

On Feldman-Ilmanen-Knopf’s conjecture for the blow-up behavior of the Kähler Ricci flow

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We consider the Ricci flow on $\mathbb{CP}^n$ blown-up at one point starting with any $U(n)$ -invariant Kähler metric. It is proved in [9, 22, 32] that the Kähler-Ricci flow must develop Type I singularities. We show that if the total volume does not go to zero at the singular time, then any Type I parabolic blow-up limit of the Ricci flow along the exceptional divisor is the unique $U(n)$ -complete shrinking Kähler-Ricci soliton on $\mathbb{C}^n$ blown-up at one point. This establishes the conjecture of Feldman-Ilmanen-Knopf [8].

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1. Introduction

The Ricci flow, first introduced by Hamilton ([11]), is the weakly parabolic equation

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g)$$

evolving the Riemannian metrics by its Ricci curvature. It has become a fundamental tool to study geometry and topology. The Kähler-Ricci flow is the Ricci flow on a Kähler manifold starting with a Kähler metric. The Kähler Ricci flow has developed into a vast field and has made important progress in recent years (e.g. [6, 15–21, 23–25, 29] this list is far from complete).

In this paper, we study the unnormalized Kähler Ricci flow

$$(1.1) \quad \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega), \quad \omega(0) = \omega_0$$

on $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$, i.e., $\mathbb{CP}^n$ blown-up at one point. We will always assume that the initial Kähler metric ω_0 is invariant under the action of a maximal compact subgroup $U(n)$ of the automorphism group of X . It is proved ([26]) that the flow (1.1) must develop finite time singularity and it either shrinks to a point, collapses to $\mathbb{CP}^{n-1}$ or contracts an exceptional divisor, in the Gromov-Hausdorff topology.

The Ricci flow solution $g(t)$ is said to develop type I singularity on X at the finite singular time T if there exists $C > 0$ such that

$$\sup_{X \times [0, T)} (T - t) |Rm(g(t))| \leq C.$$

It is proved in [7, 14] that if the Ricci flow develops type I singularity on a closed manifold, then the type I blow-up limit along essential singularities must be a nontrivial complete shrinking Ricci soliton.

$\mathbb{CP}^n$ blown-up at one point is in fact a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ given by

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)) = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}).$$

Let D_0 be the exceptional divisor of X defined by the image of the section $(1, 0)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ and D_∞ be the divisor of X defined by the image of the section $(0, 1)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}$ (see page 517 in [10]). Both the 0-section D_0 and the ∞ -section are complex hypersurfaces in X

isomorphic to $\mathbb{CP}^{n-1}$. The Kähler cone on X is given by

$$\mathcal{K} = \{-a[D_0] + b[D_\infty] \mid 0 < a < b\}.$$

In particular, when $n = 2$, D_0 is a holomorphic S^2 with self-intersection number -1 . We will write the exceptional divisor of X as E and it is in fact equal to D_0 .

Let ω_0 be the initial $U(n)$ invariant Kähler metric of the Ricci flow (1.1) on X . We let $[\omega_0] \in b_0[D_\infty] - a_0[D_0]$ with $0 < a_0 < b_0$. Then the limiting behavior of the Ricci flow can be summarized in the following three cases.

When the initial Kähler class is proportional to the first Chern class, i.e.

$$a_0(n-1) = b_0(n+1),$$

the flow shrinks to a point at the singular time $T = a_0/(n-1)$ ([26]). It is shown in [32] that the flow must develop Type I singularities and the rescaled Ricci flow converges in the Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler Ricci soliton on X constructed in [3, 12, 30].

When the initial Kähler class satisfies

$$a_0(n-1) > b_0(n+1),$$

the flow collapses to $\mathbb{CP}^{n-1}$ at $T = (b_0 - a_0)/2$ ([26]). It is shown in [9] that the flow must develop Type I singularities and the rescaled flow converges in Cheeger-Gromov-Hamilton sense to the ancient solution that splits isometrically as $\mathbb{C}^{n-1} \times \mathbb{CP}^1$.

The initial Kähler class condition of

$$a_0(n-1) < b_0(n+1),$$

is equivalent to the limiting total volume being strictly positive at the singular time $T = a_0/(n-1)$, i.e.,

$$(1.2) \quad \liminf_{t \rightarrow T^-} \text{Vol}(X, g(t)) > 0,$$

and the flow contracts the exceptional divisor D_0 at T ([26]). It is then shown in [22] that the flow (1.1) must develop Type I singularities and the parabolic blow up of the Type I Ricci flow along the exceptional divisor converges to a complete non-flat shrinking Kähler Ricci soliton on a complete manifold *diffeomorphic* to $\mathbb{C}^n$ blown-up at one point.

Theorem 1.1 ([22]). *Let X be $\mathbb{CP}^n$ blown-up at one point and E be the exceptional divisor. Let $g(t)$ be the $U(n)$ -invariant solution to (1.1) on X on $[0, T)$, where $T \in (0, \infty)$ is the singular time of the flow. If*

$$\liminf_{t \rightarrow T^-} \text{Vol}(X, g(t)) > 0,$$

the flow develops type I singularity. Moreover, for any sequence $t_j \rightarrow T$, we consider the type I parabolic rescaled flows $(X, p, g_j(t))$ defined on $[-\frac{t_j}{T-t_j}, 1)$ by

$$(1.3) \quad g_j(t) = \frac{1}{T-t_j} g(t_j + t(T-t_j))$$

with a fixed base point $p \in E$. Then there exist a subsequence converging in Cheeger-Gromov-Hamilton sense (C^∞ -topology) to a complete shrinking non-flat gradient Kähler Ricci soliton on a complete Kähler manifold diffeomorphic to $\mathbb{C}^n$ blown-up at one point.

It is proved by Feldman-Ilmanen-Knopf [8]) that there exists a unique $U(n)$ invariant complete Kähler-Ricci gradient shrinking soliton on $\mathbb{C}^n$ blown-up at one point (FIK soliton) and they further made the following conjecture.

Conjecture 1.1. *Let $g(t)$ be the $U(n)$ invariant metrics satisfying the Kähler-Ricci flow on $X = \mathbb{CP}^n$ blown-up at one point for $t \in [0, T)$. Let $T \in (0, \infty)$ be the singular time and*

$$\liminf_{t \rightarrow T^-} \text{Vol}(X, g(t)) > 0.$$

Then the flow develops type I singularities and any type I parabolic blow-up limit of $g(t)$ with a fixed base point in the exceptional divisor E is the unique FIK soliton on $\mathbb{C}^n$ blown-up at one point.

This conjecture was partially established by Maximo ([13]) when the dimension $n = 2$ under certain open conditions on the initial metric. Our main result in this paper is to show that in the non-collapsed case, the blow-up limit of the Kähler Ricci flow is *biholomorphic* to $\mathbb{C}^n$ blown-up at one point and the limit Kähler Ricci soliton is the FIK soliton constructed in [8] on $\mathbb{C}^n$ blown-up at one point, hence establishing Conjecture 1.1. Our main theorem is

Theorem 1.2. *Let X be $\mathbb{CP}^n$ blown-up at one point and E be the exceptional divisor. Let $g(t)$ be the $U(n)$ -invariant solution to (1.1) on X on $[0, T)$. If*

$$\liminf_{t \rightarrow T^-} \text{Vol}(X, g(t)) > 0,$$

we fix any base point $p \in E$ and let $(X_\infty, p_\infty, g_\infty)$ be the Cheeger-Gromov-Hamilton limit of $(X, p, g_j(t))$, where $g_j(t)$ is defined by (1.3). Then $(X_\infty, p_\infty, g_\infty)$ is biholomorphic to $\mathbb{C}^n$ blown-up at one point and g_∞ is a complete, $U(n)$ symmetric Kähler Ricci soliton metric, hence is one of the FIK solitons constructed in [8].

For $n \geq 2$, there exist infinitely many distinct complex structures on $\mathbb{R}^{2n}$ and so on its complex blow-up at a point. $U(n)$ symmetry in the complex setting is more complicated than $O(2n)$ symmetry in the real setting due to the complex structures, in particular, the complex structures might possibly degenerate or jump in the limit. For example, the manifolds $\mathcal{O}_{\mathbb{CP}^n}(-k)$ with odd $1 \leq k < n$ are all diffeomorphic, but as complex manifolds they admit different complex structures and hence different $U(n)$ -invariant complete shrinking Kähler Ricci soliton metrics ([8]). Our strategy is (1) to construct a $U(n)$ -action on the limit manifold X_∞ , which is holomorphic with respect to the limit complex structure on X_∞ , (2) to construct a holomorphic fiber bundle map $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$, and (3) to show this fiber bundle is in fact $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ and the limit metric g_∞ is $U(n)$ -invariant. Our proof can also be applied to the Kähler-Ricci flow on $\mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k))$ with $U(n)$ -invariant initial Kähler metric for $1 \leq k \leq n-1$ and as long as the total volume does not tend to 0 at the singular time, the flow must develop type I singularities and the type I blow-up limit along the exceptional divisor must be the unique $U(n)$ -invariant complete shrinking gradient Kähler-Ricci soliton on $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ constructed in [8].

This paper is organized as follows. In Section 2, we collect some known facts about the Kähler Ricci flow with $U(n)$ symmetry on X . In Section 3, we prove some a priori estimates and construct the limit map $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ and limit holomorphic vector field V_∞ on X_∞ . In Section 4, we show that the $U(n)$ actions on X_j can pass to the limit X_∞ and we can define a $U(n)$ -action on X_∞ , and prove that X_∞ is either the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ or the disk subbundle of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$. In Section 5 we finish the proof of Theorem 1.2 by showing that if X_∞ is the disk bundle in $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ then the limit metric g_∞ cannot be complete, hence X_∞ is $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$.

Throughout this paper, we will use ω to denote the Kähler form of a Kähler metric g , without specifically mentioning this. And C will denote a uniform constant depending only on the dimension n and the initial Kähler metric, which may be different from line to line.

2. Preliminaries

In this section, we collect some backgrounds and known results about the flow (1.1).

2.1. Calabi symmetry

Let $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$ be $\mathbb{CP}^n$ blown-up at one point and it is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ given by

$$(2.1) \quad X = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)) = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}).$$

Let D_0 be the exceptional divisor of X defined by the image of the section $(1, 0)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ and D_∞ be the divisor defined by the image of the section $(0, 1)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}}(1) \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}$. Both divisors D_0 and D_∞ are complex hypersurfaces isomorphic to $\mathbb{CP}^{n-1}$. The Kähler cone on X is given by

$$\mathcal{K} = \{-a[D_0] + b[D_\infty] \mid 0 < a < b\}.$$

Let $z = (z_1, \dots, z_n)$ be the standard complex coordinates on $\mathbb{C}^n$. Define $\rho = \log |z|^2 = \log(|z_1|^2 + \dots + |z_n|^2)$ on $\mathbb{C}^n \setminus \{0\}$.

Definition 2.1. A smooth convex function $u = u(\rho)$ for $\rho \in (-\infty, \infty)$ is said to satisfy the *Calabi symmetry conditions*, if

- (1) $u''(\rho) > 0, u'(\rho) > 0$ for $\rho \in (-\infty, \infty)$,
- (2) There exist $0 < a < b$ and smooth functions $U_0, U_\infty : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\begin{aligned} U'_0(0) &> 0, & U'_\infty(0) &> 0, \\ u(\rho) &= a\rho + U_0(e^\rho) \quad \text{near } \rho = -\infty, \\ u(\rho) &= b\rho + U_\infty(e^{-\rho}) \quad \text{near } \rho = +\infty. \end{aligned}$$

It is known ([1]) that a metric $\omega = i\partial\bar{\partial}u$ which defines a smooth Kähler metric on $\mathbb{C}^n \setminus \{0\}$ extends to a Kähler metric on $X = \mathbb{CP}^n \# \overline{\mathbb{CP}^n}$ if and only

if u satisfies the Calabi symmetry condition, and it defines a Kähler metric in the class $-a[D_0] + b[D_\infty]$.

On $\mathbb{C}^n \setminus \{0\}$, the Kähler metric $\omega = i\partial\bar{\partial}u$ is given by

$$(2.2) \quad \begin{aligned} \omega &= \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j \\ &= \left(e^{-\rho}u'\delta_{ij} + e^{-2\rho}\bar{z}_iz_j(u'' - u')\right)\sqrt{-1}dz_i \wedge d\bar{z}_j. \end{aligned}$$

The metric ω is invariant under the standard unitary $U(n)$ -actions on $\mathbb{C}^n$, hence also invariant under the induced $U(n)$ -actions on X , i.e. $U(n) \subset \text{Isom}(X, \omega)$, the isometry group of ω .

