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Boundedness of composition operators associated with mixed homogeneities on Lipschitz spaces

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This note is motivated by Phong and Stein's work in [PS]. We introduce a new class of Lipschitz spaces associated with mixed homogeneities and characterize these spaces via the Littlewood-Paley theory. We prove that the composition of two Calderón-Zygmund singular integral operators with different homogeneities is bounded on these Lipschitz spaces. Our result answers a question arises in the context of Lipschitz spaces due to the constructions in [MR].

1. Introduction and statement of results

This note is motivated by Phong and Stein's work in [PS]. The purpose of this note is to introduce a new class of Lipschitz spaces associated with mixed homogeneities and characterize these spaces via the Littlewood-Paley theory. We prove that the composition of two Calderón-Zygmund singular integral operators with different homogeneities is bounded on these Lipschitz spaces.

In order to explain the question we deal with let us recall the questions of composition of operators with different homogeneities. More precisely, let $e(\xi)$ be a function on $\mathbb{R}^n$ homogeneous of degree 0 in the isotropic sense and smooth away from the origin. Similarly, suppose that $h(\xi)$ is a function on $\mathbb{R}^n$ homogeneous of degree 0 in the non-isotropic sense related to the heat equation, and also smooth away from the origin. Then it is well-known

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that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type (1, 1). Rivieré in [WW] asked the question: Is the composition $T_1 \circ T_2$ still of weak-type (1,1)? Phong and Stein in [PS] were the first to answer this question and they gave the necessary and sufficient conditions for which $T_1 \circ T_2$ is of weak-type (1,1). The operators Phong and Stein studied are in fact compositions with mixed type of homogeneities which arise naturally in the $\bar{\partial}$-Neumann problem. See [PS] for more details.

Indeed, there are other questions of this type that can be asked about composition of operators associated with different homogeneities, which cannot be answered by using the properties of these operators separately. We mention that in [HLLRS] such a question was considered for the Hardy spaces. It is well-known that any operator T_1 is bounded on the classical Lipschitz space, while T_2 is bounded on the Lipschitz space associated the non-isotropic homogeneity, which was introduced in [MR]. For more about the Lipschitz spaces, see also [C], [HSV], [JTW], [K1, K2], [P], [S1] and [S2].

Based on the results proved in [MR], however, in general the composition of $T_1 \circ T_2$ is not bounded on the classical Lipschitz space and the Lipschitz space associated the non-isotropic homogeneity either. In this note, we study such a question of the boundedness of the composition of $T_1 \circ T_2$ on the Lipschitz spaces.

To describe more precisely questions and results studied in this note, we begin with considering all functions and operators defined on $\mathbb{R}^n$. We write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ with $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We consider two kinds of homogeneities

$$\delta : (x', x_n) \to (\delta x', \delta x_n), \delta > 0$$

and

$$\delta : (x', x_n) \to (\delta x', \delta^2 x_n), \delta > 0.$$

The first are the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second are non-isotropic and related to the heat equations (also Heisenberg groups).

For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we denote $|x|_e = (|x'|^2 + |x_n|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_n|)^{\frac{1}{2}}$. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. The singular integrals considered in this note are defined by

Definition 1.1. A locally integrable function $\mathcal{K}_1$ on $\mathbb{R}^n \setminus \{0\}$ is said to be a Calderón-Zygmund kernel associated with the isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\mathcal{K}_1(x)\right| \le A|x|_e^{-n-|\alpha|} \qquad \text{for all } |\alpha| \ge 0, \tag{1.1}$$

$$\int_{r_1<|x|_e<r_2} \mathcal{K}_1(x)\; dx = 0 \tag{1.2}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_1 is a Calderón-Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(\mathcal{K}_1 * f)(x)$, where $\mathcal{K}_1$ satisfies conditions of (1.1) and (1.2).

Definition 1.2. Suppose $\mathcal{K}_2 \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$. $\mathcal{K}_2$ is said to be a Calderón-Zygmund kernel associated with the non-isotropic homogeneity if

$$\left|\frac{\partial^\alpha}{\partial(x')^\alpha}\frac{\partial^\beta}{\partial(x_n)^\beta}\mathcal{K}_2(x',x_n)\right| \le B|x|_h^{-n-1-|\alpha|-2\beta} \qquad \text{for all } |\alpha|,\ \beta \ge 0, \tag{1.3}$$

$$\int_{r_1<|x|_h<r_2} \mathcal{K}_2(x)\; dx = 0 \tag{1.4}$$

for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_2 is a Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = p.v.(\mathcal{K}_2 * f)(x)$, where $\mathcal{K}_2$ satisfies the conditions in (1.3) and (1.4).

