

The quantum divided power algebra of a finite-dimensional Nichols algebra of diagonal type

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Let \mathcal{B}_q be a finite-dimensional Nichols algebra of diagonal type corresponding to a matrix q . We consider the graded dual \mathcal{L}_q of the distinguished pre-Nichols algebra $\tilde{\mathcal{B}}_q$ from [A3] and the quantum divided power algebra \mathcal{U}_q , a suitable Drinfeld double of $\mathcal{L}_q \# \mathbf{k}\mathbb{Z}^\theta$. We provide basis and presentations by generators and relations of \mathcal{L}_q and \mathcal{U}_q , and prove that they are noetherian and have finite Gelfand-Kirillov dimension.

1. Introduction

We fix an algebraically closed field \mathbf{k} of characteristic zero. Let \mathfrak{g} be a finite-dimensional simple Lie algebra and $q \in \mathbf{k}$ a root of 1 (with some restrictions depending on \mathfrak{g}). In the theory of quantum groups, there are several Hopf algebras attached to \mathfrak{g} and q :

- The Frobenius-Lusztig kernel (or small quantum group) $u_q(\mathfrak{g})$.
- The q -divided power algebra $\mathcal{U}_q(\mathfrak{g})$, see [L1, L2].
- The quantized enveloping algebra $U_q(\mathfrak{g})$, see [DK, DKP, DP].

These Hopf algebras have the following features:

- ◊ They admit triangular decompositions, e. g. $u_q(\mathfrak{g}) \simeq u_q^+(\mathfrak{g}) \otimes u_q^0(\mathfrak{g}) \otimes u_q^-(\mathfrak{g})$.
- ◊ The 0-part of this triangular decomposition is a Hopf subalgebra, actually a group algebra.

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- ◇ The positive and negative parts are not Hopf subalgebras, but rather Hopf algebras in braided tensor categories, braided Hopf algebras for short.
- ◇ There are morphisms $\mathfrak{u}_q^+(\mathfrak{g}) \hookrightarrow \mathcal{U}_q^+(\mathfrak{g})$, $\mathcal{U}_q^+(\mathfrak{g}) \twoheadrightarrow \mathfrak{u}_q^+(\mathfrak{g})$ of braided Hopf algebras, and ditto for the full Hopf algebras.
- ◇ The full Hopf algebras can be reconstructed from the positive part by standard procedures (bosonization, the Drinfeld double).
- ◇ The positive part $\mathfrak{u}_q^+(\mathfrak{g})$ has very special properties— it is a Nichols algebra.

Indeed, $\mathfrak{u}_q^+(\mathfrak{g})$ is completely determined by the matrix $\mathfrak{q} = (q^{d_i a_{ij}})$, where (a_{ij}) is the Cartan matrix of \mathfrak{g} and $d_i \in \{1, 2, 3\}$ make $(d_i a_{ij})$ symmetric. In other words, $\mathfrak{u}_q^+(\mathfrak{g})$ is the Nichols algebra of diagonal type associated to \mathfrak{q} .

The knowledge of the finite-dimensional Nichols algebras of diagonal type is crucial in the classification program of finite-dimensional Hopf algebras [AS]. Two remarkable results on these Nichols algebras are:

- (a) The explicit classification [H2].
- (b) The determination of their defining relations [A1, A2].

Let $\mathfrak{q} \in \mathbf{k}^{\theta \times \theta}$ with Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ and assume that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. There are several reasons to consider the analogues of the braided Hopf algebras $U_q^+(\mathfrak{g})$ and $\mathcal{U}_q^+(\mathfrak{g})$, for $\mathcal{B}_{\mathfrak{q}}$, motivated by the classification of Hopf algebras with finite Gelfand-Kirillov dimension and by representation theory. The analogue $\tilde{\mathcal{B}}_{\mathfrak{q}}$ of $U_q^+(\mathfrak{g})$ was introduced in [A2] and studied in [A3] under the name of distinguished pre-Nichols algebra. The definition of $\tilde{\mathcal{B}}_{\mathfrak{q}}$ is by discarding some of the relations in [A2]. The purpose of this paper is to study the analogue $\mathcal{L}_{\mathfrak{q}}$ of $\mathcal{U}_q^+(\mathfrak{g})$; this is the graded dual of $\tilde{\mathcal{B}}_{\mathfrak{q}}$ and although it could be called the distinguished post-Nichols algebra of \mathfrak{q} , we prefer to name it the Lusztig algebra as in [A+], where mentioned in passing.

The paper is organized as follows. Section 2 is devoted to preliminaries and Section 3 to Nichols algebras of diagonal type and distinguished pre-Nichols algebras. In Section 4 we discuss Lusztig algebras: we provide a basis and a presentation by generators and relations, and prove that they are noetherian and have finite Gelfand-Kirillov dimension. In Section 5 we introduce the quantum divided power algebra $\mathcal{U}_{\mathfrak{q}}$, that is a suitable Drinfeld double of $\mathcal{L}_{\mathfrak{q}} \# \mathbf{k}\mathbb{Z}^{\theta}$; we also provide a presentation by generators and relations, and prove that it is noetherian and has finite Gelfand-Kirillov dimension.

Remark 1.1. The quantum divided power algebras were introduced and studied in [GH, Hu]; they correspond to Nichols algebras of Cartan type $A_1 \times \cdots \times A_1$.

2. Preliminaries and conventions

2.1. Conventions

If $\theta \in \mathbb{N}$, then we set $\mathbb{I}_\theta := \{1, 2, \dots, \theta\}$; or simply \mathbb{I} if no confusion arises. If Γ is a group, then $\widehat{\Gamma}$ is its group of characters, that is, one-dimensional representations.

Let \mathbb{S}_n and \mathbb{B}_n be the symmetric and braid groups in n letters, with standard generators $\tau_i = (i\ i + 1)$, respectively $\sigma_i, i \in \mathbb{I}_{n-1}$. Let $s : \mathbb{S}_\theta \rightarrow \mathbb{B}_\theta$ be the (Matsumoto) section of the projection $\pi : \mathbb{B}_\theta \rightarrow \mathbb{S}_\theta, \pi(\sigma_i) = \tau_i, i \in \mathbb{I}_{n-1}$, given by $s(\omega) = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_j}$, whenever $\omega = \tau_{i_1}\tau_{i_2}\dots\tau_{i_j} \in \mathbb{S}_\theta$ has length j .

We consider the \mathbf{q} -numbers in the polynomial ring $\mathbb{Z}[\mathbf{q}]$, $n \in \mathbb{N}, 0 \leq i \leq n$,

$$\binom{n}{\mathbf{q}} = \sum_{j=0}^{n-1} \mathbf{q}^j, \quad \binom{n}{\mathbf{q}}! = \prod_{j=1}^n (j)_{\mathbf{q}}, \quad \binom{n}{i}_{\mathbf{q}} = \frac{\binom{n}{\mathbf{q}}!}{(n-i)_{\mathbf{q}}! (i)_{\mathbf{q}}!}.$$

If $q \in \mathbf{k}$, then $(n)_q, \binom{n}{i}_q$ are the respective evaluations at q .

We use the Heynemann-Sweedler notation for coalgebras and comodules; the counit of a coalgebra is denoted by ε , and the antipode of a Hopf algebra, by \mathcal{S} . All Hopf algebras in this paper have bijective antipode.

Let H be a Hopf algebra. A *Yetter-Drinfeld module* V over H is a H -module and a H -comodule satisfying the compatibility condition

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, v \in V.$$

Morphisms of Yetter-Drinfeld modules preserve the action and the coaction. Thus Yetter Drinfeld modules over H form a braided tensor category ${}^H_H\mathcal{YD}$, with braiding $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, $V, W \in {}^H_H\mathcal{YD}, v \in V, w \in W$. The full subcategory of finite-dimensional objects is rigid.

2.2. Braided vector spaces and Nichols algebras

A braided vector space is a pair (V, c) where V is a vector space and $c \in \text{Aut}(V \otimes V)$ is a solution of the braid equation $(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$.

If V is a vector space, then we identify $V^* \otimes V^*$ with a subspace of $(V \otimes V)^*$ by $\langle f \otimes g, v \otimes w \rangle = \langle f, w \rangle \langle g, v \rangle$, for $v, w \in V, f, g \in V^*$.¹ If (V, c) is a finite-dimensional braided vector space, then (V^*, c^t) is its dual braided vector space, where $c^t : V^* \otimes V^* \rightarrow V^* \otimes V^*$ is

$$\langle c^t(f \otimes g), v \otimes w \rangle = \langle f \otimes g, c(v \otimes w) \rangle.$$

We refer to [T] for the basic theory of braided Hopf algebras. If $R = \bigoplus_{n \geq 0} R^n$ is a graded braided Hopf algebra with $\dim R^n < \infty$ for all n , then its graded dual $R^d = \bigoplus_{n \geq 0} (R^n)^*$ is again a graded braided Hopf algebra. We use the variation of the Sweedler notation $\Delta(X) = X^{(1)} \otimes X^{(2)}$ for the coproducts in braided Hopf algebras.

The *Nichols algebra* of a braided vector space (V, c) is a graded braided Hopf algebra $\mathcal{B}(V) = \bigoplus_{n \geq 0} \mathcal{B}^n(V)$ with very rigid properties. There are several alternative definitions of Nichols algebras, see [AS]. We recall now two of these definitions.

Let $T(V) = \bigoplus_{n \geq 0} T^n(V)$ be the tensor algebra of V ; it has a braiding c induced from V . Let $T(V) \underline{\otimes} T(V) = T(V) \otimes T(V)$ with the multiplication $(m \otimes m)(\text{id} \otimes c \otimes \text{id})$ and let $\Delta : T(V) \rightarrow T(V) \underline{\otimes} T(V)$ be the unique algebra map such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for all $v \in V$. Then $T(V)$ is a (graded) braided Hopf algebra with respect to Δ . Dually, consider the cotensor coalgebra $T^c(V)$ which is isomorphic to $T(V)$ as a vector space. It bears a multiplication making $T^c(V)$ a braided Hopf algebra with an analogous property, see e. g. [R, AG]. There exists only one morphism of braided Hopf algebras $\Theta : T(V) \rightarrow T^c(V)$ that it is the identity on V . The image of Θ is the Nichols algebra $\mathcal{B}(V)$ of V .

Here is the second description of $\mathcal{B}(V)$. Let \mathfrak{S} be the partially ordered set of homogeneous Hopf ideals of $T(V)$ with trivial intersection with $\mathbf{k} \oplus V$. Then \mathfrak{S} has a maximal element $\mathcal{J}(V)$ and $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ [AS].

2.3. Pre- and post-Nichols algebras

For several purposes, it is useful to consider braided Hopf algebras $T(V)/I$, for various $I \in \mathfrak{S}$. These are called *pre-Nichols algebras* [M]. Indeed, $\mathfrak{Pre}(V) = \{T(V)/I : I \in \mathfrak{S}\}$ is a poset with ordering given by the surjections; so that it is isomorphic to $(\mathfrak{S}, \subseteq)$. The minimal element in $\mathfrak{Pre}(V)$ is $T(V)$, and the maximal is $\mathcal{B}(V)$. Dually, the poset $\mathfrak{Post}(V)$ consists of

¹We prefer this identification instead of $\langle f \otimes g, v \otimes w \rangle = \langle f, v \rangle \langle g, w \rangle$ because it gives the right extension to tensor categories.

graded Hopf subalgebras $S = \bigoplus_{n \geq 0} S^n$ of $T^c(V)$ such that $S^1 = V$, ordered by the inclusion. Now the minimal element is $\mathcal{B}(V)$ and the maximal is $T^c(V)$. We shall call them *post-Nichols algebras*.