On $\mathbb{C}^n \setminus \{0\}$, $\det(g_{i\bar{j}}) = e^{-n\rho}(u')^{n-1}u''$ and the Ricci potential of $\omega = i\partial\bar{\partial}u$ is

$$v = -\log \det g_{i\bar{j}} = n\rho - (n-1)\log u' - \log u'',$$

and Ricci curvature tensor of ω is given by

$$R_{i\bar{j}} = e^{-\rho}v'\delta_{ij} + e^{-2\rho}\bar{z}_iz_j(v'' - v').$$

It is known ([26]) that the Calabi symmetry is preserved by the Kähler Ricci flow (1.1), in other words, the evolving Kähler metrics $\omega(t)$ of (1.1) is invariant under $U(n)$ -action if the initial metric ω_0 is $U(n)$ -invariant. In [26] it is shown that (1.1) can be reduced to the following parabolic equation for $u = u(\rho, t)$

$$(2.3) \quad \frac{\partial}{\partial t}u(\rho, t) = \log u''(\rho, t) + (n-1)\log u'(\rho, t) - n\rho,$$

where the evolving metrics $\omega(t)$ are given by $\omega(t) = i\partial\bar{\partial}u(\rho, t)$. If the initial Kähler metric $\omega(0) \in -a_0[D_0] + b_0[D_\infty]$, then the evolving Kähler class is given by

$$\omega(t) \in -a_t[D_0] + b_t[D_\infty], \quad \text{with } a_t = a_0 - (n-1)t, \quad b_t = b_0 - (n+1)t.$$

We will identify the zero section $D_0 \subset X$ as the exceptional divisor $E \cong \mathbb{CP}^{n-1}$ in $\mathbb{C}^n$ blown-up at the origin, and $\mathbb{C}^n \subset \mathbb{CP}^n$. Under the $U(n)$ invariant metric $g = \omega = i\partial\bar{\partial}u$, the distance from a point $z \in \mathbb{C}^n \setminus \{0\}$ to E is given by

$$d_g(z, E) = \frac{1}{2} \int_{-\infty}^{\log |z|^2} \sqrt{u''(\rho)} d\rho.$$

The Calabi symmetry condition (2) above implies this distance is finite for finite $z(\neq 0)$.

We define the tubular neighborhood $B_g(E, R)$ of E (in the following we also call $B_g(E, R)$ as metric balls centered at E) as

$$B_g(E, R) := \{q \in X \mid d_g(q, E) \leq R\},$$

which (for R small) can be identified as $\pi^{-1}(B)$ for some Euclidean ball $B \subset \mathbb{C}^n$ centered at 0 and $\pi: \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$ is the blown-up map of $\mathbb{C}^n$ at 0. The volume of $B_g(E, R)$ with respect to the metric $\omega = i\partial\bar{\partial}u$ is given by

$$(2.4) \quad \int_{B(E, R)} \omega^n = C(n) \int_{-\infty}^{\rho_R} (u'(\rho))^{n-1} u''(\rho) d\rho,$$

for some constant $C(n)$ depending only on the dimension and ρ_R is the unique constant determined by the equation

$$(2.5) \quad R = \frac{1}{2} \int_{-\infty}^{\rho_R} \sqrt{u''(\rho)} d\rho,$$

i.e., a point $z \in \mathbb{C}^n \setminus \{0\}$ with $\log |z|^2 = \rho_R$ satisfies $z \in \partial B_g(E, R)$.

We recall the following formulas of gradient and Laplacian of a $U(n)$ invariant function, which follow from direct calculations so we omit the proof.

Lemma 2.1. *Suppose f is a $U(n)$ -invariant function on X , then with respect to the metric $\omega = i\partial\bar{\partial}u$, we have*

$$|\nabla f|_\omega^2 = \frac{(f')^2}{u''}, \quad \Delta_\omega f = (n-1) \frac{f'}{u'} + \frac{f''}{u''},$$

where as usual for the function f , $f' = \frac{\partial}{\partial \rho} f$, $f'' = \frac{\partial^2 f}{\partial \rho^2}$.

2.2. Type I solutions

Recall the Ricci flow (1.1) is said to develop *Type I singularity* if

$$\sup_{(x,t) \in X \times [0,T)} (T-t)|Rm|(x,t) < \infty,$$

where $T \in (0, \infty)$ is the singular time.

Theorem 2.1 ([9, 22, 32]). *Let X be $\mathbb{CP}^n$ blown-up at one point. Then the Kähler Ricci flow (1.1) on X must develop Type I singularities for any $U(n)$ invariant initial Kähler metric.*

Let $g(t)$ be the solution on $[0, T)$. For any $t_j \rightarrow T$, we consider the rescaled flows $(X, g_j(t))$ defined on $[-\frac{t_j}{T-t_j}, 1)$ by

$$g_j(t) = \frac{1}{T-t_j} g(t_j + t(T-t_j)).$$

Then one and only one of the following must occur.

- (1) ([22]) If $\liminf_{t \rightarrow T} (T-t)^{-1} \text{Vol}(X, g(t)) = \infty$, then $(X, g_j(t), p)$ sub-converges in C^∞ Cheeger-Gromov-Hamilton (CGH) sense to a complete shrinking non-flat gradient Kähler Ricci soliton on a complete Kähler manifold diffeomorphic to $\mathbb{C}^n$, for any fixed point $p \in E$, the exceptional divisor.
- (2) ([9]) If $\liminf_{t \rightarrow T} (T-t)^{-1} \text{Vol}(X, g(t)) \in (0, \infty)$, then $(X, g_j(t), p_j)$ sub-converges in C^∞ -CGH sense to $(\mathbb{C}^{n-1} \times \mathbb{CP}^1, g_{\mathbb{C}^{n-1}} \oplus (-t)g_{FS})$, where $g_{\mathbb{C}^{n-1}}$ is the standard flat metric on $\mathbb{C}^{n-1}$ and g_{FS} is the Fubini-Study metric on $\mathbb{CP}^1$ for any sequence of points p_j .
- (3) ([32]) If $\liminf_{t \rightarrow T} (T-t)^{-1} \text{Vol}(X, g(t)) = 0$, then $(X, g_j(t))$ converges in C^∞ -CGH sense to the unique compact shrinking Kähler Ricci soliton on $\mathbb{CP}^n$ blown-up at one point.

Our main result in this paper is to show the limit Kähler Ricci soliton in case (1) is in fact one of the FIK solitons constructed in [8], and the limit space is biholomorphic to $\widetilde{\mathbb{C}^n}$, $\mathbb{C}^n$ blown-up at one point.

Suppose the initial $U(n)$ -invariant Kähler metric lies in the class $-a_0[D_0] + b_0[D_\infty]$. It is proved ([26]) that the condition in case (1) above that $\liminf_{t \rightarrow T^-} \frac{\text{Vol}(X, g(t))}{T-t} = \infty$ (see also (1.2)) is equivalent to the inequality

$$0 < a_0(n+1) < b_0(n-1).$$

And the Kähler Ricci flow (1.1) will contract the exceptional divisor D_0 at the singular time

$$(2.6) \quad T = \frac{a_0}{n-1}.$$

Throughout this paper we will assume $0 < a_0(n+1) < b_0(n-1)$.

2.3. Cheeger-Gromov convergence

Let $g_j := g_j(0) = \frac{1}{T-t_j} g(t_j)$ and $X_j = X, p_j = p \in D_0 = E$ be a fixed point, then from case (1) in Theorem 2.1, we know the pointed manifolds (X_j, p_j, g_j)

converge in C^∞ Cheeger-Gromov (CG) sense to a complete Kähler manifolds $(X_\infty, p_\infty, g_\infty)$ and g_∞ is a nontrivial complete shrinking Kähler Ricci soliton. Recall the CG convergence means that there exists a sequence of increasing relatively compact exhaustion $\{U_j\}$ of X_∞ , and diffeomorphisms (onto its image) $\phi_j : U_j \rightarrow X_j$ satisfying $\phi_j(p_\infty) = p_j$ and

$$(2.7) \quad \phi_j^* g_j \xrightarrow{C_{loc}^\infty} g_\infty, \quad \phi_j^* J_j \xrightarrow{C_{loc}^\infty} J_\infty,$$

where J_j, J_∞ are the complex structures on X_j, X_∞ , respectively, compatible with the Kähler metrics g_j, g_∞ .

The Kähler form of the metric g_j is $i\partial\bar{\partial}\left(\frac{1}{T-t_j}u(t_j)\right)$. From Calabi symmetry we see that the Kähler potential satisfies $\frac{1}{T-t_j}u(t_j) = (n-1)\rho + U_{0,j}(e^\rho)$ near $\rho = -\infty$, for some $U_{0,j}$ having similar properties as the U_0 in Definition 2.1. Written in local coordinates, it is not hard to see that $(n-1)i\partial\bar{\partial}\rho|_E = (n-1)g_{FS}$ and $i\partial\bar{\partial}U_{0,j}(e^\rho)|_E = 0$, where g_{FS} is the Fubini-Study metric on $\mathbb{CP}^{n-1}$, so we have

Lemma 2.2 (see also [22]). *The diameter of $(E, g_j|_E)$ is $D_n = \alpha_n(n-1)^{1/2}$, hence uniformly bounded. Here α_n is the diameter of $(\mathbb{CP}^{n-1}, g_{FS})$.*

For notational convenience, we will also denote the exceptional divisor $E \subset X_j = X$ by E_j .

3. A priori estimates

As we mentioned before, we will assume the initial Kähler metric lies in $-a_0[D_0] + b_0[D_\infty]$ with $0 < a_0(n+1) < b_0(n-1)$. The evolving metrics belong to the Kähler classes

$$\omega(t) \in -a_t[D_0] + b_t[D_\infty], \quad \text{with } a_t = a_0 - (n-1)t, \quad b_t = b_0 - (n+1)t.$$

The evolution equations for the potentials of the evolving metrics $\omega(t) = i\partial\bar{\partial}u(\rho, t)$ for $\rho \in (-\infty, \infty)$ and $t \in [0, T)$, where T is given in (2.6), are given by ([22, 26])

$$(3.1) \quad \frac{\partial}{\partial t} u' = \frac{u'''}{u''} + (n-1) \frac{u''}{u'} - n,$$

$$(3.2) \quad \frac{\partial}{\partial t} u'' = \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + (n-1) \frac{u'''}{u'} - (n-1) \frac{(u'')^2}{(u')^2},$$

$$(3.3) \quad \begin{aligned} \frac{\partial}{\partial t} u''' &= \frac{u^{(5)}}{u''} - \frac{3u'''u^{(4)}}{(u'')^2} + \frac{2(u''')^3}{(u'')^3} + (n-1)\frac{u^{(4)}}{u'} \\ &\quad - 3(n-1)\frac{u''u'''}{(u')^2} + 2(n-1)\frac{(u'')^3}{(u')^3}. \end{aligned}$$

Along the flow (1.1) or (2.3), we have (see ([22, 26]))

Lemma 3.1. *There exists a constant $C > 0$ such that for all $t \in [0, T)$ and $\rho \in (-\infty, \infty)$ such that*

$$(3.4) \quad (n-1)(T-t) = a_t \leq u' \leq C,$$

and

$$(3.5) \quad 0 \leq \frac{u''}{u'} \leq C, \quad -C \leq \frac{u'''}{u''} \leq C.$$

Lemma 3.2. *There exists a constant $C > 0$ such that for all $t \in [0, T)$ and $\rho \in (-\infty, \infty)$*

$$(3.6) \quad C^{-1}(u' - a_t)(b_t - u') \leq u'' \leq C(u' - a_t)(b_t - u').$$

Proof. The proof of the second inequality is given in Lemma 4.5 of [26], and the first inequality can be proved following the same argument as in [26]. For readers' convenience we include the proof below.