To study the boundedness of the composition of $T_1 \circ T_2$ on the Lipschitz spaces, we first introduce the following notations:

$$(\Delta_u f)(x) = f(x-u) - f(x); (\Delta_u^z f)(x) = f(x-u) + f(x+u) - 2f(x),$$

and similar for Δ_v and Δ_v^z.

Now the Lipschitz space associated with different homogeneities is defined by the following

Definition 1.3. A continuous function $f(x)$ defined on $\mathbb{R}^n$ belongs to the Lipschitz space Lip^α_{com} with $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 > 0$ if and only if

(i) when $0 < \alpha_1, \alpha_2 < 1$,

$$\|(\Delta_u \Delta_v) f\|_{L^\infty(\mathbb{R}^n)} \leq C|u|_e^{\alpha_1}|v|_h^{\alpha_2}; \tag{1.5}$$

(ii) when $\alpha_1 = 1, 0 < \alpha_2 < 1$,

$$\|(\Delta_u^z \Delta_v) f\|_{L^\infty(\mathbb{R}^n)} \leq C|u|_e|v|_h^{\alpha_2}; \tag{1.6}$$

(iii) when $0 < \alpha_1 < 1, \alpha_2 = 1$,

$$\|(\Delta_u \Delta_v^z) f\|_{L^\infty(\mathbb{R}^n)} \leq C|u|_e^{\alpha_1}|v|_h; \tag{1.7}$$

(iv) when $\alpha_1 = \alpha_2 = 1$,

$$\|(\Delta_u^z \Delta_v^z) f\|_{L^\infty(\mathbb{R}^n)} \leq C|u|_e|v|_h; \tag{1.8}$$

for all $u, v \in \mathbb{R}^n$ and the constant C is independent of u and v.

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 1$, we write $\alpha_1 = m_1 + r_1$ and $\alpha_2 = m_2 + r_2$ where m_1, m_2 are integers and $0 < r_1, r_2 \leq 1$.$f \in Lip_{com}^{\alpha}$ means that f is a $C^{m_1+m_2}$ function, modulo polynomials of degree not exceeding $m_1 + m_2$, such that all partial derivatives $\partial^\beta f, |\beta| = m_1 + m_2$, belong to Lip_{com}^{r} with $r = (r_1, r_2)$.

If $f \in Lip_{com}^{\alpha}$, $\|f\|_{Lip_{com}^{\alpha}}$, the norm of f, is defined by the smallest constant C in (1.5) to (1.8).

We remark that all affine functions belong to Lip_{com}^{α} with zero norm. When $\alpha_1 = 1$ or $\alpha_2 = 1$ or both $\alpha_1 = \alpha_2 = 1$, the Zygmund type conditions are used. Moreover, the different homogeneities are involved in (1.5) to (1.8) implicitly.

In order to obtain the boundedness of the composition of $T_1 \circ T_2$ on the Lipschitz space Lip_{com}^{α}, we characterize Lip_{com}^{α} via the Littlewood-Paley theory. For this purpose, let $\psi^{(1)}$ be a radial function in $\mathcal{S}(\mathbb{R}^n)$, that is, $\psi^{(1)}(x)$ depends only on $|x|_e$ and satisfy

$$\operatorname{supp} \widehat{\psi^{(1)}} \subseteq \left\{(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\right\}, \tag{1.9}$$

and

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_n)|^2 = 1 \text{ for all } (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}/\{(0,0)\}. \tag{1.10}$$

And let $\psi^{(2)}(x)$ be a function in $\mathcal{S}(\mathbb{R}^n)$ depending only on $|x|_h$ and satisfy

$$\text{supp}\ \widehat{\psi^{(2)}} \subseteq \left\{(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2\right\}, \tag{1.11}$$

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_n)|^2 = 1 \text{ for all } (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \setminus \{(0,0)\}. \tag{1.12}$$

Now set $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$ where

$$\psi_j^{(1)}(x) = 2^{jn}\psi^{(1)}(2^j x', 2^j x_n) \quad \text{and} \quad \psi_k^{(2)}(x) = 2^{k(n+1)}\psi^{(2)}(2^k x', 2^{2k} x_n).$$

We have the following

Theorem 1.4. *$f \in Lip^{\alpha}_{com}$ with $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 > 0$ if and only if $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, the tempered distributions modulo all polynomials, and*

$$\sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f(x)| \leq C < \infty.$$

Moreover,

$$\|f\|_{Lip^{\alpha}_{com}} \sim \sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f(x)|.$$

The main result of this note is

Theorem 1.5. *The composition of $T_1 \circ T_2$ is bounded on Lip^{α}_{com} with $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 > 0$.*

The proofs of Theorem 1.4 and 1.5 will be given in next two sections, respectively. Following Stein's inspiring suggestions, we will make some remarks in the last section to indicate the connection between Stein and Yung's work in [SY] and the present work.

