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# Support of local cohomology modules over hypersufaces and rings with FFRT

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Let $(R, m, K)$ be a local ring and $I$ be an ideal. It is conjectured that the local cohomology modules $H^i_I(R)$ have closed support on $\operatorname{Spec} R$. We establish this conjecture for hypersurface rings in positive characteristic. In addition, we prove that the associated primes of $H^i_I(R)$ are finite for rings with finite $F$-representation type. As a consequence, $H^i_I(R)$ has finitely many associated primes for a Stanley-Reisner ring, $R$, over any field of positive characteristic. In addition, we present results in a similar direction for certain determinantal rings.

## 1. Introduction

The local cohomology modules $H^i_I(R)$ are usually not finitely generated and this presents a challenge in studying them. However, if $R$ is a regular local ring containing a field, every $H^i_I(R)$ has finitely many associated primes [4, 11, 12]. Unfortunately, this does not hold for every Noetherian ring [6, 24, 25].

It remains an important open question whether the set of minimal primes of $H^i_I(R)$ is finite. This is equivalent to asking whether the support of $H^i_I(R)$ is a Zariski closed subset of $\operatorname{Spec} R$. Since every minimal prime of $H^i_I(R)$ is an associated prime, $\operatorname{Supp}_R H^i_I(R)$ is closed in $\operatorname{Spec}(R)$ when the associated primes are finite. This holds in other cases besides regular local rings containing a field; for instance, for smooth $\mathbb{Z}$-algebras, [1], for regular local rings of unramified mixed characteristic [13],for rings of small dimension [14], and for rings of invariants of linearly reductive algebraic groups acting on regular rings [15]. We refer to [16–18, 20–22, 27] for other results in this direction. In addition, $\operatorname{Supp}_R H^i_I(R)$ has been shown to be closed in some special cases without looking at its associated primes; for instance, for rings

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of small dimension [3], for some graded rings [7, 23], and for some top local cohomology modules [7, 10].

In this paper, we study the support of local cohomology modules over a ring of positive characteristic. We investigate the structure of $H^i_I(R)$ as an $R\langle F\rangle$-module. Using this technique we recover several results previously known for top local cohomology, regular rings, and rings of small dimension (see Corollaries 4.6, 4.10, and 4.14). Our main new result using $R\langle F\rangle$-modules is regarding the support of local cohomology over hypersurface rings:

**Theorem 1.1 (see Corollary 4.13).** *Let $S$ be a Noetherian regular ring of prime characteristic. Let $f \in S$ be a nonzero divisor, $R = S/fS$. Then* $\operatorname{Supp}_R H^i_I(R)$ *is a Zariski closed subset of* $\operatorname{Spec} R$ *for every ideal $I \subseteq S$.*

We have recently learned that this result has been obtained independently by M. Katzman and W. Zhang [9], using a different method.

In a similar direction we present reductions of this open question regarding the support of local cohomology modules to computing torsion modules of local cohomology over regular rings (see Proposition 4.15). This follows the lines initiated by Huneke, Katz and Marley [3].

We also present a finiteness result for rings with finite $F$-representation type, denoted by FFRT (see Section 5 for definition and examples). Takagi and Takahashi [27] showed that local cohomology over Gorenstein rings with FFRT have finitely many associated primes. We extend this result to all rings with FFRT:

**Theorem 1.2.** *(see Theorem 5.7) Suppose that $R$ is an $F$-finite ring with FFRT. Then* $\operatorname{Ass}_R H^i_I(R)$ *is finite for every ideal $I \subseteq R$ and $i \in \mathbb{N}$.*

Very recently, H. Dao and P.H. Quy [2] proved independently this result using filter regular sequences. In contrast, we approach this problem via the interaction of Koszul cohomology with Frobenius.

As a consequence of the previous theorem, we obtain that local cohomology modules over Stanley-Reisner rings over a field of positive characteristic have a finite set of associated primes (see Corollary 5.8).

Finally, we study the associated primes of a determinantal ring $R_t = K[X]/I_t(X)$, where $X = (x_{i,j})$ is a finite matrix of variables and $I_t(X)$ is the ideal generated by the $t \times t$-minors of $X$. In particular, we show that local cohomology modules of $R$ have a finite set of associated primes for $t = 3$ (see Theorem 6.5). Furthermore, we show that the general case follows from a conjecture made by the first author regarding smooth extensions (see Conjecture 6.1 and Theorem 6.7).

Throughout this manuscript, unless otherwise specified, $A$, $R$, and $S$ denote Noetherian rings of prime characteristic $p$, and $F$ denotes the Frobenius map $r \mapsto r^p$. The characteristic $p$ hypothesis is often repeated for emphasis in the statements of results.

## 2. Preliminaries

Given a Noetherian ring $R$, and elements $f_1, \dots, f_\ell \in R$, we define the Čech complex of $R$ with respect to $\underline{f} = f_1, \dots, f_\ell$, by

$$\check{\mathrm{C}}(\underline{f}; R) = 0 \to R \to \oplus_j R_{f_j} \to \cdots \to R_{f_1 \cdots f_\ell} \to 0,$$

where the differential in each summand is given, up to sign, by the localization map. If $I = (\underline{f})$, we define the $i$-th local cohomology by

$$H^i_I(R) = H^i(\check{\mathrm{C}}(\underline{f}; R)).$$

This module depends only on $\sqrt{I}$, but not on $f_1, \dots, f_\ell$. Since

$$R_f \cong \lim_{e \to \infty} R \overset{f^{p-1}}{\to} R \overset{f^{p^2-p}}{\to} R \overset{f^{p^3-p^2}}{\to} \cdots \to R \overset{f^{p^e-p^{e-1}}}{\to} R \to \cdots,$$

we have the inductive direct limits

$$\text{(2.0.1)} \qquad \check{\mathrm{C}}(\underline{f}, R) \cong \lim_{e \to \infty} \mathcal{K}(\underline{f}^{p^e}; R) \quad \text{and} \quad H^i_I(R) = \lim_{e \to \infty} H^i(\underline{f}^{p^e}; R),$$

where $\mathcal{K}(\underline{f}^{p^e}; R)$ and $H^i(\underline{f}^{p^e}; R)$ denote the Koszul complex and the Koszul cohomology of $R$ with respect to $\underline{f}^{p^e} = f_1^{p^e}, \dots, f_\ell^{p^e}$.