Consider the quantity $H = \log u'' - \log(u' - a_t) - \log(b_t - u')$, using the evolution equations (3.1) and (3.2) we have

$$(3.7) \quad \begin{aligned} \frac{\partial H}{\partial t} &= \frac{1}{u''} \left(\frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} + (n-1)\frac{u'''}{u'} - (n-1)\frac{(u'')^2}{(u')^2} \right) \\ &\quad - \frac{1}{u' - a_t} \left(\frac{u'''}{u''} + (n-1)\frac{u''}{u'} - 1 \right) \\ &\quad - \frac{1}{b_t - u'} \left(-\frac{u'''}{u''} - (n-1)\frac{u''}{u'} - 1 \right). \end{aligned}$$

It can be checked by Calabi symmetry condition that for each fixed $t \in [0, T)$

$$\lim_{\rho \rightarrow \pm\infty} \frac{u''(\rho, t)}{(b_t - u'(\rho, t))(u'(\rho, t) - a_t)} = \frac{1}{b_t - a_t},$$

which is uniformly bounded above and below in our case.

For any $T' \in (0, T)$, suppose the minimum of H on $X \times [0, T']$ is obtained at some (ρ_0, t_0) , then at this point we have $\frac{\partial}{\partial t} H \leq 0$, $H' = 0$, and $H'' \geq 0$, i.e.

$$(3.8) \quad \frac{u'''}{u''} - \frac{u''}{u' - a_t} + \frac{u''}{b_t - u'} = 0,$$

$$(3.9) \quad \frac{u^{(4)}}{u''} - \frac{(u''')^2}{(u'')^2} - \frac{u'''}{u' - a_t} + \frac{(u'')^2}{(u' - a_t)^2} + \frac{u'''}{b_t - u'} + \frac{(u'')^2}{(b_t - u')^2} \geq 0,$$

combining with (3.7), (3.8) and (3.9) we have at (ρ_0, t_0) ,

$$(3.10) \quad \begin{aligned} & -u'' \left(\frac{1}{(u' - a_t)^2} + \frac{1}{(b_t - u')^2} \right) \\ & - (n-1) \frac{u''}{(u')^2} + \frac{1}{u' - a_t} + \frac{1}{b_t - u'} \leq 0. \end{aligned}$$

Hence

$$\frac{u''}{(b_t - u')(u' - a_t)} + \frac{(n-1)u''(b_t - u')(u' - a_t)}{(u')^2((u' - a_t)^2 + (b_t - u')^2)} \geq \frac{b_t - a_t}{(b_t - u')^2 + (u' - a_t)^2}.$$

We observe that

$$\frac{(b_t - u')(u' - a_t)}{(u')^2((u' - a_t)^2 + (b_t - u')^2)} \leq \frac{1}{(b_t - u')(u' - a_t)},$$

and

$$(u' - a_t)^2 + (b_t - u')^2 \leq 2(b_t - a_t)^2,$$

hence at (ρ_0, t_0) ,

$$\frac{u''}{(u' - a_t)(b_t - u')} \geq \frac{1}{2n(b_t - a_t)} \geq C^{-1},$$

as $b_t - a_t$ is uniformly bounded above. The maximum principle implies the minimum of H on $X \times [0, T']$ is uniformly bounded below independent of the choice of T' , hence we conclude that $\inf_{X \times [0, T)} H \geq -C$. And we finish the proof the first inequality in (3.6). $\square$

3.1. Estimates for the sequence of metrics g_j

Recall that the Cheeger-Gromov limit $(X_\infty, g_\infty, p_\infty)$ of (X_j, g_j, p) is a complete Kähler Ricci soliton by the work of [7, 14]. Hence by a theorem of

Cao-Zhou (Theorem 1.2 in [4]), there exists a constant $C_0 > 0$ such that the volume of geodesic balls $B_{g_\infty}(p_\infty, R)$ satisfies

$$(3.11) \quad \text{Vol}_{g_\infty}(B_{g_\infty}(p_\infty, R)) \leq C_0 R^{2n}.$$

And Perelman's non-collapsing ([15]) implies there exists a $\kappa > 0$ for any $q \in X_\infty$, the volume $\text{Vol}_{g_\infty}(B_{g_\infty}(q, r_0)) \geq \kappa r_0^{2n}$, for any $r_0^2 \leq \frac{1}{C}$, where C is the constant in Type I condition. In particular, we have $\text{Vol}_{g_\infty}(B_{g_\infty}(p_\infty, R)) \rightarrow \infty$ as $R \rightarrow \infty$.

Lemma 3.3. *For any $R > 0$, there exist constants $c(n, R) > 0$ and $C(n, R) = O(R^2)$ such that for $j \geq 1$ large enough, then in the metric balls $B_{g_j}(E_j, R)$, we have*

$$(3.12) \quad \begin{aligned} (n-1)(T-t_j) &= a_{t_j} \leq u'(\rho, t_j) \leq C(n, R)(T-t_j), \\ u''(\rho, t_j) &\leq C(n, R)(T-t_j). \end{aligned}$$

Moreover, on $\partial B_{g_j}(E_j, R)$, for j large enough, we have

$$(3.13) \quad \begin{aligned} c(n, R)(T-t_j) + (n-1)(T-t_j) &\leq u'(\rho_{j,R}, t_j) \leq C(n, R)(T-t_j), \\ c(n, R)(T-t_j) &\leq u''(\rho_{j,R}, t_j) \leq C(n, R)(T-t_j) \end{aligned}$$

and $c(n, R) \rightarrow +\infty$ as $R \rightarrow \infty$, $\rho_{j,R}$ is defined in (2.5), corresponding to points on $\partial B_{g_j}(E_j, R)$.

Proof. For any fixed $R > 0$, by the C^∞ -CG convergence (2.7) we have

$$\text{Vol}_{g_j}(B_{g_j}(p_j, R)) \rightarrow \text{Vol}_{g_\infty}(B_{g_\infty}(p_\infty, R)), \quad \text{as } j \rightarrow \infty,$$

in particular, we have both $\text{Vol}_{g_j}(B_{g_j}(p_j, R))$ and $\text{Vol}_{g_j}(B_{g_j}(p_j, R + D_n))$ are uniformly bounded above and below, for j large enough, where D_n is the diameter of E_j given by Lemma 2.2. Noting that

$$B_{g_j}(p_j, R) \subset B_{g_j}(E_j, R) \subset B_{g_j}(p_j, R + D_n),$$

hence there are two constants $c_1 = c_1(n, R)$ and $C_1 = C_1(n, R)$

$$(3.14) \quad c_1(n, R) \leq \text{Vol}_{g_j}(B_{g_j}(E_j, R)) \leq C_1(n, R),$$

By (3.11), it is easy to see that when j is large enough, we can choose $C_1(n, R) = O(R^{2n})$. Moreover, by the volume formula (2.4)

$$(3.15) \quad \begin{aligned} Vol_{g_j}(B_{g_j}(E_j, R)) &= \frac{C(n)}{(T - t_j)^n} \int_{-\infty}^{\rho_{j,R}} (u'(\rho, t_j), t_j)^{n-1} u''(\rho, t_j) d\rho \\ &= \frac{C(n)}{n(T - t_j)^n} \left((u'(\rho_{j,R}, t_j))^n - a_{t_j}^n \right), \end{aligned}$$

where $a_{t_j} = (n-1)(T - t_j)$, and $\rho_{j,R}$ is a constant determined by the equation (2.5) with u'' replaced by $\frac{u''(\rho, t_j)}{T - t_j}$. Combining (3.14) and (3.15), there are constants $c_2(n, R)$ and $C_2(n, R)$ such that

$$(3.16) \quad c_2(n, R) + (n-1) \leq \frac{u'(\rho_{j,R}, t_j)}{T - t_j} \leq C_2(n, R).$$

Combining with the fact that $u'(\rho, t_j)$ is increasing in ρ and Lemmas 3.1 and 3.2, if j is large enough, (3.12) and (3.13) hold. $\square$

3.2. X_j as a $\mathbb{CP}^1$ -bundle over $\mathbb{CP}^{n-1}$

Recall the manifold X_j can be viewed as a $\mathbb{CP}^1$ -bundle over $\mathbb{CP}^{n-1}$ (see (2.1)). Let

$$F_j : X_j \rightarrow \mathbb{CP}^{n-1}$$

be the holomorphic bundle map.

Lemma 3.4. *The holomorphic maps $F_j : (X_j, \omega_j) \rightarrow (\mathbb{CP}^{n-1}, \omega_{FS})$ have uniformly bounded derivatives, i.e., there exists $C > 0$ such that for all j ,*

$$(3.17) \quad \sup_{X_j} |\nabla F_j|_{\omega_j, \omega_{FS}} \leq C$$

Furthermore, for any $R > 0$, in $B_{g_j}(E_j, R)$, the $(n-1)$ -many nonzero eigenvalues of $\omega_j^{-1} \cdot F_j^* \omega_{FS}$ are uniformly bounded below by a positive constant independent of j .

Proof. Note that $F_j^* \omega_{FS} = i\partial\bar{\partial}\rho = i\partial\bar{\partial}\log|z|^2$. The energy density of the map $F_j : (X_j, \omega_j) \rightarrow (\mathbb{CP}^{n-1}, \omega_{FS})$, $e(F_j) := \text{tr}_{\omega_j} f_j^* \omega_{FS} = \Delta_{\omega_j} \rho$, is equal to

$$\frac{(T - t_j)(n-1)}{u'(\rho, t_j)} \leq 1,$$

hence the differential of maps $F_j, dF_j : TX_j \rightarrow T\mathbb{CP}^{n-1}$ is uniformly bounded.

Moreover, by (3.12), in the balls $B_{g_j}(E_j, R)$,

$$(3.18) \quad e(F_j) \geq \frac{(n-1)}{C(n, R)}.$$

By the symmetry of F_j and ω_j , it is not hard to see the $(n-1)$ -many nonzero eigenvalues of $\omega_j^{-1} \cdot F_j^* \omega_{FS}$ are bounded below by $\frac{1}{C(n, R)}$ in $B_{g_j}(E_j, R)$. $\square$

Lemma 3.5. *There exists a constant $C = C(n) > 0$, such that for any $j \geq 1$,*

$$|\nabla \nabla F_j|_{g_j, g_{FS}}^2 \leq C.$$

Hence we have uniform C^2 bound of the maps F_j .

Proof. Since F_j is holomorphic and ω_j and ω_{FS} are Kähler metrics, $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$ is also a harmonic map. By the Bochner formula

$$(3.19) \quad \begin{aligned} \Delta e(F_j) &= |\nabla \nabla F_j|^2 + \text{Ric}_{\omega_j}(\nabla F_j, \overline{\nabla F_j}) \\ &\quad - (R_{\omega_{FS}})_{\beta\bar{\gamma}\delta}^{\alpha} (\overline{F_j})_i^{\alpha} (F_j)_i^{\delta} (F_j)_k^{\beta} \overline{(F_j)_k^{\gamma}}. \end{aligned}$$

On the other hand, by direct calculations we have

$$(3.20) \quad \begin{aligned} \Delta_{\omega_j} e(F_j) &= -\frac{(n-1)^2(T-t_j)^2 u''}{(u')^3} - \frac{(n-1)(T-t_j)^2 u'''}{u''(u')^2} \\ &\leq -\frac{(n-1)(T-t_j)^2 u'''}{u''(u')^2} \leq C(n), \end{aligned}$$

where in the last inequality we use Lemma 3.1. Combining (3.19), (3.20) and the Type I condition $|\text{Ric}_{\omega_j}| \leq C$, we have

$$|\nabla \nabla F_j|^2 \leq C(n) + Ce(F_j) + Ce(F_j)^2 \leq C(n).$$

Therefore, the maps $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$ have uniform second order estimates. $\square$

Remark 3.1. The holomorphicity and so the harmonicity of the maps F_j implies F_j satisfy uniform C^k estimates locally for any $k \in \mathbb{Z}$. But the second order estimate is enough for our applications.

The target manifold of F_j is the compact $(\mathbb{CP}^{n-1}, \omega_{FS})$, and by Lemmas 3.4 and 3.5, the maps $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$ have uniformly bounded C^1 , C^2 bounds, hence F_j converge in $C^{1,\alpha}$ topology to a limit map $F_{\infty} : X_{\infty} \rightarrow$

$\mathbb{CP}^{n-1}$, where by definition $X_j \rightarrow X_\infty$ in the C^∞ -CG sense with the Riemannian metrics and complex structures converging smoothly. Since F_j are holomorphic with the given complex structures, the limit map $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ is also holomorphic with respect to the limit complex structure J_∞ on X_∞ . And the $C^{1,\alpha}$ convergence, (3.18) and Lemma 3.4 imply that the differential map $dF_\infty : TX_\infty \rightarrow T\mathbb{CP}^{n-1}$ has full rank $n-1$ at any point in X_∞ , hence implicit function theorem implies that the fibers of F_∞ are smooth complete Riemann surfaces.