2. Proof of Theorem 1.4

We first show that if $f \in Lip^{\alpha}_{com}$ with $\alpha = (\alpha_1, \alpha_2), 0 < \alpha_1, \alpha_2 < 1$, then $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$. To see this, for each $g(x) \in \mathcal{S}_\infty(\mathbb{R}^n)(= \{g \in \mathcal{S} : \int g(x) x^\beta dx = 0\}$ for all $|\beta| \geq 0)$, by a result in [HLLRS], we have

$$g(x) = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * g(x),$$

where the series converges in $\mathcal{S}_\infty(\mathbb{R}^n)$.

Therefore, for $f \in Lip^{\alpha}_{com}$ with $\alpha = (\alpha_1, \alpha_2), 0 < \alpha_1, \alpha_2 < 1$, it suffices to show that $\sum_{j,k\in\mathbb{Z}} < f, \psi_{j,k} * \psi_{j,k} * g >$ is well defined for $g \in \mathcal{S}_\infty(\mathbb{R}^n)$. To this end, we estimate $< \psi_{j,k} * f, \psi_{j,k} * g >$ as follows. Observe that $\psi_{j,k} * f(x) = \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v)f(x-u-v)dudv$. Applying the cancellations conditions on both $\psi^{(1)}_j$ and $\psi^{(2)}_k$, we have

$$\begin{aligned}\psi_{j,k} * f(x) &= \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v) \\ &\qquad \times [f(x-u-v) - f(x-u) - f(x-v) + f(x)]dudv \\ &= \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v)(\Delta_v\Delta_u f)(x)dudv.\end{aligned}$$

Using the size condition on both $\psi^{(1)}_j$ and $\psi^{(2)}_k$ together with the fact that $f \in Lip^{\alpha}_{com}$ yields

$$|\psi_{j,k} * f(x)| \leq C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{Lip^{\alpha}_{com}}. \tag{2.1}$$

We now write $\psi_{j,k} * g(x) = \psi^{(1)}_j * \psi^{(2)}_k * g(x)$. Observe that all functions $\psi^{(1)}_j$, $\psi^{(2)}_k$ and $g(x)$ are in $\mathcal{S}_\infty(\mathbb{R}^n)$. Applying the standard almost orthogonal estimate implies that for any positive integers L, M,

$$|\psi^{(1)}_j * g(x)| \lesssim 2^{-|j|L}\frac{2^{(j\wedge 0)n}}{(1+2^{j\wedge 0}|x|_e)^{n+M}}. \tag{2.2}$$

Again, applying the almost orthogonal estimate as in [HLLRS], we have

$$\begin{aligned}|\psi_{j,k} * g(x)| &= |\psi^{(1)}_j * \psi^{(2)}_k * g(x)| \\ &\lesssim 2^{-|j|L}2^{-|k|L}\frac{2^{(j\wedge 0\wedge k\wedge 0)(n-1)}}{(1+2^{j\wedge 0\wedge k\wedge 0}|x'|)^{n+M}} \\ &\qquad \times \frac{2^{j\wedge 0\wedge 2(k\wedge 0)}}{(1+2^{j\wedge 0\wedge 2(k\wedge 0)}|x_n|)^{1+M}}.\end{aligned} \tag{2.3}$$

Therefore, choosing $L = N > \alpha_1 + \alpha_2 + 1$ gives

$$| < \psi_{j,k} * g(x), \psi_{j,k} * f(x) > | \leq C2^{-|j|(N-\alpha_1)}2^{-|k|(N-\alpha_2)}\|f\|_{Lip^{\alpha}_{com}}. \tag{2.4}$$

Thus we obtain the desired result, that is, $\langle f, g\rangle$ is well defined and hence $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$.

The estimate given in (2.1), indeed, yields that

$$|\psi_{j,k} * f(x)| \lesssim 2^{-j\alpha_1} 2^{-k\alpha_2} \|f\|_{Lip^{\alpha}_{com}},$$

which implies that $2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f(x)| \le C\|f\|_{Lip^{\alpha}_{com}}$ for any $j, k \in \mathbb{Z}, x \in \mathbb{R}^n$.