Remark 2.1. The map $\Phi : \mathfrak{Pre}(V) \rightarrow \mathfrak{Post}(V^*)$, $\Phi(R) = R^d$, is an anti-isomorphism of posets.

Proof. If $R = T(V)/I \in \mathfrak{Pre}(V)$, then $R^d = I^\perp$: hence, Φ is well-defined and it reverses the order. Also Φ is surjective, because for a given $S \in \mathfrak{Post}(V^*)$, $I = S^\perp$ is a graded Hopf ideal of $T(V)$ and $S = (T(V)/I)^d$. \square

3. Nichols algebras of diagonal type

A braided vector space (V, c) is of *diagonal type* if there exist a basis x_1, \dots, x_θ of V and a matrix $\mathfrak{q} = (q_{ij}) \in M_\theta(\mathbf{k}^\times)$ such that $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $i, j \in \mathbb{I} = \mathbb{I}_\theta$. Let $H = \mathbf{k}G$ be a group algebra, $\chi_i \in \widehat{G}$ and $g_j \in Z(G)$ such that $\chi_j(g_i) = q_{ij}$, $i, j \in \mathbb{I}$. Then (V, c) is realized in ${}^H_H\mathcal{YD}$ by $h \cdot x_i = \chi_i(h)x_i$ and $\rho(x_i) = g_i \otimes x_i$ for all $i \in \mathbb{I}$, $h \in H$. We will only consider the case when $H = \mathbf{k}\mathbb{Z}^\theta$, $g_i = \alpha_i$ and $\chi_j \in \widehat{\mathbb{Z}^\theta}$ is given by $\chi_j(\alpha_i) = q_{ij}$, $i, j \in \mathbb{I}$. Here $\alpha_1, \dots, \alpha_\theta$ is the canonical basis of \mathbb{Z}^θ .

Let $V^* \in {}^{\mathbf{k}\mathbb{Z}^\theta}{}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$; it is also a braided vector space of diagonal type, with matrix \mathfrak{q} . Indeed, if y_1, \dots, y_θ is the dual basis of x_1, \dots, x_θ , then

$$\begin{aligned} \langle c^t(y_i \otimes y_j), x_h \otimes x_k \rangle &= \langle y_i \otimes y_j, c(x_h \otimes x_k) \rangle = q_{hk} \langle y_i \otimes y_j, x_k \otimes x_h \rangle \\ &= q_{hk} \delta_{jk} \delta_{ih} = q_{ij} \langle y_j \otimes y_i, x_h \otimes x_k \rangle. \end{aligned}$$

Since $T(V)$ and $\mathcal{B}_\mathfrak{q} = \mathcal{B}(V)$ are Hopf algebras in ${}^{\mathbf{k}\mathbb{Z}^\theta}{}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$, we may consider the bosonizations $T(V) \# \mathbf{k}\mathbb{Z}^\theta$ and $\mathcal{B}_\mathfrak{q} \# \mathbf{k}\mathbb{Z}^\theta$. We refer to [AS, §1.5] for the definition of the adjoint action of a Hopf algebra, respectively the braided adjoint ad_c action of a Hopf algebra in ${}^{\mathbf{k}\mathbb{Z}^\theta}{}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. Then $\text{ad}_c x \otimes \text{id} = \text{ad}(x \# 1)$ if $x \in T(V)$ or $\mathcal{B}_\mathfrak{q}$, see [AS, (1-21)].

Now the matrix \mathfrak{q} gives rise to a \mathbb{Z} -bilinear form $\Xi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbf{k}^\times$ by $\Xi(\alpha_j, \alpha_k) = q_{jk}$ for all $j, k \in \mathbb{I}$. If $\alpha, \beta \in \mathbb{Z}^\theta$, we also set

$$(1) \quad q_{\alpha\beta} = \Xi(\alpha, \beta).$$

The algebra $T(V)$ is \mathbb{Z}^θ -graded. If $x, y \in T(V)$ are homogeneous of degrees $\alpha, \beta \in \mathbb{Z}^\theta$ respectively, then their braided commutator is

$$(2) \quad [x, y]_c = xy - \text{multiplication} \circ c(x \otimes y) = xy - q_{\alpha\beta}yx.$$

Note that $\text{ad}_c(x)(y) = [x, y]_c$ whenever x is primitive. We say that x \mathfrak{q} -commutes with a family $(y_i)_{i \in I}$ of homogeneous elements if $[x, y_i]_c = 0$, for all $i \in I$. Same considerations are valid in any braided graded Hopf algebra.

Define a matrix $(c_{ij}^{\mathfrak{q}})_{i,j \in \mathbb{I}}$ with entries in $\mathbb{Z} \cup \{-\infty\}$ by $c_{ii}^{\mathfrak{q}} = 2$,

$$(3) \quad c_{ij}^{\mathfrak{q}} := -\min \{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\}, \quad i \neq j.$$

We assume from now on that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Then $c_{ij}^{\mathfrak{q}} \in \mathbb{Z}$ for all $i, j \in \mathbb{I}$ [R, Section 3.2] and we may define the reflections $s_i^{\mathfrak{q}} \in GL(\mathbb{Z}^{\theta})$, by $s_i^{\mathfrak{q}}(\alpha_j) = \alpha_j - c_{ij}^{\mathfrak{q}}\alpha_i$, $i, j \in \mathbb{I}$. Let $i \in \mathbb{I}$ and let $\rho_i(V)$ be the braided vector space of diagonal type with matrix $\rho_i(\mathfrak{q})$, where

$$(4) \quad \rho_i(\mathfrak{q})_{jk} = \Xi(s_i^{\mathfrak{q}}(\alpha_j), s_i^{\mathfrak{q}}(\alpha_k)), \quad j, k \in \mathbb{I}.$$

The proofs of statements (a) and (b) in the Introduction have as a crucial ingredient the Weyl groupoid [H1] and the generalized root system [HY1]; the definitions involve the assignments $\mathfrak{q} \rightsquigarrow \rho_i(\mathfrak{q})$ described above. For our purposes, we just need to recall that

$$(5) \quad \Delta_{\mathfrak{q}}^+ \text{ is the set of positive roots of } \mathcal{B}_{\mathfrak{q}}.$$

3.1. Drinfeld doubles

Let (V, c) be our fixed braided vector space of diagonal type with matrix \mathfrak{q} , realized in ${}_{\mathbb{k}\mathbb{Z}^{\theta}}^{\mathbb{k}\mathbb{Z}^{\theta}}\mathcal{YD}$ as above. In this Subsection, the hypothesis on the dimension of the Nichols algebra is not needed. We describe here the Drinfeld doubles of the bosonizations $T(V)\#\mathbb{k}\mathbb{Z}^{\theta}$, $\mathcal{B}_{\mathfrak{q}}\#\mathbb{k}\mathbb{Z}^{\theta}$ with respect to suitable bilinear forms. This construction goes back essentially to Drinfeld [Dr] and was adapted to different settings in various papers; here we follow [H3].

Definition 3.1. The Drinfeld double $\mathbf{U}_{\mathfrak{q}}$ of $T(V)\#\mathbb{k}\mathbb{Z}^{\theta}$ is the algebra generated by elements $E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}$, $i \in \mathbb{I}$, with defining relations

$$\begin{aligned} XY &= YX, & X, Y &\in \{K_i^{\pm}, L_i^{\pm} : i \in \mathbb{I}\}, \\ K_i K_i^{-1} &= L_i L_i^{-1} = 1, & E_i F_j - F_j E_i &= \delta_{i,j}(K_i - L_i). \\ K_i E_j &= q_{ij} E_j K_i, & L_i E_j &= q_{ji}^{-1} E_j L_i, \\ K_i F_j &= q_{ij}^{-1} F_j K_i, & L_i F_j &= q_{ji} F_j L_i. \end{aligned}$$

Then \mathbf{U}_q is a \mathbb{Z}^θ -graded Hopf algebra, where the comultiplication and the grading are given, for $i \in \mathbb{I}$, by

$$\begin{aligned} \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, & \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(L_i^{\pm 1}) &= L_i^{\pm 1} \otimes L_i^{\pm 1}, & \Delta(F_i) &= F_i \otimes L_i + 1 \otimes F_i. \\ \deg(K_i) &= \deg(L_i) = 0, & \deg(E_i) &= \alpha_i = -\deg(F_i). \end{aligned}$$

Let \mathbf{U}_q^+ (respectively, \mathbf{U}_q^-) be the subalgebra of \mathbf{U}_q generated by E_i (respectively, F_i), $i \in \mathbb{I}$. Let $W = (V^*, \mathfrak{q}^t)$.² Moreover, \mathbf{U}_q^+ and \mathbf{U}_q^- are Hopf algebras in $\frac{\mathbf{k}\mathbb{Z}^\theta}{\mathbf{k}\mathbb{Z}^\theta} \mathcal{YD}$ via the actions and coactions

$$\begin{aligned} K_i \cdot E_j &= q_{ij} E_j, & \delta(E_i) &= K_i \otimes E_i; \\ L_i \cdot F_j &= q_{ji} F_j, & \delta(F_i) &= L_i \otimes F_i. \end{aligned}$$

Thus, there are isomorphisms $\psi^+ : T(V) \rightarrow \mathbf{U}_q^+$, $\psi^- : T(W) \rightarrow \mathbf{U}_q^-$ of Hopf algebras in $\frac{\mathbf{k}\mathbb{Z}^\theta}{\mathbf{k}\mathbb{Z}^\theta} \mathcal{YD}$ given by $\psi^+(x_i) = E_i$ and $\psi^-(y_i) = F_i$.

Let

$$\mathfrak{u}_q = \mathbf{U}_q / (\psi^-(\mathcal{J}_{\mathfrak{q}^t}) + \psi^+(\mathcal{J}_q));$$

this is the Drinfeld double of $\mathcal{B}_q \# \mathbf{k}\mathbb{Z}^\theta$. We denote by E_i, F_i, K_i, L_i the elements of \mathfrak{u}_q that are images of their homonymous in \mathbf{U}_q . Let \mathfrak{u}^0 (respectively, $\mathfrak{u}_q^+, \mathfrak{u}_q^-$) be the subalgebra of \mathfrak{u}_q generated by K_i, L_i , (respectively, by E_i , by F_i), $i \in \mathbb{I}$. Then $\mathfrak{u}^0 \simeq \mathbf{k}\mathbb{Z}^{2\theta}$;

- there is a triangular decomposition $\mathfrak{u}_q \simeq \mathfrak{u}_q^+ \otimes \mathfrak{u}^0 \otimes \mathfrak{u}_q^-$;
- $\mathfrak{u}_q^+ \simeq \mathcal{B}_q, \mathfrak{u}_q^- \simeq \mathcal{B}_{\mathfrak{q}^t}$.