The cohomological dimension of $I$ is defined by

$$\begin{aligned} \mathrm{cd}(I) &= \mathrm{Max}\{i \in \mathbb{N} \mid H^i_I(R) \neq 0\} \\ &= \mathrm{Max}\{i \in \mathbb{N} \mid H^i_I(M) \neq 0 \text{ for an } R-\text{module } M\}. \end{aligned}$$

Given a map of rings $\varphi : R \to S$ and an $R$-module $N$, we have an induced functor from the category of $R$-modules to the category of $S$-modules given by $\varphi^* N = N \otimes_R S$. We pay particular attention to this functor when $\varphi$ is the $e$-th iterated Frobenius map $F^e : R \to R$, $r \mapsto r^{p^e}$. If $I \subseteq R$ is an ideal, we have that $F^{e*} R/I \cong R/I^{[p^e]}$, where $I^{[p^e]} = (f^{p^e} \mid f \in I)$.

We point out that $F^*H^i(\underline{f};R) \cong H^i(\underline{f}^p;R)$, when $R$ is a regular ring. In particular, the limit in Equation 2.0.1 induces an isomorphism $H^i_I(R) \to F^*H^i_I(R)$. Modules with this property are called $F$-modules [12].

## 3. The class of rings for which local cohomology has finitely many associated primes

In this section we do not need to make any restriction on the characteristic. We note the following general facts, which are already known. We include proofs and references for the sake of completeness.

**Theorem 3.1.** *Let $R$ denote a Noetherian ring, let $I$ denote an ideal of $R$, and let $S$ be a Noetherian $R$-algebra.*

*(a) If $H^i_I(R)$ has finitely many associated (respectively, minimal) primes, then so does $H^i_{IS}(S)$ for every Noetherian ring $S$ that is flat over $R$.*

*(b) If $S$ is faithfully flat over $R$ and $H^i_I(S)$ has finitely many associated (respectively, minimal) primes in $S$, then $H^i_I(R)$ has finitely many associated (respectively, minimal) primes in $R$.*

*(c) The class of rings $R$ such that $H^i_I(R)$ has finitely many associated (respectively, minimal) primes for all $i$ and all ideals $I$ of $R$ is closed under localization.*

*(d) If $R$ is pure in $S$, e.g., if $R$ is a direct summand of $S$ as an $R$-module, and the set of associated primes of $H^i_I(S)$ is finite, then the set of associated primes of $H^i_I(R)$ is finite. Hence, if $H^i_J(S)$ has finitely many associated primes for all $i$ and ideals $J \subseteq S$, then so does $H^i_I(R)$ for all $i$ and ideals $I \subseteq R$.*

*(e) If the affine open sets $\operatorname{Spec}(R_{f_1}), \ldots, \operatorname{Spec}(R_{f_h})$ cover $\operatorname{Spec}(R)$ except for finitely many closed points corresponding to maximal ideals $m_1, \ldots, m_k$ of $R$, and $H^i_I(R_{f_j})$ has only finitely many associated (respectively, minimal) primes for each $j$, $1 \leq j \leq h$, then $H^i_I(R)$ has only finitely many associated (respectively, minimal) primes.*

*(f) If the affine open sets $\operatorname{Spec}(R_{f_1}), \ldots, \operatorname{Spec}(R_{f_h})$ cover $\operatorname{Spec}(R)$ except for finitely many closed points corresponding to maximal ideals $m_1, \ldots, m_k$ of $R$, and $H^i_I(R_{f_j})$ has only finitely many associated (respectively, minimal) primes for all $I$ and $i$ and for all $j$, $1 \leq j \leq h$, then $H^i_I(R)$ has only finitely many associated (respectively, minimal) primes for all $I$ and $i$.*

*Proof.* (a) and (b). The set of associated primes of $H^i_{IS}(S) \cong S \otimes_R H$, where $H = H^i_I(R)$, is the union of the sets $\mathrm{Ass}_S(S/PS)$ as $P$ varies in $\mathrm{Ass}_R(H)$. The statement about associated primes in (a) is immediate. If $Q$ is minimal in the support of $S \otimes H$, it lies over some prime $P$ in $R$. To complete the proof of the parenthetical statement in (a), it will suffice to show that $P$ is a minimal prime of $H$. If not, choose a strictly smaller prime $P_0$ of $R$ that is a minimal prime of $H$. Since $R_P \to S_Q$ is faithfully flat, there is a prime $Q_0$ strictly contained in $Q$ that lies over $P_0$. Then $R_{P_0} \to S_{Q_0}$ is faithfully flat, and since $H_{P_0} \neq 0$, we have that $H_{P_0} \otimes_{R_{P_0}} S_{Q_0} \neq 0$. But this is the same as $(S \otimes_R H)_{Q_0}$, contradicting the minimality of $Q$. If $S$ is faithfully flat over $R$, the statements in part (b) follow because if $P$ is an associated (respectively, minimal) prime of $H$, it is the contraction of any minimal prime $Q$ of $PS$ (and if $P$ is a minimal prime of $H$, this choice of $Q$ will be a minimal prime of $S \otimes_R H$). (c) follows from (a) and the fact that when $S$ is a localization of $R$, every ideal is the expansion of an ideal of $R$. (d) is proved for associated primes in [15, Theorem 1.1], where it is shown that $H = H^i_I(R)$ injects into $N = H^i_I(S)$. The hypothesis of (e) implies that $V(f_1, \ldots, f_h) = \{m_1, \ldots, m_k\}$, and the result is clear because any associated (respectively, minimal) prime of $H^i_I(R)$ that is not one of the $m_j$ must fail to contain one of the $f_j$ and so will correspond to one of the finitely many associated (respectively, minimal) primes of $H^i_I(R_{f_j})$. Part (f) is immediate from part (e). □

## 4. Local cohomology results via $R\langle F\rangle$-modules

In this section we proof our main result for hypersurfaces. We start by recalling the definition and properties of $R\langle F\rangle$-modules.