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = 1, 0 < \alpha_2 < 1$, observing first that $\psi^{(1)}_j$ is a radial function and applying the cancellation conditions on $\psi^{(1)}_j$ first and then on $\psi^{(2)}_k$, we have

$$\begin{aligned}
\psi_{j,k} * f(x) &= \frac{1}{2} \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v)[f(x-u-v) + f(x+u-v)]dudv \\
&= \frac{1}{2} \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v) \\
&\qquad \times \big\{[f(x-u-v) + f(x+u-v) - 2f(x-v)] \\
&\qquad\quad - [f(x-u) + f(x+u) - 2f(x)]\big\}dudv \\
&= \frac{1}{2} \iint \psi^{(1)}_j(u)\psi^{(2)}_k(v)(\Delta^z_u \Delta_v f)(x)dudv.
\end{aligned}$$

Applying the same proof gives the desired estimate for this case. All other cases $\alpha = (\alpha_1, \alpha_2)$ where $0 < \alpha_1 < 1, \alpha_2 = 1$ or $\alpha_1 = \alpha_2 = 1$ can be handled similarly. For the case where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 > 1$, set $\alpha_1 = m_1 + r_1, \alpha_2 = m_2 + r_2$ with $0 < r_1, r_2 \le 1$ and $\widehat{\widetilde{\psi}^{(1)}_j}(\xi) = \frac{\widehat{\psi}^{(1)}_j(\xi)}{(-2\pi i\xi)^{m_1}}$ and $\widehat{\widetilde{\psi}^{(2)}_k}(\xi) = \frac{\widehat{\psi}^{(2)}_k(\xi)}{(-2\pi i\xi)^{m_2}}$ then $\psi_{j,k} * f = \partial^{m_1+m_2}\widetilde{\psi}_{j,k} * f = (-1)^{m_1+m_2}\widetilde{\psi}_{j,k} * \partial^{m_1+m_2} f$, where $\widetilde{\psi}_{j,k} = \widetilde{\psi}^{(1)}_j * \widetilde{\psi}^{(2)}_k$. Note that $2^{jm_1}2^{km_2}\widetilde{\psi}_{j,k}$ satisfy the similar smoothness, size and cancellation conditions as $\psi_{j,k}$. Thus, repeating the same proof gives

$$\begin{aligned}
|\psi_{j,k} * f| &= |2^{-jm_1}2^{-km_2}(2^{jm_1}2^{km_2}\widetilde{\psi}_{j,k}) * \partial^{m_1+m_2} f| \\
&\le C2^{-jm_1}2^{-km_2}2^{-jr_1}2^{-kr_2}\|f\|_{Lip^{\alpha}_{com}} \le C2^{-j\alpha_1}2^{-k\alpha_2}\|f\|_{Lip^{\alpha}_{com}}.
\end{aligned}$$

Therefore, this case can be also handled similarly.

We now prove the converse implication of Theorem 1.4. Suppose that $f(x) \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ and $|\psi_{j,k} * f(x)| \le C2^{-j\alpha_1}2^{-k\alpha_2}$, where $\alpha_1, \alpha_2 > 0$. We first show that f is a continuous function. To do this, as in [HLLRS], $f(x) = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x)$, where the series converges in $\mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$. Splitting $\sum_{j,k\in\mathbb{Z}}$ by the sums over $(i) j,k > 0; (ii) j \le 0, k > 0; (iii) j > 0, k \le 0;$ and $(iv) j,k \le 0$, and writing $f = f_1 + f_2 + f_3 + f_4$ in $\mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ for corresponding j and k. Observe that $|\psi_{j,k} * \psi_{j,k} * f(x)| \le C2^{-j\alpha_1}2^{-k\alpha_2}$ and hence the series for f_1 is converges uniformly in x. This implies that f_1 is a continuous function on $\mathbb{R}^n$. Applying the cancellation condition on g in the distribution sense, we write

$$\begin{aligned}&\langle \psi_{j,k} * \psi_{j,k} * f, g\rangle \\ &= \left\langle \iint \psi_j^{(1)}(v-u)[\psi_k^{(2)}(x-v) - T_{2N}(\psi_k^{(2)})(-v)]\psi_{j,k} * f(u)dudv, g(x)\right\rangle\end{aligned}$$

where $T_{2N}(\psi_k^{(2)})(-v)$ is the Taylor polynomial of $\psi_k^{(2)}(x-v)$, as the function of x at $x=0$, with the degree $2N$, $N > \alpha_1+\alpha_2+1$. Then by the smoothness condition on $\psi_k^{(2)}$ and the size condition on $\psi_j^{(1)}$, we obtain that in the distribution sense,

$$|\psi_{j,k} * \psi_{j,k} * f(x)| \leq C2^{-j\alpha_1}2^{k(2N-\alpha_2)}|x|_h^{2N}$$

and thus for any given large $R > 0$ the series for f_2 is converges uniformly for $|x|_h^{2N} \leq R$. This means that f_2 is a continuous function on any compact subset in $\mathbb{R}^n$. Similarly, write

$$\begin{aligned}&\langle \psi_{j,k} * \psi_{j,k} * f, g\rangle \\ &= \left\langle \iint [\psi_j^{(1)}(x-u) - T_{2N}(\psi_j^{(1)}(-u)]\psi_k^{(2)}(u-v)\psi_{j,k} * f(v)dudv, g(x)\right\rangle,\end{aligned}$$

then $|\psi_{j,k} * \psi_{j,k} * f(x)| \leq C2^{j(2N-\alpha_1)}2^{-k\alpha_2}|x|_e^{2N}$ and thus f_3 is a continuous function.