We remark that the convergence of $F_j \rightarrow F_\infty$ is in the Cheeger-Gromov sense, that is, the maps $\phi_j^* F_j$ converge to F_∞ in uniform $C_{loc}^{1,\alpha}$ topology on any compact subset of X_∞ , where $\phi_j : U_j \rightarrow X_j$ is the diffeomorphism we chose in Section 2.3 realizing the C^∞ -Cheeger-Gromov convergence.

3.3. Holomorphic vector fields

Let $V = \sum_i z_i \frac{\partial}{\partial z_i}$ be a holomorphic vector field on $\mathbb{C}^n \setminus \{0\}$, which extends to a holomorphic vector field on X , and vanishes on the exceptional divisor E . Clearly V is tangential to the fibers of $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$.

Lemma 3.6. *With respect to a Kähler metric $\omega = i\partial\bar{\partial}u$ with Calabi symmetry, the imaginary part $\text{Im}(V)$ of V is a Killing vector field. Moreover, $\text{Im}(V)$ is also Killing with respect to the restriction of the metric on each fiber of $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$.*

Proof. This follows from straightforward calculations. Observe that $Vu = u'$ and

$$L_V \omega = d(\iota_V i\partial\bar{\partial}u) = d(i\bar{\partial}(u')) = i\partial\bar{\partial}u',$$

taking conjugate on both sides we have $L_{\bar{V}} \omega = i\partial\bar{\partial}u'$, hence it holds that $L_{V-\bar{V}} \omega = 0$ and this implies the imaginary part of V , $\text{Im}(V)$, is a Killing vector field with respect to the metric ω , i.e.

$$(3.21) \quad L_{\text{Im}(V)} \omega = 0.$$

On the other hand, for any fiber F_p of $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$, we denote the restriction of the metric ω on this fiber by $\omega|$ and $i : F_p \rightarrow X$ the inclusion map of the fiber in X . Using the fact that $\text{Im}(V)$ is tangential to F_p and $L_{\text{Im}(V)} \omega = 0$, when pulled back by the map i , we have

$$L_{\text{Im}(V)}(\omega|) = 0,$$

hence $\text{Im}(V)$ is also a Killing vector field on $(F_p, \omega|)$. □

Remark 3.2. We note that the equation (3.21) only involves the first order derivatives of V .

Lemma 3.7. *For any $R > 0$, there exist $c(n, R) > 0$ which goes to ∞ as $R \rightarrow \infty$ and $C(n, R) > 0$, such that if j is large enough, then*

$$\inf_{\partial B_{g_j}(E_j, R)} |V|_{g_j}^2 \geq c(n, R),$$

and

$$\sup_{B_{g_j}(E_j, R)} |V|_{g_j}^2 \leq C(n, R).$$

Proof. Applying the expansion formula (2.2) of the metric g_j we have

$$(3.22) \quad |V|_{g_j}^2 = \frac{u''(\rho, t_j)}{T - t_j},$$

so by Lemma 3.3, we have

$$(3.23) \quad |V|_{g_j}^2 \leq C(n, R) = O(R^2), \quad \text{in } B_{g_j}(E_j, R)$$

and

$$(3.24) \quad \text{on } \partial B_{g_j}(E_j, R), \quad |V|_{g_j}^2 \geq c(n, R) \rightarrow \infty, \quad \text{as } R \rightarrow \infty.$$

□

Lemma 3.7 implies in $B_{g_j}(E_j, R)$, V is a nontrivial holomorphic vector field vanishing exactly at E_j and having uniform positive lower bound on the boundary $\partial B_{g_j}(E_j, R)$.

We will estimate the bound of the derivatives of V with respect to g_j .

Lemma 3.8. *There exists a constant $C = C(n) > 0$ such that $|\nabla_j V|_{g_j}^2 \leq C$ for any j , where $\nabla_j V$ denote the covariant derivative of V with respect to the metric g_j .*

Proof. We will write $V = V^i \frac{\partial}{\partial z_i}$ for $V^i = z_i$, then $|\nabla V|^2 = V_{,k}^i \overline{V_{,k}^i}$, where

$$V_{,k}^i = \frac{\partial}{\partial z_k} V^i + \Gamma_{kl}^i V^l$$

is the covariant derivative of V and $\Gamma_{kl}^i = g^{i\bar{p}} \frac{\partial}{\partial z_l} g_{k\bar{p}}$ is the Levi-Civita connection of a Kähler metric g . Use the expansion formula (2.2) (multiplied

by $(T - t_j)^{-1}$ of the metric g_j , we have

$$V_{,k}^i = \frac{u''}{u'} \delta_{ik} + \left(\frac{u'''}{u''} - \frac{u''}{u'} \right) z_i \bar{z}_k e^{-\rho},$$

observe that when restricted to the exceptional divisor $E = (\rho = -\infty)$ the matrix $(V_{,k}^i)$ is of the form (hence has rank 1)

$$(3.25) \quad \nabla V|_E = \text{diag}(1, 0, \dots, 0).$$

We calculate the norm of ∇V :

$$(3.26) \quad \begin{aligned} |\nabla V|_{g_j}^2 &= n \left(\frac{u''}{u'} \right)^2 + 2 \frac{u''}{u'} \left(\frac{u'''}{u''} - \frac{u''}{u'} \right) + \left(\frac{u'''}{u''} - \frac{u''}{u'} \right)^2 \\ &= (n-1) \left(\frac{u''}{u'} \right)^2 + \left(\frac{u'''}{u''} \right)^2. \end{aligned}$$

Hence Lemma 3.8 follows from the estimates in Lemma 3.1. $\square$

So we have uniform C^1 bounds of V with respect to g_j . Next we would derive the C^2 bounds of V with respect to the metrics g_j on any metric balls $B_{g_j}(E_j, R)$.

Proposition 3.1. *For any $R > 0$, there is a constant $C(n, R) > 0$ such that for j large enough we have*

$$\sup_{B_{g_j}(E_j, R)} \left(|\nabla \nabla V|_{\omega_j}^2 + |\bar{\nabla} \nabla V|_{\omega_j}^2 \right) \leq C(n, R),$$

i.e., the C^2 bounds of V with respect to the Kähler metrics ω_j hold uniformly on any metric ball $B_{g_j}(E_j, R)$.

To prove Proposition 3.1, we need the following Bochner type identity.

Lemma 3.9. *We have the Bochner type identity: for a Kähler metric ω ,*

$$(3.27) \quad \begin{aligned} \Delta_\omega |\nabla V|^2 &= |\nabla \nabla V|_\omega^2 + |\bar{\nabla} \nabla V|_\omega^2 + R_{l\bar{m}} V_{,m}^i \bar{V}_{,l}^i - R_{m\bar{i}} V_{,l}^m \bar{V}_{,l}^i \\ &\quad - 2\text{Re} \left(R_{i\bar{m}k\bar{l}} V_{,k}^m \bar{V}_{,l}^i + R_{m\bar{i},l} V^m \bar{V}_{,l}^i \right). \end{aligned}$$

Proof. This is a direct calculation.

$$(3.28) \quad \begin{aligned} \Delta |\nabla V|^2 &= (V_{,l}^i \overline{V_{,l}^i})_{k\bar{k}} \\ &= V_{lk\bar{k}}^i \overline{V_{,l}^i} + V_{,lk}^i \overline{V_{,lk}^i} + V_{,l\bar{k}}^i \overline{V_{,l\bar{k}}^i} + V_{,l}^i \overline{V_{,l\bar{k}k}^i}. \end{aligned}$$

By changing the indices, we have

$$(3.29) \quad \begin{aligned} V_{,lk\bar{k}}^i &= V_{,l\bar{k}k}^i + V_{,m}^i R_{l\bar{m}k\bar{k}} - V_{,l}^m R_{m\bar{k}k}^i \\ &= V_{,l\bar{k}k}^i + V_{,m}^i R_{l\bar{m}} - V_{,l}^m R_{m,}^i, \end{aligned}$$

and

$$(3.30) \quad V_{,l\bar{k}k}^i = \left(V_{,kl}^i - V^m R_{m\bar{k}l}^i \right)_k = -V_{,k}^m R_{m\bar{k}l}^i - V^m R_{m,l}^i,$$

where we use the fact that V is a holomorphic vector field and the second Bianchi identity. Combining the formulas (3.28), (3.29) and (3.30), we can see (3.27). $\square$

Lemma 3.10. *On the balls $B_{g_j}(E_j, R) \subset X_j$, there exists a constant $C(n, R) > 0$ such that*

$$\Delta_{\omega_j} |\nabla V|_{\omega_j}^2 \leq C(n, R), \quad \forall j \gg 1.$$

Proof. From Lemma 2.1 and (3.26) we have

$$(3.31) \quad \Delta_{\omega_j} |\nabla V|^2 = (n-1)(T-t_j) \frac{(|\nabla V|^2)'}{u'} + \frac{T-t_j}{u''} \left(|\nabla V|^2 \right)'',$$

where as before $u' = \frac{\partial}{\partial \rho} u(\rho, t_j)$, etc. Our goal is to show that both terms on RHS of (3.31) are uniformly bounded on the balls $B_{g_j}(E_j, R)$. To begin with, we need to estimate $u^{(4)}$.

Claim. *There is a uniform constant $C = C(n) > 0$ such that*

$$|u^{(4)}| \leq C \frac{(u'')^2}{T-t} + \frac{(u''')^2}{u''}.$$

*Proof of the **Claim**.* By the formula of scalar curvature (see [22]), we have

$$(3.32) \quad R(\omega(t)) = -\frac{u^{(4)}}{(u'')^2} + \frac{(u''')^2}{(u'')^3} - 2(n-1)\frac{u'''}{u'u''} \\ - (n-1)(n-2)\frac{u''}{(u')^2} + \frac{n(n-1)}{u'}.$$

And by Type I condition we have $|R| \leq \frac{C}{T-t}$. Combining with Lemma 3.1, it is easy to see the bound on $|u^{(4)}|$. $\square$

The first term on RHS of (3.31) is equal to

$$\begin{aligned} & \frac{(n-1)(T-t_j)}{u'} \left(2(n-1)\frac{u''}{u'} \frac{u'''u' - (u'')^2}{(u')^2} + 2\frac{u'''}{u''} \frac{u^{(4)}u'' - (u''')^2}{(u'')^2} \right) \\ &= \frac{2(n-1)^2(T-t_j)}{u'} \frac{u''}{u'} \cdot \frac{u'''u' - (u'')^2}{(u')^2} \\ &+ \frac{2(n-1)(T-t_j)}{u'} \frac{u'''}{u''} \cdot \frac{u^{(4)}u'' - (u''')^2}{(u'')^2}, \end{aligned}$$

by examining the terms above using Lemma 3.1 and **Claim** we see that the first term on RHS of (3.31) is uniformly bounded above by $C = C(n) > 0$.

The second term in RHS of (3.31) is a little complicated, after some calculations and replacing the $u^{(4)}$ by the scalar curvature (3.32), the second term in RHS of (3.31) is equal to

$$(3.33) \quad \begin{aligned} & 4(n-1)(n-3)\frac{T-t_j}{u''} \left(\frac{((u''')^2 + u''u^{(4)})u' - u'''(u'')^2}{(u')^3} \right) \\ & - 6(n-1)\frac{T-t_j}{u''} \left(\frac{u''}{u'} \right)^2 \frac{u'''u' - (u'')^2}{(u')^2} - \frac{T-t_j}{u''} \left(2R'u''' + 2Ru^{(4)} \right) \\ & - 2(n-1)\frac{T-t_j}{u''} \left(\frac{2u'u''u'''u^{(4)} - (u''')^2(u'u''' + (u'')^2)}{(u'u'')^2} \right) \\ & + 2n(n-1)\frac{T-t_j}{u''} \frac{u^{(4)}u' - u'''u''}{(u')^2}. \end{aligned}$$

We look at the third term in (3.33). By the Type I condition and Shi's derivative estimate along Ricci flow, we know $|\nabla R(\omega(t_j))| \leq \frac{C}{(T-t_j)^{3/2}}$, and

also we know $|\nabla R|^2 = \frac{(R')^2}{u''}$, hence

$$|R'| \leq C \frac{\sqrt{u''}}{(T - t_j)^{3/2}},$$

so we have

$$\begin{aligned} & \left| -\frac{T - t_j}{u''} \left(2R'u''' + 2Ru^{(4)} \right) \right| \\ & \leq C \frac{T - t_j}{u''} \left(\frac{\sqrt{u''}}{(T - t_j)^{3/2}} |u'''| + \frac{(u'')^2}{(T - t_j)^2} + \frac{1}{T - t_j} \frac{(u''')^2}{u''} \right) \\ & \leq C(n, R), \end{aligned}$$

by the Lemmas 3.1, 3.3 and the **Claim**.