**Definition 4.1.** Let $R$ be a Noetherian ring of prime characteristic $p$. The ring $R\langle F\rangle$ is defined by $\frac{R\langle F\rangle}{R(r^pF - Fr \mid r \in R)}$, the non-commutative $R$-algebra generated by $F$ with relations $r^p \cdot F = F \cdot r$, for $r \in R$.

We note that an $R$-module is a left $R\langle F\rangle$-module if and only if $M$ has a *Frobenius action* (an additive map $F : M \to M$ such that $F(ru) = r^pF(u)$ for $u \in M$ and $r \in R$).

We now focus on $R\langle F\rangle$-modules that are finitely generated. This is because these modules have a closed support over $R$ (see Corollary 4.6).

**Remark 4.2.** The finite direct sum of finitely generated $R\langle F\rangle$-modules is a finitely generated $R\langle F\rangle$-module. The image of a finitely generated $R\langle F\rangle$-module under a map of $R\langle F\rangle$-modules is again a finitely generated $R\langle F\rangle$-module.

**Example 4.3.** For every element $f \in R$, $R_f$ is generated by $\frac{1}{f}$ as an $R\langle F\rangle$-module.

**Lemma 4.4.** *Let $R$ be a Noetherian ring of prime characteristic $p$ and $W \subseteq R$ be a multiplicative system. Suppose that $M$ is finitely generated as an $R\langle F\rangle$-module by $v_1, \dots, v_\ell$. Then, $W^{-1}M$ is finitely generated as a $W^{-1}R\langle F\rangle$-module by the images of $v_1, \dots, v_\ell$ under the localization map.*

*Proof.* We have that for every element in $\frac{v}{s} \in W^{-1}M$, there exist $\delta_i \in R\langle F\rangle$ such that

$$v = \delta_1 v_1 + \cdots + \delta_\ell v_\ell.$$

Then,

$$\frac{v}{s} = \frac{1}{s}\delta_1 v_1 + \cdots + \frac{1}{s}\delta_\ell v_\ell,$$

where $\frac{1}{s}\delta_i \in W^{-1}R\langle F\rangle$. □

**Lemma 4.5.** *Let $R$ be a Noetherian ring of prime characteristic $p$. Suppose that $M$ is finitely generated as an $R\langle F\rangle$-module by $v_1, \dots, v_\ell$. Let $N$ be the $R$-submodule of $M$ generated by $v_1, \dots, v_\ell$. Then,* $\operatorname{Supp}_R M = \operatorname{Supp}_R N$.

*Proof.* Since $N \subseteq M$, $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. We will prove that

$$\operatorname{Spec} R \setminus \operatorname{Supp}_R N \subseteq \operatorname{Spec} R \setminus \operatorname{Supp}_R M.$$

Suppose that $P \notin \operatorname{Supp}_R N$. We have that $N_P = 0$. By Lemma 4.4, we have that $0 = R_P\langle F\rangle \cdot N_P = M_P$. Hence, $P \notin \operatorname{Supp}_R M$. □

Using the previous lemma, we can give a different proof of a known fact about local cohomology [8].

**Corollary 4.6.** *Let $R$ be a Noetherian ring of prime characteristic $p$. Let $f_1, \dots, f_\ell \in R$ and $I = (f_1, \dots, f_\ell)$. Then,* $\operatorname{Supp} H^\ell_I(R)$ *is a Zariski closed set.*

*Proof.* We have by Example 4.2 that $R_{f_1 \cdots f_\ell}$ is a finitely generated $R\langle F\rangle$-module. Since $H^\ell_I(R)$ is a quotient of $R_{f_1 \cdots f_\ell}$ by an $R\langle F\rangle$-submodule, $H^\ell_I(R)$

is also finitely generated as an $R\langle F\rangle$-module. The result follows from Lemma 4.5. ☐

We point out that, to the best of our knowledge, the corresponding result is not known in equal characteristic zero even when $\ell = 3$.

**Remark 4.7.** Given a morphism $\varphi : R \to S$ of rings and a sequence $\underline{f} = f_1, \ldots, f_\ell \in R$, we have morphisms

$$H^i(\underline{f};R) \to H^i(\underline{f};R) \otimes_R S \to H^i(\underline{f};S). \tag{4.7.1}$$

If $\varphi$ is the Frobenius endomorphsim, the composition map in 4.7.1 induces maps

$$H^i(\underline{f}^{p^e};R) \to H^i(\underline{f}^{p^{e+1}};R).$$

for all $e$. These maps induce the Frobenius action $H^i_I(R) \to H^i_I(R)$ under the limit in 2.0.1.

We now show a lemma that implies that local cohomology over regular rings are finitely generated $R\langle F\rangle$-modules.

**Lemma 4.8.** *Let $R$ be a Noetherian ring of prime characteristic $p$. Let $I \subseteq R$ be an ideal generated by $\underline{f} = f_1, \ldots, f_\ell$. Suppose that the natural map $F^{e*}H^i(\underline{f};R) \to H^i_I(\underline{f}^{p^e};R)$ is surjective for every positive integer $e$. Then, the image of $H^i(\underline{f};R) \to H^i_I(R)$ generates $H^i_I(R)$ as an $R\langle F\rangle$-module. In particular, $H^i_I(R)$ is a Zariski closed set.*

*Proof.* Let $v \in H^i_I(R)$. then, there exists $e \in \mathbb{N}$ and $v_e \in H^i(\underline{f}^{p^e};R)$ such that $v_e \mapsto v$, because $H^i_I(R) = \lim\limits_{e\to\infty} H^i(\underline{f}^{p^e};R)$.