3.2. Lusztig isomorphisms and PBW bases

G. Lusztig defined automorphisms of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a simple Lie algebra \mathfrak{g} , see [L2]. These automorphisms satisfy the relations of the braid group covering the Weyl group of \mathfrak{g} ; they are instrumental in the construction of Poincaré-Birkhoff-Witt (PBW) bases of $U_q(\mathfrak{g})$. These results were extended to the Drinfeld double of a finite-dimensional Nichols algebra of diagonal type in [H3], with the role of the Weyl group played here by the Weyl groupoid \mathcal{W}_q . The definition of the Lusztig isomorphisms in [H3]

²Here and in Section 5 below, \mathfrak{q}^t corresponds to V^* when realized as Yetter-Drinfeld module over the dual Hopf algebra.

requires some hypotheses on the matrix \mathfrak{q} , that are always satisfied in the finite-dimensional case. So, let (V, c) and \mathfrak{q} as above; recall that we assume that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Fix $i \in \mathbb{I}$. We first recall the definition of the isomorphisms $\mathfrak{u}_{\mathfrak{q}} \rightarrow \mathfrak{u}_{\rho_i(\mathfrak{q})}$ [H3]. For $i \neq j \in \mathbb{I}$ and $n \in \mathbb{N}_0$, define the elements of $\mathfrak{u}_{\mathfrak{q}}$

$$E_{j,n} = (\text{ad } E_i)^n E_j, \quad F_{j,n} = (\text{ad } F_i)^n F_j.$$

Let $\underline{E}_j, \underline{F}_j, \underline{K}_j, \underline{L}_j$ be the generators of $\mathfrak{u}_{\rho_i(\mathfrak{q})}$. Set

$$(6) \quad a_j(\mathfrak{q}) := (-c_{ij}^{\mathfrak{q}})_{q_{ii}}! \prod_{s=0}^{-c_{ij}^{\mathfrak{q}}-1} (q_{ii}^s q_{ij} q_{ji} - 1), \quad j \neq i.$$

Theorem 3.2. [H3, 6.11] *There are algebra isomorphisms $T_i : \mathfrak{u}_{\mathfrak{q}} \rightarrow \mathfrak{u}_{\rho_i(\mathfrak{q})}$ uniquely determined, for $h, j \in \mathbb{I}, j \neq i$, by*

$$\begin{aligned} T_i(K_h) &= \underline{K}_i^{-c_{ih}^{\mathfrak{q}}} \underline{K}_h, & T_i(E_i) &= \underline{F}_i \underline{L}_i^{-1}, & T_i(E_j) &= \underline{E}_{j, -c_{ij}^{\mathfrak{q}}}, \\ T_i(L_h) &= \underline{L}_i^{-c_{ih}^{\mathfrak{q}}} \underline{L}_h, & T_i(F_i) &= \underline{K}_i^{-1} \underline{E}_i, & T_i(F_j) &= \frac{1}{a_j(\rho_i(\mathfrak{q}))} \underline{F}_{j, -c_{ij}^{\mathfrak{q}}}. \end{aligned}$$

Let $w \in \mathcal{W}_{\mathfrak{q}}$ be an element of maximal length and fix a reduced expression $w = \sigma_{i_1}^{\mathfrak{q}} \sigma_{i_2} \cdots \sigma_{i_M}$. If $k \in \mathbb{I}_M$ and $\mathbf{h} = (h_1, \dots, h_M) \in \mathbb{N}_0^M$, set

$$\begin{aligned} (7) \quad & \beta_k = s_{i_1}^{\mathfrak{q}} \cdots s_{i_{k-1}}(\alpha_{i_k}), \\ (8) \quad & E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}) \in (\mathfrak{u}_{\mathfrak{q}}^+)_{\beta_k}, \\ (9) \quad & \mathbf{E}^{\mathbf{h}} = E_{\beta_M}^{h_M} E_{\beta_{M-1}}^{h_{M-1}} \cdots E_{\beta_1}^{h_1}. \end{aligned}$$

By [CH, Prop. 2.12], $\Delta_+^{\mathfrak{q}} = \{\beta_k | 1 \leq k \leq M\}$. Thus, we set

$$(10) \quad N_{\beta} = N_k = \text{ord } q_{\beta\beta} \in \mathbb{N} \cup \{\infty\}, \quad \text{if } \beta = \beta_k \in \Delta_+^{\mathfrak{q}}.$$

Theorem 3.3. [HY2, 4.5, 4.8, 4.9] *The following set is a basis of $\mathfrak{u}_{\mathfrak{q}}^+$:*

$$\{\mathbf{E}^{\mathbf{h}} \mid \mathbf{h} \in \mathbb{N}_0^M, 0 \leq h_k < N_k, k \in \mathbb{I}_M\}.$$

3.3. Distinguished pre-Nichols algebra

We now recall the definition of the distinguished pre-Nichols algebra from [A3]. Let \mathfrak{q}, V be as above. First, $i \in \mathbb{I}$ is a *Cartan vertex* of \mathfrak{q} if

$$(11) \quad q_{ij} q_{ji} = q_{ii}^{c_{ij}^{\mathfrak{q}}}, \quad \text{for all } j \neq i,$$

recall (3). Then the set of Cartan roots of \mathfrak{q} is

$$\mathfrak{D}_{\mathfrak{q}} = \{s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}} : i \in \mathbb{I} \text{ is a Cartan vertex of } \rho_{i_k} \dots \rho_{i_2} \rho_{i_1}(\mathfrak{q})\}.$$

A set of defining relations of the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$, i. e. generators of the ideal $\mathcal{J}_{\mathfrak{q}}$, was given in [A2, Theorem 3.1]. We now consider the ideal $\mathcal{I}_{\mathfrak{q}} \subset \mathcal{J}_{\mathfrak{q}}$ of $T(V)$ generated by all the relations in *loc. cit.*, but

- we exclude the power root vectors $E_{\alpha}^{N_{\alpha}}$, $\alpha \in \mathfrak{D}_{\mathfrak{q}}$,
- we add the quantum Serre relations $(\text{ad}_c E_i)^{1-c_{ij}^{\mathfrak{q}}} E_j$ for those $i \neq j$ such that $q_{ii}^{c_{ij}^{\mathfrak{q}}} = q_{ij} q_{ji} = q_{ii}$.

Definition 3.4. [A3, 3.1] The *distinguished pre-Nichols algebra* of V is

$$\tilde{\mathcal{B}}_{\mathfrak{q}} = T(V)/\mathcal{I}_{\mathfrak{q}}.$$

Let $\tilde{\mathfrak{u}}_{\mathfrak{q}} = \mathbf{U}_{\mathfrak{q}}/(\psi^-(\mathcal{I}_{\mathfrak{q}^t}) + \psi^+(\mathcal{I}_{\mathfrak{q}}))$; this is the Drinfeld double of $\tilde{\mathcal{B}}_{\mathfrak{q}} \# \mathbf{k}\mathbb{Z}^{\theta}$. It was shown in [A3] that there is a triangular decomposition $\tilde{\mathfrak{u}}_{\mathfrak{q}} \simeq \tilde{\mathfrak{u}}_{\mathfrak{q}}^+ \otimes \tilde{\mathfrak{u}}^0 \otimes \tilde{\mathfrak{u}}_{\mathfrak{q}}^-$ as above, with $\tilde{\mathfrak{u}}^0 \simeq \mathfrak{u}^0 \simeq \mathbf{k}\mathbb{Z}^{2\theta}$.

If β_k is as in (7), $k \in \mathbb{I}_M$, then we set

$$\tilde{N}_k = \begin{cases} N_k & \text{if } \beta_k \notin \mathfrak{D}_{\mathfrak{q}}, \\ \infty & \text{if } \beta_k \in \mathfrak{D}_{\mathfrak{q}}. \end{cases}$$

For simplicity, we introduce

$$(12) \quad \mathbb{H} = \{\mathbf{h} \in \mathbb{N}_0^M : 0 \leq h_k < \tilde{N}_k, \text{ for all } k \in \mathbb{I}_M\}.$$

Theorem 3.5.

- (a) [A3, 3.4] *There exist algebra isomorphisms $\tilde{T}_i : \tilde{\mathfrak{u}}_{\mathfrak{q}} \rightarrow \tilde{\mathfrak{u}}_{\rho_i(\mathfrak{q})}$ inducing the isomorphisms $T_i : \mathfrak{u}_{\mathfrak{q}} \rightarrow \mathfrak{u}_{\rho_i(\mathfrak{q})}$.*
- (b) [A3, 3.6] *Let \tilde{E}_{β_k} , $\tilde{\mathbf{E}}^{\mathbf{h}}$ be the elements of $\tilde{\mathfrak{u}}_{\mathfrak{q}}$ defined as in (8), (9) with \tilde{T}_i instead of T_i . Then $\{\tilde{\mathbf{E}}^{\mathbf{h}} \mid \mathbf{h} \in \mathbb{H}\}$ is a basis of $\tilde{\mathfrak{u}}_{\mathfrak{q}}^+$.*

As before, we have an isomorphism $\tilde{\psi} : \tilde{\mathcal{B}}_{\mathfrak{q}} \rightarrow \tilde{\mathfrak{u}}_{\mathfrak{q}}^+$ of Hopf algebras in $\frac{\mathbf{k}\mathbb{Z}^{\theta}}{\mathbf{k}\mathbb{Z}^{2\theta}}\mathcal{YD}$, so we define

$$x_{\beta_k} = \tilde{\psi}^{-1}(\tilde{E}_{\beta_k}), \quad k \in \mathbb{I}_M; \quad \mathbf{x}^{\mathbf{h}} = \tilde{\psi}^{-1}(\tilde{\mathbf{E}}^{\mathbf{h}}), \quad \mathbf{h} \in \mathbb{H}.$$

Note that \tilde{E}_{β_k} is a well-defined sequence of braided commutators in the elements E_i , $i \in \mathbb{I}$; then x_{β_k} is the same sequence of braided commutators in

the x_i 's. Also, $\mathbf{x}^{\mathbf{h}} = x_{\beta_M}^{h_M} x_{\beta_{M-1}}^{h_{M-1}} \cdots x_{\beta_1}^{h_1}$ and

$$\mathbf{B} = \{\mathbf{x}^{\mathbf{h}} \mid \mathbf{h} \in \mathbf{H}\}$$

is a basis of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. The Hilbert series of a graded vector space $V = \bigoplus_{n \in \mathbb{N}_0} V^n$ is $\mathcal{H}_V = \sum_{n \in \mathbb{N}_0} \dim V^n T^n \in \mathbb{Z}[[T]]$. It follows from Theorem 3.5 (b) that

$$(13) \quad \text{GKdim } \widetilde{\mathcal{B}}_{\mathfrak{q}} = |\mathfrak{D}_{\mathfrak{q}}|, \quad \mathcal{H}_{\widetilde{\mathcal{B}}_{\mathfrak{q}}} = \prod_{\beta_k \in \mathfrak{D}_{\mathfrak{q}}} \frac{1}{1 - T^{\deg \beta}} \cdot \prod_{\beta_k \notin \mathfrak{D}_{\mathfrak{q}}} \frac{1 - T^{N_{\beta} \deg \beta}}{1 - T^{\deg \beta}}.$$

4. Lusztig algebras

Let $\mathfrak{q} = (q_{ij}) \in M_{\theta}(\mathbf{k}^{\times})$, (V, c) the corresponding braided vector space of diagonal type and (V^*, \mathfrak{q}) the dual braided vector space. We still assume that $\mathcal{B}_{\mathfrak{q}}$ is finite-dimensional. As in [A+, 3.3.4], we define the *Lusztig algebra* $\mathcal{L}_{\mathfrak{q}}$ of (V, c) as the graded dual of the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ of (V^*, \mathfrak{q}) ; thus, $\mathcal{B}_{\mathfrak{q}} \subseteq \mathcal{L}_{\mathfrak{q}}$. In this Section we establish some basic properties of this algebra.