Taking the geometric means of these two estimates shows that f_4 is a continuous function.

Now we estimate $\|f\|_{Lip^{\alpha}_{com}}$ as follows. First, if $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 < 1$, then

$$\begin{aligned}|(\Delta_v\Delta_u f)(x)| &= |f(x-u-v) - f(x-u) - f(x-v) + f(x)| \\ &= \Bigg|\sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}^n} [\psi_{jk}(x-u-v-\omega) - \psi_{jk}(x-u-\omega) \\ &\qquad - \psi_{jk}(x-v-\omega) + \psi_{jk}(x-\omega)]\psi_{j,k} * f(\omega)d\omega\Bigg|.\end{aligned}$$

We choose n_1 and n_2 such that $2^{-n_1} \leq |u|_e < 2^{-n_1+1}$ and $2^{-n_2} \leq |v|_h < 2^{-n_2+1}$. Observe that

$$\begin{aligned}A &= \int_{\mathbb{R}^n} [\psi_{jk}(x-u-v-\omega) - \psi_{jk}(x-u-\omega) \\ &\qquad - \psi_{jk}(x-v-\omega) + \psi_{jk}(x-\omega)]\psi_{j,k} * f(\omega)d\omega \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [\psi_j^{(1)}(x-u-z-\omega) - \psi_j^{(1)}(x-z-\omega)] \\ &\qquad \times [\psi_k^{(2)}(z-v) - \psi_k^{(2)}(z)]\psi_{j,k} * f(\omega)dzd\omega.\end{aligned}$$

Now we split the above series by

$$\begin{aligned}&\sum_{j\ge n_1}\sum_{k\ge n_2} A + \sum_{j<n_1}\sum_{k\ge n_2} A + \sum_{j\ge n_1}\sum_{k<n_2} A + \sum_{j<n_1}\sum_{k<n_2} A \\ &:= A_1 + A_2 + A_3 + A_4.\end{aligned}$$

To deal with the first series, applying the size conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ yields

$$\Big|A\Big| \lesssim \sup_{j,k,\omega} |\psi_{j,k} * f(\omega)| \le C2^{-j\alpha_1}2^{-k\alpha_2}$$

and hance the first series A_1 is dominated by

$$\Big|A_1\Big| \le C\sum_{j\ge n_1}\sum_{k\ge n_2} 2^{-j\alpha_1}2^{-k\alpha_2} \le C2^{-n_1\alpha_1}2^{-n_2\alpha_2} \le C|u|_e^{\alpha_1}|v|_h^{\alpha_2}.$$

To estimate the second series A_2, applying the smoothness condition on $\psi_j^{(1)}$ and the size condition on $\psi_k^{(2)}$ implies

$$|A| \lesssim (2^j|u|_e)\sup_{j,k,\omega}|\psi_{j,k} * f(\omega)| \lesssim 2^{j(1-\alpha_1)}2^{-k\alpha_2}|u|_e.$$

This implies that the second series A_2 is bounded by

$$\begin{aligned}\Big|A_2\Big| &\le C\sum_{j<n_1}\sum_{k\ge n_2} 2^{j(1-\alpha_1)}2^{-k\alpha_2}|u|_e \lesssim 2^{n_1(1-\alpha_1)}2^{-n_2\alpha_2}|u|_e \\ &\qquad \lesssim |u|_e^{\alpha_1-1}|v|_h^{\alpha_2}|u|_e \lesssim |u|_e^{\alpha_1}|v|_h^{\alpha_2}.\end{aligned}$$

The estimate for third series A_3 is similar to the estimate for A_2. Finally, to handle with the last series A_4, applying the smoothness conditions on both $\psi_j^{(1)}$ and $\psi_k^{(2)}$ we obtain that $|A| \lesssim (2^j|u|_e)(2^k|v|_h) \sup_{j,k,\omega} |\psi_{j,k} * f(\omega)| \lesssim 2^{j(1-\alpha)}2^{k(1-\alpha)}|u|_e|v|_h$. Hence this implies that A_4 is dominated by

$$\begin{aligned}\Big|A_4\Big| &\leq C \sum_{j<n_1} \sum_{k<n_2} 2^j|u|_e 2^k|v|_h 2^{-j\alpha_1}2^{-k\alpha_2} \\ &\lesssim 2^{n_1(1-\alpha_1)}2^{n_2(1-\alpha_2)}|u|_e|v|_h \lesssim |u|_e^{\alpha_1}|v|_h^{\alpha_2}.\end{aligned}$$

