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Weyl modules for $\mathfrak{osp}(1,2)$ and nonsymmetric Macdonald polynomials

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The main goal of our paper is to establish a connection between the Weyl modules of the current Lie superalgebras (twisted and untwisted) attached to $\mathfrak{osp}(1,2)$ and the nonsymmetric Macdonald polynomials of types $A_2^{(2)}$ and $A_2^{(2)\dagger}$. We compute the dimensions and construct bases of the Weyl modules. We also derive explicit formulas for the $t=0$ and $t=\infty$ specializations of the nonsymmetric Macdonald polynomials. We show that the specializations can be described in terms of the Lie superalgebras action on the Weyl modules.

Introduction

The Weyl modules play important role in the representation theory of infinite-dimensional Lie algebras (see [CP, CL, FoLi1, FeLo1]). These representations pop up in various problems of representation theory and its applications [CFK, FeLo2, FoLi1, FoLi2, SVV]. The most important for us is the established in many cases isomorphism between the Weyl modules and Demazure modules (at least, in types A, D, E) for integrable representations of the affine Kac-Moody algebras. The Demazure modules are known to provide finite-dimensional approximation of the infinite-dimensional modules of the affine Kac-Moody Lie algebras and hence so do the Weyl modules. Yet another consequence from the link between the Demazure and Weyl modules is that the characters of both (in the level one case) can be expressed in terms of the nonsymmetric Macdonald polynomials (see [I, S]). The nonsymmetric Macdonald polynomials (see [M1, M2]) are rational functions in parameters q and t and the characters of the Demazure modules are equal to the $t=0$ specialization. It was conjectured recently (see [CO1, CO2, FM]) that the $t=\infty$ specialization also has representation theoretic realization in terms of the PBW filtration (see [FFL1, FFL2] for the case of simple Lie algebras).

Now let us turn to the case of superalgebras. It is not clear what should be an appropriate definition of a Demazure module for superalgebras. However, the Weyl modules are perfectly well defined (see e.g. [CLS]). So there are two natural questions here. The first one is to compute the characters of the Weyl modules for affine superalgebras and to find a connection with some super analogues of the nonsymmetric Macdonald polynomials. The second question is to figure out if a limit of the Weyl modules (when the highest weight grows) does exist. In this paper we consider the smallest Lie superalgebra $\mathfrak{osp}(1,2)$, which plays in the super theory a role similar to that of the Lie algebra $\mathfrak{sl}_2$ in the theory of classical simple Lie algebras. The Weyl modules in this case are parametrized by a non-negative integer n; we denote the corresponding $\mathfrak{osp}(1,2)\otimes\mathbb{C}[t]$ module by W_{-n}. We prove the following theorem.

Theorem 0.1. *W_{-n} as $\mathfrak{osp}(1,2)$-module is isomorphic to the tensor product of n copies of 3-dimensional irreducible $\mathfrak{osp}(1,2)$-module. Moreover, the $\mathfrak{osp}(1,2)\otimes\mathbb{C}[t]$-module structure is given by the graded tensor product (fusion product) construction.*

We show that W_{-n} can be filtered by the Weyl modules for $\mathfrak{sl}_2\otimes\mathbb{C}[t]$. This allows us to construct bases and compute the characters of W_{-n}. Our next goal is to relate the characters of the Weyl modules to the nonsymmetric Macdonald polynomials ([Ch1, Ch2]). Using the Ram-Yip formula for the nonsymmetric Macdonald polynomials (see [RY, OS]), we prove the following theorem.

Theorem 0.2. *Let $E_{-n}^{A_2^{(2)\dagger}}(x,q,t)$ be the nonsymmetric Macdonald polynomials of type $A_2^{(2)\dagger}$. Then the character of W_{-n} is given by $E_{-n}^{A_2^{(2)\dagger}}(x,q,0)$ and the specialization $E_{-n}^{A_2^{(2)\dagger}}(x,q,\infty)$ coincides with the PBW twisted character of W_{-n}.*

We close the introduction with several remarks.

First, the $\mathfrak{osp}(1,2)$ current algebra has the twisted analogue $\mathfrak{osp}(1,2)[t]^\sigma$. We study all the questions described above in the twisted case as well. In particular, we establish a connection with the specializations of the nonsymmetric Macdonald polynomials of type $A_2^{(2)}$. We note that both algebras $\mathfrak{osp}(1,2)[t]$ and $\mathfrak{osp}(1,2)[t]^\sigma$ are Borel's subalgebras in the affine superalgebra $\widehat{\mathfrak{osp}(1,2)}$.

