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On canonically polarized Gorenstein 3-folds satisfying the Noether equality

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We study canonically polarized Gorenstein minimal 3-folds satisfying $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$. We characterize their canonical maps, describe a structure theorem for such 3-folds and completely classify the smooth ones. New examples of canonically polarized smooth 3-folds with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$ are constructed. These examples are natural extensions of those constructed by M. Kobayashi.

1. Introduction

Throughout the article, we work over the complex number field $\mathbb{C}$.

Let S be a smooth minimal projective surface of general type. We have the classical Noether inequality $K_S^2 \geq 2p_g(S) - 4$ (cf. [18]). In [12] Horikawa classified surfaces satisfying the equality $K_S^2 = 2p_g(S) - 4$.

Let X be a projective 3-fold of general type. There have been many works dedicated to proving the 3-dimensional version of the Noether inequality (cf. [15], [6], [7], [2] and [5]). In [5], the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ is proved under the assumption that X is Gorenstein minimal. This inequality is sharp according to Kobayashi's examples (cf. [15]).

In this article, a normal projective 3-fold X is called Gorenstein minimal if X has at most $\mathbb{Q}$-factorial terminal singularities, the canonical divisor K_X is a Cartier divisor and K_X is nef. A Gorenstein minimal 3-fold X is said to be on the Noether line if X is of general type and satisfies $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$.

We aim to study Gorenstein minimal 3-folds on the Noether line. We restrict our attention to the case where the 3-folds are canonically polarized and investigate such 3-folds in two aspects: the classification and the construction of examples. We first state the results of the construction of examples, though it is the classification that leads to the discovery of the new ones.

Theorem 1.1. *Let (e, a) be a pair of integers such that $a \geq e \geq 3$; or $1 \leq e \leq 2$, $a \geq e+1$; or $e = 0$, $a \geq 2$. Then there are smooth 3-folds X with $K_X^3 = 8a - 4e - 6$ and $p_g(X) = 6a - 3e - 2$. Moreover, the canonical divisor K_X is ample and the canonical image of X is the image of the embedding of the Hirzebruch surface Σ_e into $\mathbb{P}^{p_g(X)-1}$ induced by the linear system $|s + (3a - e - 2)l|$.*

We recall that Σ_e is the projective bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ over $\mathbb{P}^1$. Here l stands for a fiber of the natural ruling $\Sigma_e \to \mathbb{P}^1$ and s stands for the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(-e)$. It is clear that the corresponding 3-folds X with different pairs of (e, a) are not isomorphic. Our construction in Section 2 essentially follows the same method of [15, (3.2)–(3.5)] and all these 3-folds are finite double covers of certain $\mathbb{P}^1$-bundles over the Hirzebruch surfaces. But we do construct more examples than [15]. The 3-folds given in [15] correspond to the pairs (e, a) with $a = e \geq 3$. Also observe that $p_g(X) \geq 7$ for any X in Theorem 1.1. See Remark 2.3 for examples of 3-folds with $K_X^3 = 2$ and $p_g = 4$, having exactly one singularity and canonically fibred by curves.

Now we analyse the canonical maps of canonically polarized Gorenstein minimal 3-folds on the Noether line and with $p_g(X) \geq 7$. Assertions (a) and (b) in the following theorem characterize the base locus of $|K_X|$. They are the key points in the classification.

Theorem 1.2 (cf. [15, (3.1) Theorem]). *Let X be a Gorenstein minimal 3-fold satisfying $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$. Assume that K_X is ample.*

(a) *The base locus of $|K_X|$ consists of a smooth rational curve $\overline{\Gamma}$, which is contained in the smooth locus of X.*

(b) *Let $\pi\colon Y \to X$ be the blowup along $\overline{\Gamma}$ and let E_0 be the exceptional divisor. Then $|\pi^*K_X - E_0|$ is base point free and it induces a fibration $\phi\colon Y \to \Sigma$, where Σ is a surface in $\mathbb{P}^{p_g(X)-1}$ with $\deg \Sigma = p_g(X) - 2$.*

(c) *Let C be the general fiber of ϕ. Then $g(C) = 2$ and $\pi^*K_X.C = 1$.*

(d) *The restriction $\phi|_{E_0}\colon E_0 \to \Sigma$ is birational.*

To prove Theorem 1.2, we need to reproduce the proofs of the Noether inequality in [6], [7] and [5], and to get some geometric properties of the 3-folds on the Noether line (see Section 3). We remark that Theorem 1.2 no longer holds for the case where $p_g(X) = 4$. See Remark 2.3 for counterexamples and see Remark 3.7 for further remarks.

To classify the 3-folds in Theorem 1.2, we shall determine the surface Σ explicitly and analyse the fibration $\phi\colon Y \to \Sigma$. From the classification of surfaces of minimal degree (see for instance [1, Exercises IV.18-4)] or [10, p. 380, Corollary 2.19]), we know that Σ is a Hirzebruch surface or a cone over a rational normal curve. By showing the flatness of ϕ and by applying the semi-positivity theorem to ϕ (see [19]), we obtain a structure theorem for X and a complete classification when X is smooth (see Section 4).

Theorem 1.3. *Keep the same assumptions and notation of Theorem 1.2.*

(a) *The surface Σ is isomorphic to a Hirzebruch surface Σ_e and $K_X^3 = 8a - 4e - 6$ for a pair of integers (e, a) such that $a \geq e \geq 3$; or $1 \leq e \leq 2$, $a \geq e + 1$; or $e = 0$, $a \geq 2$.*

(b) *The fibration $\phi\colon Y \to \Sigma$ is flat with irreducible fibers and E_0 is a section of ϕ.*

Identify $\mathrm{Pic}(\Sigma)$ *with* $\mathrm{Pic}(\Sigma_e) = \mathbb{Z}s \oplus \mathbb{Z}l$ *via* $\Sigma \cong \Sigma_e$ *in* (a).

(c) *Then $\phi_*\mathcal{O}_Y(2E_0) = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-2s - 2al)$ and $\phi_*\omega_{Y/\Sigma} = \mathcal{O}_\Sigma(3s + 3al) \oplus \mathcal{O}_\Sigma(s + al)$, where $\omega_{Y/\Sigma} := K_Y - \phi^*K_\Sigma$.*

Set $P := \mathbb{P}_\Sigma(\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-2s - 2al))$. Denote by $\tau\colon P \to \Sigma$ the natural projection and by E the section of τ corresponding to the projection $\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-2s - 2al) \to \mathcal{O}_\Sigma(-2s - 2al)$.

(d) *There is a finite double cover $\psi\colon Y \to P$ branched along $E + T$ such that $\phi = \tau \circ \psi$, where $T \in |5E + \tau^*(10s + 10al)|$ and $T \cap E = \emptyset$.*

Moreover, if X is smooth, then X is one of the smooth 3-folds constructed in Section 2.

We remark that a Gorenstein minimal 3-fold is indeed locally factorial (see [13, Lemma 5.1]). In the proof of Theorem 1.3, we need this fact in order to apply intersection theory (cf. Remark 4.2). In generality, one would like to classify Gorenstein minimal 3-folds on the Noether line. However, this problem does not seem possible to resolve with the methods and the techniques of the present article.

2. Examples

Kobayashi constructed examples of 3-folds on the Noether line in [15]. Using the same method of [15], we construct more examples of canonically polarized smooth 3-folds with $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$. See the diagram (2.1) for the process. We start with the Hirzebruch surfaces and certain $\mathbb{P}^1$-bundles over them (cf. [10, p. 162, Proposition 7.11-7.12; p. 253, Exercise 8.4; Chapter V, Section 2]).

$$
\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[d] \arrow[r, "\psi"] & P \arrow[d, "\tau"] & \\
\mathbb{P}(H^0(Y, \mathcal{O}_Y(mH))^*) & X \arrow[l, hook'] & \Sigma_e \arrow[r] & \mathbb{P}^1
\end{tikzcd}
\tag{2.1}
$$

Denote by Σ_e the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ for $e \geq 0$. Denote by l the fiber of the natural ruling $\Sigma_e \to \mathbb{P}^1$ and by s the section corresponding to the projection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \to \mathcal{O}_{\mathbb{P}^1}(-e)$. Then

$$
\begin{aligned}
&s^2 = -e, \ \mathcal{O}_{\Sigma_e}(s) = \mathcal{O}_{\Sigma_e}(1), \\
&\mathrm{Pic}(\Sigma_e) = \mathbb{Z}s \oplus \mathbb{Z}l \quad \text{and} \quad K_{\Sigma_e} = -2s - (e+2)l
\end{aligned}
\tag{2.2}
$$

The curve s is called the negative section when $e > 0$.

Let $\mathcal{E}$ be a locally free sheaf of rank 2, sitting in the exact sequence

$$
0 \to \mathcal{O}_{\Sigma_e} \to \mathcal{E} \to \mathcal{O}_{\Sigma_e}(-2s - 2al) \to 0
\tag{2.3}
$$

where a is an integer, and let $P := \mathbb{P}_{\Sigma_e}(\mathcal{E})$. Denote by $\tau \colon P \to \Sigma_e$ the natural projection and by E the section of τ corresponding to the morphism $\mathcal{E} \to \mathcal{O}_{\Sigma_e}(-2s - 2al)$ in (2.3). By abuse of notation, we identify $\mathrm{Pic}(E)$ with $\mathbb{Z}s \oplus \mathbb{Z}l$ via $\tau|_E \colon E \cong \Sigma_e$.