The other terms in (3.33) can be estimated similarly using the lemmas above, and we can see they are all uniformly bounded. Hence we finish the proof of Lemma 3.10. $\square$

Proof of Proposition 3.1. Combining with the Bochner identity (3.27), Type I condition and Shi's derivative estimates, i.e., $|Rm(g_j)|_{g_j}, |\nabla Rm(g_j)|_{g_j} \leq C$, and Lemma 3.10, we can get the bound on $|\nabla \nabla V|_{\omega_j}^2 + |\bar{\nabla} \nabla V|_{\omega_j}^2$. $\square$

Proposition 3.2. *There exists a nontrivial holomorphic vector field V_∞ as the subsequential limit of V along the Cheeger-Gromov convergence, such that V_∞ is tangential to the fibers of $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ and $\text{Im}(V_\infty)$ is a nontrivial Killing vector field on each fiber of F_∞ .*

Proof. Along the Cheeger-Gromov convergence (2.7), by the (locally) uniform C^0, C^1, C^2 bound of the holomorphic vector fields $V_j = V$ with respect to the metrics ω_j , up to a subsequence, V_j converge in $C_{loc}^{1,\alpha}$ norm (in the Cheeger-Gromov sense) to a vector field V_∞ on X_∞ , which is holomorphic with respect to the complex structure J_∞ . The holomorphic vector field V_∞ satisfies similar C^0, C^1, C^2 bounds as V_j , when restricted on the balls $B_{g_\infty}(p_\infty, R)$.

To see V_∞ is nontrivial, there exists a sequence of points $x_j \in \partial B_{g_j}(E_j, R)$ converging to an $x_\infty \in X_\infty$, by (3.24), we see that $|V_\infty|(x_\infty) \geq c(n, R) > 0$, hence V_∞ is nontrivial.

On the other hand, the vector fields V_j vanish identically on the exceptional divisors E_j in $B_{g_j}(E_j, R)$, and by taking limits, V_∞ also has zero points, e.g. $V_\infty(p_\infty) = 0$. Hence the zero set of V_∞ is a nonempty analytic

set, since V_∞ is a holomorphic vector field, and we denote this zero set by E_∞ . It's clear that if a sequence of points $x_j \in E_j$ converges to $x_\infty \in X_\infty$, then $x_\infty \in E_\infty$.

Since V_j is tangential to the fibers of $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$, $dF_j(V_j) = 0$, from the $C^{1,\alpha}$ convergence of F_j, V_j , the limit vector field V_∞ satisfies $dF_\infty(V_\infty) = 0$, i.e., V_∞ is tangential to the fibers of $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$.

Choose a fiber $F_\infty^{-1}(y)$ of F_∞ (here $y \in \mathbb{CP}^{n-1}$). There exists a sequence of points $x_j \in F_j^{-1}(y) \cap E_j$ which converge up to a subsequence to $x_\infty \in X_\infty \cap F_\infty^{-1}(y)$, such that $V_\infty(x_\infty) = 0$. On the other hand, for any other point $x'_\infty \in F_\infty^{-1}(y)$, we may assume $d_\infty(x_\infty, x'_\infty) = R > 0$ and there exists a subsequence of $x'_j \in X_j \cap F_j^{-1}(y)$ with $d_{g_j}(x_j, x'_j) > R/2 > 0$ which converges to x'_∞ , then by (3.24), we see $|V_\infty|_{g_\infty}(x'_\infty) \geq c(n, R) > 0$. We remark that (3.26) implies $|\nabla V_\infty|_\infty^2(x_\infty) = 1$.

Thus on each fiber $F_\infty^{-1}(y)$ of F_∞ , V_∞ is a holomorphic vector field with simple single zero point. From (3.21) and $C^{1,\alpha}$ convergence of V_j , the imaginary part $\text{Im}(V_\infty)$ of V_∞ is a Killing vector field of g_∞ . Since $\text{Im}(V_\infty)$ is tangential to the fiber, it follows that on the fiber $F_\infty^{-1}(y)$, with respect to the restriction metric of g_∞ to $F_\infty^{-1}(y)$, the vector field $\text{Im}(V_\infty)$ is also Killing. $\square$

Corollary 3.1. *The fibers of $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ are biholomorphic to either $\mathbb{C}$ or the disk $D \subset \mathbb{C}$.*

Proof. Fix any fiber $F_\infty^{-1}(y)$ of F_∞ , take the unique point $x_\infty \in F_\infty^{-1}(y)$ where $V_\infty(x_\infty) = 0$. There exists a sequence of points $x_j \in F_j^{-1}(y) \cap E_j$ such that up to a subsequence $x_j \rightarrow x_\infty$ in the Cheeger-Gromov sense. For any $R > 0$, there exist $x'_j \in F_j^{-1}(y)$ such that $d_j(x_j, x'_j) = R$, and up to a subsequence $x'_j \rightarrow x'_\infty \in F_\infty^{-1}(y)$, and $d_\infty(x_\infty, x'_\infty) = R$. This implies that $\text{diam}_{g_\infty}(F_\infty^{-1}(y)) = \infty$ since $R > 0$ is arbitrary. In particular, the fiber $F_\infty^{-1}(y)$ is a complete noncompact Riemann surface.

From the proof of Proposition 3.2, we know the vector field $\text{Im}(V_\infty)$ is Killing in $F_\infty^{-1}(y)$ and has a single zero point in $F_\infty^{-1}(y)$. By Lemma 0.1 in [5], we conclude that topologically $F_\infty^{-1}(y)$ is $\mathbb{R}^2$, which in particular is simply connected. By the uniformization theorem for Riemann surfaces, $F_\infty^{-1}(y)$ is either $\mathbb{C}$ or the holomorphic disk $D \subset \mathbb{C}$. $\square$

4. $U(n)$ -actions on the limit space X_∞

4.1. $U(n)$ -actions

We first define a metric on the compact Lie group $U(n)$ by

$$(4.1) \quad d_0(\sigma_1, \sigma_2) := \max\{d_{\mathbb{C}^n}(\sigma_1(x), \sigma_2(x)) \mid \text{for all } x \in S^{2n-1} \subset \mathbb{C}^n\}$$

where $d_{\mathbb{C}^n}$ is the Euclidean distance on $\mathbb{C}^n$ and $\sigma_1, \sigma_2 \in U(n)$ act in the standard way on $S^{2n-1} \subset \mathbb{C}^n$. We remark that the metrics on the compact group $U(n)$ are all equivalent, so any other metrics on $U(n)$ will play the same role.

Lemma 4.1. d_0 defines a metric on the compact group $U(n)$.

Proof. We only need to prove that d_0 satisfies the triangle inequality, since the $U(n)$ -action on S^{2n-1} is effective. For any $\sigma_1, \sigma_2, \sigma_3 \in U(n)$, any $\epsilon > 0$, there exists an $x_\epsilon \in S^{2n-1}$ such that $d_0(\sigma_1, \sigma_2) \leq d_{\mathbb{C}^n}(\sigma_1(x_\epsilon), \sigma_2(x_\epsilon)) + \epsilon$, then

$$\begin{aligned} d_0(\sigma_1, \sigma_2) &\leq d_{\mathbb{C}^n}(\sigma_1(x_\epsilon), \sigma_3(x_\epsilon)) + d_{\mathbb{C}^n}(\sigma_2(x_\epsilon), \sigma_3(x_\epsilon)) + \epsilon \\ &\leq d_0(\sigma_1, \sigma_3) + d_0(\sigma_2, \sigma_3) + \epsilon, \end{aligned}$$

then letting $\epsilon \rightarrow 0$ we can get the triangle inequality. $\square$

For each $\sigma \in U(n)$, we consider the map $\chi_{j,\sigma}$

$$(4.2) \quad \chi_{j,\sigma} : (X_j, g_j, J_j) \rightarrow (X_j, g_j, J_j),$$

defined by $\chi_{j,\sigma}(x) = \sigma(x)$. Recall the $U(n)$ -action on $X_j = X$ is induced from the standard $U(n)$ -action on $\mathbb{C}^n \setminus \{0\}$. σ acts isometrically and holomorphically on (X_j, g_j, J_j) , so $\chi_{j,\sigma}$ is a holomorphic isometry. Thus the energy density of $\chi_{j,\sigma}$, $|\nabla_j \chi_{j,\sigma}|_{g_j}^2 = n$, where ∇_j is the connection induced from g_j and $\chi_{j,\sigma}^* g_j$. Since $\chi_{j,\sigma}$ is holomorphic, hence also harmonic. For notation convenience we denote $F = \chi_{j,\sigma}$, then by Bochner formula,

$$\begin{aligned} (4.3) \quad 0 &= \Delta_j |\nabla_j F|^2 \\ &= |\nabla \nabla_j F|^2 + \text{Ric}_{g_j}(\nabla_j F, \overline{\nabla_j F}) - R(F^* g_j)_{\bar{\alpha}\beta\bar{\gamma}\delta} \overline{F_{,i}^\alpha} F_{,i}^\delta \overline{F_{,k}^\beta} F_{,k}^\gamma, \end{aligned}$$

where $\Delta_j = \Delta_{g_j}$ and $R(F^* g_j)_{\bar{\alpha}\beta\bar{\gamma}\delta}$ denotes the sectional curvature of the pulled-back metric $F^* g_j$, which is uniformly bounded by the Type I condition, so is the Ricci curvature of g_j . Hence by (4.3) and $|\nabla_j F|^2 = n$, we see

that $|\nabla \nabla_j F|^2 \leq C$ for a uniform constant $C = C(n)$. Therefore, we get the uniform C^2 bound of the maps $\chi_{j,\sigma}$, independent of j, σ .

Since $\chi_{j,\sigma}$ is an isometry and maps E_j to itself, which has fixed diameter D_n under the metric g_j , we have for any $R > 0$, the image of $B_{g_j}(p_j, R)$ under $\chi_{j,\sigma}$ is contained in $B_{g_j}(p_j, R + D_n)$. Therefore, the maps $\chi_{j,\sigma}$ are locally uniformly bounded, and satisfy uniform C^1, C^2 bounds, so along the Cheeger-Gromov convergence (2.7), up to a subsequence of j , $\chi_{j,\sigma}$ converge to a limit map

$$\chi_{\infty,\sigma} : (X_\infty, g_\infty, J_\infty) \rightarrow (X_\infty, g_\infty, J_\infty),$$

which preserves the metric g_∞ and complex structure J_∞ , hence an isometry and holomorphic map. The map $\chi_{\infty,\sigma}$ is defined through a subsequence of $\chi_{j,\sigma}$. For different $\sigma \in U(n)$, the subsequence might be different. Our next lemma will show that there exists a subsequence of j , such that for all $\sigma \in U(n)$, $\chi_{j,\sigma}$ converge to limit maps $\chi_{\infty,\sigma}$.