When $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = \alpha_2 = 1$ observe that

$$\begin{aligned}(\Delta_v^z\Delta_u^z f)(x) &= [f(x-u-v) + f(x+u-v) - 2f(x-v)] \\ &\quad + [f(x-u+v) + f(x+u+v) - 2f(x+v)] \\ &\quad - 2[f(x-u) + f(x+u) - 2f(x)] \\ &= \sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [\psi_j^{(1)}(x-u-z-\omega) + \psi_j^{(1)}(x+u-z-\omega) \\ &\qquad - 2\psi_j^{(1)}(x-z-\omega)] \\ &\qquad \times [\psi_k^{(2)}(z-v) + \psi_k^{(2)}(z+v) - 2\psi_k^{(2)}(z)] \\ &\qquad \times \psi_{j,k} * f(\omega) dz d\omega.\end{aligned}$$

Repeating a similar calculation gives the desired result for this case. The other two cases where $\alpha_1 = 1, 0 < \alpha_2 < 1$ and $0 < \alpha_1 < 1, \alpha_2 = 1$ can be handled similarly. Lastly, when $1 < \alpha_1 = m_1 + r_1, 1 < \alpha_2 = m_2 + r_2$ with $0 < r_1, r_2 \leq 1$, note that

$$\begin{aligned}&\partial^{m_1+m_2} f(x-u-v) - \partial^{m_1+m_2} f(x-u) \\ &\quad - \partial^{m_1+m_2} f(x-v) + \partial^{m_1+m_2} f(x) \\ &= \sum_{j,k\in\mathbb{Z}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [\partial^{m_1}\psi_j^{(1)}(x-u-z-\omega) - \partial^{m_1}\psi_j^{(1)}(x-z-\omega)] \\ &\qquad \times [\partial^{m_2}\psi_k^{(2)}(z-v) - \partial^{m_2}\psi_k^{(2)}(z)]\psi_{j,k} * f(\omega) dz d\omega.\end{aligned}$$

Again observe that the properties of $\partial^{m_1}\psi_j^{(1)}$ and $\partial^{m_2}\psi_k^{(2)}$ are similar to $2^{jm_1}\psi_j^{(1)}$ and $2^{km_2}\psi_k^{(2)}$, respectively, and hence the estimate for this case is the same as the proof for the case where $\alpha = (\alpha_1, \alpha_2)$ with $0 < \alpha_1, \alpha_2 \leq 1$. We leave the details to the reader. The proof of Theorem 1.4 is concluded.

3. Proof of Theorem 1.5

We first show that if $f \in Lip^{\alpha}_{com}$ with $\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 > 0$, then there exists a sequence $\{f_n\}$ such that $f_n \in L^2 \cap Lip^{\alpha}_{com}$ and f_n converges to f in the distribution sense. Moreover, $\|f_n\|_{Lip^{\alpha}_{com}} \leq C\|f\|_{Lip^{\alpha}_{com}}$, where the constant C is independent of f_n and f. To do this, note that

$$f = \sum_{j,k\in\mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x)$$

in the distribution sense. Set

$$f_n = \sum_{|j|,|k|\leq n} \psi_{j,k} * \psi_{j,k} * f(x).$$

Obviously, $f_n \in L^2$ and converges to f in the distribution sense. To see that $f_n \in Lip^{\alpha}_{com}$, by Theorem 1.4,

$$\|f_n\|_{Lip^{\alpha}_{com}} \leq C \sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f_n(x)|.$$

Note that

$$\begin{aligned}\psi_{j,k} * f_n(x) &= \psi_{j,k} * \sum_{|j'|,|k'|\leq n} \psi_{j',k'} * \psi_{j',k'} * f(x)\\ &= \sum_{|j'|,|k'|\leq n} \psi_{j,k} * \psi_{j',k'} * \psi_{j',k'} * f(x).\end{aligned}$$

By an estimate given in [HLLRS], there exist two positive integers $L, M > \alpha_1 + \alpha_2 + 1$, such that

$$\begin{aligned}|\psi_{j,k} * \psi_{j',k'}(x)| \lesssim 2^{-|j-j'|L} 2^{-|k-k'|L} & \frac{2^{(j\wedge j'\wedge k\wedge k')(n-1)}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|x'|)^{n+M}}\\ &\times \frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1 + 2^{j\wedge j'\wedge 2(k\wedge k')}|x_n|)^{1+M}}.\end{aligned}$$

Therefore, again by Theorem 1.4, it follows that

$$2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * f_n(x)| \lesssim \sup_{j',k',x} 2^{j'\alpha_1} 2^{k'\alpha_2} |\psi_{j',k'} * f(x)| \lesssim \|f\|_{Lip^{\alpha}_{com}}.$$