Second, in both twisted and untwisted cases we define the positive n versions of the Weyl modules W_n. We also make a link to the Macdonald polynomials.

Third, we show that there exist embeddings of $\mathfrak{osp}(1,2)\otimes\mathbb{C}[t]$-modules $W_{-n}\subset W_{-n-1}$ and we compute the character of the (infinite-dimensional) limit. We note that we do not know if there is a structure of the representation of a larger algebra on this limit.

Finally, let us mention that in the Macdonald polynomials part of our paper we follow the ideas and methods of the paper [OS]. In Appendix A we describe the most important for us ingredients of the approach of Orr and Shimozono.

The paper is organized as follows:

In Section 1 we study the Weyl modules for $\mathfrak{osp}(1,2)\otimes\mathbb{C}[t]$ and their twisted version. In Section 2 we derive explicit formulas for the types $A_2^{(2)}$ and $A_2^{(2)\dagger}$ Macdonald polynomials. In Section 3 we establish a connection between the two parts of the story.

1. Weyl modules

1.1. The classical case

Let D_{-n}, $n\geq 0$ be the Weyl modules for the current algebra $\mathfrak{sl}_2[t]=\mathfrak{sl}_2\otimes\mathbb{C}[t]$. They are defined as finite-dimensional cyclic modules with cyclic vector d_{-n}, subject to the conditions

$$hd_{-n}=-nd_{-n},\quad f\otimes\mathbb{C}[t].d_{-n}=0,\quad h\otimes t\mathbb{C}[t].d_{-n}=0,$$

where e,h,f for the standard basis of $\mathfrak{sl}_2$. These modules are known to be 2^n-dimensional with a monomial basis

$$e_{a_1}\cdots e_{a_k}d_{-n},\quad 0\leq a_1\leq\cdots\leq a_k\leq n-k.$$

For a graded vector space $M=\bigoplus_{s\geq 0}M_s$ with an action of the operator h We define the character $\mathrm{ch}M(x,q)$ as $\sum_{s\geq 0}q^s\mathrm{tr}(x^h|_{M_s})$. The character of D_{-n} is equal to $\sum_{k=0}^n x^{-n+2k}\binom{n}{k}_q$, where the q-binomial coefficients are given by the formula

$$\binom{n}{m}_q=\frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^m)(1-q)\cdots(1-q^{n-m})}.$$

The modules D_{-n} are known to be isomorphic to the graded tensor product (the fusion product [FeLo1]) of n copies of standard 2-dimensional $\mathfrak{sl}_2$-modules. Moreover, D_{-n} is isomorphic to a Demazure module in the basic level one representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. In particular, there exist embeddings of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules

$$D_0 \subset D_{-2} \subset D_{-4} \subset \cdots, \qquad D_{-1} \subset D_{-3} \subset D_{-5} \subset \cdots$$

and the inductive limits are isomorphic to the integrable $\widehat{\mathfrak{sl}}_2$ modules of level 1. We have the explicit formula for the characters of the limits:

$$\mathrm{ch} \lim_{n\to\infty} D_{-2n} = \sum_{k\in\mathbb{Z}} x^{2k} \frac{q^{k^2}}{(q)_\infty}, \quad \mathrm{ch} \lim_{n\to\infty} D_{-2n-1} = \sum_{k\in\mathbb{Z}} x^{2k+1} \frac{q^{k(k+1)}}{(q)_\infty}.$$

1.2. Weyl modules for superalgebras

Our references here are [P, Mus1, Mus2]. The Lie superalgebra $\mathfrak{osp}(1,2)$ is isomorphic to the direct sum $\mathfrak{sl}_2 \oplus \pi_1$, where $\mathfrak{sl}_2$ is the even part and the two-dimensional $\mathfrak{sl}_2$ module π_1 if the odd part. Let e, h, f be the standard basis of $\mathfrak{sl}_2$ and let g^+, g^- be the basis of π_1. One has the nontrivial commutation relations

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f,$$
$$[h,g^+] = g^+, \quad [h,g^-] = -g^-, \quad [g^+,g^-]_+ = h,$$
$$[g^+,g^+]_+ = 2e, \quad [g^-,g^-]_+ = -2f, \quad [f,g^+] = g^-, \quad [e,g^-] = -g^+.$$

We have the Cartan decomposition

$$\mathfrak{osp}(1,2) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{n}^- = \mathrm{span}(f, g^-),$$
$$\mathfrak{n}^+ = \mathrm{span}(e, g^+), \quad \mathfrak{h} = span(h).$$