We have a commutative diagram

$$\begin{array}{ccccccc} H^i(\underline{f};R) & \longrightarrow & H^i(\underline{f}^{p};R) & \longrightarrow & H^i(\underline{f}^{p^2};R) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \downarrow{\scriptstyle F^e} & & \\ H^i(\underline{f}^{p^e};R) & \longrightarrow & H^i(\underline{f}^{p^{e+1}};R) & \longrightarrow & H^i(\underline{f}^{p^{e+2}};R) & \longrightarrow & \cdots, \end{array}$$

where the morphisms on the rows are given by the limit in equation 2.0.1. By taking limits we obtain the natural action of the $i$-th iterated Frobenius map in $F^e : H^i_I(R) \to H^i_I(R)$.

Since $F^{e*}H^i(\underline{f};R) \to H^i(\underline{f}^{p^e};R)$ is surjective, we have that the image of

$$H^i(\underline{f};R) \to F^{e*}H^i(\underline{f};R) \to H^i(\underline{f}^{p^e};R) \tag{4.8.1}$$

generates $H^i(\underline{f}^{p^e};R)$ as $R$-module. We note that the map in 4.8.1 is the Frobenius map $H^i(\underline{f};R) \to H^i(\underline{f}^{p^e};R)$ seen in Remark 4.7. There exist $w_1$, $\ldots, w_r \in H^i(\underline{f};R)$ and $g_1, \ldots, g_r \in R$ such that

$$v_e = g_1 F^e w_1 + \cdots + g_r F^e w_r.$$

Therefore, $v \in R\langle F\rangle \cdot M$, where $M$ is the image of $H^i(\underline{f};R)$ in $H^i_I(R)$. □

**Proposition 4.9.** *Let $S$ be a Noetherian regular ring of prime characteristic $p$. Let $\underline{f} = f_1, \ldots, f_\ell \in S$ be a sequence in $S$. Let $I \subseteq S$ be the ideal generated by $\underline{f}$. The image of $H^i(\underline{f};S) \to H^i_I(S)$ generates $H^i_I(S)$ as an $S\langle F\rangle$-module. In particular, $H^i_I(S)$ is finitely generated as an $S\langle F\rangle$-module.*

*Proof.* Since $S$ is regular, $F^*H^i(\underline{f}^{p^e};S) = H^i(\underline{f}^{p^{e+1}};S)$ for every positive integer $e$. The rest follows from Lemma 4.8. □

Thanks to Proposition 4.9, we can give a different proof of a well-known fact about local cohomology over regular rings of positive characteristic [4, 12].

**Corollary 4.10.** *Let $S$ be a Noetherian regular ring of prime characteristic $p > 0$. $\operatorname{Supp}_S H^i_I(S)$ is a Zariski closed set for every ideal $I \subseteq S$.*

*Proof.* This is an immediate consequence of Lemma 4.5 and Proposition 4.9. □

**Remark 4.11.** The idea of using finitely generated $R\langle F\rangle$-modules can also be used to prove the following statement [3, Theorem 4.1]: "Let $I \subseteq R$ be an ideal generated by $\underline{f} = f_1, \ldots, f_\ell$. Suppose that $\operatorname{pd} H^i(\underline{f};R) < \infty$ for $i > j$. Then $H^i_I(R)$ is a Zariski closed set for $i \geq j$." The Claim 1, the first part of their proof, implies that $F^{e*}H^i(\underline{f};R) = H^i(\underline{f}^{p^e};R)$. The result then follows from Lemmas 4.5 and 4.8.

We now present a sufficient condition on an ideal $I$ in a regular ring $S$ to guarantee that local cohomology modules over $R = S/I$ have closed support. This condition holds, in particular, for principal ideals.

**Theorem 4.12.** *Let $S$ be a Noetherian regular ring of prime characteristic $p > 0$. Let $J \subseteq S$, $R = S/J$, and $I \subseteq S$. Suppose that $\operatorname{Ass}_S H_I^{i+1}(J)$ is finite. Then, $\operatorname{Supp}_R H_I^i(R)$ is a Zariski closed set. Hence, if $\operatorname{Ass}_S H_I^{i+1}(J)$ is finite for all $i$ and all ideals $I$ of $S$, then $\operatorname{Supp}_R H_I^i(R)$ is closed for all $i$ and $I$.*

*Proof.* Since $S$ is a finite product of domains, we reduce at once to the domain case. We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow F & & \downarrow F & & \\
0 & \longrightarrow & J & \xrightarrow{f} & S & \longrightarrow & R & \longrightarrow & 0.
\end{array}$$

This induces a commutative diagram in the local cohomology

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_I^i(J) & \longrightarrow & H_I^i(S) & \longrightarrow & H_I^i(R) & \longrightarrow & H_I^{i+1}(J) & \longrightarrow \cdots \\
 & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & \\
\cdots \longrightarrow & H_I^i(J) & \xrightarrow{f} & H_I^i(S) & \longrightarrow & H_I^i(R) & \longrightarrow & H_I^{i+1}(J) & \xrightarrow{f} \cdots .
\end{array}$$

We have that $H_I^i(S) \to H_I^i(R)$ is a map of $S\langle F\rangle$-modules. Let $M = \operatorname{Im}(H_I^i(S) \to H_I^i(R))$. We have that $M$ is finitely generated as an $S\langle F\rangle$-module by Proposition 4.9. Then, $M$ has a closed support by Proposition 4.5. Let $N = \operatorname{Im}(H_I^i(R) \to H_I^{i+1}(J))$. We know that $\operatorname{Ass}_S N \subseteq \operatorname{Ass}_S H_I^{i+1}(J)$ is finite by assumption, and so, $\operatorname{Supp}_S N$ is a Zariski closed set.