Lemma 4.2. *For any $R > 0$, there exists a $C(n, R) > 0$ such that for j large enough, we have*

$$d_{g_j}(\sigma_1(x), \sigma_2(x)) \leq C(n, R)d_0(\sigma_1, \sigma_2), \quad \forall \sigma_1, \sigma_2 \in U(n)$$

and $x \in B_{g_j}(E_j, R) \subset (X_j, g_j, J_j)$, where d_0 is the metric on $U(n)$ defined in (4.1)

Proof. By the expansion formula of $g_j = \frac{1}{T-t_j}g(t_j)$ in (2.2), and Lemma 3.3 we have on $B_{g_j}(E_j, R) \setminus E_j \subset \mathbb{C}^n \setminus \{0\}$ (here we identify $B_{g_j}(E_j, R) \setminus E_j$ as a punctured ball in $\mathbb{C}^n \setminus \{0\}$)

$$(4.4) \quad \begin{aligned} g_j &= \frac{u'}{T-t_j} \left(\frac{\delta_{ik}}{|z|^2} - \frac{\bar{z}_i z_k}{|z|^4} \right) dz_i \wedge d\bar{z}_k + \frac{u''}{T-t_j} \frac{\bar{z}_i z_k}{|z|^4} dz_i \wedge d\bar{z}_k \\ &\leq \frac{C(n, R)}{|z|^2} \omega_{\mathbb{C}^n}, \end{aligned}$$

and $\omega_{\mathbb{C}^n}$ is the Euclidean metric on $\mathbb{C}^n$, so for any $x \in B_{g_j}(E_j, R) \setminus E_j \subset \mathbb{C}^n \setminus \{0\}$, and $\sigma_1, \sigma_2 \in U(n)$, $\sigma_1(x), \sigma_2(x)$ remain in $B_{g_j}(E_j, R) \setminus E_j \subset \mathbb{C}^n \setminus \{0\}$ and the Euclidean norm $|\sigma_1(x)| = |\sigma_2(x)| = |x|$. Choose a curve $\gamma \subset S_{|x|}^{2n-1}$, the Euclidean sphere in $\mathbb{C}^n \setminus \{0\}$ with radius $|x|$, connecting $\sigma_1(x)$ and $\sigma_2(x)$

and the Euclidean length $L_{\mathbb{C}^n}(\gamma) \leq 2d_{\mathbb{C}^n}(\sigma_1(x), \sigma_2(x))$. Hence by the estimate (4.4), we have

$$\begin{aligned}
 (4.5) \quad d_{g_j}(\sigma_1(x), \sigma_2(x)) &\leq d_{g_j}(\gamma) \\
 &\leq \frac{C(n, R)}{|x|} L_{\mathbb{C}^n}(\gamma) \\
 &\leq \frac{2C(n, R)}{|x|} d_{\mathbb{C}^n}(\sigma_1(x), \sigma_2(x)) \\
 &= 2C(n, R) d_{\mathbb{C}^n}\left(\sigma_1\left(\frac{x}{|x|}\right), \sigma_2\left(\frac{x}{|x|}\right)\right) \\
 &\leq 2C(n, R) d_0(\sigma_1, \sigma_2).
 \end{aligned}$$

By continuity, (4.5) also holds for $x \in E_j$. □

If we define maps

$$(4.6) \quad \chi_j : (X_j, g_j, J_j) \times (U(n), d_0) \rightarrow (X_j, g_j, J_j)$$

by $\chi_j(x, \sigma) = \chi_{j, \sigma}(x)$, which are holomorphic in x and satisfy

$$d_{g_j}(\chi_j(x, \sigma), \chi_j(y, \sigma)) = d_{g_j}(x, y), \quad \forall x, y \in X_j, \sigma \in U(n),$$

and by Lemma 4.2 we also have

$$\begin{aligned}
 d_{g_j}(\chi_j(x, \sigma_1), \chi_j(x, \sigma_2)) &\leq C(n, R) d_0(\sigma_1, \sigma_2), \\
 \forall x \in B_{g_j}(E_j, R), \sigma_1, \sigma_2 &\in U(n).
 \end{aligned}$$

Hence for any $x, y \in B_{g_j}(E_j, R)$ and $\sigma_1, \sigma_2 \in U(n)$

$$\begin{aligned}
 (4.7) \quad d_{g_j}(\chi_j(x, \sigma_1), \chi_j(y, \sigma_2)) &\leq d_{g_j}(\chi_j(x, \sigma_1), \chi_j(y, \sigma_1)) + d_{g_j}(\chi_j(y, \sigma_1), \chi_j(y, \sigma_2)) \\
 &\leq d_{g_j}(x, y) + C(n, R) d_0(\sigma_1, \sigma_2)
 \end{aligned}$$

which implies the maps χ_j defined in (4.6) are locally uniformly bounded and locally equi-continuous with respect to the given product metrics. Moreover the maps $\chi_j(\cdot, \sigma)$ satisfy uniform C^1, C^2 bounds for any $\sigma \in U(n)$, hence by

Arzela-Ascoli theorem, up to a subsequence of j , χ_j converge to a map

$$(4.8) \quad \chi_\infty : (X_\infty, g_\infty, J_\infty) \times (U(n), d_0) \rightarrow (X_\infty, g_\infty, J_\infty),$$

and for each $\sigma \in U(n)$, the map

$$\chi_\infty(\cdot, \sigma) : (X_\infty, g_\infty, J_\infty) \rightarrow (X_\infty, g_\infty, J_\infty)$$

is an isometry and J_∞ -holomorphic.

Lemma 4.3. *The map χ_∞ defined in (4.8) satisfies*

$$(4.9) \quad \chi_\infty(x, \sigma_1 \sigma_2) = \chi_\infty(\chi_\infty(x, \sigma_2), \sigma_1), \quad \forall x \in X_\infty, \sigma_1, \sigma_2 \in U(n).$$

Proof. For any $x \in X_\infty$ and $\sigma_1, \sigma_2 \in U(n)$, choose a sequence of $x_j \in X_j$ converging to x . For each j from the definition we have

$$\begin{aligned} \chi_j(x_j, \sigma_1 \sigma_2) &= \chi_{j, \sigma_1 \sigma_2}(x_j) = \sigma_1 \sigma_2(x_j) = \sigma_1(\sigma_2(x_j)) \\ &= \chi_j(\sigma_2(x_j), \sigma_1) = \chi_j(\chi_j(x_j, \sigma_2), \sigma_1), \end{aligned}$$

taking $j \rightarrow \infty$ and by the definition of χ_∞ we have

$$\chi_\infty(x, \sigma_1 \sigma_2) = \chi_\infty(\chi_\infty(x, \sigma_2), \sigma_1).$$

□

Remark 4.1. If we define the “action” of $\sigma \in U(n)$ on X_∞ , $\sigma : X_\infty \rightarrow X_\infty$ by $\sigma \cdot x = \chi_\infty(x, \sigma)$, then Lemma 4.3 means that for any $\sigma_1, \sigma_2 \in U(n)$, $(\sigma_1 \sigma_2) \cdot x = \sigma_1 \cdot (\sigma_2 \cdot x)$, for any $x \in X_\infty$.

It is clear that the identity element $e \in U(n)$ satisfies $\chi_\infty(x, e) = x$, i.e., $e \cdot x = x$ for any $x \in X_\infty$. Hence the $U(n)$ -action on X_∞ defined above is a group action.

4.2. $U(n)$ -action and fiber map F_∞

Recall in Section 3.2, we define a holomorphic map $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$, as the limit map of $F_j : X_j \rightarrow \mathbb{CP}^{n-1}$. It is clear that F_j is $U(n)$ -equivariant with respect to the $U(n)$ -action on $X_j = \mathbb{CP}^n \setminus \overline{\mathbb{CP}^n}$ and the standard action

on $\mathbb{CP}^{n-1}$, i.e.

$$F_j(\sigma \cdot x_j) = \sigma \cdot F_j(x_j), \quad \forall x_j \in X_j \quad \forall \sigma \in U(n).$$

Now for any $x \in X_\infty$, there is a sequence $x_j \in X_j$ converging to x , taking $j \rightarrow \infty$ and by the smooth convergence of F_j to F_∞ , we have

$$F_\infty(\sigma \cdot x) = \sigma \cdot F_\infty(x),$$

i.e. F_∞ is $U(n)$ -equivariant. Hence for any $y \in \mathbb{CP}^{n-1}$, $\sigma \in U(n)$ maps the fiber $F_\infty^{-1}(y)$ to $F_\infty^{-1}(\sigma \cdot y)$.

Lemma 4.4. *The restriction of $\sigma : X_\infty \rightarrow X_\infty$ to the fiber $F_\infty^{-1}(y)$*

$$\sigma|_{F_\infty^{-1}(y)} : F_\infty^{-1}(y) \rightarrow F_\infty^{-1}(\sigma \cdot y)$$

is a biholomorphic map.

Proof. This follows from the fact that

$$\sigma\sigma^{-1} = e = id : F_\infty^{-1}(\sigma \cdot y) \rightarrow F_\infty^{-1}(\sigma \cdot y),$$

and

$$\sigma^{-1}\sigma = e = id : F_\infty^{-1}(y) \rightarrow F_\infty^{-1}(y).$$

And both σ and σ^{-1} are holomorphic maps. □

Corollary 4.1. *The fibers of $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ are all biholomorphic to each other.*

This follows from the previous lemma and the fact that $U(n)$ action on $\mathbb{CP}^{n-1}$ is transitive.

Fix $p = [1 : 0 : \cdots : 0] \in \mathbb{CP}^{n-1}$, and denote the fiber $F_\infty^{-1}(p)$ by F_p . We know from Corollary 4.1 all fibers of F_∞ are isomorphic. It is expected that F_∞ is in fact a fiber bundle over $\mathbb{CP}^{n-1}$ with fiber F_p .

Proposition 4.1. *The map $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ is a fiber bundle with fibers isomorphic to F_p .*

Proof. The compact group $SU(n)$ -action on X_∞ induces an action of the complexified group $SL(n, \mathbb{C})$ of $SU(n)$, which is defined through the infinitesimal action: for any $\xi + \sqrt{-1}\eta \in \mathfrak{su}(n) \oplus \sqrt{-1}\mathfrak{su}(n) = \mathfrak{sl}(n, \mathbb{C})$, we define $\exp(\xi + \sqrt{-1}\eta) \cdot x = \exp(\xi) \exp(J_\infty \eta) \cdot x$, where J_∞ is the complex structure on X_∞ .

Define a map

$$\pi : SL(n, \mathbb{C}) \times F_p \rightarrow X_\infty, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

This is indeed a *surjective* map by the property of group actions. If $\pi(\sigma_1, x_1) = \pi(\sigma_2, x_2)$ for some $\sigma_1, \sigma_2 \in SL(n, \mathbb{C})$ and $x_1, x_2 \in F_p$. Then $\sigma_1 \cdot x_1 = \sigma_2 \cdot x_2$, and $\sigma_1 \cdot p = \sigma_1 \cdot F_\infty(x_1) = F_\infty(\sigma_1 \cdot x_1) = F_\infty(\sigma_2 \cdot x_2) = \sigma_2 \cdot p$, therefore, $\sigma_1^{-1} \circ \sigma_2 \in$ isotropic subgroup B of $SL(n, \mathbb{C})$ acting on $\mathbb{CP}^{n-1}$, which is given by the matrices of the form

$$\sigma_1^{-1} \circ \sigma_2 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}$$

where $a \in \mathbb{C}^*$ and $A \in GL(n-1, \mathbb{C})$ such that $a \det A = 1$ and $*$ denotes a vector in $\mathbb{C}^{n-1}$. Hence we have $x_1 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix} \cdot x_2$.

We define an equivalence relation on $SL(n, \mathbb{C}) \times F_p$ as

$$(\sigma_1, x_1) \sim (\sigma_2, x_2)$$

if there exists a matrix $\begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix} \in B$ such that $\sigma_2 = \sigma_1 \circ \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}$ and $x_2 = \begin{pmatrix} a & * \\ \mathbf{0} & A \end{pmatrix}^{-1} \cdot x_1$. Then we can see that if $(\sigma_1, x_1) \sim (\sigma_2, x_2)$, then $\pi(\sigma_1, x_1) = \pi(\sigma_2, x_2)$. Hence the quotient map

$$\bar{\pi} : SL(n, \mathbb{C}) \times F_p / \sim \rightarrow X_\infty$$

is bijective and also a biholomorphic map, since each action $\sigma \in SL(n, \mathbb{C})$ on X_∞ is holomorphic and $SL(n, \mathbb{C})$ is a complex manifold.

Claim. $SL(n, \mathbb{C}) \times F_p / \sim$ is a fiber bundle over $\mathbb{CP}^{n-1}$ with fibers isomorphic to F_p .

Proof of the Claim: Define the projection map $pr : SL(n, \mathbb{C}) \times F_p / \sim \rightarrow SL(n, \mathbb{C}) / B \cong \mathbb{CP}^{n-1}$, by $pr(\sigma, x) = Q(\sigma)$, where $Q : SL(n, \mathbb{C}) \rightarrow SL(n, \mathbb{C}) / B$ is the quotient map. pr is clearly well-defined and we want to show pr is locally trivial. The principal B -bundle Q is locally trivial, so around any point in $\mathbb{CP}^{n-1} \cong SL(n, \mathbb{C}) / B$, there is an open set U such that $Q^{-1}(U) \cong U \times B$, i.e. there is a local trivialization $\varphi : Q^{-1}(U) \rightarrow U \times B$, and we denote $\varphi = (\varphi_1, \varphi_2)$. By the definition of quotient map Q , it is clear that $\varphi_1(\sigma \mathbf{b}) = \varphi_1(\sigma)$ for any $\sigma \in Q^{-1}(U)$ and $\mathbf{b} \in B$. Thus we can define a local section $s : U \rightarrow Q^{-1}(U)$ of Q by $s(y) = \varphi^{-1}(y, \mathbf{e})$ with $\mathbf{e} \in B$ being the identity matrix.