4.1. Presentation

In the rest of the section we consider the bilinear form $\langle \cdot, \cdot \rangle : \widetilde{\mathcal{B}}_{\mathfrak{q}} \times \widetilde{\mathcal{B}}_{\mathfrak{q}}^* \rightarrow \mathbf{k}$ carried from the identification $V^* \otimes V^* \simeq (V \otimes V)^*$ in Section 2.2 which satisfies for all $x, x' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$, $y, y' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}^*$

$$\langle y, xx' \rangle = \langle y^{(2)}, x \rangle \langle y^{(1)}, x' \rangle \quad \text{and} \quad \langle yy', x \rangle = \langle y, x^{(2)} \rangle \langle y', x^{(1)} \rangle.$$

If $\mathbf{h} \in \mathbf{H}$, then define $\mathbf{y}_{\mathbf{h}} \in \widetilde{\mathcal{B}}_{\mathfrak{q}}^*$ by $\langle \mathbf{y}_{\mathbf{h}}, \mathbf{x}^{\mathbf{j}} \rangle = \delta_{\mathbf{h}, \mathbf{j}}$, $\mathbf{j} \in \mathbf{H}$. Then $\mathbf{y}_{\mathbf{h}} \in \mathcal{L}_{\mathfrak{q}}$ and $\{\mathbf{y}_{\mathbf{h}} \mid \mathbf{h} \in \mathbf{H}\}$ is a basis of $\mathcal{L}_{\mathfrak{q}}$.

Let $(\mathbf{h}_k)_{k \in \mathbb{I}_M}$ denote the canonical basis of \mathbb{Z}^M . If $k \in \mathbb{I}_M$ and $\beta = \beta_k \in \Delta_+^{\mathfrak{q}}$, then we denote the element $\mathbf{y}_{n\mathbf{h}_k}$ by $y_{\beta}^{(n)}$.

We recall some notation and results from [A3] and [AY]. For $i \in \mathbb{I}_M$, let

$$\begin{aligned} B^i &= \langle \{x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1} \mid 0 \leq h_j < N_j\} \rangle \subseteq \mathcal{B}_{\mathfrak{q}}, \\ \mathbf{B}^i &= \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_i}^{h_i} \mid 0 \leq h_j < N_j\} \rangle \subseteq \mathcal{B}_{\mathfrak{q}}, \\ \widetilde{B}^i &= \langle \{x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1} \mid 0 \leq h_j < \widetilde{N}_j\} \rangle \subseteq \widetilde{\mathcal{B}}_{\mathfrak{q}}, \\ \widetilde{\mathbf{B}}^i &= \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_i}^{h_i} \mid 0 \leq h_j < \widetilde{N}_j\} \rangle \subseteq \widetilde{\mathcal{B}}_{\mathfrak{q}}. \end{aligned}$$

We also denote by \tilde{L}^i and $\tilde{\mathbf{L}}^i$ the analogous subspaces of \mathcal{L}_q :

$$\begin{aligned} \tilde{L}^i &= \langle \{y_{\beta_1}^{(h_1)} \cdots y_{\beta_i}^{(h_i)} \mid 0 \leq h_j < \tilde{N}_j\} \rangle \subseteq \mathcal{L}_q, \\ \tilde{\mathbf{L}}^i &= \langle \{y_{\beta_i}^{(h_i)} \cdots y_{\beta_M}^{(h_M)} \mid 0 \leq h_j < \tilde{N}_j\} \rangle \subseteq \mathcal{L}_q. \end{aligned}$$

Proposition 4.1. • [AY, 4.2, 4.11] B^i (respectively \mathbf{B}^i) is a right (respectively left) coideal subalgebra of \mathcal{B}_q .

- [A3, 4.1] If $\beta \in \mathfrak{D}_q$, then $x_\beta^{N_\beta}$ q -commutes with every element of $\tilde{\mathcal{B}}_q$.
- [A3, 4.9] If $\beta_i \in \mathfrak{D}_q$, then there exist $X(n_1, \dots, n_{i-1}) \in \tilde{\mathcal{B}}_q$ such that

$$\begin{aligned} \Delta(x_{\beta_i}^{N_{\beta_i}}) &= x_{\beta_i}^{N_{\beta_i}} \otimes 1 + 1 \otimes x_{\beta_i}^{N_{\beta_i}} \\ &\quad + \sum_{n_k \in \mathbb{N}_0} x_{\beta_{i-1}}^{n_{i-1}N_{\beta_{i-1}}} \cdots x_{\beta_1}^{n_1N_{\beta_1}} \otimes X(n_1, \dots, n_{i-1}). \end{aligned}$$

Corollary 4.2. \tilde{B}^i is a right coideal subalgebra of $\tilde{\mathcal{B}}_q$.

Let Z_q^+ be the subalgebra of $\tilde{\mathcal{B}}_q$ generated by $x_\beta^{N_\beta}$, $\beta \in \mathfrak{D}_q$.

Theorem 4.3. [A3, 4.10, 4.13] Z_q^+ is a braided normal Hopf subalgebra of $\tilde{\mathcal{B}}_q$. Moreover $Z_q^+ = {}^{\text{cop}}\tilde{\mathcal{B}}_q$, where π denotes the canonical projection of $\tilde{\mathcal{B}}_q$ onto \mathcal{B}_q .

Lemma 4.4. Let x , x_1 and x_2 be elements in the PBW basis \mathbf{B} of $\tilde{\mathcal{B}}_q$. Write $\Delta(x)$ as a linear combination of $\{a \otimes b \mid a, b \in \mathbf{B}\}$. Assume that $x_1 \otimes x_2$ has a non-zero coefficient in $\Delta(x)$ (in this combination) and x_1x_2 (the concatenation of x_1 and x_2) is in \mathbf{B} . Then $x = x_1x_2$.

Proof. Suppose that $x = x_{\beta_i}^{h_i} \cdots x_{\beta_1}^{h_1}$ with $h_i > 0$. Let

$$\begin{aligned} m(x) &= \min\{j \in \mathbb{N} : h_j \neq 0\}, \\ \mathcal{D}(x) &= \sum_{j=1}^i \sum_{t=1}^{h_j} \binom{h_j}{t}_{q_{\beta_j \beta_j}} x_{\beta_i}^{h_i} \cdots x_{\beta_j}^t \otimes x_{\beta_j}^{h_j-t} \cdots x_{\beta_1}^{h_1} + 1 \otimes x, \\ \tilde{C}^i &= \langle \{x_{\beta_M}^{h_M} \cdots x_{\beta_1}^{h_1} \in \mathbf{B} \mid \exists j > i \text{ s.t. } h_j \neq 0\} \rangle. \end{aligned}$$

Observe that if $x_1 \otimes x_2$ appears in $\mathcal{D}(x)$, then $x = x_1x_2$. However, if $x_1 \otimes x_2 \in \sum_{u \in \tilde{\mathcal{B}}^i} u \otimes \tilde{C}^{m(u)}$, then $x_1x_2 \notin \mathbf{B}$. Therefore the proof is completed

by showing that

$$\Delta(x) \in \mathcal{D}(x) + \sum_{u \in \tilde{B}^i} u \otimes \tilde{C}^{m(u)}.$$

We proceed by induction on i . If $i = 1$, then $x = x_{\beta_1}^h$ and x_{β_1} is primitive, so $\Delta(x_{\beta_1}^h) = \sum_{0 \leq k \leq h} \binom{h}{k}_{q_{\beta_1 \beta_1}} x_{\beta_1}^k \otimes x_{\beta_1}^{h-k} = \mathcal{D}(x_{\beta_1}^h)$. Let $i > 1$. Now we proceed by induction on h_i . Set $x' = x_{\beta_i}^{h_i-1} x_{\beta_{i-1}}^{h_{i-1}} \cdots x_{\beta_1}^{h_1}$, so $x = x_{\beta_i} x'$. Notice that

$$(14) \quad \Delta(x_{\beta_i}) \in x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i} + \tilde{B}^{i-1} \otimes \tilde{C}^i.$$

Indeed the analogous statement for \mathcal{B}_q was proved in [AY, 4.3], but the same argument applies for $\tilde{\mathcal{B}}_q$. By the inductive hypothesis and (14)

$$\begin{aligned} \Delta(x) &= \Delta(x_{\beta_i})\Delta(x') \\ &\in (x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i} + \tilde{B}^{i-1} \otimes \tilde{C}^i) \left(\mathcal{D}(x') + \sum_{u \in \tilde{B}^i} u \otimes \tilde{C}^{m(u)} \right). \end{aligned}$$

Notice that $(x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i})\mathcal{D}(x') \in \mathcal{D}(x) + \sum_{u \in \tilde{B}^i} u \otimes \tilde{C}^{m(u)}$, since

$$\begin{aligned} &(x_{\beta_i} \otimes 1 + 1 \otimes x_{\beta_i}) \left(\sum_{t=1}^{h_i-1} \binom{h_i-1}{t}_{q_{\beta_i \beta_i}} x_{\beta_i}^t \otimes x_{\beta_i}^{h_i-1-t} \cdots x_{\beta_1}^{h_1} + 1 \otimes x' \right) \\ &= x_{\beta_i} \otimes x' + \sum_{t=2}^{h_i} \binom{h_i-1}{t-1}_{q_{\beta_i \beta_i}} x_{\beta_i}^t \otimes x_{\beta_i}^{h_i-t} \cdots x_{\beta_1}^{h_1} \\ &\quad + \sum_{t=1}^{h_i-1} q_{\beta_i \beta_i}^t \binom{h_i-1}{t}_{q_{\beta_i \beta_i}} x_{\beta_i}^t \otimes x_{\beta_i}^{h_i-t} \cdots x_{\beta_1}^{h_1} + 1 \otimes x_{\beta_i} x' \end{aligned}$$

and for $h_i > 1$, $1 \leq t < h_i$, we have $\binom{h_i-1}{t-1}_{q_{\beta_i \beta_i}} + q_{\beta_i \beta_i}^t \binom{h_i-1}{t}_{q_{\beta_i \beta_i}} = \binom{h_i}{t}_{q_{\beta_i \beta_i}}$. Also, $\tilde{B}^{i-1} \subset \tilde{B}^i$, \tilde{B}^i is a subalgebra and $\tilde{C}^i z \subset \tilde{C}^i$ for all $z \in \tilde{\mathcal{B}}_q$, by [A3, 3.15], so

$$(\tilde{B}^{i-1} \otimes \tilde{C}^i)\mathcal{D}(x') \subset \tilde{B}^{i-1} \tilde{B}^i \otimes \tilde{C}^i \tilde{B}^i \subset \tilde{B}^i \otimes \tilde{C}^i.$$

As $x_{\beta_i} u \in \tilde{B}^i$ for all $u \in \tilde{B}^i$ and $m(u) = m(x_{\beta_i} u)$, then

$$x_{\beta_i} u \otimes \tilde{C}^{m(u)} = x_{\beta_i} u \otimes \tilde{C}^{m(x_{\beta_i} u)} \quad \text{and} \quad u \otimes x_{\beta_i} \tilde{C}^{m(u)} \subset u \otimes \tilde{C}^{m(u)}.$$

Finally, $\tilde{B}^{i-1} u \otimes \tilde{C}^i \tilde{C}^{m(u)} \subset \tilde{B}^i \otimes \tilde{C}^i \subset \sum_{v \in \tilde{B}^i} v \otimes \tilde{C}^{m(v)}$ for all $u \in \tilde{B}^i$. From these considerations the proof of the inductive step follows directly. \square

Corollary 4.5. *If $\beta \in \Delta_+^q$, then*

$$(15) \quad y_\beta^{(r)} = \frac{y_\beta^r}{(r)_{q_{\beta\beta}}!}, \quad r < N_\beta = \text{ord } q_{\beta\beta};$$

$$(16) \quad y_\beta^{(n)} = \frac{(y_\beta^{(N_\beta)})^s}{s!} y_\beta^{(r)}, \quad \beta \in \mathfrak{D}_q, n = sN_\beta + r, r < N_\beta.$$

Proof. Arguing inductively, we may suppose that $y_\beta^{r-1} = (r-1)_{q_{\beta\beta}}! y_\beta^{(r-1)}$. If $x = \mathbf{x}^h \in \tilde{\mathcal{B}}_q$ such that

$$\langle y_\beta^r, x \rangle = \langle y_\beta^{r-1}, x^{(1)} \rangle \langle y_\beta, x^{(2)} \rangle \neq 0,$$

then by Lemma 4.4, $x = x_\beta^r$. Then

$$\langle y_\beta^r, x_\beta^r \rangle = \langle y_\beta^{r-1}, (x_\beta^r)^{(1)} \rangle \langle y_\beta, (x_\beta^r)^{(2)} \rangle = (r-1)_{q_{\beta\beta}}! (r)_{q_{\beta\beta}} = (r)_{q_{\beta\beta}}!.$$

The second equation follows immediately since $\langle y_\beta^{(N_\beta)} y_\beta^{(r)}, x_\beta^{N_\beta+r} \rangle = 1$. \square

The next lemma is crucial for the presentation of the algebra \mathcal{L}_q by generators and relations.