Lemma 2.1. *Keep the same notation as above. Then*

(a) $\mathcal{O}_P(E) = \mathcal{O}_P(1)$, $\quad K_P = \tau^*(-4s - (2a + e + 2)l) - 2E$ *and* $\mathcal{O}_E(E) = \mathcal{O}_E(-2s - 2al)$;

(b) *the exact sequence* (2.3) *splits if* $a \geq e$;

(c) *the linear system* $|E + \tau^*(2s + 2al)|$ *is base point free if* $a \geq e$.

Proof. The first two formulae of (a) are standard. Since $K_E = -2s - (e + 2)l$, the third one follows from the second one and the adjunction formula.

For (b), it suffices to show $H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(2s + 2al)) = 0$. According to [10, p. 371, Lemma 2.4 and p. 162, Proposition 7.11],

$$
\begin{aligned}
H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(2s + 2al)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a)) &\oplus H^1(\mathbb{P}^1, \mathcal{O}^1_{\mathbb{P}}(2a - e)) \\
&\oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a - 2e)),
\end{aligned}
$$

which is 0 since $a \geq e$.

If $a \geq e$, then $|2s + 2al|$ is base point free by [10, p. 380, Corollary 2.8] and so is $\tau^*(2s + 2al)$. From the third formula in (a), we have $\mathcal{O}_P(E + \tau^*(2s + 2al))|_E = \mathcal{O}_E$. So the sequence

$$0 \to \mathcal{O}_P(\tau^*(2s + 2al)) \to \mathcal{O}_P(E + \tau^*(2s + 2al)) \to \mathcal{O}_E \to 0$$

is exact. Since $R^i\tau_*\mathcal{O}_P = 0$ for $i \geq 1$, we have

$$H^1(P, \mathcal{O}_P(\tau^*(2s + 2al))) = H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(2s + 2al))) = 0.$$

So the trace of $|E + \tau^*(2s + 2al)|$ on E is base point free. This completes the proof of (c). $\square$

In the remaining of this section, we fix a pair (e, a) as in the assumption of Theorem 1.1. We use the theory of double covers to construct 3-folds and prove Theorem 1.1.

Let T be any smooth effective divisor in $|5E + \tau^*(10s + 10al)|$. Such T exists by Lemma 2.1 (c) and the Bertini theorem. Then $T \cap E = \emptyset$ since $\mathcal{O}_P(T)|_E = \mathcal{O}_E$ (see the proof of Lemma 2.1 (c)). We also have

$$E + T \sim 2\mathcal{L}, \text{ where } \mathcal{L} = 3E + \tau^*(5s + 5al) \tag{2.4}$$

Therefore there is a finite smooth double cover $\psi\colon Y \to P$ branched along $E + T$ such that $\psi_*\mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{L}^\vee$. Set $E_0 := \psi^{-1}(E)$. Then $\psi^*E = 2E_0$. We identify $\mathrm{Pic}(E_0)$ with $\mathbb{Z}s \oplus \mathbb{Z}l$ via $\tau|_E \circ \psi|_{E_0}\colon E_0 \cong E \cong \Sigma_e$.

Proposition 2.2. *Let $H := K_Y - E_0$. Then*

(a) $K_Y = \psi^*\tau^*(s + (3a - e - 2)l) + 2E_0$ *and* $p_g(Y) = 6a - 3e - 2$;

(b) H *is nef and* $H^3 = 8a - 4e - 6$;

(c) *for an irreducible curve C in Y, $H.C = 0$ holds if and only if C is a fiber of the ruling of E_0 induced by $|l|$;*

(d) $3H - K_Y$ *is ample.*

Proof. For (a), we have $K_Y = \psi^*(K_P + \mathcal{L})$ and

$$p_g(Y) = p_g(P) + h^0(P, \mathcal{O}_P(K_P + \mathcal{L})) = h^0(P, \mathcal{O}_P(K_P + \mathcal{L}))$$

since $p_g(P) = 0$. Note that $K_P + \mathcal{L} = E + \tau^*(s + (3a - e - 2)l)$ by the second formula of Lemma 2.1 (a) and (2.4). The formula for K_Y follows. By

Lemma 2.1 (a)–(b), we have

$$\begin{aligned}\tau_*\mathcal{O}_P(K_P+\mathcal{L}) &= \mathcal{E}\otimes\mathcal{O}_{\Sigma_e}(s+(3a-e-2)l)\\ &= \mathcal{O}_{\Sigma_e}(s+(3a-e-2)l)\oplus\mathcal{O}_{\Sigma_e}(-s+(a-e-2)l).\end{aligned}$$

As in the proof of Lemma 2.1 (b), a similar calculation yields

$$\begin{aligned}&h^0(\Sigma_e,\mathcal{O}_{\Sigma_e}(s+(3a-e-2)l)) = 6a-3e-2\\ \text{and}\quad &h^0(\Sigma_e,\mathcal{O}_{\Sigma_e}(-s+(a-e-2)l)) = 0.\end{aligned}$$

Therefore $p_g(Y) = 6a-3e-2$.

Let $M := \psi^*\tau^*(s+(3a-e-2)l)$. We have

$$H = M+E_0, \quad K_Y = M+2E_0 \quad \text{and} \quad M|_{E_0} = s+(3a-e-2)l.$$

So $E_0|_{E_0} = \frac{1}{3}(K_{E_0}-M|_{E_0}) = -s-al$ by the adjunction formula and then $H|_{E_0} = (2a-e-2)l$. It follows that

$$\begin{aligned}&H^2.E_0 = 0, \quad H.M.E_0 = H|_{E_0}.M|_{E_0} = 2a-e-2\\ \text{and}\quad &M^2.E_0 = (M|_{E_0})^2 = 6a-3e-4.\end{aligned}$$

Therefore $H^3 = 8a-4e-6$.

Now assume $H.C\le 0$ for an irreducible curve C. If $\tau\psi(C)$ is a point, then $H.C = \frac{1}{2}\psi^*E.C > 0$ since E is a section of τ. So $\tau\psi(C)$ is a curve. Since $s+(3a-e-2)l$ is very ample by [10, p. 380, Corollary 2.18], $M.C>0$ and thus $E_0.C = (H-M).C<0$. Therefore C is contained in E_0. Since $H|_{E_0} = (2a-e-2)l$ and $2a-e-2>0$, we conclude that $H.C=0$ and C is a fiber of the ruling of E_0. This completes the proof of (b) and (c).

Because both H and M are nef and $H^3>0$, $3H-K_Y = H+M$ is nef and big. Since $M|_{E_0}$ is very ample, it follows by (c) that $(3H-K_Y).C>0$ for any curve C. Assume $(3H-K_Y)^2.S = (H+M)^2.S = 0$ for some irreducible surface S. Then $H^2.S = H.M.S = M^2.S = 0$ and thus $M.E_0.S = M.(H-M).S = 0$. Because $M|_{E_0}$ is very ample, we conclude $S\cap E_0 = \emptyset$. Then the equality $H.M.S=0$ yields a contradiction to (c), since $|M|$ is base point free. We have shown $(3H-K_Y)^2.S>0$ for any irreducible surface S. Therefore $3H-K_Y$ is ample. □

By Proposition 2.2 (c)–(d), the base-point-free theorem [14, Theorem 3-1-1] implies that the image X of the morphism $\pi\colon Y\to\mathbb{P}(H^0(Y,\mathcal{O}_Y(mH))^*)$ for $m\gg 0$ can be identified with the blowdown of Y along the ruling of E_0. Therefore X is a 3-fold birational to Y.

We now show that X satisfies the properties in Theorem 1.1. Observe that

$$(3H - K_Y) + tK_Y|_{E_0} = 3l + (t-1)(-s + (a-e-2)l)$$

for any $t \in \mathbb{R}$. Hence $\max\{t \in \mathbb{R} | 3H - K_Y + tK_Y \text{ is nef}\} = 1$. Since $\mathcal{O}_{E_0}(E_0) \cong \mathcal{O}_{E_0}(-s - al)$, X is smooth by [17, Theorem (3.3) (3.3.1)]. Also K_X is ample by the base-point-free theorem. From the construction, we have $\pi^* K_X \cong H$ and thus $K_X^3 = H^3 = 8a - 4e - 6$. Since X is birational to Y, $p_g(X) = p_g(Y) = 6a - 3e - 2$. We have seen $H = E_0 + \psi^*\tau^*(s + (3a - e - 2)l)$ in the proof of Proposition 2.2. Because the movable part of $|\pi^* K_X| = |H|$ is $\psi^*\tau^*(s + (3a - e - 2)l)$, we see that the base locus of $|K_X|$ consists of the smooth rational curve $\pi(E_0)$ and the canonical image of X is the image of the embedding of Σ_e into $\mathbb{P}^{p_g(X)-1}$ induced by $|s + (3a - e - 2)l|$.

Remark 2.3. If $(e, a) = (0, 1)$, the construction of Y still works and Proposition 2.2 still holds except the assertion (c). Indeed, $H|_{E_0} = 0$ in this case and $|mH|$ ($m \gg 0$) contracts exactly the whole divisor E_0. The 3-fold X obtained by the base-point-free theorem has ample canonical divisor and it is still on the Noether line. Indeed, $K_X^3 = 2$ and $p_g(X) = 4$. But X is no longer smooth. It has exactly one singularity (cf. [17, Theorem (3.3) (3.3.3)]). Moreover, $|K_X|$ has this singularity as base locus and the canonical image of X is a smooth quadric.

Remark 2.4. In the construction above, if we allow T ($\in |5E + \tau^*(10s + 10al)|$) to be singular, then so is X. However, it is difficult to explicitly construct T with prescribed singularities in the linear system $|5E + \tau^*(10s + 10al)|$.