Now we claim that if $f \in L^2$, then

$$\|T_1 \circ T_2(f)\|_{Lip^{\alpha}_{com}} \lesssim \|f\|_{Lip^{\alpha}_{com}}. \tag{3.1}$$

Indeed, by Theorem 1.4,

$$\|T_1\circ T_2(f)\|_{Lip^{\alpha}_{com}}\lesssim \sup_{j,k,x} 2^{j\alpha_1}2^{k\alpha_2}|\psi_{j,k}*T_1\circ T_2(f)(x)|.$$

Observe that $T_2\circ T_1$ is bounded on L^2 and hance

$$T_1\circ T_2 f(x)=\sum_{j',k'\in\mathbb{Z}}(T_1\circ T_2\psi_{j',k'})*\psi_{j',k'}*f(x).$$

Therefore,

$$\psi_{j,k}*T_1\circ T_2 f(x)=\sum_{j',k'\in\mathbb{Z}}(\psi_{j,k}*T_1\circ T_2\psi_{j',k'})*\psi_{j',k'}*f(x).$$

Applying again an estimate in [HLLRS],

$$\begin{aligned}|\psi_{j,k}*T_1\circ T_2\psi_{j',k'}(x)|\lesssim{}& 2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{(j\wedge j'\wedge k\wedge k')(n-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{n+M}}\\ &\times\frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_n|)^{1+M}},\end{aligned}$$

and repeating the same proof as above give

$$\sup_{j,k,x}2^{j\alpha_1}2^{k\alpha_2}|\psi_{j,k}*T_1\circ T_2 f(x)|\lesssim \sup_{j',k',x}2^{j'\alpha_1}2^{k'\alpha_2}|\psi_{j',k'}*f(x)|\lesssim\|f\|_{Lip^{\alpha}_{com}}.$$

We now extend $T_1\circ T_2$ to Lip^{α}_{com} as follows. First, if $f\in Lip^{\alpha}_{com}$ then there exists a sequence $\{f_n\}_{n\in\mathbb{Z}}\in L^2\cap Lip^{\alpha}_{com}$ such that f_n converges to f in the distribution sense and $\|f_n\|_{Lip^{\alpha}_{com}}\le C\|f\|_{Lip^{\alpha}_{com}}$. It follows from the claim in (3.1) that

$$\|T_1\circ T_2(f_n)-T_1\circ T_2(f_m)\|_{Lip^{\alpha}_{com}}\le C\|f_n-f_m\|_{Lip^{\alpha}_{com}}$$

and hence $T_1\circ T_2(f_n)$ converges in the distribution sense. We define

$$T_1\circ T_2(f)=\lim_{n\to\infty}T_1\circ T_2(f_n)$$

in the distribution sense. We obtain, by Theorem 1.4,

$$\begin{aligned}
\|T_1 \circ T_2(f)\|_{Lip^{\alpha}_{com}} &\lesssim \sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * T_1 \circ T_2(f)(x)| \\
&\lesssim \sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} | \lim_{n\to\infty} \psi_{j,k} * T_1 \circ T_2(f_n)(x)| \\
&\lesssim \liminf_{n\to\infty} \sup_{j,k,x} 2^{j\alpha_1} 2^{k\alpha_2} |\psi_{j,k} * T_1 \circ T_2(f_n)(x)| \\
&\lesssim \liminf_{n\to\infty} \|f_n\|_{Lip^{\alpha}_{com}} \\
&\lesssim \|f\|_{Lip^{\alpha}_{com}}.
\end{aligned}$$

The proof of Theorem 1.5 is concluded.

4. Some remarks

In this last section, we mention some antecedents of our results. As mentioned in the introduction, Phong and Stein were the first to study compositions of singular integrals with different homogeneities. Such a question naturally appeared in the study of the boundedness of Marcinkiewicz multipliers on the Heisenberg group. Indeed, on the Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$, the simplest flag singular integral with the flag $\{0\} \subset \mathbb{C}^n \subset \mathbb{H}^n$ can be regarded as the composition of the classical singular integral on $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and the singular integral on $\mathbb{R}$. Müller, Ricci and Stein in [MRS1] and [MRS2] characterized such flag kernels on $\mathbb{H}^n$ and as applications, they proved the $L^p, 1 < p < \infty$, boundedness of Marcinkiewicz multipliers on $\mathbb{H}^n$. It has been later refined by Nagel, Ricci and Stein in [NRS] that a certain class of Fourier multiplier operators can be equivalently defined by classifying between the corresponding multipliers and flag kernels under the Fourier transform. Nagel, Ricci, Stein and Wainger in [NRSW] further studied singular integrals with flag kernels on homogeneous groups and an algebra of pseudolocal operators.