Lemma 4.6. *Let $i \in \mathbb{I}_M$, $h_i < \tilde{N}_{\beta_i}$ and $\mathbf{h} = (h_1, \dots, h_M) \in \mathbb{N}_0^M$, then*

$$(17) \quad \mathbf{y}_\mathbf{h} = y_{\beta_1}^{(h_1)} \cdots y_{\beta_M}^{(h_M)}.$$

Hence $\{y_{\beta_1}^{(h_1)} \cdots y_{\beta_M}^{(h_M)} \mid 0 \leq h_i < \tilde{N}_{\beta_i}\}$ is a basis of \mathcal{L}_q .

Proof. The proof is by induction on $\text{ht}(\mathbf{h}) := \sum_{i \in \mathbb{I}_M} h_i$. If $\text{ht}(\mathbf{h}) = 1$ then $\mathbf{y}_\mathbf{h} = y_\beta$ for some $\beta \in \Delta_+^q$ and the claim follows by definition.

Let $1 \leq i_1 < \dots < i_j \leq M$, $n_k < \tilde{N}_{\beta_{i_k}}$ and $n_1 = sN_{\beta_{i_1}} + r \neq 0$ where $r < N_{\beta_{i_1}}$. Let $y = y_{\beta_{i_1}}^{(n_1)} \cdots y_{\beta_{i_j}}^{(n_j)} \in \mathcal{L}_q$. Since $\{\mathbf{y}_\mathbf{h} \mid \mathbf{h} \in \mathbb{H}\}$ is a basis of \mathcal{L}_q , we can express y as the linear combination $y = \sum_{\mathbf{h} \in \mathbb{H}} c_\mathbf{h} \mathbf{y}_\mathbf{h}$. Notice that $c_\mathbf{h} \neq 0$ if and only if $\langle y, x^\mathbf{h} \rangle \neq 0$.

If $r \neq 0$, then we write $y = \frac{1}{(r)_q} y_{\beta_{i_1}} y'$ where $y' = y_{\beta_{i_1}}^{(n_1-1)} \cdots y_{\beta_{i_j}}^{(n_j)}$ and $q = q_{\beta_{i_1}\beta_{i_1}}$. Then $\langle y, x^\mathbf{h} \rangle = \frac{1}{(r)_q} \langle y_{\beta_{i_1}}, (x^\mathbf{h})^{(2)} \rangle \langle y', (x^\mathbf{h})^{(1)} \rangle$. By inductive hypothesis and Lemma 4.4, $c_\mathbf{h} \neq 0$ if and only if $\mathbf{h} = (0, \dots, n_1, \dots, n_k, 0, \dots)$. Moreover, the nonzero $c_\mathbf{h}$ is equal to 1 and the proof in this case is completed.

If $r = 0$, $n_1 = sN_{\beta_{i_1}}$, then we write $y = y_{\beta_{i_1}}^{(N_{\beta_{i_1}})} y'$. Arguing as above, (17) follows. Hence $\{y_{\beta_1}^{(h_1)} \cdots y_{\beta_M}^{(h_M)} \mid 0 \leq h_i < \tilde{N}_{\beta_i}\}$ is a basis of \mathcal{L}_q because so is $\{\mathbf{y}_\mathbf{h} : \mathbf{h} \in \mathbb{H}\}$ by definition. \square

We seek for a presentation of \mathcal{L}_q . Let us consider the algebra \mathbb{L} presented by generators $y_\beta^{(n)}$, $\beta \in \Delta_+^q$, $n \in \mathbb{N}$ with relations

$$\begin{aligned}
 (18) \quad & y_\beta^{(N_\beta)} = 0, & \beta \in \Delta_+^q - \mathfrak{D}_q; \\
 (19) \quad & y_\beta^{(h)} y_\beta^{(j)} = \binom{h+j}{j}_{q_{\beta\beta}} y_\beta^{(h+j)}, & \beta \in \Delta_+^q, \\
 & & h, j \in \mathbb{N}; \\
 (20) \quad & [y_\beta^{(h)}, y_\alpha^{(j)}]_c = \sum_{\mathbf{m} \in M(\alpha, \beta, h, j)} \kappa_{\mathbf{m}} \mathbf{m}, & \alpha < \beta \in \Delta_+^q, \\
 & & 0 < h < N_\alpha, \\
 & & 0 < j < N_\beta; \\
 (21) \quad & [y_\beta^{(N_\beta)}, y_\alpha^{(N_\alpha)}]_c = \kappa_\gamma y_\gamma^{(N_\gamma)} + \sum_{\substack{0 < l < N_\beta, 0 < i < N_\alpha \\ \mathbf{m} \in M(\alpha, \beta, N_\alpha - i, N_\beta - l)}} \kappa_{\mathbf{m}}^{i,l} y_\alpha^{(i)} \mathbf{m} y_\beta^{(l)}, & \alpha, \beta, \gamma \in \mathfrak{D}_q, \\
 & & \alpha < \gamma < \beta; \\
 (22) \quad & [y_\beta^{(j)}, y_\alpha^{(N_\alpha)}]_c = \sum_{\substack{0 < i < N_\alpha, \\ \mathbf{m} \in M(\alpha, \beta, N_\alpha - i, j)}} \kappa_{\mathbf{m}}^{i,0} y_\alpha^{(i)} \mathbf{m}, & \alpha \in \mathfrak{D}_q, \\
 & & \beta \in \Delta_+^q, \\
 & & 0 < j < N_\beta.
 \end{aligned}$$

Here we set

$$\begin{aligned}
 M(\alpha, \beta, h, j) &= \{ \mathbf{m} = y_{\beta_r}^{(h_r)} \cdots y_{\beta_k}^{(h_k)} \in \tilde{L}^\beta \cap \tilde{\mathbf{L}}^\alpha : \deg \mathbf{m} = \deg y_\alpha^{(h)} + \deg y_\beta^{(j)} \}; \\
 \kappa_{\mathbf{m}}^{i,l} &= \langle y_\beta^{(h)} y_\alpha^{(j)}, x_\beta^l x_{\beta_k}^{h_k} \cdots x_{\beta_r}^{h_r} x_\alpha^i \rangle; \\
 \kappa_\gamma &= \langle y_\beta^{(N_\beta)} y_\alpha^{(N_\alpha)}, x_\gamma^{N_\gamma} \rangle, \quad \deg y_\gamma^{(N_\gamma)} = \deg y_\alpha^{(N_\alpha)} + \deg y_\beta^{(N_\beta)}.
 \end{aligned}$$

Theorem 4.7. *There is an algebra isomorphism $\Upsilon : \mathbb{L} \rightarrow \mathcal{L}_q$ given by*

$$\Upsilon(y_\beta^{(n)}) = y_\beta^{(n)}, \quad \beta \in \Delta_+^q, n < \tilde{N}_\beta.$$

Proof. We first prove that Υ is well-defined, i. e. that (18), ..., (22) are satisfied by the elements $y_\beta^{(n)} \in \mathcal{L}_q$. Relation (18) is trivial since $x_\beta^{N_\beta} = 0$ if $\beta \notin \mathfrak{D}_q$ and (19) is clear from (15).

For the other relations, given $\alpha < \beta$ and $h, j \in \mathbb{N}$, we write $y_\beta^{(h)} y_\alpha^{(j)} = \sum_{\mathbf{h} \in \mathbb{H}} c_{\mathbf{h}} \mathbf{y}_{\mathbf{h}}$. Then

$$c_{\mathbf{h}} = \langle y_\beta^{(h)} y_\alpha^{(j)}, \mathbf{x}^{\mathbf{h}} \rangle = \langle y_\alpha^{(j)}, (\mathbf{x}^{\mathbf{h}})^{(1)} \rangle \langle y_\beta^{(h)}, (\mathbf{x}^{\mathbf{h}})^{(2)} \rangle$$

is the coefficient of $x_\alpha^j \otimes x_\beta^h$ in the expression of $\Delta(\mathbf{x}^{\mathbf{h}})$ as linear combination of elements of the PBW basis in both sides of the tensor product.

If $j < N_\alpha$ and $h < N_\beta$, then $y_\alpha^{(j)}, y_\beta^{(h)} \in \mathcal{B}_q$. If $c_{\mathbf{h}} \neq 0$ then $\mathbf{x}^{\mathbf{h}}$ appears in the expression of $x_\alpha^j x_\beta^h$ in elements of the PBW basis, see [A1, Section 3]. Hence, by [HY2, 4.8] $\mathbf{x}^{\mathbf{h}} \in \mathbf{B}^\alpha \cap B^\beta$, and relation (20) is clear.

Let $\alpha, \beta \in \mathfrak{D}_q$, $j = N_\alpha$ and $h = N_\beta$. Suppose that there is

$$\mathbf{h} = (h_1, \dots, h_M)$$

such that $c_{\mathbf{h}} \neq 0$ and $h_i \geq N_i$ for some $i \in \mathbb{I}_M$. As $x_{\beta_i}^{N_i}$ q -commutes with every element of $\tilde{\mathcal{B}}_q$, we have $\mathbf{x}^{\mathbf{h}} = c x_{\beta_i}^{N_i} \mathbf{x}^{\mathbf{h}'}$, where $\mathbf{h}' = (h_1, \dots, h_i - N_i, \dots, h_M)$ and $c = \Xi(h_M \beta_M + \dots + h_{i+1} \beta_{i+1}, N_i \beta_i) \in \mathbf{k}$. Then $\Delta(\mathbf{x}^{\mathbf{h}}) = c \Delta(x_{\beta_i}^{N_i}) \Delta(\mathbf{x}^{\mathbf{h}'})$ and hence $\mathbf{x}^{\mathbf{h}} = x_{\beta_i}^{N_i}$ by Proposition 4.1. For the remaining \mathbf{j} such that $c_j \neq 0$ we have $j_i < N_i$ for all $i \in \mathbb{I}_M$. We write

$$x_\alpha^{N_\alpha} \otimes x_\beta^{N_\beta} = \xi(1 \otimes x_\beta^n)(x_\alpha^{N_\alpha - m} \otimes x_\beta^{N_\beta - n})(x_\alpha^m \otimes 1)$$

where $\xi = \Xi^{-1}((N_\alpha - m)\alpha, n\beta)\Xi^{-1}(m\alpha, (N_\beta - n)\beta)$. Therefore, arguing as in the proof of (20) for $y_\beta^{(N_\beta - n)} y_\alpha^{(N_\alpha - m)}$, we obtain that $\mathbf{y}_j = y_\alpha^{(m)} \mathbf{m} y_\beta^{(n)}$, $\mathbf{m} \in \tilde{L}^\beta \cap \tilde{L}^\alpha$. Here, either $m = N_\alpha, n = N_\beta$ so $\mathbf{y}_j = \Xi(N_\alpha \alpha, N_\beta \beta) y_\alpha^{(N_\alpha)} y_\beta^{(N_\beta)}$, or else $m < N_\alpha, n < N_\beta$. Hence relation (21) follows up to consider the correct degree for \mathbf{y}_h .