3. Base locus of the canonical linear systems

We prove Theorem 1.2 in this section. Throughout this section, we denote by X a canonically polarized Gorenstein minimal 3-fold. According to [13, Lemma 5.1], X is locally factorial. In order to prove Theorem 1.2, we need to reproduce the proofs of the Noether inequality in [6], [7] and [5].

3.1. Setting

This subsection is devoted to study the canonical map of X. Write

$$|K_X| = |\overline{M}| + \overline{Z} \tag{3.1}$$

where $|\overline{M}|$ is the movable part and $\overline{Z}$ is the fixed part.

We shall resolve the base locus of $|\overline{M}|$ in two steps. For a linear system Υ, we denote by $\mathrm{Bs}\Upsilon$ the base locus of Υ. Roughly speaking, the first step is to resolve $\mathrm{Bs}|\overline{M}| \cap \mathrm{Sing}(X)$.

Proposition 3.1 (cf. [5, Section 2]). *There is a birational morphism $\alpha\colon X_0 \to X$ satisfying the following properties.*

(a) *The morphism α is a composition of successive divisorial contractions to points and X_0 is a Gorenstein 3-fold with locally factorial terminal singularities.*

(b) *Denote by $|M_0|$ the movable part of $|\alpha^*\overline{M}|$. Then $\mathrm{Bs}|M_0| \cap \mathrm{Sing}(X_0) = \emptyset$.*

(c) *The following formulae*

$$\begin{aligned} K_{X_0} &= \alpha^* K_X + \sum_{t=1}^{m} c_t D_t, \\ \alpha^*(\overline{M}) &= M_0 + \sum_{t=1}^{m} d_t D_t, \quad \alpha^*(\overline{Z}) = Z_0 + \sum_{t=1}^{m} e_t D_t \end{aligned} \tag{3.2}$$

hold, where

(i) *Z_0 is the strict transform of $\overline{Z}$,*

(ii) *D_t is a prime divisor such that $\alpha(D_t)$ is a point for $1 \le t \le m$, and*

(iii) *c_t, d_t and e_t are non-negative integers such that $0 < c_t \le d_t$ for $1 \le t \le m$.*

(d) *Each connected component of $\cup_{t=1}^{m} D_t$ is a fiber of α.*

Proof. The birational morphism α is constructed in [5, p. 4–p. 5], using explicit resolutions of terminal singularities (see [4] and [5, Definition 2.2]). Then (a) and (b) follow from the construction and [13, Lemma 5.1]. Since both X_0 and X are locally factorial, c_t, d_t and e_t are non-negative integers. The inequality $c_t \le d_t$ follows by [5, Corollary 2.4]. Assertion (d) follows from the fact that α is a composition of successive divisorial contractions to points. □

We fix a birational morphism $\alpha\colon X_0 \to X$ as in Proposition 3.1. We may assume that the number of divisorial contractions in the construction of α is minimal. The second step is to resolve the base locus of $|M_0|$ without changing the singularities of X_0. This is possible by Proposition 3.1 (b) and Hironaka's Theorem (cf. [11]).

Proposition 3.2 (cf. [6, Lemma 4.2]). *There are successive blowups*

$$\beta\colon Y = X_{n+1} \overset{\pi_n}{\to} X_n \to \cdots \to X_{i+1} \overset{\pi_i}{\to} X_i \to \cdots \to X_1 \overset{\pi_0}{\to} X_0$$

such that π_i is a blowup along a smooth irreducible center W_i, W_i is contained in the base locus of the movable part of $|(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1})^* M_0|$ and $W_i \cap \operatorname{Sing}(X_i) = \emptyset$. Moreover, the morphism $\beta = \pi_n \circ \cdots \circ \pi_0$ satisfies the following properties.

(a) *Denote by $|M|$ the movable part of $|\beta^* M_0|$. Then $|M|$ is base point free.*

(b) *The following formulae*

$$K_Y = \beta^* K_{X_0} + \sum_{i=0}^{n} a_i E_i, \quad \beta^* M_0 = M + \sum_{i=0}^{n} b_i E_i \tag{3.3}$$

hold, where E_i is the strict transform of the exceptional divisor of π_i for $0 \le i \le n$, a_i and b_i are positive integers such that $a_i \le 2b_i$ for $0 \le i \le n$.

(c) *If $a_k = b_k = 1$ for some k such that $0 \le k \le n$, then W_k is a curve of X_k and the general member of $|M_0|$ is smooth at a general point of the curve $\pi_0 \circ \cdots \circ \pi_{k-2} \circ \pi_{k-1}(W_k)$.*

(d) *If $a_k = 2b_k$ for some k such that $0 \le k \le n$, then W_k is a point of X_k and the general member of $|M_0|$ is smooth at the point $\pi_0 \circ \cdots \circ \pi_{k-2} \circ \pi_{k-1}(W_k)$.*

Proof. The construction of the blowups π_i and (a) follow by Proposition 3.1 (b) and Hironaka's Theorem (cf. [11]). We remark that the assertion $a_i \le 2b_i$ in (b) is exactly [6, Lemma 4.2]. What really involved here is the assertions (c) and (d).

We introduce some notation. For $0 \le j < i \le n$, let $E_j^{(i)}$ ($\subset X_i$) be the strict transform of the exceptional divisor of π_j and let $E_j^{(n+1)} := E_j$. According to the definitions of a_i and b_i, for $0 \le i \le n$, we have

$$K_{X_{i+1}} = (\pi_0 \circ \pi_1 \circ \cdots \circ \pi_i)^* K_{X_0} + \sum_{j=0}^{i} a_j E_j^{(i+1)},$$

$$(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_i)^*(M_0) = M_{i+1} + \sum_{j=0}^{i} b_j E_j^{(i+1)}, \tag{3.4}$$

where $|M_{i+1}|$ is the movable part of $|K_{X_{i+1}}|$.

Considering the single blowup $\pi_k\colon X_{k+1} \to X_k$ for $1 \le k \le n$, we have for $j < k$,

$$\begin{aligned}
&\pi_k^* E_j^{(k)} = E_j^{(k+1)} + r_j E_k^{(k+1)}, \quad K_{X_{k+1}} = \pi_k^* K_{X_k} + a'_k E_k^{(k+1)},\\
&\pi_k^* M_k = M_{k+1} + b'_k E_k^{(k+1)},
\end{aligned}$$

where r_j is a nonnegative integer such that $r_j > 0$ if and only if $W_k \subset E_j^{(k)}$, $a'_k = 1$ if W_k is a curve or $a'_k = 2$ if W_k is a point, and b'_k is a positive integer since $W_k \subset \mathrm{Bs}|M_k|$.

Comparing these formulae with (3.4) when $i = k$ and $i = k-1$, we obtain

$$a_k = a'_k + \sum_{j=0}^{k-1} r_j a_j, \quad b_k = b'_k + \sum_{j=0}^{k-1} r_j b_j \quad \text{for } 1 \le k \le n \tag{3.5}$$

We conclude that $a_i \le 2b_i$ for $0 \le i \le n$ by induction on i and (b) follows.

For (c), the case $k = 0$ is trivial. If $k > 0$, by (3.5), we have $a'_k = 1$, $b'_k = 1$ and $r_j = 0$ for any $j < k$. Therefore W_k is a smooth curve of X_k, the general member of $|M_k|$ is smooth at a general point of W_k and $W_k \not\subset E_j^{(k)}$ for $j < k$. Assertion (c) follows.

We proceed to prove (d) by induction on k. Assume $a_0 = 2b_0$. Note that $a_0 = 1$ if W_0 is a curve or $a_0 = 2$ if W_0 is a point. Therefore $a_0 = 2$ and $b_0 = 1$. It follows that W_0 is a point of X and the general member of $|M_0|$ is smooth at W_0.

Now assume $a_k = 2b_k$ for some $k > 0$. Since $a_j \le 2b_j$ for any $j < k$, by (3.5), we have $a'_k = 2$, $b'_k = 1$ and $a_j = 2b_j$ for those j such that $j < k$ and $r_j > 0$.

Therefore W_k is a smooth point of X_k and the general member of M_k is smooth at W_k. If $r_j = 0$ for any $j < k$, then the point $W_k \notin E_j^{(k)}$ for any $j < k$ and thus the second statement of (b) clearly holds. If $r_j > 0$ for some $j < k$, then $a_j = 2b_j$ and $\pi_0 \circ \cdots \circ \pi_{k-2} \circ \pi_{k-1}(W_k) = \pi_0 \circ \cdots \circ \pi_{j-2} \circ \pi_{j-1}(W_j)$. Assertion (d) follows by induction. $\square$

From now on, we fix a birational morphism β as in Proposition 3.2 such that the number $n+1$ of blowups is minimal. Denote by ϕ_{K_X} the canonical map of X and by Σ the image of ϕ_{K_X}. Let ϕ be the morphism induced by $|M|$. Then $\phi = \phi_{K_X} \circ \pi$, where $\pi = \alpha \circ \beta$. Let $Y \xrightarrow{f} B \xrightarrow{\delta} \Sigma$ be the Stein factorization of ϕ. We have the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[r, "f"] \arrow[dl, "\beta"'] \arrow[d, "\pi"] \arrow[dr, "\phi"] & B \arrow[d, "\delta"] \\
X_0 \arrow[r, "\alpha"] & X \arrow[r, dashed, "\phi_{K_X}"'] & \Sigma
\end{tikzcd}$$

Note that B is normal. We have the following known results: if $\dim B = 3$, then $K_X^3 \geq 2p_g(X) - 6$ (cf. [15, Main Theorem]); if $\dim B = 2$, then $K_X^3 \geq \left\lceil \frac{2}{3}(g(C) - 1) \right\rceil (p_g(X) - 2)$ where $g(C)$ is the genus of a general fiber C of f (cf. [6, Theorem 4.1 (ii)]); if $\dim B = 1$, then $K_X^3 \geq \frac{7}{5}p_g(X) - 2$ (cf. [6, Theorem 4.1 (iii)] and [2, Theorem 4.1]).