Since $0 \to M \to H_I^i(R) \to N \to 0$ is a short exact sequence, we have that

$$\operatorname{Supp}_R H_I^i(R)) = \operatorname{Supp}_S H_I^i(R)) = \operatorname{Supp}_S M \bigcup \operatorname{Supp}_S N,$$

which is a Zariski closed set. □

As a consequence of the previous theorem, we obtain our first main result.

**Corollary 4.13.** *Let $S$ be a Noetherian regular ring of prime characteristic $p$. Let $f \in S$ be a nonzero divisor, $R = S/fS$ and $I \subseteq R$. Then $\operatorname{Supp}_R H_I^i(R)$ is a Zariski closed subset of $\operatorname{Spec} R$ for every ideal $I \subseteq R$.*

*Proof.* Since $S$ is a regular ring in positive characteristic, we have that $H_I^i(fS) \cong H_I^i(S)$ has a finite set of associated primes [4, 12]. Then, the claim follows immediately from Theorem 4.12. □

Theorem 4.12 also gives a different proof of a known result for rings of small embedding dimension [3, Proposition 3.4].

**Corollary 4.14.** *Let $(R, m, K)$ be a local ring of prime characteristic $p$ and embedding dimension at most 5. Then, $\operatorname{Supp}_R H^i_I(R)$ is a Zariski closed set for every ideal $I \subseteq S$.*

*Proof.* We may assume that $R$ is complete. Then there exists a surjective map $\pi : S = K[[x_1, \ldots, x_5]] \to R$, where $K$ is a field of prime characteristic. Let $J = \pi^{-1}(0)$. Then $J$ is a torsion-free $S$-module and $\operatorname{Ass}_S H^{i+1}_I(J)$ is finite for every $i \geq 0$ [14, Theorem 2.11]. The result follows from Theorem 4.12. □

We now reduce the problem of closed support of local cohomology over Cohen-Macaulay ring to study the one specific torsion module.

**Proposition 4.15.** *Let $(R, m, K)$ be a complete Cohen-Macaulay local ring of prime characteristic $p > 0$. Let $S \to R$ be any surjection from a regular local ring $S$, and $I$ be the kernel of this map. Suppose that $\operatorname{Tor}^S_1(R, H^\ell_J(S))$ has finite support for every ideal, $J$, generated by $\ell$-elements for every $\ell \geq 1$. Then $H^i_{\mathfrak{a}}(R)$ has finite support for every ideal $\mathfrak{a} \subseteq R$.*

*Proof.* It suffices to prove that $H^{\ell-1}_J(R)$ has closed support for every ideal $J$ generated by $\ell$ elements [3, Corollary 5.6].

The inclusion map $I \to S$ induces a map $H^\ell_J(I) \to H^\ell_J(S)$. Since $J$ is generated by $\ell$ elements, we have that $H^\ell_J(I) = H^\ell_J(S) \otimes_S I$. In addition, $H^\ell_J(I) \to H^\ell_J(S)$ is the map given by tensoring $I \to S$ with $H^\ell_J(S)$. From the short exact sequence $0 \to I \to S \to R \to 0$, we obtain a long exact sequence

$$0 \to \operatorname{Tor}^S_1(R, H^\ell_J(S)) \to H^\ell_J(S) \otimes_S I \to H^\ell_J(S) \otimes_S S \to H^\ell_J(S) \otimes_S R \to 0.$$

This is the same sequence as

$$0 \to \operatorname{Tor}^S_1(R, H^\ell_J(S)) \to H^\ell_J(I) \to H^\ell_J(S) \to H^\ell_J(R) \to 0.$$

Then,

$$\operatorname{Tor}^S_1(R, H^\ell_J(S)) = \operatorname{Ker}(H^\ell_J(I) \to H^\ell_J(S)).$$

From the short exact sequence $0 \to I \to S \to R \to 0$, we also get a long exact sequence

$$\cdots \to H^i_J(I) \to H^i_J(S) \to H^i_J(R) \to \cdots$$

Let $N = \operatorname{Im}(H^{\ell-1}_J(S) \to H^{\ell-1}_J(R))$. We have that $N$ has a closed support because it is a finitely generated $R\langle F\rangle$-module, since it is a quotient of a finitely generated $R\langle F\rangle$-module by Lemma 4.5. We have a short exact sequence

$$0 \to N \to H^{\ell-1}_J(R) \to \operatorname{Tor}^S_1(R, H^{\ell}_J(S)) \to 0.$$

Hence,

$$\operatorname{Supp}_S H^{\ell-1}_J(R) = \operatorname{Supp}_S N \bigcup \operatorname{Supp}_S \operatorname{Tor}^S_1(R, H^{\ell}_J(S)),$$

which is a Zariski closed subset of $\operatorname{Spec} S$ and so, closed in $\operatorname{Spec} R$. □

## 5. Rings with finite *F*-representation type

In this section we present our main result regarding rings with finite $F$-representation type. We start by recalling basic definitions and properties of a functor which, roughly speaking, takes the $p^e$-roots of a module.

**Notation 5.1.** Let $R$ be a Noetherian ring of prime characteristic $p$ and $M$ an $R$-module. We denote by ${}^eM$ the module in which the underlying set is $M$. The addition in ${}^eM$ is the same as in $M$, but $R$ acts via Frobenius. This is $r \cdot v = r^{p^e} v$ for every $v \in {}^eM$.

We now recall a well-known result, which we include for the sake of completeness.

**Lemma 5.2.** *Let $R$ be a Noetherian ring of characteristic $p$ and $M$ an $R$-module. Then,* $\operatorname{Ass}_R {}^eM = \operatorname{Ass}_R M$.