For (22), $c_{\mathbf{h}} \neq 0$ implies $\mathbf{x}^{\mathbf{h}} \in \mathcal{B}_q$ by the same argument above, since Z_q^+ is a braided Hopf subalgebra by Theorem 4.3.

Hence, Υ is a morphism of algebras. By the presentation of \mathbb{L} we can prove that $\{y_{\beta_1}^{(h_1)} \dots y_{\beta_M}^{(h_M)} : h_i < \tilde{N}_i\}$ is a basis of \mathbb{L} . So, Υ maps a basis to a basis by Lemma 4.6 and then it is bijective. □

Example 4.8. Let $\theta = 3 \leq N, q \in \mathbf{k}^\times, \text{ord } q = N$. We consider a diagonal braiding (of super type A) given by a matrix $q = (q_{ij})_{i,j \in \mathbb{I}_3}$ such that

$$q_{11} = q_{23}q_{32} = q, \quad q_{12}q_{21} = q^{-1}, \quad q_{22} = q_{33} = -1, \quad q_{13}q_{31} = 1.$$

Let $\alpha_{jk} = \sum_{j \leq i \leq k} \alpha_i$; then $\Delta_q^+ = \{\alpha_{jk} : 1 \leq j \leq k \leq 3\}, \mathfrak{D}_q^+ = \{\alpha_1, \alpha_{23}, \alpha_{13}\}$.

The Lusztig algebra \mathcal{L}_q is presented by generators $y_{jk}^{(n)}, 1 \leq j \leq k \leq 3, n \in \mathbb{N}$ and relations:

$$\begin{aligned}
 y_{12}^{(2)} &= y_2^{(2)} = y_3^{(2)} = 0, \\
 y_{jk}^{(n)} y_{jk}^{(m)} &= \binom{n+m}{n}_{q_{jk}} y_{jk}^{(n+m)}, \quad n, m \in \mathbb{N}, \\
 [y_{12}, y_1]_c &= [y_{13}, y_1]_c = [y_3, y_1]_c = [y_{13}, y_{12}]_c = [y_2, y_{12}]_c = [y_{23}, y_{12}]_c = 0, \\
 [y_2, y_{13}]_c &= [y_{23}, y_{13}]_c = [y_3, y_{13}]_c = [y_{23}, y_2]_c = [y_3, y_{23}]_c = 0, \\
 [y_2, y_1]_c &= (1 - q^{-1})y_{12}, \quad [y_3, y_{12}]_c = (1 - q)y_{13}, \\
 [y_{23}, y_1]_c &= (1 - q^{-1})y_{13}, \quad [y_3, y_2]_c = (1 - q)y_{23}, \\
 [y_{23}^{(N)}, y_1]_c &= (1 - q^{-1})(q_{21}q_{31})^{N-1} y_{13} y_{23}^{(N-1)}, \\
 [y_{23}, y_1^{(N)}]_c &= (1 - q^{-1})(q_{21}q_{31})^{N-1} y_1^{(N-1)} y_{13}, \\
 [y_2, y_1^{(N)}]_c &= (1 - q^{-1})q_{21}^{N-1} y_1^{(N-1)} y_{12}, \\
 [y_{12}, y_1^{(N)}]_c &= [y_{13}, y_1^{(N)}]_c = [y_3, y_1^{(N)}]_c = 0, \\
 [y_{13}^{(N)}, y_1]_c &= [y_{13}^{(N)}, y_{12}]_c = [y_2, y_{13}^{(N)}]_c = [y_{23}, y_{13}^{(N)}]_c = [y_3, y_{13}^{(N)}]_c = 0, \\
 [y_{23}^{(N)}, y_{12}]_c &= [y_{23}^{(N)}, y_{13}]_c = [y_{23}^{(N)}, y_2]_c = [y_3, y_{23}^{(N)}]_c = 0, \\
 [y_{13}^{(N)}, y_1^{(N)}]_c &= [y_{23}^{(N)}, y_{13}^{(N)}]_c = 0, \\
 [y_{23}^{(N)}, y_1^{(N)}]_c &= (1 - q^{-1})^N (q_{21}q_{31})^{N \frac{N-1}{2}} y_{13}^{(N)} \\
 &\quad + \sum_{k=1}^{N-1} (1 - q^{-1})^k (q_{21}q_{31})^{k \frac{2N-k-1}{2}} y_1^{(N-k)} y_{13}^{(k)} y_{23}^{(N-k)}.
 \end{aligned}$$

Indeed, to compute $y_{23}^{(N)} y_1^{(N)}$ in \mathcal{L}_q , we need to describe all $\mathbf{h} \in \mathbb{H}$, cf. (12), such that $x_1^N \otimes x_{23}^N$ appears in $\Delta(\mathbf{x}^{\mathbf{h}})$ with non-zero coefficient (also to be determined), where (for some numeration of Δ_q^+)

$$\mathbf{x}^{\mathbf{h}} = x_3^{h_1} x_{23}^{h_2} x_2^{h_3} x_{123}^{h_4} x_{12}^{h_5} x_1^{h_6}.$$

One of these $\mathbf{x}^{\mathbf{h}}$ is $x_{23}^N x_1^N$, with coefficient $q_{N\alpha_1, N\alpha_2 + N\alpha_3}$. Let \mathbf{h} be as needed. We use the coproduct formulas in [A3, 5.1]. Clearly $h_1 = 0$. From $\Delta(x_{23}^{h_2})$, the only contribution is $(1 \otimes x_{23})^{h_2}$. Then we deduce easily that $h_3 = h_5 = 0$, and $h_6 = h_2 = N - h_4$. In this case, set $h_4 = k$ to simplify the notation, so

$$(1 \otimes x_{23})^{N-k} (x_1 \otimes x_{23})^k (x_1 \otimes 1)^{N-k} = (q_{21}q_{31})^{k \frac{2N-k-1}{2}} x_1^N \otimes x_{23}^N.$$

This gives the last relation, and the others are deduced analogously.

Corollary 4.9. *The algebra \mathcal{L}_q is finitely generated.*

Proof. By (19), it is generated by $\{y_\beta : \beta \in \Delta_+^q\} \cup \{y_\alpha^{(N_\alpha)} : \alpha \in \mathfrak{D}_q\}$. \square

Remark 4.10. Actually, the subalgebra $\mathcal{B}_q \subset \mathcal{L}_q$ is generated by its primitive elements $\{y_\alpha : \alpha \in \Pi_q\}$ where Π_q denotes the set of simple roots $\alpha_1, \dots, \alpha_\theta$. Moreover, $y_\gamma^{(N_\gamma)} \in \mathbf{k}^\times [y_\beta^{(N_\beta)}, y_\alpha^{(N_\alpha)}]_c$ if and only if $x_\alpha^{N_\alpha} \otimes x_\beta^{N_\beta}$ appears with nonzero coefficient in $\Delta(x_\gamma^{N_\gamma})$. Hence,

$$\{y_\alpha : \alpha \in \Pi_q\} \cup \{y_\alpha^{(N_\alpha)} : \alpha \in \mathfrak{D}_q, x_\alpha^{N_\alpha} \in \mathcal{P}(\tilde{\mathcal{B}}_q)\}$$

generates \mathcal{L}_q as an algebra.

Proposition 4.11. $\tilde{\mathbf{L}}^i$ is a right coideal subalgebra of \mathcal{L}_q .

Proof. From Theorem 4.7 we have that $y_{\beta_j}^{(n)} y_{\beta_i}^{(m)} \in \tilde{\mathbf{L}}^i$ for $i < j$, thus $\tilde{\mathbf{L}}^i$ is a subalgebra of \mathcal{L}_q . On the other hand, we know that

$$\langle y_\beta^{(n)}, xx' \rangle = \langle (y_\beta^{(n)})^{(2)}, x \rangle \langle (y_\beta^{(n)})^{(1)}, x' \rangle.$$

Therefore $y_j \otimes y_h$ appears with nonzero coefficient in $\Delta(y_\beta^{(n)})$ if and only if x_β^n appears with nonzero coefficient in the expression of $x^{\mathbf{h}} x^{\mathbf{j}}$ in the PBW basis. The last condition implies that $x^{\mathbf{h}} \in \tilde{B}^\beta$ and $x^{\mathbf{j}} \in \tilde{\mathbf{B}}^\beta$. Hence,

$$\Delta(y_\beta^{(n)}) \in \sum_{i=0}^n y_\beta^{(i)} \otimes y_\beta^{(n-i)} + \tilde{\mathbf{L}}^\beta \otimes \tilde{L}^\beta.$$

Hence $\Delta(y_{\beta_i}^{(n_i)} \dots y_{\beta_M}^{(n_M)}) = \Delta(y_{\beta_i}^{(n_i)}) \Delta(y_{\beta_{i+1}}^{(n_{i+1})} \dots y_{\beta_M}^{(n_M)}) \in \tilde{\mathbf{L}}^i \otimes \mathcal{L}_q$ and the proof is complete. \square

4.2. Noetherianity and Gelfand-Kirillov dimension

We argue as in the pre-Nichols case [A3, Section 3.4], cf. [DP]. Let us consider the lexicographic order in \mathbb{N}_0^M , so that $\mathbf{h}_M < \dots < \mathbf{h}_1$, where $(\mathbf{h}_j)_{j \in \mathbb{I}_M}$ denotes the canonical basis of \mathbb{Z}^M .

Lemma 4.12. Let $\mathcal{L}_q(\mathbf{h})$ be the subspace of \mathcal{L}_q generated by \mathbf{y}_j , with $\mathbf{j} \leq \mathbf{h}$. Then $\mathcal{L}_q(\mathbf{h})$ is an \mathbb{N}_0^M -algebra filtration of \mathcal{L}_q .