3.2. Some basic results

From now on, we assume further that X is on the Noether line $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ and $p_g(X) \geq 7$.

Lemma 3.3 (cf. [15, Section 3]). *Keep the same notation as above.*

(a) *The morphism f is a fibration over the normal surface B.*

(b) *Let C be a general fiber of f. Then C is a smooth curve of genus 2 with $\pi^*K_X.C = 1$.*

(c) *Let d_Σ be the degree of Σ in $\mathbb{P}^{p_g(X)-1}$. Then $d_\Sigma \geq p_g(X) - 2$ and $M^2 \equiv d_\Sigma C$.*

(d) *The morphism δ is birational.*

Here the symbol $\equiv$ in (c) stands for numerical equivalence.

Proof. From the discussion at the end of the last subsection, we see that B is a normal surface and $g(C) = 2$. In particular, we have $M^2 \equiv d_\Sigma \cdot \deg \delta \cdot C$.

Because Σ is non-degenerate, we have $d_\Sigma \geq p_g(X) - 2$. Since both π^*K_X and M are nef, we conclude that

$$K_X^3 \geq \pi^*K_X.M^2 = d_\Sigma \cdot \deg \delta \cdot \pi^*K_X.C \geq (p_g(X) - 2) \deg \delta \cdot \pi^*K_X.C.$$

If $\pi^*K_X.C \geq 2$ or $\deg \delta \geq 2$, then $K_X^3 \geq 2p_g(X) - 4$, a contradiction to the Noether equality. Therefore $\pi^*K_X.C = 1$, $\deg \delta = 1$ and thus $M^2 \equiv d_\Sigma C$. $\square$

In what follows, we shall figure out the geometric information hidden in the Noether equality $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$. For this purpose, we use the techniques in the proofs of [6, Theorem 4.3] and [5, Theorem 3.1]. Recall from (3.1), (3.2) and (3.3) that

$$
\begin{aligned}
K_Y &= \pi^* K_X + \left(\sum_{t=1}^{m} c_t \beta^* D_t + \sum_{i=0}^{n} a_i E_i\right), \\
\pi^* K_X &= M + \left(\sum_{t=1}^{m} (d_t + e_t)\beta^* D_t + \sum_{i=0}^{n} b_i E_i + \beta^* Z_0\right)
\end{aligned}
\tag{3.6}
$$

Note that $K_X^3 = (\pi^* K_X)^3 = (\pi^* K_X)^2.M + K_X^2.\overline{Z}$. We aim to bound $(\pi^* K_X)^2.M$ from below.

For this purpose, by Bertini's theorem, we choose a general member S of $|M|$ such that S is smooth and consider the fibration $f|_S$. By abuse of notation, we still denote by C the general fiber of $f|_S$. We remark that S is of general type since so is X. Also the divisors $\beta^* D_t|_S$ and $E_i|_S$ are effective for $1 \le t \le m$ and $0 \le i \le n$. The following lemma is mainly due to [6, Theorem 4.3]. We basically follow the proof in [6] with slight modification.

Lemma 3.4 (See the proof of [6, Theorem 4.3]). *We have*

$$
\begin{aligned}
\left(\sum_{t=1}^{m} c_t \beta^* D_t + \sum_{i=0}^{n} a_i E_i\right)|_S &= \Gamma + D_V + E_V, \\
\left(\sum_{t=1}^{m} (d_t + e_t)\beta^* D_t + \sum_{i=0}^{n} b_i E_i + \beta^* Z_0\right)|_S &= \Gamma + D'_V + E'_V
\end{aligned}
\tag{3.7}
$$

where Γ is an irreducible reduced curve with $\Gamma.C = 1$, while E_V, D_V, E'_V and D'_V are effective divisors contained in the fibers of $f|_S$. Moreover, exactly one of the following two cases occurs:

(i) $\Gamma \subseteq \mathrm{Supp}(\sum_{t=1}^{m} \beta^* D_t|_S)$ *; in this case,* $D_V = \sum_{t=1}^{m} c_t \beta^* D_t|_S - \Gamma$, $E_V = \sum_{i=0}^{n} a_i E_i|_S$, $D'_V = \sum_{t=1}^{m} (d_t + e_t)\beta^* D_t|_S - \Gamma$ *and* $E'_V = \sum_{i=0}^{n} b_i E_i|_S + \beta^* Z_0|_S$.

(ii) $\Gamma \le E_l|_S$ *for a unique integer l; in this case,* $a_l = b_l = 1$, $D_V = \sum_{t=1}^{m} c_t \beta^* D_t|_S$, $E_V = \sum_{i=0, i\neq l}^{n} a_i E_i|_S + (E_l|_S - \Gamma)$, $D'_V = \sum_{t=1}^{m} (d_t + e_t)\beta^* D_t|_S$ *and* $E'_V = (\sum_{i=0, i\neq l}^{n} b_i E_i|_S + \beta^* Z_0|_S) + (E_l|_S - \Gamma)$.

In either case, we have $D_V \le D'_V$ and $E_V \le 2E'_V$.

Proof. Since $g(C) = 2$, $K_Y.C = 2$ by the adjunction formula. According to Lemma 3.3 (b) and (3.6), we have

$$\left(\sum_{t=1}^{m} c_t\beta^* D_t + \sum_{i=0}^{n} a_i E_i\right)|_S.C = 1$$
$$\text{and} \quad \left(\sum_{t=1}^{m}(d_t+e_t)\beta^* D_t + \sum_{i=0}^{n} b_i E_i + \beta^* Z_0\right)|_S.C = 1.$$

We conclude that the horizontal part of $(\sum_{t=1}^{m} c_t\beta^* D_t + \sum_{i=0}^{n} a_i E_i)|_S$ consists of an irreducible reduced curve Γ, which is also the horizontal part of $(\sum_{t=1}^{m}(d_t+e_t)\beta^* D_t + \sum_{i=0}^{n} b_i E_i + \beta^* Z_0)|_S$ and $\Gamma.C = 1$.

From Proposition 3.1 and Proposition 3.2, we see that $c_t, d_t + e_t, a_i$ and b_i are positive integers such that $c_t \le d_t + e_t$ and $a_i \le 2b_i$.

If $E_l.C > 0$ for some l, then $E_l.C = 1$ since E_i is Cartier, and then $\Gamma \le E_l|_S$. We see that the case (ii) occurs. If $E_i.C = 0$ for all i, then

$$\left(\sum_{t=1}^{m} c_t\beta^* D_t\right).C = 1 \quad \text{and} \quad \Gamma \le \operatorname{Supp}\left(\sum_{t=1}^{m}\beta^* D_t|_S\right),$$

and the case (i) occurs.

The last statement follows from (i), (ii) and the inequalities $c_t \le d_t + e_t$ and $a_i \le 2b_i$. □

Remark 3.5. We remark that if $\overline{Z} = 0$, then $\operatorname{Supp}(E_V) = \operatorname{Supp}(E'_V)$ in either case. This is indeed true by the following proposition. Also one sees that $\operatorname{Supp}(D_V) = \operatorname{Supp}(D'_V)$ in the case (ii). We also remark that the case (ii) occurs by the following proposition.

Proposition 3.6 (See the proofs of [6, Theorem 4.3] and [5, Theorem 3.1]). *In the above setting, we have*

(a) $\overline{Z} = 0$*;*

(b) $\operatorname{Supp}(E_V) = \operatorname{Supp}(E'_V)$ *and* $D_V.\Gamma = D'_V.\Gamma = (2E'_V - E_V).\Gamma = 0$*;*

(c) $\delta\colon B \to \Sigma$ *is an isomorphism and* Σ *is a normal rational surface with* $d_\Sigma = p_g(X) - 2$*;*

(d) Γ *is a smooth rational curve and* $\pi^* K_X|_S.\Gamma = \frac{1}{3}p_g(X) - \frac{4}{3}$*;*

(e) $\pi^* K_X|_S.E_V = \pi^* K_X|_S.E'_V = \pi^* K_X|_S.D_V = \pi^* K_X|_S.D'_V = 0.$

(f) $\Gamma \le E_l|_S$ *for a unique integer* l, *and for this integer* l, $a_l = b_l = 1$, $E_l.C = 1$ *and* $E_i.C = 0$ *for* $i \neq l$.