*Proof.* We have that ${}^e(M_Q) \cong ({}^eM)_Q$ for every prime ideal $Q \subseteq R$. Then, we have that ${}^e(H^0_I(M)) = H^0_I({}^eM)$ for every ideal $\subseteq R$. In addition, ${}^eM \neq 0$ if and only if $M \neq 0$. Then

$$\begin{aligned}\operatorname{Ass}_R {}^eM &= \{Q \in \operatorname{Spec} R | H^0_Q({}^eM)_Q \neq 0\} \\ &= \{Q \in \operatorname{Spec} R | {}^e(H^0_Q(M)_Q) \neq 0\} \\ &= \{Q \in \operatorname{Spec} R | H^0_Q(M)_Q \neq 0\} = \operatorname{Ass}_R M.\end{aligned}$$

□

**Remark 5.3.** If $R$ is a reduced ring, we denote by $R^{1/p^e} = \{r^{1/p^e} \mid r \in R\}$ the rings of $p^e$-roots. If $M$ is an $R$-module, then $M^{1/p^e} = \{v^{1/p^e} \mid v \in M\}$ is an $R^{1/p^e}$-module with the following rules:

$$v^{1/p^e} + w^{1/p^e} = (v+w)^{1/p^e} \quad \text{and} \quad r^{1/p^e} v^{1/p^e} = (rv)^{1/p^e}.$$

Furthermore, $M^{1/p^e}$ is an $R$-module via restriction of scalars, and $M^{1/p^e} \cong {}^eM$ as $R$-modules.

We now recall the definition of rings with finite $F$-representation type, which are the main objects for this section. The original definition requires that $R$ satisfies the Krull-Schmidt condition. However, by an small variation of the definition one can drop this assumption [28].

**Definition 5.4 ([26, 28]).** Let $R$ be an $F$-finite ring. We say an $R$-module, $M$, has finite $F$-representation type, FFRT for short, if there exist a finite set of finitely generated $R$-modules, $N_1, \ldots, N_\ell$ such that for every $e \in \mathbb{N}$ there exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{N}$ such that

$${}^eM = \{{}^ev \mid v \in M\} \cong N_1^{\alpha_1} \oplus \cdots \oplus N_\ell^{\alpha_\ell}.$$

If $R$ has FFRT as $R$-module, we say that $R$ has FFRT.

**Example 5.5.** The following are examples of rings that have FFRT, provided they are $F$-finite (see [27, Example 1.3]):

- Complete regular local rings,
- Quotients of polynomial rings over monomial ideals,
- Artinian local rings,
- All normal monoid rings,
- The determinantal ring of $2 \times 2$ minors of a generic matrix,
- Polynomial or power series rings over rings with FFRT, and
- Suppose that $R \to S$ splits, $S$ is finitely generated as an $R$-module. If $S$ has FFRT, then $R$ also has FFRT.

We now give a preparation lemma needed for our main result on rings with FFRT. This lemma requires that $R$ is reduced. However, this assumption is not needed for Theorem 5.7.

**Lemma 5.6.** *Let $R$ be an $F$-finite Noetherian reduced ring of positive characteristic $p$, and let $M$ be an $R$-module with FFRT. Then, $\operatorname{Ass}_R H^i_I(M)$ is finite for every ideal $I \subseteq R$ and $i \in \mathbb{N}$.*

*Proof.* Let $\underline{f} = f_1, \ldots, f_s \in R$ be a sequence of elements such that $I = (f_1, \ldots, f_s)$. Let $\underline{f}^t$ denote the sequence $f_1^t, \ldots, f_s^t$ and $\mathcal{K}(\underline{f}^t; R)$ the associated Koszul complex. We have a ring isomorphism $R^{1/p^e} \to R$ defined by $r^{1/p^e} \to r$. This induces an equivalence of categories $G : R^{1/p^e} - \mathrm{Mod} \to R - \mathrm{Mod}$. In particular, $G(M^{1/p^e}) = M$. We have that

$$G(H^i(\mathcal{K}(\underline{f}; M^{1/p^e}))) = H^i(\mathcal{K}(\underline{f}^{p^e}; M)).$$

We have that for any $R^{1/p}$-module, $N$, $\mathrm{Ass}_R\, G(N) = \mathrm{Ass}_R\, N$ by Lemma 5.2 and Remark 5.3.

Let $N_1, \ldots, N_\ell$ be finitely generated $R$-modules such that for all $e \in \mathbb{N}$ there exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{N}$ with

$$M^{1/p^e} = \{v^{1/p^e} \mid v \in M\} \cong N_1^{\alpha_1} \oplus \cdots \oplus N_\ell^{\alpha_\ell}.$$

We have that

$$\mathrm{Ass}_R\, H^i(\mathcal{K}(\underline{f}; M^{1/p^e})) \subseteq \bigcup_{j=1}^{\ell} \mathrm{Ass}_R\, H^i(\mathcal{K}(\underline{f}; N_j)),$$

and

$$H_I^i(M) = \lim_{e\to\infty} H^i(\mathcal{K}(\underline{f}^{p^e}; M)).$$

Then,

$$\begin{aligned} \mathrm{Ass}_R\, H_I^i(M) \subseteq \bigcup_{e\in\mathbb{N}} \mathrm{Ass}_R\, H^i(\mathcal{K}(\underline{f}^{p^e}; M) &= \bigcup_{e\in\mathbb{N}} \mathrm{Ass}_R\, H^i(\mathcal{K}(\underline{f}; M^{1/p^e}) \\ &= \bigcup_{j=1}^{\ell} \mathrm{Ass}_R\, H^i(\mathcal{K}(\underline{f}; N_j)). \end{aligned}$$

Hence, $\mathrm{Ass}_R\, H_I^i(R)$ is finite. □

We are now ready to show our main result in this section.