Define a map $\tilde{\varphi} : pr^{-1}(U) = Q^{-1}(U) \times F_p / \sim \rightarrow U \times F_p$ by

$$\tilde{\varphi}(\sigma, x) = (\varphi_1(\sigma), s(\varphi_1(\sigma))^{-1} \cdot \sigma \cdot x)$$

which by the property of φ_1 is clearly well-defined. We want to show $\tilde{\varphi}$ is bijective. $\tilde{\varphi}$ is clearly surjective. To see that it is also injective, suppose $\tilde{\varphi}(\sigma_1, x_1) = \tilde{\varphi}(\sigma_2, x_2)$, then $\varphi_1(\sigma_1) = \varphi_1(\sigma_2)$, so there exists a matrix $\mathbf{b} \in B$ such that $\sigma_2 = \sigma_1 \mathbf{b}$. Since $s(\varphi_1(\sigma_1))^{-1} : F_{\sigma_1 \cdot p} \rightarrow F_p$ is an isomorphism, we must have $\sigma_1 \cdot x_1 = \sigma_2 \cdot x_2$, and this implies $x_2 = \mathbf{b}^{-1} \cdot x_1$, and hence $(\sigma_1, x_1) \sim (\sigma_2, x_2)$, and the map $\tilde{\varphi}$ is injective. In the definition of $\tilde{\varphi}$, all maps are holomorphic hence $\tilde{\varphi}$ is also holomorphic, and $\tilde{\varphi}$ provides the local trivialization of $SL(n, \mathbb{C}) \times F_p / \sim$ over $U \subset \mathbb{CP}^{n-1}$. $\square$

For the fixed point $p = [1 : 0 : \cdots : 0] \in \mathbb{CP}^{n-1}$, it is well-known that the isotropic subgroup U_p at p of the $U(n)$ -action on $\mathbb{CP}^{n-1}$ is isomorphic to $U(1) \times U(n-1)$ and given by the the matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}, \quad \text{for some } A \in U(n-1), \quad e^{i\theta} \in U(1).$$

Each $\sigma \in U_p$ induces an isomorphism of the fiber F_p , which is either $\mathbb{C}$ or the unit disk $D \subset \mathbb{C}$.

Lemma 4.5. *There exists an $x_0 \in F_p$ such that for any $\sigma \in U_p$, $\sigma \cdot x_0 = x_0$. Moreover, if $\sigma \in U_p$ fixes all $x \in F_p$, then $\sigma \in \{1\} \times U(n-1)$, i.e., σ is of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad \text{for some } A \in U(n-1).$$

Proof. It is clear that $\sigma \in U_p$ also induces an isomorphism of the fibers $F_j^{-1}(p)$. For each j , there exists an $x_{0,j} \in F_j^{-1}(p) \cap E_j$ which is fixed by all $\sigma \in U_p$. Therefore we can assume that $x_{0,j}$ converges to x_0 as $j \rightarrow \infty$. We then have that $x_0 \in F_\infty^{-1}(p) = F_p$ is the fixed point of all $\sigma \in U_p$.

Suppose there exists a $\sigma \in U_p$ such that $\sigma \cdot x = x$ for all $x \in F_p$. Fix a large $R > 0$ and for any $x_j \in F_j^{-1}(p) \cap B_{g_j}(E_j, R)$ with $x_j \rightarrow x_\infty \in F_p$, $d_{g_j}(\sigma \cdot x_j, x_j) \leq \epsilon_j \rightarrow 0$ as $j \rightarrow \infty$, since $d_{g_j}(\sigma \cdot x_j, x_j) \rightarrow d_{g_\infty}(\sigma \cdot x_\infty, x_\infty) = 0$. On $\partial B_{g_j}(E_j, R)$, by Lemma 3.3 and the expansion formula of g_j in (4.4)

there exists a constant $c(n, R) > 0$ such that

$$(4.10) \quad g_j \geq c(n, R) \frac{\omega_{\mathbb{C}^n}}{|z|^2}.$$

For any $x_j \in F_j^{-1}(p) \cap \partial B_{g_j}(E_j, R)$, the minimal geodesic γ_j (with respect to g_j) connecting x_j and $\sigma \cdot x_j$ must be contained in the annulus $B_{g_j}(E_j, R + \epsilon_j) \setminus B_{g_j}(E_j, R - \epsilon_j) \subset \mathbb{C}^n \setminus \{0\}$, where the estimate (4.10) still holds with some different $c(n, R) > 0$, therefore we have

$$(4.11) \quad \begin{aligned} \epsilon_j &\geq d_{g_j}(\sigma \cdot x_j, x_j) = L_{g_j}(\gamma_j) \\ &\geq c(n, R) d_{\frac{\omega_{\mathbb{C}^n}}{|z|^2}}(\gamma_j) \\ &\geq c(n, R) d_{g_{S^{2n-1}}} \left(\sigma \left(\frac{x_j}{|x_j|} \right), \frac{x_j}{|x_j|} \right), \end{aligned}$$

where $g_{S^{2n-1}}$ is the standard metric on the unit sphere $S^{2n-1} \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ and we use the fact that the metric

$$\frac{\omega_{\mathbb{C}^n}}{|z|^2} = (d \log |z|)^2 + g_{S^{2n-1}},$$

is a product metric on $\mathbb{C}^n \setminus \{0\}$, so the distance of x_j and $\sigma \cdot x_j \in \mathbb{C}^n \setminus \{0\}$ with respect to $\frac{\omega_{\mathbb{C}^n}}{|z|^2}$ is equal to $d_{g_{S^{2n-1}}} \left(\sigma \left(\frac{x_j}{|x_j|} \right), \frac{x_j}{|x_j|} \right)$, since the Euclidean norms $|x_j| = |\sigma \cdot x_j|$. Suppose $\sigma \in U_p \subset U(n)$ is given by $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & A \end{pmatrix}$ for some $A \in U(n-1)$, and it acts on the big circle $F_j^{-1}(p) \cap S^{2n-1}$ by rotation by angle θ . Then (4.11) means that for any $x \in F_j^{-1}(p) \cap S^{2n-1}$, $d_{g_{S^{2n-1}}}(\sigma \cdot x, x)$ is arbitrarily small, hence equals to zero, so the rotation angle $\theta = 0$, and $\sigma \in U_p$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ for some $A \in U(n-1)$. $\square$

Remark 4.2. Noting that the automorphism groups of D and $\mathbb{C}$ are given by

$$\begin{aligned} \text{Aut}(D) &= \left\{ f_{a,\theta} | f_{a,\theta}(\zeta) = e^{i\theta} \frac{\zeta - a}{1 - \bar{a}\zeta}, \quad \theta \in S^1, a \in D \right\}, \\ \text{Aut}(\mathbb{C}) &= \{a\zeta + b | a, b \in \mathbb{C}, a \neq 0\}, \end{aligned}$$

respectively. The action of each nonidentity $\sigma \in U_p$ on F_p is of one of the above, hence has one and only one fixed point in F_p .

We know holomorphic line bundles over $\mathbb{CP}^{n-1}$ are given by $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ for some $k \in \mathbb{Z}$. And each fiber F_p can be embedded in the complex line

$\mathbb{C}$ with the fixed point x_0 identified as $0 \in \mathbb{C}$ hence the fiber bundle $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ can be embedded into some line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$, so that X_∞ is either the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ or the disk bundle as a portion of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$.

Lemma 4.6. *We have $k = -1$.*

Proof. We have known from Theorem 2.1 (1) (see also [22]) that X_∞ is diffeomorphic to $\widetilde{\mathbb{C}^n}$, $\mathbb{C}^n$ blown-up at one point, so k must be negative and odd. On the other hand, if $k \neq -1$, then the U_p actions on the fiber of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ over $p \in \mathbb{CP}^{n-1}$ are not “effective” in the sense that a matrix of the form $\begin{pmatrix} e^{2\pi i/k} & 0 \\ 0 & A \end{pmatrix}$ inducing the identity action on the fiber of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ over $p \in \mathbb{CP}^{n-1}$, and inducing the identity action on F_p . This contradicts Lemma 4.5. $\square$

Corollary 4.2. *X_∞ is either the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ or the holomorphic disk bundle as a portion of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$.*

5. Proof of Theorem 1.2

We first show that the limit metric g_∞ on $X_\infty \subset \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ is $U(n)$ invariant with respect to the natural coordinates of $\mathbb{C}^n \setminus \{0\} = \mathcal{O}_{\mathbb{CP}^{n-1}}(-1) \setminus E_\infty$.

Lemma 5.1. *There exists a smooth function U_∞ on X_∞ , such that*

$$g_\infty = (n-1)F_\infty^* \omega_{FS} + i\partial\bar{\partial}U_\infty,$$

where $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$ is the map constructed in Section 3, and ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^{n-1}$.

Proof. Let $R > 0$ be large number. On $B_{g_j}(E_j, R)$ the metrics

$$(5.1) \quad g_j = i\partial\bar{\partial}\left(\frac{u(t_j, \rho)}{T - t_j}\right) = (n-1)F_j^* \omega_{FS} + i\partial\bar{\partial}\left(\frac{u(t_j, \rho)}{T - t_j} - (n-1)\rho\right).$$

By the Calabi symmetry condition, the Kähler potentials $u(t, \rho) = (n-1)(T-t)\rho + U_0(t, e^\rho)$ near $\rho = -\infty$, and we can normalize for each $t \in [0, T)$, $U_0(t, 0) = 0$, hence the smooth functions $\left(\frac{u(t_j, \rho)}{T - t_j} - (n-1)\rho\right)|_{E_j} = \left(\frac{u(t_j, \rho)}{T - t_j} - (n-1)\rho\right)|_{\rho=-\infty} = 0$ for any $j \geq 1$. Set $U_j = \frac{u(t_j, \rho)}{T - t_j} - (n-1)\rho$. The

gradient of U_j with respect to g_j is

$$\begin{aligned}
 |\nabla U_j|_{g_j}^2 &= (T - t_j) \frac{(U_j')^2}{u''(t_j, \rho)} \\
 &= (T - t_j) \frac{\left(\frac{u'(\rho, t_j)}{T - t_j} - (n - 1) \right)^2}{u''(t_j, \rho)} \\
 &\leq \frac{(u'(t_j, \rho) - (n - 1)(T - t_j))}{T - t_j} \frac{u'(\rho, t_j) - a_{t_j}}{u''(t_j, \rho)} \\
 &\leq C(n, R) \quad \text{on } B_{g_j}(E_j, R)
 \end{aligned}$$

for j large enough, where in the last inequality we use Lemmas 3.1 and 3.3. Hence $\|U_j\|_{C^0(B_{g_j}(E_j, R))} \leq C(n, R)$ for some $C(n, R) > 0$. Moreover, the Laplacian of U_j

$$\Delta_{g_j} U_j = n - (n - 1) \operatorname{tr}_{\omega_j} f_j^* \omega_{FS} = n - (n - 1) \frac{(n - 1)(T - t_j)}{u'(t_j, \rho)}$$

satisfies $\Delta_{g_j} U_j|_{E_j} = 1$ and

$$|\nabla \Delta_j U_j|_{g_j}^2 = C(n) \frac{(T - t_j)^3 u''(t_j)}{(u'(t_j))^4} \leq C(n),$$

so

$$\|\Delta_j U_j\|_{C^1(g_j, B_{g_j}(E_j, R))} \leq C(n, R).$$

Hence by elliptic estimate

$$\|U_j\|_{C^{2, \alpha}(g_j, B_{g_j}(E_j, R/2))} \leq C(n, R).$$

Therefore the functions U_j are locally uniformly bounded in $C^{2, \alpha}$ norm on any compact subset $B_{g_j}(E_j, R)$ of X_j . Taking a subsequence and using a diagonal argument, U_j converge (in the Cheeger-Gromov sense) locally uniformly in $C^{2, \alpha}$ topology to some $C^{2, \alpha}$ function U_∞ on X_∞ , therefore from (5.1), $C_{loc}^{1, \alpha}$ convergence of the holomorphic maps F_j to F_∞ and smooth convergence of complex structures, the metrics g_j converge in C^α norm to

$$(5.2) \quad g_\infty = (n - 1) F_\infty^* \omega_{FS} + i \partial \bar{\partial} U_\infty.$$