Proof. Lemma 3.4 yields that $D'_V - D_V \ge 0$, $2E'_V - E_V \ge 0$ and both divisors are contained in fibers of of $f|_S$. Because Γ is a section of $f|_S$,

$$\Gamma.(D'_V - D_V) \ge 0, \quad \Gamma.(2E'_V - E_V) \ge 0 \tag{3.8}$$

The adjunction formula yields

$$K_S.\Gamma + \Gamma^2 = 2p_a(\Gamma) - 2 \ge -2 \tag{3.9}$$

Note that

$$K_Y|_S = \pi^* K_X|_S + \Gamma + D_V + E_V \quad \text{and} \quad \pi^* K_X|_S = M|_S + \Gamma + D'_V + E'_V$$

by (3.6) and Lemma 3.4. By (3.8), one has

$$\begin{aligned} -2 \le (K_S + \Gamma).\Gamma &= (K_Y + M)|_S.\Gamma + \Gamma^2 \\ &= (\pi^* K_X|_S + M|_S + D_V + E_V + 2\Gamma).\Gamma \\ &\le (\pi^* K_X|_S + M|_S + 2D'_V + 2E'_V + 2\Gamma).\Gamma \\ &= (3\pi^* K_X|_S - M|_S).\Gamma \\ &= 3\pi^* K_X|_S.\Gamma - d_\Sigma. \end{aligned} \tag{3.10}$$

The last equality holds by $M^2 \equiv d_\Sigma C$ (see Lemma 3.3 (c)) and $\Gamma.C = 1$. Also $d_\Sigma \ge p_g(X) - 2$, we obtain

$$\pi^* K_X|_S.\Gamma \ge \frac{1}{3}(d_\Sigma - 2) \ge \frac{1}{3}(p_g(X) - 4) \tag{3.11}$$

Finally, we have

$$\begin{aligned} K_X^3 = (\pi^* K_X)^3 &= (\pi^* K_X|_S)^2 + K_X^2.\overline{Z} \\ &= \pi^* K_X|_S.M|_S + \pi^* K_X|_S.\Gamma + \pi^* K_X|_S.D'_V \\ &\quad + \pi^* K_X|_S.E'_V + K_X^2.\overline{Z} \\ &\ge d_\Sigma + \left(\frac{1}{3}p_g(X) - \frac{4}{3}\right) + \pi^* K_X|_S.D'_V \\ &\quad + \pi^* K_X|_S.E'_V + K_X^2.\overline{Z} \\ &\ge \frac{4}{3}p_g(X) - \frac{10}{3} \end{aligned} \tag{3.12}$$

By assumption, we see that all the equalities in the inequalities (3.8)–(3.12) hold.

Since K_X is ample, (a) follows by the equality $K_X^2.\overline{Z} = 0$ in (3.12). Therefore we have $Z_0 = 0$ and $e_t = 0$ for all $1 \le t \le m$. Then (b) follows by Remark 3.5, the equalities $\Gamma.(D'_V - D_V) = \Gamma.(2E'_V - E_V) = 0$ in (3.8) and the one $D_V.\Gamma = 2D'_V.\Gamma$ in (3.10).

For (c), we conclude $d_\Sigma = p_g(X) - 2$ from (3.11). This implies that Σ is of minimal degree. Since Σ is non-degenerate, Σ is a normal rational surface (cf. [1, Exercises IV. 18.-4)]). Because B is also normal and δ is finite by Stein factorization, δ is an isomorphism by Lemma 3.3 (d).

Assertion (d) follows by the equalities in (3.9) and (3.11). For (e), the equality in (3.12) implies $\pi^*K_X|_S.D'_V = \pi^*K_X|_S.E'_V = 0$. Since $\pi^*K_X|_S$ is nef, $D_V \le D'_V$ and $E_V \le 2E'_V$ by Lemma 3.4, we conclude $\pi^*K_X|_S.D_V = \pi^*K_X|_S.E_V = 0$.

Since $p_g(X) \ge 7$, $\pi(\Gamma)$ is a curve by (d). Because $\alpha(D_t)$ is a point for $1 \le t \le m$ (see Proposition 3.1) and $\pi = \alpha \circ \beta$, we conclude that the case (ii) of Lemma 3.4 occurs. It remains to show $E_l.C = 1$ and $E_i.C = 0$ for $i \ne l$ in (f). By Lemma 3.4, $E_V.C = 0$ and $E_V = (\sum_{i=0, i\ne l}^n a_iE_i)|_S + (E_l|_S - \Gamma)$. Because a_i is positive and $\Gamma.C = 1$, we obtain the required equalities. □

Remark 3.7. In the discussion above, we use the assumption $p_g(X) \ge 7$ only in the proof of Lemma 3.3, to show that the canonical image of X is a surface, and in the proof of Proposition 3.6 (f), to show that $\pi(\Gamma)$ is curve. The fact that $\pi(\Gamma)$ is a curve plays an important role in the proof of Theorem 1.2. For example, it is the key point to show that we actually do not need the first step to resolve $\mathrm{Bs}|\overline{M}| \cap \mathrm{Sing}(X)$ (see Lemma 3.9 and Proposition 3.1).

It is proved in [15] that $K_X^3 \ge 2p_g(X) - 6$ if the 3-fold X has at most canonical singularities and $\dim \mathrm{Im}\phi_{K_X} = 3$. Note that the lines $K_X^3 = 2p_g(X) - 6$ and $K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3}$ intersect at a point $(K_X^3, p_g(X)) = (2, 4)$.

Assume that X is a canonically polarized Gorenstein minimal 3-fold and with $(K_X^3, p_g(X)) = (2, 4)$. If $\dim \mathrm{Im}\phi_{K_X} = 3$, then $\phi_{K_X} : X \to \mathbb{P}^3$ is a finite double cover of $\mathbb{P}^3$ branched along a surface of degree 10, according to the results of Fujita [9] and [8].

If $\dim \mathrm{Im}\phi_{K_X} = 2$, then Lemma 3.3, Lemma 3.4 and Proposition 3.6 except the last assertion (f) still hold. However, $\pi(\Gamma)$ is a point according to Proposition 3.6 (d) and the same argument in the proof of Lemma 3.9 breaks down.

3.3. Proof of Theorem 1.2

We are able to complete the proof of Theorem 1.2. Let X be a 3-fold as in the assumption of Theorem 1.2. We stick to the same notation as above.

We have seen that $|K_X|$ has no fixed part, i.e., $\overline{Z}=0$ by Proposition 3.6 (a). Note that $\mathrm{Bs}|K_X|\neq\emptyset$ because $|K_X|$ is ample and $\dim \mathrm{Im}\phi_{K_X}=2$. Also recall from Lemma 3.6 (f) and Lemma 3.4 that $\Gamma\leq E_l|_S$ and

$$
\begin{aligned}
E_V &= \left(\sum_{i=1,i\neq l}^{n} a_iE_i\right)|_S+(E_l|_S-\Gamma),\\
E'_V &= \left(\sum_{i=1,i\neq l}^{n} b_iE_i\right)|_S+(E_l|_S-\Gamma)
\end{aligned}
\tag{3.13}
$$

Lemma 3.8. *The base locus of $|K_X|$ consists of a unique irreducible curve $\overline{\Gamma}:=\pi(\Gamma)$.*

Proof. Note that $\pi(\Gamma)$ is an irreducible curve and $\pi(\Gamma)=\pi(E_l)$ by Proposition 3.6 (d)–(f). Let $\overline{\Gamma}:=\pi(\Gamma)$. Then $\overline{\Gamma}\subseteq \mathrm{Bs}|K_X|$.

Assume that $\overline{\Lambda}$ is an irreducible curve such that $\overline{\Lambda}\subseteq\mathrm{Bs}|K_X|$ and $\overline{\Lambda}\neq\overline{\Gamma}$. Recall from Proposition 3.1 and Proposition 3.2 the construction of $\pi=\alpha\circ\beta$ and the fact that $\alpha(D_t)$ is a point. We see that there exists an irreducible curve Λ contained in some $E_i|_S$ such that $\overline{\Lambda}=\pi(\Lambda)$. Since $\overline{\Lambda}\neq\overline{\Gamma}$, we conclude that $i\neq l$ and Λ is contained in $\mathrm{Supp}E_V$. Hence $K_X.\overline{\Lambda}=0$ by Proposition 3.6 (e). Therefore $\overline{\Lambda}=0$ since K_X is ample, a contradiction.

Suppose p is an isolated point of $\mathrm{Bs}|K_X|$ and $p\notin\overline{\Gamma}$. Note that $\pi^*K_X|_S=M|_S+\Gamma+D'_V+E'_V$ by (3.6) and (3.7). Therefore $\pi^{-1}(p)\cap S\subseteq\mathrm{Supp}(D'_V+E'_V)$. We may write $D'_V+E'_V=A+B$, where A and B are effective divisors such that $\pi(\mathrm{Supp}A)=p$ and $p\notin\pi(\mathrm{Supp}(B))$. Then $A.B=0$ and $A.\Gamma=0$ since $p\notin\overline{\Gamma}$. Because $D'_V+E'_V$ is contained in the fibers of $f|_S$ by Lemma 3.4, we have $M|_S.A=0$. We also conclude from Proposition 3.6 (e) that $\pi^*K_X.A=0$. Then $A^2=A.(\pi^*K_X|_S-M|_S-\Gamma-B)=0$. But $\pi^*K_X|_S$ is nef and big, this contradicts the algebraic index theorem. This completes the proof. □

Lemma 3.9. *The birational morphism $\alpha\colon X_0\to X$ in Proposition 3.1 is indeed the identity morphism and $\overline{\Gamma}$ is contained in the smooth locus of X.*

Proof. Because $\overline{\Gamma}$ is a curve and $\alpha(D_t)$ is a point for $1\leq t\leq m$, we conclude that Γ is not contained in β^*D_t and then, by Proposition 3.6 (b) and

Lemma 3.4, $\Gamma \cap \beta^* D_t = \emptyset$ for any t. This yields $\beta(\Gamma) \cap D_t = \emptyset$ for any t. Because $\cup_{t=1}^m D_t$ is a union of fibers of α by Proposition 3.1 (d), we thus get $\alpha(\beta(\Gamma)) \cap \alpha(\cup_{t=1}^m D_t) = \emptyset$. Consequently $\alpha(D_t) \notin \pi(\Gamma)$ for any t, since $\pi = \alpha \circ \beta$ and $\alpha(D_t)$ is a point. Since $\overline{\Gamma} = \mathrm{Bs}|K_X| = \mathrm{Bs}|\overline{M}|$ by Lemma 3.8, we see that $\alpha(D_t) \notin \mathrm{Bs}|\overline{M}|$ for any t. According to Proposition 3.1 (b), we conclude that $\alpha\colon X_0 \to X$ is the identity morphism and $\overline{\Gamma} = \mathrm{Bs}|\overline{M}|$ is contained in the smooth locus of X. $\square$

It follows from the previous lemma that $\pi = \beta$ (see Proposition 3.2) and (3.6) becomes

$$\begin{aligned} K_Y &= \pi^* K_X + \sum_{i=1}^n a_i E_i = \pi^* K_X + \Gamma + E_V, \\ \pi^* K_X &= M + \sum_{i=1}^n b_i E_i = M + \Gamma + E'_V \end{aligned} \tag{3.14}$$

For a general member $S \in |M|$, denote by $\overline{S} := \pi(S)$ and by $\sigma := \pi|_S\colon S \to \overline{S}$. Note that $\overline{S} \in |K_X|$.