We consider the current algebra $\mathfrak{osp}(1,2)[t] = \mathfrak{osp}(1,2) \otimes \mathbb{C}[t]$ and its Weyl module W_{-n} defined as the cyclic module with a generator w_{-n} subject to the relations

$$(\mathfrak{n}^- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t].w_{-n} = 0, \quad (\mathfrak{n}^- \otimes 1).w_{-n} = 0,$$
$$(h \otimes 1).w_{-n} = -nw_{-n}, \quad (e \otimes 1)^{n+1}.w_{-n} = 0.$$

For $x \in \mathfrak{osp}(1,2)$ we denote by $x_i \in \mathfrak{osp}(1,2)[t]$ the element $x \otimes t^i$.

Lemma 1.1. *One has* $\mathfrak{osp}(1,2)\otimes t^n\mathbb{C}[t].w_{-n}=0$ *and* W_{-n} *is spanned by the monomials of the form*

$$e_{a_1}\cdots e_{a_s}g^+_{b_1}\cdots g^+_{b_k}w_{-n},\qquad 0\le b_1<\cdots<b_k\le n-1,\quad 0\le a_1\le\cdots\le a_s\le n-k-s. \tag{1.1}$$

Proof. The condition $\mathfrak{sl}_2\otimes t^n\mathbb{C}[t]w_{-n}=0$ follows from the results on the Weyl modules for $\mathfrak{sl}_2$ (see e.g. [CL]). Now if $e_iw_{-n}=0$, then $g^-_0e_iw_{-n}=g^+_iw_{-n}=0$ and similarly for $g^-_iw_{-n}$.

We note that since $[g^+_i,g^+_j]_+=2e_{i+j}$, we have

$$W_{-n}=\sum_{0\le b_1<\cdots<b_k\le n-1}U(\mathfrak{sl}_2\otimes\mathbb{C}[t])g^+_{b_1}\cdots g^+_{b_k}w_{-n}.$$

We introduce a partial order on the monomials $g^+_{b_1}\cdots g^+_{b_k}$, $0\le b_1<\cdots<b_k\le n-1$, saying that for two different monomials $g^+_{b_1}\cdots g^+_{b_k}\le g^+_{c_1}\cdots g^+_{c_l}$ if $k<l$ or ($k=l$ and $b_i\ge c_i$, $i=1,\dots,k$). Let us totally order the monomials $g^+_{b_1}\cdots g^+_{b_k}$, $0\le b_1<\cdots<b_k\le n-1$ as $m_1,m_2,\dots,m_N$ in such a way that if $m_i<m_j$ with respect to the partial order then $i<j$.

Now let us introduce an increasing filtration F_i on W_{-n} by

$$F_i=\sum_{j=1}^{i}U(\mathfrak{sl}_2\otimes\mathbb{C}[t])m_iw_{-n}.$$

We claim that the monomials (1.1) span the associated graded space $\mathrm{gr}F_\bullet$. Indeed, let $\overline{m_iw_{-n}}$ be the image of m_iw_{-n} in the associated graded space. Then for $m_i=g^+_{b_1}\cdots g^+_{b_k}$ we need to show that

$$(\mathfrak{n}^-\oplus\mathfrak{h})\otimes t\mathbb{C}[t].\overline{m_iw_{-n}}=0,\quad(\mathbb{C}f).\overline{m_iw_{-n}}=0,$$
$$(h\otimes 1).\overline{m_iw_{-n}}=(-n+k)w_{-n}.$$

The last equality is obvious. Let us prove that for $m>0$ the vector $h_m.\overline{m_iw_{-n}}$ vanishes. Commuting h_m through $g^+_{b_1}\cdots g^+_{b_k}$ and using $h_mw_{-n}=0$, we obtain

$$h_mg^+_{b_1}\cdots g^+_{b_k}w_{-n}=\sum_{i=1}^k g^+_{b_1}\cdots g^+_{b_{i-1}}g^+_{b_i+m}g^+_{b_{i+1}}\cdots g^+_{b_k}w_{-n}.$$

We note that if $b_i+m=b_j$ for some j, then the product $g^+_{b_i+m}g^+_j=e_{2j}$ and hence

$$\overline{g^+_{b_1}\cdots g^+_{b_{i-1}}g^+_{b_i+m}g^+_{b_{i+1}}\cdots g^+_{b_k}w_{-n}}=0. \tag{1.2}$$

If $b_i + m$ is different from all the b_j, then the positivity of m implies that (1.2) vanishes in $\mathrm{gr}F_\bullet$. Finally, similarly one shows that for $m \geq 0$ the vector $f_m.\overline{m_i w_{-n}}$ vanishes.