Since g_∞ and $F_\infty^* \omega_{FS}$ are both smooth, U_∞ is also a smooth function on X_∞ . $\square$

Take coordinates of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1) \cong \widetilde{\mathbb{C}^n}$, $\mathbb{C}^n$ blown-up at the origin, $\zeta = z_1 (\neq 0)$, $w_2 = z_2/z_1, \dots, w_n = z_n/z_1$, where $z_1, \dots, z_n$ are the natural coordinates on $\mathbb{C}^n$, and ζ is the coordinate of fibers and $w_2, \dots, w_n$ are coordinates of $\mathbb{CP}^{n-1}$. Set $\rho = \log |z|^2 = \log (|\zeta|^2(1 + |w|^2))$, our goal in this subsection is to show

Lemma 5.2. *The function U_∞ constructed in (5.2) can be modified to depend only on ρ . That is, $U_\infty(\zeta, w) = \tilde{U}_\infty(|\zeta|^2(1 + |w|^2))$ for some single-variable function $\tilde{U}_\infty(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$. Hence U_∞ is $U(n)$ invariant on $\mathbb{C}^n \setminus \{0\}$.*

Proof. By construction the limit metric g_∞ is invariant under the $U(n)$ -action defined in section 4, so we have

$$\sigma^* g_\infty = g_\infty, \quad \forall \sigma \in U(n).$$

By (5.2) we have

$$\sigma^*(i\partial\bar{\partial}U_\infty) = i\partial\bar{\partial}\sigma^*U_\infty = i\partial\bar{\partial}U_\infty, \quad \forall \sigma \in U(n).$$

By averaging the function U_∞ over the compact group $U(n)$ using the Haar measure, we may assume $\sigma^*U_\infty = U_\infty$ for all $\sigma \in U(n)$. Since we identify the unique fixed point of the U_p action in the fiber F_p with the origin in $\mathbb{C}$, the zero section E_∞ (which is locally given by $\zeta = 0$) of the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ coincide with the fixed point of the $U(n)$ -actions in each fiber of $F_\infty : X_\infty \rightarrow \mathbb{CP}^{n-1}$. Since $U(n)$ -action is transitive on $\mathbb{CP}^{n-1}$, for any $w = (\zeta, w_2, \dots, w_n) \in X_\infty$, there is some $\sigma_w \in U(n)$ mapping w to $(\zeta', 0, \dots, 0) \in X_\infty$ for some $\zeta' \in \mathbb{C}$ satisfying $|\zeta'|^2 = |\zeta|^2(1 + |w|^2)$. So (writing $Z = (w_2, \dots, w_n)$)

$$(5.3) \quad U_\infty(\zeta, \bar{\zeta}, Z, \bar{Z}) = U_\infty(\sigma_w \cdot (\zeta, Z)) = U_\infty(\zeta', \bar{\zeta}', 0, 0).$$

On the other hand, for $p = (0, \dots, 0) \in \mathbb{CP}^{n-1}$, the isotopy group $U_p \subset U(n)$ at p preserves the fiber $F_\infty^{-1}(p)$, which is either $D \subset \mathbb{C}$ or $\mathbb{C}$. The subgroup U_p fixes the point $(\zeta = 0, 0, \dots, 0) \in E_\infty$, which can be viewed as the origin in the fiber $F_\infty^{-1}(p)$. Noting that the automorphism groups of D and $\mathbb{C}$ are given by

$$\begin{aligned} \text{Aut}(D) &= \left\{ f_{a,\theta} | f_{a,\theta}(\zeta) = e^{i\theta} \frac{\zeta - a}{1 - \bar{a}\zeta}, \quad \theta \in S^1, a \in D \right\}, \\ \text{Aut}(\mathbb{C}) &= \{ a\zeta + b | a, b \in \mathbb{C}, a \neq 0 \}, \end{aligned}$$

respectively. We see from both cases that the U_p action on the fiber $F_\infty^{-1}(p)$ is given by $\sigma_\theta(\zeta) = e^{i\theta}\zeta$ for $\theta \in S^1$, which means that the U_p action on the fiber is the rotation action of S^1 on $\mathbb{C}$. The property that U_∞ is invariant under the U_p action implies that

$$U_\infty(\zeta', \bar{\zeta}', 0, 0) = U_\infty(|\zeta'|, |\zeta'|, 0, 0), \quad \forall \zeta' \in \mathbb{C},$$

combining with (5.3), we see for any $(\zeta, w_2, \dots, w_n) \in X_\infty$

$$U_\infty(\zeta, w_2, \dots, w_n) = \tilde{U}_\infty(|\zeta|^2(1 + |w|^2))$$

for some single variable function $\tilde{U}_\infty$. □

5.1. Proof of Theorem 1.2

So far we have shown that $X_\infty \subset \mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$ is a fiber bundle with fibers either disk D or the line $\mathbb{C}$ and the metric g_∞ is $U(n)$ -invariant on $\mathbb{C}^n \setminus \{0\} \cap (X_\infty \setminus E_\infty)$. We know (Theorem 2.1, or [22]) that the metric g_∞ is a *complete* gradient Kähler Ricci soliton, i.e., for some $f_\infty \in C^\infty(X_\infty)$ such that

$$(5.4) \quad \text{Ric}(g_\infty) + i\partial\bar{\partial}f_\infty = g_\infty, \quad \nabla\nabla f_\infty = 0$$

Without loss of generality, we can choose f_∞ such that it is invariant under the $U(n)$ -action on $\mathbb{C}^n \setminus \{0\}$, since both g_∞ and $\text{Ric}(g_\infty)$ are invariant under $U(n)$ -action. $X_\infty \setminus E_\infty$ can be identified with either a punctured ball $B^* \subset \mathbb{C}^n \setminus \{0\}$, or $\mathbb{C}^n \setminus \{0\}$, on which the metric g_∞ can be written as $g_\infty = i\partial\bar{\partial}u_\infty$ satisfying the Calabi symmetry condition near $z = 0 \in \mathbb{C}^n$, i.e.,

$$u_\infty = u_\infty(\rho) = (n-1)\rho + U_0(e^\rho), \quad \text{near } \rho = -\infty$$

for some smooth $U_0 : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ such that $U_0'(0) > 0$, $U_0''(0) > 0$, where $\rho = \log|z|^2$, and $u_\infty', u_\infty'' > 0$. The system of equations (5.4) is equivalent to the following equation on $X_\infty \setminus E_\infty$ (by taking $f_\infty = u_\infty + \log((u_\infty')^{n-1}u_\infty''e^{-n\rho})$ and observing that ∇f_∞ defines a holomorphic vector field from the second equation in (5.4). For more details, one may refer to [8]),

$$u_\infty^{(4)} - 2\frac{(u_\infty''')^2}{u_\infty''} + nu_\infty''' - (n-1)\frac{(u_\infty'')^3}{(u_\infty')^2} - (u_\infty'''u_\infty' - (u_\infty'')^2) = 0,$$

where $u'_\infty = \frac{d}{d\rho} u_\infty$. Denote $\phi = u'_\infty$, then by some calculations we see that the above equation is equivalent to

$$(5.5) \quad (\log \phi')' + (n-1)(\log \phi)' - \mu \phi' + \phi - n = 0, \quad \text{for some } \mu \in \mathbb{R}.$$

Lemma 5.3. $\mu \neq 0$.

Proof. If $\mu = 0$, then for $Q := \log \det g_\infty + u_\infty = -n\rho + (n-1)\log \phi + \log(\phi') + u_\infty$, we have $Q' = 0$, and this implies the metric g_∞ is KE with $\text{Ric}(g_\infty) = g_\infty$. Myers' theorem from Riemannian geometry implies the diameter of (X_∞, g_∞) is bounded, however, from previous arguments we know the diameter of (X_∞, g_∞) is infinity, hence a contradiction. Thus $\mu \neq 0$. $\square$

As in [8], since $\phi' = u''_\infty > 0$, we may write $\phi' = F(\phi)$ for some smooth function F on $\mathbb{R}^+$, in terms of which (5.5) can be written as

$$(5.6) \quad F' + \left(\frac{n-1}{\phi} - \mu \right) F - (n-\phi) = 0,$$

and one can solve this first order ODE

$$(5.7) \quad \phi' = F(\phi) = \frac{\nu e^{\mu\phi}}{\phi^{n-1}} + \frac{\phi}{\mu} - \frac{\mu-1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j \phi^{j+1-n},$$

for some constant $\nu \in \mathbb{R}$.

At any zero point ϕ_0 of $F(\phi)$, by (5.6), we know $F(\phi_0)' = n - \phi_0$, and by the intermediate value theorem this implies that $F(\phi)$ has at most two positive zeros $0 < a \leq b$ satisfying $0 < a \leq n \leq b$. By the Calabi symmetry, we have

$$\lim_{\rho \rightarrow -\infty} \phi(\rho) = n-1, \quad \lim_{\rho \rightarrow -\infty} \phi'(\rho) = 0,$$

and $0 = \lim_{\rho \rightarrow -\infty} \phi' = \lim_{\rho \rightarrow -\infty} F(\phi) = F(n-1)$, so $a = n-1$ is a zero of F . Plugging $a = n-1$ into (5.7) we get

$$(5.8) \quad \frac{\nu e^{\mu a}}{a^{n-1}} + \frac{a}{\mu} - \frac{\mu-1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j a^{j+1-n} = 0$$

Proposition 5.1. *We must have $\mu > 0$ and $\nu = 0$.*

Proof. Suppose $\mu < 0$, then for large $\phi > 0$, the leading term on the RHS of (5.7) is ϕ/μ , hence the solution to (5.7) exists for all $\rho \in (-\infty, \infty)$, and

$\phi(\rho)$ is uniformly bounded for $\rho \in \mathbb{R}$, we have

$$\lim_{\rho \rightarrow \infty} \phi(\rho) = b, \quad \lim_{\rho \rightarrow \infty} \phi'(\rho) = 0, \quad \text{for some } b > 0.$$

So we have $a \leq \phi \leq b$. However, the volume of (X_∞, g_∞) is given by

$$\text{Vol}(X_\infty, g_\infty) = C(n) \int_{-\infty}^{\infty} (\phi)^{n-1} \phi' d\rho = C(n) \left(\left(\lim_{\rho \rightarrow \infty} \phi(\rho) \right)^n - a^n \right),$$

and we know $\text{Vol}(X_\infty, g_\infty)$ is unbounded, hence $\lim_{\rho \rightarrow \infty} \phi(\rho)$ is not bounded, and we get a contradiction.

Suppose $\nu < 0$, then for large ϕ , $F(\phi)$ is dominated by $\nu \phi^{1-n} e^{\mu\phi} < 0$, and this implies $F(\phi)$ has another zero $b > a$, which contradicts the unboundedness of the volume of (X_∞, g_∞) as before. If $\nu > 0$, F is controlled by the term $\nu \phi^{1-n} e^{\mu\phi} > 0$ when ϕ is large, so there is no second zero b of F , and $F > 0$ on $\phi \in (a, \infty)$, $\phi(\rho) \rightarrow \infty$ as ρ converges to a maximal value $\rho_0 < \infty$.

For ϕ large enough, we have $\phi' \geq c e^{2\mu\phi/3}$ for some small constant $c = c(\nu) > 0$, integrating over $[\rho, \rho_0]$, we have

$$e^{\mu\phi(\rho)} \leq \frac{1}{c\mu(\rho_0 - \rho)^{3/2}},$$

and hence for ϕ large

$$u''_\infty = \phi' \leq \frac{2\nu}{a^{n-1}} \frac{1}{c\mu(\rho_0 - \rho)^{3/2}},$$

then the integral

$$\int_0^{\rho_0} \sqrt{u''_\infty} d\rho \leq C \int_0^{\rho_0} \frac{1}{(\rho_0 - \rho)^{3/4}} d\rho < \infty$$

contradicting the completeness of the metric g_∞ on X_∞ . □

Hence from (5.7) we know that the solution ϕ exists for all $\rho \in (-\infty, \infty)$ since the leading term on RHS of (5.7) is the linear ϕ/μ when ϕ is large and this implies X_∞ is the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-1)$, and from (5.8) we have

$$\frac{a}{\mu} - \frac{\mu-1}{\mu^{n+1}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j a^{j+1-n} = 0,$$

which must have a positive root $\mu = \mu(n)$ for the given $a = n-1$ by the intermediate value theorem, and for this root μ , the solution ϕ to (5.7) defines

a complete Kähler Ricci soliton, which must be one of the FIK solutions constructed in [8].

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