Lemma 3.10. *We may choose a general $S \in |M|$ such that both S and $\overline{S}$ are smooth. Moreover,*

(a) *the exceptional divisors of $\sigma\colon S \to \overline{S}$ is contained in E_V;*

(b) *the formula $K_X|_{\overline{S}} = (p_g(X) - 2)\overline{C} + \sigma_*(\Gamma)$ holds, where $|\overline{C}|$ is a base-point-free pencil of curves induced by the fibration $f|_S$ such that $\sigma^*\overline{C} = C$.*

Proof. According to Bertini's theorem, we may choose $S \in |M|$ such that

(1) $\overline{S}$ is smooth outside $\overline{\Gamma}$ by Lemma 3.8;

(2) $\overline{S}$ is smooth at a general point of $\overline{\Gamma}$ by Proposition 3.6 (f) and Proposition 3.2 (c);

(3) $\overline{S}$ is smooth at the point $\pi_0 \circ \cdots \circ \pi_{k-1}(W_k)$ for those k such that $a_k = 2b_k$ by Proposition 3.2 (d).

In particular, $\overline{S}$ is a normal surface.

Note that $\sigma (= \pi|_S)$ is isomorphic at the points outside $\Gamma \cup \mathrm{Supp} E_V$. To show the smoothness of $\overline{S}$, it suffices to show that $\overline{S}$ is smooth at the points where σ^{-1} is not defined. Let q be such a point. Then $q \in \overline{\Gamma}$ by the choice of S. According to Zariski's main theorem, $\sigma^{-1}(q)$ is a connected curve. Since

$q \in \overline{\Gamma}$, there is an irreducible component Q of $\sigma^{-1}(q)$ such that $Q.\Gamma > 0$. Note that $Q \leq E_V$ and recall that $(2E'_V - E_V).\Gamma = 0$ by Proposition 3.6 (b). It follows from (3.13) that $Q \leq E_k|_S$ for some $k \neq l$ and $a_k = 2b_k$. Hence $\overline{S}$ is smooth at q by the choice of S.

Because both S and $\overline{S}$ are smooth, σ is a composition of blowups and the support of the exceptional divisors of σ coincides with $\mathrm{Supp}E_V$. Since C is a general fiber of $f|_S$ and E_V is contained in the fibers of $f|_S$, there is a base-point-free pencil $|\overline{C}|$ of curves such that $\sigma^*\overline{C} = C$.

Note that $\dim|K_X||_{\overline{S}} = p_g(X) - 2$. Recall from Lemma 3.3 and Proposition 3.6 that $d_\Sigma = p_g(X) - 2$ and $M|_S \equiv d_\Sigma C$. Note that the target space of the fibration $f|_S$ is a hyperplane section of Σ, which is a rational normal curve. Therefore $M|_S$ is indeed linearly equivalent to $d_\Sigma C$. By (3.14), we have

$$\sigma^*(K_X|_{\overline{S}}) = \pi^*K_X|_S = M|_S + \Gamma + E'_V = d_\Sigma C + \Gamma + E'_V.$$

Since $\mathrm{Supp}E'_V = \mathrm{Supp}E_V$ and $\sigma_*E'_V = 0$, we obtain $K_X|_{\overline{S}} = (p_g(X) - 2)\overline{C} + \sigma_*(\Gamma)$. □

Proof of Theorem 1.2. We choose S and $\overline{S}$ as Lemma 3.10.

For (a), by Lemma 3.8 and Proposition 3.6 (d), it remains to show that $\overline{\Gamma}$ is smooth. Let d be the degree of $\sigma|_\Gamma\colon \Gamma \to \overline{\Gamma}$. Then $\sigma_*(\Gamma) = d\overline{\Gamma}$. By Lemma 3.10 (b), the projection formula yields $d\overline{C}.\overline{\Gamma} = \overline{C}.\sigma_*(\Gamma) = \sigma^*\overline{C}.\Gamma = C.\Gamma = 1$. Hence $d = 1$ and $\overline{C}.\overline{\Gamma} = 1$. Because $|\overline{C}|$ is base point free, $\overline{\Gamma}$ is smooth.

For (b), let $\pi'\colon X' \to X$ be the blowup of X along the curve $\overline{\Gamma}$. We have

$$K_{X'} = \pi'^*(K_X) + E, \quad \pi'^*(K_X) = M' + E',$$

where E' is the exceptional divisor of the blowup and $|M'|$ is the movable part of $|\pi'^*(K_X)|$. To prove (b), it suffices to prove that $|M'|$ is base point free.

Let S' be the strict transform of $\overline{S}$ under π' and let $\sigma' := \pi'|_{S'}\colon S' \to \overline{S}$. Because both $\overline{S}$ and $\overline{\Gamma}$ are smooth, S' is also smooth, $\pi'^*\overline{S} = S' + E$ and σ' is an isomorphism. Moreover, $E'|_{S'}$ is a smooth rational curve and $E'|_{S'} = \sigma'^*(\overline{\Gamma})$.

It suffices to show that the trace of $|M'|$ on S' is base point free. Note that

$$M'|_{S'} = \pi'^*(K_X)|_{S'} - E'|_{S'} = \sigma'^*(K_X|_{\overline{S}}) - \sigma'^*(\overline{\Gamma}).$$

We have seen $\sigma_*(\Gamma) = \overline{\Gamma}$ and $K_X|_{\overline{S}} = (p_g(X)-2)\overline{C} + \overline{\Gamma}$ in Lemma 3.10. Therefore $M'|_{S'} = \sigma'^*((p_g(X)-2)\overline{C})$. Because $|\overline{C}|$ is base point free and $\dim |M'||_{S'} = p_g(X) - 2$, we conclude that $|M'||_{S'}$ is composed with the pencil $|\sigma'^*\overline{C}|$. So $|M'||_{S'}$ is base point free.

Assertions (c) and (d) follow by Lemma 3.3 (b)–(c) and Proposition 3.6 (c). $\square$

4. Classification

The whole section is devoted to prove Theorem 1.3. We remark again that X is locally factorial by [13, Lemma 5.1] (see Remark 4.2). We stick to the same notation of Theorem 1.2. In particular, the left triangle of the following diagram

(4.1)
$$\begin{tikzcd} Y \arrow[r, dashed, "\overline{\phi}"] \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \Sigma_e \arrow[d, "r"] & \\ X \arrow[r, dashed, "\phi_{K_X}"'] & \Sigma \arrow[r, hook] & \mathbb{P}^{p_g(X)-1} \end{tikzcd}$$

is commutative and Y is also locally factorial. And we have

(4.2)
$$K_Y = \pi^* K_X + E_0,\ \pi^* K_X = M + E_0$$

where $|M|$ is base point free and ϕ is induced by $|M|$. Also $\phi|_{E_0} \colon E_0 \to \Sigma$ is a birational morphism.

According to Theorem 1.2 (c), $\deg \Sigma = p_g(X) - 2 \geq 5$. So Σ is obtained from a Hirzebruch surface Σ_e for some $e \geq 0$ via the birational morphism $r \colon \Sigma_e \to \Sigma$ induced by the linear system $|s + (e+k)l|$, where k is a nonnegative integer such that

(4.3)
$$p_g(X) = 2k + e + 2 \quad \text{and} \quad \deg \Sigma = 2k + e$$

(see the notation of Section 2). More precisely, we have two possibilities as follows (cf. [1, Exercises IV.18-4)] or [10, p. 380, Corollary 2.19]).

(1) If $k \geq 1$, then r is an isomorphism and Σ is indeed smooth. In this case, $\phi|_{E_0} \colon E_0 \to \Sigma$ is indeed an isomorphism since E_0 is also a Hirzebruch surface. If $e > 0$, then the ruling $|l|$ of Σ_e coincides with $\pi|_{E_0} \colon E_0 \to \overline{\Gamma}$ via $r^{-1}\phi|_{E_0}$ because Σ_e has a unique ruling. We may assume that this still holds when $e = 0$ by possibly exchanging the two rulings of Σ_0.

(2) If $k=0$, then $e \geq 5$ and Σ is a cone over a rational normal curve. Moreover, r contracts the negative section s to the vertex v of the cone Σ.

To prove Theorem 1.3, we shall exclude the case (2). The following lemma allows us to treat both cases in a unified way.

Lemma 4.1. *The rational map $\overline{\phi} = r^{-1}\phi$ is indeed a morphism and $\overline{\phi}|_{E_0} : E_0 \to \Sigma_e$ is an isomorphism.*

Proof. The lemma is nontrivial only for $k=0$. In this case, since $\rho(E_0)=2$ and $\rho(\Sigma)=1$, the birational morphism $\phi|_{E_0} \colon E_0 \to \Sigma$ contracts exactly the negative section s_{E_0} of the ruling $\pi|_{E_0} \colon E_0 \to \overline{\Gamma}$ and maps any fiber l_{E_0} of this ruling to a line on Σ. Since $\phi|_{E_0}$ is induced by $|M||_{E_0}$, we conclude that $M|_{E_0} = s_{E_0} + el_{E_0}$.

