}_k(n, r)$$

defined by sending $\sum_{A\in\widetilde{\Theta}(n)} \beta_A[A]_\varepsilon$ to $\sum_{A\in\Theta(n,r)} \beta_A[A]_\varepsilon$. It is easy to see that

$$\zeta_{r,k} = \widehat{\dot{\zeta}}_{r,k} \circ \xi \tag{6.1.1}$$

where ξ is given in (5.0.1). This together with (5.0.2) implies that $\zeta_{r,k}(\widetilde{\mathfrak{u}}_k(n)_h) = \widehat{\dot{\zeta}}_{r,k}(\mathcal{V}_k(n)_h)$. Clearly, for $A \in \Theta^\pm(n)_h$ and $\bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n$, we have $\widehat{\dot{\zeta}}_{r,k}([\![A + \mathrm{diag}(\bar{\lambda})]\!]_h) = [\![A + \mathrm{diag}(\bar{\lambda}), r]\!]_h$. Combining these facts with 5.1 and 5.3 gives the result. □

Let $\mathtt{s}_k(n, r)_h$ be the the infinitesimal q-Schur algebra introduced in [2, 3]. The algebra $\mathtt{s}_k(n, r)_h$ is a certain k-subalgebra of the q-Schur algebra $\mathcal{S}_k(n, r)$. According to [2, 5.3.1], we have the following result.

Lemma 6.2. *The set* $\{[A]_\varepsilon \mid A \in \Theta(n, r)_h\}$ *forms a* k*-basis of* $\mathtt{s}_k(n, r)_h$*.*

For $h \geqslant 1$ let $\mathtt{s}_k(n)_h$ be the k-subalgebra of $U_k(n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\left[{K_j;0 \atop t} \right]$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, $t \in \mathbb{N}$ and $0 \leqslant m < lp^{h-1}$. We will prove in 6.4 that the algebra $\mathtt{s}_k(n, r)_h$ is a homomorphic image of $\mathtt{s}_k(n)_h$.

Lemma 6.3. *Each of the following set forms a* k*-basis for* $\mathtt{s}_k(n)_h$ *:*

(1) $\{E^{(A^+)} \prod_{1\leqslant i\leqslant n} K_i^{\delta_i} \left[{K_i;0 \atop \lambda_i} \right] F^{(A^-)} \mid A \in \Theta^\pm(n)_h,\ \delta \in \mathbb{N}^n,\ \delta_i \in \{0, 1\}, \forall i,\ \lambda \in \mathbb{N}^n\}$;

(2) $\{A(\delta, \lambda) \mid A \in \Theta^\pm(n)_h,\ \delta, \lambda \in \mathbb{N}^n,\ \delta_i \in \{0, 1\}, \forall i\}$.

Proof. The assertion can be proved in a way similar to the proof of 3.7. □

Proposition 6.4. *We have* $\zeta_{r,k}(\mathtt{s}_k(n)_h) = \mathtt{s}_k(n, r)_h$*.*

Proof. From 6.1.1 we see that

$$\zeta_{r,k}(A(\delta,\lambda)) = \hat{\dot{\zeta}}_{r,k}(A(\delta,\lambda)_\varepsilon) = A(\delta,\lambda,r)_\varepsilon$$

for all A,δ,λ, where $A(\delta,\lambda,r)_\varepsilon = \sum_{\mu\in\Lambda(n,r-\sigma(A))} \varepsilon^{\mu\centerdot\delta} \left[{\mu \atop \lambda}\right]_\varepsilon [A+\mathrm{diag}(\mu)]_\varepsilon \in \mathcal{S}_k(n,r)$. Thus by 6.2 and 6.4 we conclude that

$$\begin{aligned}\zeta_{r,k}(\mathtt{s}_k(n)_h) &= \mathrm{span}_k\{A(\delta,\lambda,r)_\varepsilon \mid A\in\Theta^\pm(n)_h,\ \delta,\lambda\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\ \forall i\}\\ &\subseteq \mathtt{s}_k(n,r)_h.\end{aligned}$$

On the other hand, for $A\in\Theta^\pm(n)_h$ and $\mu\in\Lambda(n,r-\sigma(A))$ we have $[A+\mathrm{diag}(\mu)] = A(\mathbf{0},\mu,r)\in\zeta_{r,k}(\mathtt{s}_k(n)_h)$. This implies that $\mathtt{s}_k(n,r)_h\subseteq\zeta_{r,k}(\mathtt{s}_k(n)_h)$. The assertion follows. □

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