Proof. It is enough to prove that $\mathbf{y}_h \mathbf{y}_j \in \mathcal{L}_q(\mathbf{h} + \mathbf{j})$ for all $\mathbf{h}, \mathbf{j} \in \mathbb{H}$. First we consider the case when $\mathbf{h} = n\mathbf{h}_k, \mathbf{j} = m\mathbf{h}_l, k, l \in \mathbb{I}_M, n, m \in \mathbb{N}$. We claim that

$y_{\beta_k}^{(n)} y_{\beta_l}^{(m)} \in \mathcal{L}_q(n\mathbf{h}_k + m\mathbf{h}_l)$. This follows by definition when $k \leq l$. If $l < k$, then $[y_{\beta_k}^{(n)}, y_{\beta_l}^{(m)}]_c \in \sum_{j < m} y_{\beta_l}^{(j)} \cdot \tilde{\mathbf{L}}^{l+1}$ by Theorem 4.7, thus

$$y_{\beta_k}^{(n)} y_{\beta_l}^{(m)} \in \mathcal{L}_q(n\mathbf{h}_k + m\mathbf{h}_l) \quad \text{since} \quad \sum_{j=l+1}^M a_j \mathbf{h}_j < n\mathbf{h}_k + m\mathbf{h}_l.$$

The Lemma follows by reordering the factors of $\mathbf{y}_h \mathbf{y}_j$, for any $\mathbf{h}, \mathbf{j} \in \mathbb{N}_0^M$. \square

We now consider the corresponding graded algebra

$$\text{gr } \mathcal{L}_q = \bigoplus_{\mathbf{h} \in \mathbb{N}_0^M} \text{gr}^{\mathbf{h}} \mathcal{L}_q, \quad \text{where} \quad \text{gr}^{\mathbf{h}} \mathcal{L}_q = \mathcal{L}_q(\mathbf{h}) / \sum_{\mathbf{j} < \mathbf{h}} \mathcal{L}_q(\mathbf{j}).$$

Lemma 4.13. *The algebra $\text{gr } \mathcal{L}_q$ is presented by generators $y_k^{(n)}$, $k \in \mathbb{I}_M$, $n \in \mathbb{N}$, and relations*

$$\begin{aligned} y_k^{(N_k)} &= 0, \quad \beta_k \notin \mathfrak{D}_q, \\ y_k^{(n)} y_k^{(m)} &= \binom{n+m}{m}_{q_{\beta_k \beta_k}} y_k^{(n+m)}, \\ [y_k^{(n)}, y_l^{(m)}]_c &= 0, \quad l < k. \end{aligned}$$

Proof. Let \mathcal{G} be the algebra presented by the generators and relations above and $\pi : \mathcal{G} \rightarrow \text{gr } \mathcal{L}_q$ given by $y_k^{(n)} \mapsto y_{\beta_k}^{(n)}$. By Theorem 4.7, the relations above hold in $\text{gr } \mathcal{L}_q$. By a direct computation, \mathcal{G} has a basis

$$\{y_1^{(h_1)} \dots y_M^{(h_M)} : h_i < \tilde{N}_i\}.$$

On the other hand, $\mathbf{y}_h \in \mathcal{L}_q(\mathbf{h}) - \sum_{\mathbf{j} < \mathbf{h}} \mathcal{L}_q(\mathbf{j})$. Hence the projection of the PBW basis of \mathcal{L}_q is a basis of $\text{gr } \mathcal{L}_q$ and π is an isomorphism. \square

Proposition 4.14. *The algebra \mathcal{L}_q is noetherian.*

Proof. Let \mathcal{Z}^+ be the subalgebra of $\text{gr } \mathcal{L}_q$ generated by $\{y_{\beta}^{(N_{\beta})} : \beta \in \mathfrak{D}_q\}$. Then \mathcal{Z}^+ is a quantum affine space and $\text{gr } \mathcal{L}_q$ is a finitely generated free \mathcal{Z}^+ -module. Hence $\text{gr } \mathcal{L}_q$ is noetherian and so is \mathcal{L}_q . \square

We compute either from Lemma 4.6 or else from Lemma 4.13 the Gelfand-Kirillov dimension of \mathcal{L}_q .

Proposition 4.15. $\text{GKdim } \mathcal{L}_q = |\mathfrak{D}_q|$.

5. Quantum divided power algebras

5.1. Definition

Let $\mathfrak{q}, (V, c)$ be as above with $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Let $W = V^*$, with matrix \mathfrak{q}^t , see footnote 2, and let $\{z_{\beta}^{(n)} : \beta \in \Delta_+^{\mathfrak{q}}, n \in \mathbb{N}\}$ be the generators of $\mathcal{L}_{\mathfrak{q}^t}$. Here we consider $W \in \frac{\mathbf{k}^{\mathbb{Z}^{\theta}}}{\mathbf{k}^{\mathbb{Z}^{\theta}}} \mathcal{YD}$ via the equivalence of categories between $\frac{(\mathbf{k}^{\mathbb{Z}^{\theta}})^*}{(\mathbf{k}^{\mathbb{Z}^{\theta}})^*} \mathcal{YD}$ and $\frac{\mathbf{k}^{\mathbb{Z}^{\theta}}}{\mathbf{k}^{\mathbb{Z}^{\theta}}} \mathcal{YD}$. Then we have a natural evaluation map such that $\langle w \otimes w', v \otimes v' \rangle = \langle w \otimes v' \rangle \langle w' \otimes v \rangle$. In this section we define the *quantum divided power algebra* $\mathcal{U}_{\mathfrak{q}}$ of (V, c) and we establish some of its basic properties.

Let Γ and Λ be two copies of \mathbb{Z}^{θ} , generated by $(K_i)_{i \in \mathbb{I}}$ and $(L_i)_{i \in \mathbb{I}}$ respectively; so that $(K_i^{\pm 1})_{i \in \mathbb{I}}$ and $(L_i^{\pm 1})_{i \in \mathbb{I}}$ are the generators of $\mathbf{k}\Gamma$ and $\mathbf{k}\Lambda$, respectively. Set $K_{\alpha} = K_1^{a_1} \dots K_{\theta}^{a_{\theta}}$ and $L_{\alpha} = L_1^{a_1} \dots L_{\theta}^{a_{\theta}}$ for $\alpha = (a_1, \dots, a_{\theta}) \in \mathbb{Z}^{\theta}$. Then $\mathcal{L}_{\mathfrak{q}} \in \frac{\mathbf{k}\Gamma}{\mathbf{k}\Gamma} \mathcal{YD}$, $\mathcal{L}_{\mathfrak{q}^t} \in \frac{\mathbf{k}\Lambda}{\mathbf{k}\Lambda} \mathcal{YD}$ with structure determined by the formulae

$$\begin{aligned} K_{\alpha}^{\pm 1} \cdot y_{\beta}^{(n)} &= q_{\alpha\beta}^{\pm n} y_{\beta}^{(n)}, & \rho(y_{\beta}^{(n)}) &= K_{\beta}^n \otimes y_{\beta}^{(n)}; \\ L_{\alpha}^{\pm 1} \cdot z_{\beta}^{(n)} &= q_{\beta\alpha}^{\pm n} z_{\beta}^{(n)}, & \rho(z_{\beta}^{(n)}) &= L_{\beta}^n \otimes y_{\beta}^{(n)}. \end{aligned}$$

Therefore, we can consider the bosonizations $\mathcal{L}_{\mathfrak{q}} \# \mathbf{k}\Gamma$ and $\mathcal{L}_{\mathfrak{q}^t} \# \mathbf{k}\Lambda$.

We define next the quantum double of $\mathcal{L}_{\mathfrak{q}} \# \mathbf{k}\Gamma$ and $\mathcal{L}_{\mathfrak{q}^t} \# \mathbf{k}\Lambda$ following [J, 3.2.2]. For this we need a Hopf pairing between them.

Lemma 5.1. *There is a unique bilinear form $(|) : T^c(V) \times (T^c(W))^{\text{cop}} \rightarrow \mathbf{k}$ such that $(1|1) = 1$,*

$$\begin{aligned} (y_i | z_j) &= \delta_{ij}, & i, j &\in \mathbb{I}; \\ (y | z z') &= (y^{(1)} | z)(y^{(2)} | z'), & y &\in T^c(V), z, z' \in T^c(W); \\ (y y' | z) &= (y | z^{(1)})(y' | z^{(2)}), & y, y' &\in T^c(V), z \in T^c(W); \\ (y | z) &= 0, & |y| &\neq |z|, y \in T^c(V), z \in T^c(W). \end{aligned}$$

Proof. Let $\mathbf{T}^n = \sum_{\sigma \in \mathbb{S}_n} s(\sigma) : (T^c)^n(W) \rightarrow T^n(W)$, where $s : \mathbb{S}_n \rightarrow \mathbb{B}_n$ is the Matsumoto section, see [AG, §3.2]. Let $\langle, \rangle : T^c(V) \otimes T(W)^{\text{op}} \rightarrow \mathbf{k}$ be the evaluation map. We define $(1|1) = 1$,

$$\begin{aligned} (y | z) &= \langle y, \mathbf{T}^n(z) \rangle, & y &\in (T^c)^n(V), z \in (T^c)^n(W) \\ (y | z) &= 0, & y &\in (T^c)^n(V), z \in (T^c)^m(W), n \neq m. \end{aligned}$$

Note that $\mathbf{T}^{i+j} = \mathbf{T}_{i,j}(\mathbf{T}^i \otimes \mathbf{T}^j)$ with $\mathbf{T}_{i,j} = \sum s(\sigma^{-1})$ where the sum is over all (i, j) -shuffles σ . Then, for $y \in (T^c)^n(V)$, $z \in (T^c)^{n-i}(W)$, $z' \in$

$(T^c)^i(W)$,

$$\begin{aligned} \langle y|zz' \rangle &= \langle y, \mathbf{T}^n(z'z) \rangle = \langle y, \mathbf{T}_{i,n-i}(\mathbf{T}^i \otimes \mathbf{T}^{n-i})(z'z) \rangle \\ &= \langle y, \mathbf{T}_{i,n-i}(\mathbf{T}^i(z') \otimes \mathbf{T}^{n-i}(z)) \rangle = \langle y^{(1)}, \mathbf{T}^{n-i}(z) \rangle \langle y^{(2)}, \mathbf{T}^i(z') \rangle \\ &= (y^{(1)}|z)(y^{(2)}|z') \end{aligned}$$

The other conditions are clear. □

This bilinear form restricts to $\mathcal{L}_q \times (\mathcal{L}_{q^t})^{\text{cop}}$ and then it can be extended to a bilinear form between their bosonizations. Then we may define a skew-Hopf pairing between $\mathcal{L}_q \# \mathbf{k}\Gamma$ and $\mathcal{L}_{q^t} \# \mathbf{k}\Lambda$, or equivalently:

Corollary 5.2. *There is a unique Hopf pairing*

$$(|) : \mathcal{L}_q \# \mathbf{k}\Gamma \times (\mathcal{L}_{q^t} \# \mathbf{k}\Lambda)^{\text{cop}} \rightarrow \mathbf{k}$$

such that for all $Y, Y' \in \mathcal{L}_q \# \mathbf{k}\Gamma$, $Z, Z' \in (\mathcal{L}_{q^t} \# \mathbf{k}\Lambda)^{\text{cop}}$, $y_\alpha^{(n)} \in \mathcal{L}_q$, $K_\alpha \in \mathbf{k}\mathbb{Z}^\theta$, $z_\beta^{(m)} \in \mathcal{L}_{q^t}$ and $L_\beta \in \mathbf{k}\mathbb{Z}^\theta$

$$\begin{aligned} (Y|ZZ') &= (Y_{(1)}|Z)(Y_{(2)}|Z'), & (YY'|z) &= (Y|Z_{(1)})(Y'|Z_{(2)}), \\ (y_\alpha|z_\beta) &= \delta_{\alpha,\beta}, & (y_\alpha^{(N_\alpha)}|z_\beta^{(N_\beta)}) &= \delta_{\alpha,\beta}, \\ (y_\alpha^{(n)}|L_\beta) &= 0, & (K_\alpha|z_\beta^{(m)}) &= 0, & (K_\alpha|L_\beta) &= q_{\alpha\beta}. \end{aligned}$$

Moreover, this pairing satisfies the equation $(yK|zL) = (y|z)(K|L)$.