Assume by contradiction that $r^{-1}\phi$ is not a morphism. Then the locus where $r^{-1}\phi$ is not defined is contained in $\phi^{-1}(\mathrm{v})$. Let $\widehat{\pi} \colon \widehat{Y} \to Y$ be the resolution of the indeterminacy of $r^{-1}\phi$ and let $\widehat{\phi} \colon \widehat{Y} \to \Sigma_e$ be the induced morphism such that $\phi\widehat{\pi} = r\widehat{\phi}$:

$$\begin{array}{ccccc} \widehat{Y} & \xrightarrow{\widehat{\phi}} & \Sigma_e & \qquad & s = r^{-1}(\mathrm{v}) \\ \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle r} & & \downarrow \\ Y & \xrightarrow{\phi} & \Sigma & & \mathrm{v} \end{array}$$

We may assume that $\widehat{\pi}|_{\widehat{\pi}^{-1}(Y \setminus \phi^{-1}(\mathrm{v}))} \colon \widehat{\pi}^{-1}(Y \setminus \phi^{-1}(\mathrm{v})) \to Y \setminus \phi^{-1}(\mathrm{v})$ is an isomorphism and that $\widehat{\pi}^{-1}(\phi^{-1}(\mathrm{v}))$ is contained in the smooth locus of $\widehat{Y}$.

From the commutative diagram, we have

$$\widehat{\pi}^* M = \widehat{\pi}^*\phi^*\mathcal{O}_\Sigma(1) = \widehat{\phi}^* r^*\mathcal{O}_\Sigma(1) = \widehat{\phi}^*(s+el) = \widehat{\Delta} + e\widehat{L},$$

where $\widehat{\Delta} := \widehat{\phi}^* s$, $\widehat{L} := \widehat{\phi}^* l$ and $|\widehat{L}|$ is a base-point-free pencil of divisors. It follows that $M = \Delta + eL$, where $\Delta := \widehat{\pi}_*\widehat{\Delta}$, $L := \widehat{\pi}_*\widehat{L}$ and $|L|$ has no fixed part.

Recall that C is a general fiber of ϕ and $E_0.C = 1$. Since $M.C = 0$, we have $\Delta.C = L.C = 0$. Then $\Delta \ngeq E_0$ and $L \ngeq E_0$ since $E_0.C = 1$,

We now show that $|L|$ is a base-point-free pencil. Note that

$$e+2 = h^0(Y, \mathcal{O}_Y(M)) \geq h^0(Y, \mathcal{O}_Y(eL))) \geq eh^0(Y, \mathcal{O}_Y(L)) - e + 1.$$

Therefore $\dim |L| = 1$. Since $\Delta|_{E_0} + eL|_{E_0} = M|_{E_0} \sim s_{E_0} + el_{E_0}$ and $\Delta \ngeq E_0$, we have $\Delta|_{E_0} > 0$ and thus $\Delta > 0$. Moreover, from the commutativity of the diagram above and the definition of Δ, we see that $\phi(\mathrm{Supp}\Delta) = \mathrm{v}$. Because $\phi|_{E_0}$ contracts exactly the curve s_{E_0}, $\Delta|_{E_0} = bs_{E_0}$ for some integer $b \geq 1$. Then

$$\begin{aligned} e^2\pi^*K_X.L^2 &= (M + E_0).(M - \Delta)^2 \\ &= M|_{E_0}^2 - 2M|_{E_0}.\Delta|_{E_0} + \Delta|_{E_0}^2 = e(1 - b^2). \end{aligned}$$

Since K_X is ample and $|L|$ has no fixed part, we obtain $\pi^*K_X.L^2 = 0$, $b = 1$ and thus $L|_{E_0} = \frac{1}{e}(M - \Delta)|_{E_0} = l_{E_0}$. Then the trace of the pencil $|L|$ on E_0 is $|l_{E_0}|$ since $L \not\geq E_0$. This implies $\mathrm{Bs}|L| \cap E_0 = \emptyset$. If $\mathrm{Bs}|L| \neq \emptyset$, since $\dim |L| = 1$ and Y is locally factorial, $\mathrm{Bs}|L| = L_1 \cap L_2$ for any two distinct members $L_1, L_2 \in |L|$ and $\mathrm{Bs}|L|$ is of pure dimension one. Then the ampleness of K_X, $\pi^*K_X.L^2 = 0$ and $\mathrm{Bs}|L| \cap E_0 = \emptyset$ imply that $|L|$ is base point free.

Because both $|L|$ and $\widehat{L}$ are base point free and $L = \widehat{\pi}_*\widehat{L}$, we have $\widehat{\pi}^*L = \widehat{L}$.

Let F be any irreducible and reduced curve contracted by $\widehat{\pi}$. Then $\widehat{L}.F = \widehat{\pi}^*L.F = 0$. On one hand, since $\widehat{L} = \widehat{\phi}^*(l)$, $\widehat{\phi}(F)$ is contained in one of the fiber of ruling induced by $|l|$. On the other hand, $r^{-1}\phi$ is defined outside $\phi^{-1}(\mathrm{v})$, so $r\widehat{\phi}(F) = \phi\widehat{\pi}(F) = \mathrm{v}$ and thus $\widehat{\phi}(F)$ is contained in $r^{-1}(\mathrm{v}) = s$. Therefore $\widehat{\phi}(F)$ is a point in Σ_e. This means $\widehat{\phi}$ factors though Y.

Hence $\overline{\phi}$ is a morphism. Its restriction $\overline{\phi}|_{E_0} \colon E_0 \to \Sigma_e$ is birational because so is $\phi|_{E_0}$. Then it is an isomorphism because both E_0 and Σ_e are Hirzebruch surfaces. □

Remark 4.2. In the definition of a Gorenstein minimal 3-fold X (see the introduction), X is assumed to be $\mathbb{Q}$-factorial. Then X is indeed locally factorial by [13, Lemma 5.1]. We need this assumption to apply intersection theory.

For example, if this assumption is dropped, then we do not know whether $\Delta := \widehat{\pi}_*\widehat{\Delta}$ is $\mathbb{Q}$-Cartier. In this situation, $\Delta|_{E_0}$ might be not well-defined. For another example, we need the fact that Y is locally factorial to conclude that $\mathrm{Bs}|L|$ is of pure dimension one from $\mathrm{Bs}|L| = L_1 \cap L_2$.

This assumption is also important in the proof of Lemma 4.3.

From now on, we denote by j the inverse of the isomorphism $\overline{\phi}|_{E_0} \colon E_0 \to \Sigma_e$. By abuse of notation, we identify $\mathrm{Pic}(E_0)$ with $\mathrm{Pic}(\Sigma_e) = \mathbb{Z}l \oplus \mathbb{Z}s$. Since $M = \phi^*\mathcal{O}_\Sigma(1)$ and $r^*\mathcal{O}_\Sigma(1) = s + (e + k)l$, we have

$$M = \overline{\phi}^*(s + (e + k)l), \quad M|_{E_0} = s + (e + k)l \tag{4.4}$$

Since $r\overline{\phi} = \phi$, we still denote by C the general fiber of $\overline{\phi}$.

Lemma 4.3. *Every fiber of $\overline{\phi}$ is* 1*-dimensional, reduced and irreducible. In particular, $\overline{\phi}$ is flat.*

Proof. Assume that Φ is a 2-dimensional irreducible component of a fiber of $\overline{\phi}$. Recall that $\overline{\phi}|_{E_0}\colon E_0 \to \Sigma_e$ is an isomorphism by Lemma 4.1. It follows that $E_0 \cap \Phi = \emptyset$ or $E_0 \cap \Phi$ consists of a single point. Since Y is locally factorial, every divisor of Y is Cartier and thus $\dim E_0 \cap \Phi = 1$ if $E_0 \cap \Phi \neq \emptyset$. We conclude that $E_0 \cap \Phi = \emptyset$. Therefore $(\pi^* K_X)^2.\Phi = (M + E_0)^2.\Phi = 0$ according to (4.2). Since K_X is ample, it follows that $\dim \pi(\Phi) \leq 1$. Then $\pi(\Phi) = \overline{\Gamma}$ and $\Phi = E_0$, a contradiction. Hence every fiber of $\overline{\phi}$ is 1-dimensional and $\overline{\phi}$ is flat (cf. [16, p. 179, Theorem 23.1 and Corollary]).

Because K_X is ample and $\pi^* K_X.C = 1$ by Theorem 1.2 (c), if $\overline{\phi}$ has reducible fibers, then some irreducible component of some reducible fiber is contained in E_0. This contradicts that E_0 is a section of $\overline{\phi}$. Therefore every fiber is irreducible. Also $\pi^* K_X.C = 1$ implies that every fiber is reduced. $\square$

Lemma 4.4. *Let $\mathcal{E} := \overline{\phi}_* \mathcal{O}_Y(2E_0)$.*

(a) *Then $\mathcal{E}$ is a locally free sheaf of rank* 2 *and the natural morphism $\overline{\phi}^* \mathcal{E} \to \mathcal{O}_Y(2E_0)$ is surjective.*

(b) *Let $P := \mathbb{P}_{\Sigma_e}(\mathcal{E})$ and let $\tau\colon P \to \Sigma_e$ be the natural projection. Then the Σ_e-morphism $\psi\colon Y \to P$ associated to the surjective morphism $\overline{\phi}^* \mathcal{E} \to \mathcal{O}_Y(2E_0)$ is finite of degree* 2*.*

(c) *Let $E = \psi(E_0)$. Then E is a section of τ and it is an irreducible connected component of the branch divisor of ψ.*

(d) *The section E of τ corresponds to the surjective morphism $\mathcal{E} = j^* \overline{\phi}^* \mathcal{E} \to j^* \mathcal{O}_Y(2E_0)$, whose kernel is $\mathcal{O}_{\Sigma_e}$.*

Proof. Let C be any fiber of $\overline{\phi}$. Then $p_a(C) = 2$ by Theorem 1.2 (c). Because $\overline{\phi}$ is a flat morphism, C is Gorenstein. Since C is reduced and irreducible by the previous lemma, $|\omega_C|$ is base point free by [3, Theorem 3.3]. We have $K_C = K_Y|_C = 2E_0|_C$ from (4.2). So $\mathcal{O}_Y(2E_0)|_C$ is generated by global sections and $h^0(C, \mathcal{O}_Y(2E_0)|_C) = p_a(C) = 2$. Then (a) follows by Grauert's Theorem.