More recently, to study the phenomena that arise when one combines the standard pseudodifferential operators with those appeared in the study of subelliptic estimates and on strongly pseudoconvex domains, Stein and Yung in [SY] introduced a class of pseudodifferential operators of mixed type adapted to distributions of k-planes. This class forms an algebra of pseudoclocal operators and is geometrically invariant. Moreover, It consists of a 2-parameter family $S^{m,n}$ that contains the standard "isotropic" pseudodifferential operators of order m and "non-isotropic" pseudodifferential operators of order n. See [SY] for many other main results. One of main results, namely Theorem 15 on page 1202 in [SY], that is the boundedness of operators in this class on Lischitz space, is particularly connected to Theorem 1.5 in this note. To see this, following [SY], let $\Lambda^\alpha, \alpha > 0$ be the ordinary "isotropic" Lipschitz space on $\mathbb{R}^N$. For $0 < \alpha < 1, f \in \Gamma^\alpha$, a "non-isotropc" Lipschitz space, if and only if

$$|f(x) - f(y)| \leq Cd(x,y)^\alpha$$

for all $x, y \in \mathbb{R}^N$. One defines

$$\|f\|_{\Gamma^\alpha} = \|f\|_{L^\infty} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Stein and Yung in [SY] proved

Theorem 4.1. *Let $a \in S^{\epsilon,-2\epsilon}$ or $S^{-\epsilon,\epsilon}$ for some $\epsilon > 0$. Then operators T_a preserve Λ^α and Γ^α for all $\alpha > 0$.*

Note that S^0 does not preserve Γ^α for $\alpha > 0$, and S^0_D does not preserve Λ^α for $\alpha > 0$. The proof of this theorem used the Littlewood-Paley decomposition which was developed by Stein and Yung in [SY]. This new version of the Littlewood-Paley decomposition is of interesting in their own right. See [SY] for more details.

Observe that operators of type $S^{m,n}$ with $m = n = 0$ include in a natural way the local compositions of two different kinds of Calderón-Zygmund operators. Therefore, it is natural to ask if Theorem 1.5 in this note can extend to operators in $S^{0,0}$. We would like to sketch indications of this possible extension.

We first, following the present note, introduce following Lipschitz space involved the mixed homogeneities.

For $0 < \alpha_1, \alpha_2 < 1$, we say $f \in Lip^{(\alpha_1,\alpha_2)}$ if and only if

$$\begin{aligned}\|f\|_{Lip^{(\alpha_1,\alpha_2)}} := \|f\|_{L^\infty} + \sup_{u\neq 0,x} \frac{|f(x-u) - f(x)|}{|u|^{\alpha_1}} + \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^{\alpha_2}} \\ + \sup_{u\neq 0, x\neq y} \frac{|f(x) - f(y) - f(x-u) + f(y-u)|}{|u|^{\alpha_1} d(x,y)^{\alpha_2}} < \infty.\end{aligned}$$

Suitable modifications can be made for the definition of $Lip^{(\alpha_1,\alpha_2)}$ when either $\alpha_1 \geq 1$ or $\alpha_2 \geq 1$.

In regarding Theorem 4.1, one has

Theorem 4.2. *Let* $a \in S^{0,0}$. *Then* T_a *preserve* $Lip^{(\alpha_1,\alpha_2)}$ *for* $0<\alpha_1,\alpha_2<1$.

To see the proof, one needs to use two versions of Littlewood-Paley decompositions, as Stein and Yung developed in [SY], one of which is adapted to the standard "isotropic" dilations and another is the "non-isotropic" dilations. Then one needs to combine these two decompositions to establish the Littlewood-Paley decomposition adapted to the mixed homogeneities as in [SY]. See Lemma 11 and 12 in [SY] for details. One also needs to characterize Lipschitz space $Lip^{(\alpha_1,\alpha_2)}$ via the Littlewood-Paley decomposition adapted to the mixed homogeneities, as in Proposition 14, Lemma 16 and 17 in [SY]. We omit the details.

Finally, we remark that Theorem 1.5 holds for $0 < \alpha_1 < \epsilon_1$ and $0 < \alpha_2 < \epsilon_2$ if

$$\left|\mathcal{K}_1(x) - \mathcal{K}_1(x')\right| \leq A|x-x'|_e^{\epsilon_1}|x|_e^{-n-\epsilon_1} \qquad \text{for } \ |x-x'|_e \leq \frac{1}{2}|x|_e$$

and

$$\left|\mathcal{K}_2(x) - \mathcal{K}_2(x')\right| \leq B|x-x'|_h^{\epsilon_2}|x|_h^{-n-1-\epsilon_2} \qquad \text{for all } |x-x'|_h \leq \frac{1}{2}|x|_h.$$

Moreover, Theorem 1.5 still holds for non-convolution Calderón-Zygmund operators with appropriate conditions. We leave the details of these proofs to the reader.

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