Let \mathcal{U}_q be the Drinfeld double of $\mathcal{L}_q \# \mathbf{k}\Gamma$ and $(\mathcal{L}_{q^t} \# \mathbf{k}\Lambda)^{\text{cop}}$ with respect to the Hopf pairing in Corollary 5.2. In other words:

Definition 5.3. Let \mathcal{U}_q be the unique Hopf algebra such that

- 1) $\mathcal{U}_q = (\mathcal{L}_q \# \mathbf{k}\Gamma) \otimes (\mathcal{L}_{q^t} \# \mathbf{k}\Lambda)$ as vector spaces,
- 2) the maps $Y \mapsto Y \otimes 1$ and $Z \mapsto 1 \otimes Z$ are Hopf algebra morphisms,
- 3) the product is given by

$$(Y \otimes Z)(Y' \otimes Z') = (Y'_{(1)}|\mathcal{S}(Z_{(1)}))YY'_{(2)} \otimes Z_{(2)}Z'(Y'_{(3)}|Z_{(3)})$$

for all $Y, Y' \in \mathcal{L}_q \# \mathbf{k}\Gamma$ and $Z, Z' \in (\mathcal{L}_{q^t} \# \mathbf{k}\Lambda)^{\text{cop}}$.

By the construction of \mathcal{U}_q , there is a triangular decomposition, via the multiplication, $\mathcal{U}_q \simeq \mathcal{U}_q^+ \otimes \mathcal{U}^0 \otimes \mathcal{U}_q^-$ where

$$\mathcal{U}_q^+ \simeq \mathcal{L}_q, \quad \mathcal{U}_q^- \simeq \mathcal{L}_{q^t}, \quad \mathcal{U}^0 \simeq \mathbf{k}(\mathbb{Z}^\theta \times \mathbb{Z}^\theta).$$

We give a presentation of the algebra \mathcal{U}_q by generators and relations. The tensor product signs in elements of \mathcal{U}_q will be omitted.

Proposition 5.4. *The algebra \mathcal{U}_q is generated by the elements $y_\beta^{(n)}, z_\beta^{(n)}, K_\beta^{\pm 1}, L_\beta^{\pm 1}$ for $\beta \in \Delta_+^q, n \in \mathbb{N}$; and relations (18), ..., (22) between the $y_\beta^{(n)}$'s, similar relations for the $z_\beta^{(n)}$'s plus the relations*

$$\begin{aligned} (23) \quad & K_\beta K_\beta^{-1} = L_\beta^{-1} L_\beta = 1, \quad K_\beta^{\pm 1} L_\alpha^{\pm 1} = L_\alpha^{\pm 1} K_\beta^{\pm 1} \\ (24) \quad & K_\alpha y_\beta^{(n)} = q_{\alpha\beta}^n y_\beta^{(n)} K_\alpha, \quad L_\alpha y_\beta^{(n)} = q_{\beta\alpha}^{-n} y_\beta^{(n)} L_\alpha, \\ (25) \quad & K_\alpha z_\beta^{(n)} = q_{\alpha\beta}^{-n} z_\beta^{(n)} K_\alpha, \quad L_\alpha z_\beta^{(n)} = q_{\beta\alpha}^n z_\beta^{(n)} L_\alpha, \\ (26) \quad & zy = (y^{(1)} | \mathcal{S}(z^{(3)})) (K_2 K_3 | L_3^{-1}) (y^{(3)} | z^{(1)}) y^{(2)} K_3 z^{(2)} L_3, \end{aligned}$$

for all $\alpha, \beta \in \Delta_+^q, n, m \in \mathbb{N}$. Here in (26) $y = y_\beta^{(n)} \in \mathcal{L}_q, z = z_\alpha^{(m)} \in \mathcal{L}_{q^t}$, and denote $K_i = (y^{(i)})_{(-1)}$ and $L_i = (z^{(i)})_{(-1)}$ for the coactions of $\mathbf{k}\Gamma$ and $\mathbf{k}\Lambda$ respectively.

Note that if $y = y_{\alpha_i}, z = z_{\alpha_j}$ with $\alpha_i, \alpha_j \in \Pi_q$, then y, z are primitives and relation (26) is $zy - yz = \delta_{ij}(K_i - L_i)$.

5.2. Basic properties

Proceeding as in [DP, A3], we will prove that the algebra \mathcal{U}_q is noetherian. For each $\mathbf{h}, \mathbf{j} \in \mathbb{H}, K \in \Gamma, L \in \Lambda$, set

$$\begin{aligned} d_1(\mathbf{y}_\mathbf{h} K L \mathbf{z}_\mathbf{j}) &= \sum_{i \in \mathbb{1}_M} (h_i + j_i) \text{ht}(\beta_i), \\ d(\mathbf{y}_\mathbf{h} K L \mathbf{z}_\mathbf{j}) &= \left(d_1(\mathbf{y}_\mathbf{h} K L \mathbf{z}_\mathbf{j}), h_1, \dots, h_M, j_1, \dots, j_M \right) \in \mathbb{N}_0^{2M+1}. \end{aligned}$$

Consider the lexicographic order in \mathbb{N}_0^{2M+1} . If $\mathbf{u} \in \mathbb{N}_0^{2M+1}$, then we set

$$\mathcal{U}_q(\mathbf{u}) = \text{span of } \{ \mathbf{y}_\mathbf{h} K L \mathbf{z}_\mathbf{j} : \mathbf{h}, \mathbf{j} \in \mathbb{H}, K \in \Gamma, L \in \Lambda, d(\mathbf{y}_\mathbf{h} K L \mathbf{z}_\mathbf{j}) \leq \mathbf{u} \}.$$

Lemma 5.5. *$(\mathcal{U}_q(\mathbf{u}))_{\mathbf{u} \in \mathbb{N}_0^{2M+1}}$ is an \mathbb{N}_0^{2M+1} -algebra filtration of \mathcal{U}_q .*

Proof. It is enough to prove that $(\mathbf{y}_h K L \mathbf{z}_j)(\mathbf{y}_{h'} K' L' \mathbf{z}_{j'}) \in \mathcal{U}_q(\mathbf{u} + \mathbf{u}')$ for all $\mathbf{h}, \mathbf{j}, \mathbf{h}', \mathbf{j}' \in \mathbb{H}$, $K, K' \in \Gamma$ and $L, L' \in \Lambda$ where

$$d(\mathbf{y}_h K L \mathbf{z}_j) = \mathbf{u} \quad \text{and} \quad d(\mathbf{y}_{h'} K' L' \mathbf{z}_{j'}) = \mathbf{u}'.$$

First we claim that

$$(27) \quad d_1(z_\beta^{(n)} y_\alpha^{(m)} - y_\alpha^{(m)} z_\beta^{(n)}) < m \operatorname{ht}(\alpha) + n \operatorname{ht}(\beta).$$

Indeed, since the coproduct in \mathcal{L}_q (resp. \mathcal{L}_{q^t}) is graded, we have that

$$d_1((y_\alpha^{(m)})^{(2)}) < m \operatorname{ht}(\alpha)$$

if $(y_\alpha^{(m)})^{(1)} \neq 1$ (resp. $d_1((z_\beta^{(n)})^{(2)}) < n \operatorname{ht}(\beta)$ if $(z_\beta^{(n)})^{(1)} \neq 1$). Hence, for $K \in \Gamma$ and $L \in \Lambda$ we have

$$d_1((y_\alpha^{(m)})^{(2)} K L (z_\beta^{(n)})^{(2)}) \leq m \operatorname{ht}(\alpha) + n \operatorname{ht}(\beta)$$

and by Proposition 5.4 the claim follows.

Since K, L q -commutes with all elements of \mathcal{L}_q and \mathcal{L}_{q^t} for all $K \in \Gamma$ and $L \in \Lambda$. We proceed as in Lemma 4.12 and we reduce the proof to the product between $z_{\beta_i}^{(n)}$ and $y_{\beta_j}^{(m)}$. Then

$$z_{\beta_i}^{(n)} y_{\beta_j}^{(m)} \in \mathcal{U}_q(m \operatorname{ht}(\beta_j) + n \operatorname{ht}(\beta_i), \delta_j, \delta_i)$$

follows directly by (27). □

We consider the associated graded algebra $\operatorname{gr} \mathcal{U}_q = \bigoplus_{\mathbf{v} \in \mathbb{N}_0^{2M+1}} \mathcal{U}_q^{\mathbf{v}}$ where $\mathcal{U}_q^{\mathbf{v}} = \mathcal{U}_q(\mathbf{v}) / \sum_{\mathbf{u} < \mathbf{v}} \mathcal{U}_q(\mathbf{u})$.

Corollary 5.6. *The algebra $\operatorname{gr} \mathcal{U}_q$ is presented by generators $\mathbf{y}_j^{(n)}, \mathbf{z}_j^{(n)}, K_j^{\pm 1}, L_j^{\pm 1}, j \in \mathbb{I}_M, n \in \mathbb{N}$ and relations*

$$\begin{aligned} RS &= SR, & R, S &\in \{K_j^{\pm 1}, L_j^{\pm 1} : j \in \mathbb{I}_M\} \\ K_\beta K_\beta^{-1} &= L_\beta L_\beta^{-1} = 1, & \mathbf{y}_k^{(n)} \mathbf{z}_l^{(m)} &= \mathbf{z}_l^{(m)} \mathbf{y}_k^{(n)} \\ \mathbf{y}_k^{(N_k)} &= 0, \quad \beta_k \notin \mathfrak{D}_q, & \mathbf{z}_k^{(N_k)} &= 0, \quad \beta_k \notin \mathfrak{D}_q, \end{aligned}$$

$$\begin{aligned}
 y_k^{(n)} y_k^{(m)} &= \binom{n+m}{m}_{q_{\beta_k \beta_k}} y_k^{(n+m)}, & z_k^{(n)} z_k^{(m)} &= \binom{n+m}{m}_{q_{\beta_k \beta_k}} z_k^{(n+m)}, \\
 [y_k^{(n)}, y_l^{(m)}]_c &= 0, \quad l < k, & [z_k^{(n)}, z_l^{(m)}]_c &= 0, \quad l < k, \\
 K_\alpha y_\beta^{(n)} &= q_{\alpha\beta}^n y_\beta^{(n)} K_\alpha, & K_\alpha z_\beta^{(n)} &= q_{\alpha\beta}^{-n} z_\beta^{(n)} K_\alpha, \\
 L_\alpha y_\beta^{(n)} &= q_{\beta\alpha}^{-n} y_\beta^{(n)} L_\alpha, & L_\alpha z_\beta^{(n)} &= q_{\beta\alpha}^n z_\beta^{(n)} L_\alpha.
 \end{aligned}$$

Proof. The proof of this statement is similar to the proof of Lemma 4.13 if we check that $y_k^{(n)} z_l^{(m)} = z_l^{(m)} y_k^{(n)}$ for all $y_k^{(n)} \in \mathcal{L}_q$ and $z_l^{(m)} \in \mathcal{L}_{q^t}$; but this follows by (27). □

Proposition 5.7. *The algebra \mathcal{U}_q is noetherian and*

$$\text{GKdim } \mathcal{U}_q = 2|\mathfrak{D}_q| + 2\theta.$$

Proof. Let \mathcal{Z} be the subalgebra of $\text{gr } \mathcal{U}_q$ generated by $\{y_\beta^{(N_\beta)}, z_\beta^{(N_\beta)} : \beta \in \mathfrak{D}_q\}$ and $\{K_i, L_i : i \in \mathbb{I}\}$. Then \mathcal{Z} is the localization of a quantum affine space and $\text{gr } \mathcal{U}_q$ is a free \mathcal{Z} -module of rank $\prod_{i \in \mathbb{I}_M} N_i$. Therefore $\text{gr } \mathcal{U}_q$ is noetherian and so is \mathcal{U}_q . Moreover, by [KL, Prop. 6.6],

$$\text{GKdim } \mathcal{U}_q = \text{GKdim } \text{gr } \mathcal{U}_q = \text{GKdim } \mathcal{Z} = 2|\mathfrak{D}_q| + 2\theta.$$

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