For (b), we have the following diagram such that $\tau\psi = \overline{\phi}$.

$$\begin{tikzcd} & P \arrow[d, "\tau"] \\ Y \arrow[ur, "\psi"] \arrow[r, "\overline{\phi}"] & \Sigma_e \arrow[l, bend left, "j"] \end{tikzcd}$$

Note the the restriction $\psi|_C$ is indeed the canonical map of C, which is a finite morphism of degree 2 to the projective line $\mathbb{P}^1$. This proves (b).

For (c), first note that the branch divisor of ψ is pure of dimension one because Y is normal and P is smooth. Since Y has only finite many singularities and Y is locally factorial, we conclude that the irreducible components and the connected components of the branch divisor coincide. Because E_0 is a section of $\overline{\phi}$, E is a section of τ. Moreover, since $E_0|_C$ consists one point and $K_C = 2E_0|_C$, we conclude that $E_0|_C$ is a ramification point of the canonical morphism of C. Hence E is an irreducible component and thus a connected component of the branch divisor.

Note that $E = \psi(E_0) = \psi j(\Sigma_e)$. From the construction of ψ, ψj corresponds the pullback of the surjective morphism $\overline{\phi}^*\mathcal{E} \to \mathcal{O}_Y(2E_0)$ by j^*, which is $\mathcal{E} = j^*\overline{\phi}^*\mathcal{E} \to j^*\mathcal{O}_Y(2E_0)$. Denote by $\mathcal{K}$ its kernel. Then $\mathcal{O}_P(E) \otimes \tau^*\mathcal{K} = \mathcal{O}_P(1)$ (see the proof of [10, p. 371, Propostion 2.6]). Applying $(\psi j)^*$ to this equality, since $\psi^*E = 2E_0$ by (c), $\tau(\psi j) = \mathrm{id}_{\Sigma_e}$ and $(\psi j)^*\mathcal{O}_P(1) = j^*\mathcal{O}_Y(2E_0)$, we conclude that $\mathcal{K} = \mathcal{O}_{\Sigma_e}$. $\square$

Let D be the branch divisor of the double cover $\psi\colon Y \to P$. Then

$$D \sim 2\mathcal{L} \quad \text{and} \quad K_Y = \psi^*(K_P + \mathcal{L}) \tag{4.5}$$

for some $\mathcal{L} \in \mathrm{Pic}(P)$. Since $P = \mathbb{P}_{\Sigma_e}(\mathcal{E})$ with $\mathcal{E}$ in the following exact sequence

$$0 \to \mathcal{O}_{\Sigma_e} \to \mathcal{E} \to j^*\mathcal{O}_Y(2E_0) \to 0 \tag{4.6}$$

we have $\mathrm{Pic}(P) = \mathbb{Z}E \oplus \tau^*\mathrm{Pic}(\Sigma_e)$,

$$\mathcal{O}_P(E) = \mathcal{O}_P(1) \quad \text{and} \quad K_P = \tau^*(K_{\Sigma_e} + j^*\mathcal{O}_Y(2E_0)) - 2E \tag{4.7}$$

We now determine $j^*\mathcal{O}_Y(2E_0)$ and $\mathcal{L}$ in terms of $\mathrm{Pic}(\Sigma_e)$ and $\mathcal{O}_P(E)$.

Lemma 4.5. *We have*

$$j^*\mathcal{O}_Y(2E_0) \cong \mathcal{O}_{\Sigma_e}(-2s-2al) \quad \textit{and} \quad \mathcal{L} = 3E + \tau^*(5s+5al), \tag{4.8}$$

where a is an integer such that

$$k = 3a - 2e - 2 \tag{4.9}$$

Proof. By (4.2), (4.4) and the adjunction formula, we have $\mathcal{O}_{E_0}(E_0) = \frac{1}{3}(K_{E_0} - M|_{E_0}) = -s - al$ with integer $a = \frac{1}{3}(k+2e+2)$. Since $j\colon \Sigma_e \to Y$ factors through E_0, $j^*\mathcal{O}_Y(2E_0) \cong \mathcal{O}_{\Sigma_e}(-2s-2al)$.

Since $\psi^*E = 2E_0$, according to (4.2) and (4.4), $K_Y = \psi^*(\tau^*(s+(e+k)l) + E)$. On the other hand, by (4.5) and (4.7),

$$K_Y = \psi^*(\tau^*(-4s-(e+2a+2)l) - 2E + \mathcal{L}).$$

It follows that $\psi^*(\mathcal{L}_0) = \mathcal{O}_Y$, where $\mathcal{L}_0 = \mathcal{L} - (3E + \tau^*(5s+5al))$. Note that $\psi_*\mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{L}^\vee$. The projection formula yields $\mathcal{L}_0 \oplus (\mathcal{L}_0 \otimes \mathcal{L}^\vee) = \mathcal{O}_P \oplus \mathcal{L}^\vee$. It is clear that $H^0(P, \mathcal{L}^\vee) = 0$ and $H^0(P, \mathcal{L}_0 \otimes \mathcal{L}^\vee) = 0$. We obtain $\mathcal{L}_0 = \mathcal{O}_P$ and the required formula for $\mathcal{L}$. $\square$

Proposition 4.6. *Let $\omega_{Y/\Sigma_e} = K_Y - \overline{\phi}^* K_{\Sigma_e}$.*

(a) *Then $\overline{\phi}_*\omega_{Y/\Sigma_e} = \mathcal{E} \otimes \mathcal{O}_{\Sigma_e}(3s+3al)$.*

(b) *The pair (e,a) with a defined by (4.9) satisfies $a \geq e \geq 3$; or $1 \leq e \leq 2$, $a \geq e+1$; or $e = 0$, $a \geq 2$.*

Proof. Let C be any fiber of $\overline{\phi}$. Then $\omega_{Y/\Sigma_e}|_C = 2E_0|_C = K_C$ by (4.2). Therefore $\overline{\phi}_*\omega_{Y/\Sigma_e}$ is a locally free sheaf of rank 2 by Grauert's theorem. It follows that $\overline{\phi}_*\omega_{Y/\Sigma_e}$ is semi-positive by [19, Theorem III and (1.3) Remark (iii)].

We have seen $K_Y = \psi^*(\tau^*(s+(3a-e-2)l) + E)$ and $\mathcal{O}_P(E) = \mathcal{O}_P(1)$. Also $\psi_*\mathcal{O}_Y = \mathcal{O}_P \oplus \mathcal{L}^\vee$. Applying the projection formula to ψ and then to τ, we obtain $\overline{\phi}_*(\omega_{Y/\Sigma_e}) = \mathcal{E} \otimes \mathcal{O}_{\Sigma_e}(3s+3al)$. It has $\mathcal{O}_{\Sigma_e}(s+al)$ as a quotient by (4.6) and (4.8). So $a - e = \deg \mathcal{O}_{\Sigma_e}(s+al)|_s \geq 0$ by the semi-positivity of $\overline{\phi}_*\omega_{Y/\Sigma_e}$.

Note that $p_g(X) = 6a - 3e - 2$ by (4.3) and (4.9). Since $p_g(X) \geq 7$ by the assumption of Theorem 1.3, it is clear that the pair (e,a) satisfies the required inequalities. $\square$

Proof of Theorem 1.3. By Proposition 4.6, we see that $k = 3a - 2e - 2 \geq 1$ and thus Σ is the embedding of Σ_e in $\mathbb{P}^{p_g(X)-1}$ by the discussion at the beginning of this section. Moreover, we have seen $p_g(X) = 6a - 3e - 2$ in the proof of the proposition above and thus $K_X^3 = 8a - 6e - 4$ by the Noether equality. Hence (a) is established.

Assertion (b) follows from Lemma 4.1 and Lemma 4.3 since $r \circ \overline{\phi} = \phi$.

Because $j^*\mathcal{O}_Y(2E_0) \cong \mathcal{O}_{\Sigma_e}(-2s - 2al)$ by Lemma 4.5 and $a \geq e$ by Proposition 4.6, the exact sequence (4.6) splits by Lemma 2.1 (b) and thus $\mathcal{E} = \mathcal{O}_{\Sigma_e} \oplus \mathcal{O}_{\Sigma_e}(-2s - 2al)$. By (4.5), (4.8) and Lemma 4.4 (c), we see that $D = E + T$, with $T \in |5E + \tau^*(10s + 10al)|$ and $E \cap T = \emptyset$. We conclude (c) and (d) directly from the discussion above, by identifying $\overline{\phi}$ with ϕ via the isomorphism r.

If X is smooth, so is Y. Therefore the branch locus of ψ is smooth. Comparing (4.5), (4.6) and (4.8) with (2.3)–(2.4), we conclude that X is one of the 3-folds constructed in Section 2 and complete the proof of Theorem 1.3. □

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