**Theorem 5.7.** *Let $R$ be an $F$-finite Noetherian ring of prime characteristic $p$, and let $M$ be an $R$-module with FFRT. Then $\mathrm{Ass}_R\, H_I^i(M)$ is finite for every ideal $I \subseteq R$ and $i \in \mathbb{N}$. In particular, if $R$ has FFRT, then $\mathrm{Ass}_R\, H_I^i(R)$ is finite for every ideal $I \subseteq R$ and $i \in \mathbb{N}$.*

*Proof.* Let $R^{p^e}$ denote the image of $F^e : R \to R$. Since $R$ is Noetherian, there exists an $\ell \in \mathbb{N}$ such that $R^{p^\ell}$ is a reduced ring. We note that $M$ is also an

$R^{p^\ell}$-module with FFRT. Let $J = F^\ell(I) \subseteq R^{p^\ell}$, which is an ideal. We have that $\operatorname{Ass}_{R^{p^\ell}} H^i_J(M)$ is finite by Lemma 5.6. We note that $H^i_J(M) \cong H^i_I(M)$ as $R^{p^\ell}$-modules. For every $R$-module, $N$, there is a bijection between $\operatorname{Ass}_R M$ and $\operatorname{Ass}_{R^{p^\ell}} M$ given by $Q \mapsto Q \cap R^{p^\ell}$. Then, $\operatorname{Ass}_R H^i_I(M)$ is finite. □

Takagi and Takahashi [27] prove that local cohomology modules of rings with FFRT have a finite set of associated primes under the additional assumption that $R$ is a Gorenstein ring.

**Corollary 5.8.** *Let $K$ be any field of positive characteristic. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $I$ be a monomial ideal, and $R = S/I$. Then, the set of associated primes of $H^i_J(R)$ is finite for every ideal $J \subseteq R$ and $i \in \mathbb{N}$.*

*Proof.* Let $\overline{K}$ denote the algebraic closure of $K$. Let $T = \overline{K}[x_1, \ldots, x_n]$ and $R' = T/IT$. Any inclusion $K \to \overline{K}$ induces a map of rings $R \to R'$ such that $R'$ is a free $R$-module. We have that $R'$ is an $F$-finite graded ring with FFRT, and all local cohomology modules $H^i_{J'}(R')$ have finitely many associated primes for every ideal $J' \subseteq R'$ by Theorem 5.7 and Example 5.5 (see also [5, Example 2.36]). Hence, all modules $H^i_J(R)$ has also finitely many primes for every ideal $J \subseteq R$, because $R'$ faithfully flat of $R$-algebra (see Theorem 3.1(b)). □

## 6. Determinantal rings

### 6.1. A relative conjecture on finiteness of associated primes

In this section we focus on a conjecture made by the first author regarding extensions with regular fibers.

**Conjecture 6.1.** *Let $S$ be smooth over a Noetherian ring $R$ and let $P$ be a prime ideal of $R$. Suppose that $S_P$ is flat over $R_P$ and the fiber $(R_P/PR_P) \otimes_R S$ is regular. Then for all integers $i$ and all ideals $J$ of $S$, the set of associated primes $\operatorname{Ass}_S\big(H^i_J(S)\big)$ that lie over $P$ is finite.*

We refer to the case where $S$ is assumed to be a polynomial ring over $R$ (which immediately implies the case where $S$ is a localization of a polynomial ring over $R$) as the *polynomial case* of Conjecture 6.1.

One may replace $R \to S$ by $R_P \to S_P$ here, one may assume that $(R, P)$ is local. Thus, it suffices if $S_P$ is smooth over $R_P$.

If $(R, P)$ is local or, more generally, if $P$ is maximal, it is equivalent to assert that the set of associate primes of $H^0_P\big(H^i_J(S)\big)$ is finite.

There are some positive results about this conjecture [16, 18–20, 22].

In the special case where $R$ has isolated singularities and is either a ring of characteristic $p$ or an algebra essentially of finite type over a field of characteristic 0, this would imply that when $S$ is a localization of a polynomial ring over $R$, then $\mathrm{Ass}_S\big(H^i_J(S)\big)$ is finite if the conjecture holds. The argument is simple: let $(g_1, \ldots, g_h)R$ be the defining ideal of the singular locus in $R$, so that $V(g_1, \ldots, g_h) = \{m_1, \ldots, m_k\}$. For each $g_i$, there are only finitely many associated primes of $H^i_J(S)$ that do not contain $g_i$, because $S_{g_i}$ is regular. Any other associated prime contains $(g_1, \ldots, g_h)$ and so must lie over one of the $m_t$, and for each $m_t$ there are only finitely many of these by the special case of Conjecture 6.1 that we are assuming.

In the next section we show that Conjecture 6.1, in the case where $S$ is a localization of a polynomial ring over $R$, implies that the set of associated primes of the local cohomology of a generic determinantal ring over a field is finite.

## 6.2. Local cohomology over determinantal rings

Let $L$ denote any ring and let $X$ be an $r \times s$ matrix of variables over $L$ Let $L[X]$ be a polynomial ring over $L$ with the entries of $X$ as variables. Given a matrix $A$ with entries in a ring, $I_t(A)$ denotes the ideal generated by the $t \times t$ minors of $A$.

**Remark 6.2.** Let $L$ be a field. In the characteristic 0 case, $R = L[X]/I_t(X)$ is a direct summand of a polynomial ring, and this is true in characteristic $p > 0$ as well if $t = 2$. In fact, when $t = 2$, we have not only that $L[X]/I_2(X)$ is a direct summand of a polynomial ring: it is, in fact, given by a Segre product of two polynomial rings. In characteristic 0 or when $t = 2$, $R$ remains a direct summand of a polynomial ring if we adjoin some indeterminates or if we do that and localize. Note also that if $B$ be any regular algebra finitely generated over $L$. Then, $T = B[X]/I_t(X) = B \otimes_L L[X]/I_t(X)$ is a direct summand of a polynomial ring, $S$ over $B$ (in characteristic 2 or if $t = 2$), and that $S$ is also a regular algebra finitely generated over $L$. Hence, for every ideal $J \subseteq T$, $\mathrm{Ass}_T\, H^i_J(T)$ is finite [15].

**Discussion 6.3.** Let $L$ be any base ring and let $X = (x_{i,j})$ be an $r \times s$ matrix of indeterminates over $L$. Let $Y$ be an $(r-1) \times (s-1)$ matrix of inderterminates over $L$. Suppose that $2 \le t \le \min\{r, s\}$. Let $R = L[X]/I_t(X)$

and let $x$ be any entry of $X$. Then $R_x$ is isomorphic to the localization at one of the variables, namely $x$, of a polynomial ring over $L[Y]/I_{t-1}(Y)$. By symmetry, it suffices to see this when $x = x_{11}$. Multiply the first row of $X$ by $1/x$, and then subtract multiples of the first column from the others so that as to make the rest of the entries of the first row equal to 0. Then subtract multiples of the first row from the the other rows so as to make the rest of the first column 0. The matrix then has the block form $\begin{bmatrix}1 & 0\\ 0 & Y\end{bmatrix}$ where $y_{i,j} = x_{i+1,j+1} - x_{i+1,1}\frac{x_{1,j+1}}{x}$. Then $R_x \cong S = L[Y][x_{1,j}, x_{i,1} : i, j]_x$, where the $y_{ij}$ and the variables $x_{1,j}$, $x_{i,1}$ in the first row and column are algebraically independent, since one can recover the original matrix $X$ from them by reversing the row and column operations. Moreover, $I_t(X)S = I_{t-1}(Y)S$, so that $S$ is a localized polynomial ring over $R_0 = L[Y]/I_{t-1}(Y)$.

We next observe that if one has Conjecture 6.1, then it follows that when $L$ is a field of characteristic $p$, $H^i_I\big(L[X]/I_t(X)\big)$ has finitely many associated primes for every integers $i, t$, and ideal $I$. Moreover, when $t = 3$, we can prove this claim without assuming Conjecture 6.1. Note that we only need to give this argument in positive characteristic. However, the argument works

**Lemma 6.4.** *Let notation be as in Discussion 6.3, and assume that L is a field. Let $R_{r,s,t} = L[X]/I_t(X)$. If the set of associated primes of any local cohomology module of a ring that is locally polynomial over $R_{r-1,s-1,t-1}$ is finite, then the $\operatorname{Ass}_{R_{r,s,t}} H^i_I(R_{r,s,t})$is finite for every ideal $I \subseteq R_{r,s,t}$ and $i \in \mathbb{N}$. If Conjecture 6.1 holds for localized polynomial rings over $R_{r,s,t}$, then the set of associated primes of any local cohomology module of a localized polynomial ring over $R_{r,s,t}$ is finite.*

*Proof.* We may assume $t \geq 3$. In the case of $R = R_{r,s,t}$ itself, either the associated prime contains all the $x_{i,j}$, in which case it is the maximal ideal they generate, or it fails to contains some $x_{i,j}$. But for each choice of $i, j$ there are only finitely many of these, because when we localize at $x = x_{i,j}$, $R$ becomes isomorphic to a localized polynomial ring over $R_{r-1,s-1,t-1}$, and so there are only finitely many associated prime that do not contain a given $x_{ij}$.

If $S$ is a localized polynomial ring over $R_{r,s,t}$, it follows just as above that there are only finitely many associated primes that do not contain a given $x_{i,j}$. Therefore, we only need to show the finiteness of the set of associated primes which contain all the $x_{i,j}$, and so lie over the maximal ideal that the $x_{i,j}$ generate. This is immediate from Conjecture 6.1 for localized polynomial rings over $R_{r,s,t}$. □

By Remark 6.2, we have at once:

**Theorem 6.5.** *Let $L$ be any field, and let $X$ be a matrix of indeterminates over $L$. Let $R = L[X]/I_3(X)$. Then,* $\operatorname{Ass}_R H^i_J(R)$ *is finite for every ideal $J \subseteq R$.* □

*Proof.* This follows immediately from Remark 6.2 and Lemma 6.4. □

**Remark 6.6.** This is new in positive characteristic. Note that we do not know whether this is true for polynomial rings over $R = L[X]/I_3(X)$ in positive characteristic, although that is true in equal characteristic 0.

By induction on $t$, we also have:

**Theorem 6.7.** *Let $L$ be any field, and let $X$ be a matrix of indeterminates over $L$. Assume that Conjecture 6.1 holds when $S$ is a localized polynomial ring over $R$. Let $R = L[X]/I_t(X)$, and let $S$ be a localized polynomial ring over $R$. Then,* $\operatorname{Ass}_R H^i_J(R)$ *is finite for every ideal $J \subseteq R$.* □

*Proof.* This follows from an straightforward induction on $t$ given Remark 6.2 and Lemma 6.4. □

In fact, we can generalize this as follows. By recursion on $k$, define a Noetherian ring $R$ to have singularities that are $k$-isolated as follows. $R$ has singularities that are *1-isolated* if it is has isolated singularities, and $R$ has singularities that are $k+1$-isolated if, after omitting a set $T$ finitely many closed points, $\operatorname{Spec}(R) \setminus T$ has an open cover (which may always be taken to be finite) by $\operatorname{Spec}(S_i)$ such that $S_i$ is smooth over a ring $R_i$ that has singularities that are $k$-isolated.

For any field $L$, $L[X]/I_t(X)$ has singularities that are $(t-1)$-isolated by Discussion 6.3. Moreover, by Theorem 3.1 part (e), Conjecture 6.1, and a straightforward induction, we have the following more general result:

**Theorem 6.8.** *If Conjecture 6.1 holds, $R$ is an affine algebra over a field, and $R$ has $k$-isolated singularities for some choice of $k$, then* $\operatorname{Ass}_R H^i_J(R)$ *is finite for every $i$ and every ideal $J \subseteq R$.*

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