We have shown that all the relations for the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ Weyl module with lowest weight $-n+k$ are satisfied, which implies the claim of the lemma. In particular, one sees that the $\mathfrak{sl}_2[t]$ module generated by $\overline{m_i w_{-n}}$ is finite-dimensional. $\square$

Lemma 1.2. *The number of elements in the set* (1.1) *is equal to* 3^n *and its character is given by*

$$\sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q \sum_{s=0}^{n-k} x^{-n+k+2s} \binom{n-k}{s}_q.$$

Proof. We note that the character of the set of vectors (1.1) with fixed indices $b_1, \ldots, b_k$ is equal to $\sum_{s=0}^{n-k} x^{-n+k+2s} \binom{n-k}{s}_q$. Since the character of the space

$$g^+_{b_1} \cdots g^+_{b_k} w_{-n}, \quad 0 \leq b_1 < \cdots < b_k \leq n-1$$

is equal to $\sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q$, we deduce the claim of the lemma. $\square$

Now let us give the twisted version of the lemma above. We replace the current algebra $\mathfrak{osp}(1,2)[t]$ with its twisted analogue

$$\mathfrak{osp}(1,2)[t]^\sigma = \bigoplus_{i=0}^{\infty} \mathfrak{sl}_2 \otimes t^{2i} \oplus \bigoplus_{i=0}^{\infty} \pi_1 \otimes t^{2i+1}.$$

This is again a Lie superalgebra. We define its Weyl module W^σ_{-n} as the cyclic integrable with respect to $\mathfrak{sl}_2$ module with a generator w^σ_{-n} subject to the relations

$$f_{2i}.w^\sigma_{-n} = 0, \quad g^-_{2i+1}.w^\sigma_{-n} = 0, \quad h_{2i+2}.w^\sigma_{-n} = 0, \quad \text{for } i \geq 0$$

and $h_0.w^\sigma_{-n} = -nw^\sigma_{-n}$, $e_0^{n+1}.w_{-n} = 0$.

Lemma 1.3. *One has* $\mathfrak{sl}_2 \otimes t^{2n}\mathbb{C}[t^2].w_{-n} = 0$ *and* $\pi_1 \otimes t^{2n+1}\mathbb{C}[t^2].w_{-n} = 0$. W^σ_{-n} *is spanned by the monomials of the form*

$$(1.3) \qquad e_{a_1} \cdots e_{a_s} g^+_{b_1} \cdots g^+_{b_k} w_{-n},$$
$$1 \leq b_1 < \cdots < b_k \leq 2n-1, \quad 0 \leq a_1 \leq \cdots \leq a_s \leq 2(n-k-s),$$

where a_i *are even and* b_j *are odd.*

Lemma 1.4. *The number of elements in the set* (1.3) *is equal to* 3^n *and its character is given by*

$$\sum_{k=0}^{n} q^{k^2} \binom{n}{k}_{q^2} \sum_{s=0}^{n-k} x^{-n+k+2s} \binom{n-k}{s}_{q^2}.$$

1.3. The graded tensor products for superalgebras

We start with the untwisted case.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with $\mathfrak{g}_{\bar{0}}$ being the even part and $\mathfrak{g}_{\bar{1}}$ being the odd part. For a $\mathfrak{g}$-module X we denote by $X_{\bar{0}}$ its even part and by $X_{\bar{1}}$ its odd part. Let V and W be cyclic $\mathfrak{g}$ modules with cyclic vectors v and w; in what follows we always assume that the cyclic vectors are even. The tensor product of V and W is defined by the formula

$$g(x \otimes y) = gx \otimes y + (-1)^{ab} x \otimes gy, \quad g \in \mathfrak{g}_{\bar{a}}, x \in V_{\bar{b}}.$$

Let $z_1, \dots, z_n$ be a collection of pairwise distinct complex numbers and let $V^1, \dots, V^n$ be cyclic representations of $\mathfrak{g}$ with cyclic vectors $v^1, \dots, v^n$. Let $V^i(z_i)$ be the evaluation representations of $\mathfrak{g} \otimes \mathbb{C}[t]$, where $x \otimes t^k$ acts as $z_i^k x$.

Lemma 1.5. *The tensor product* $\bigotimes_{i=1}^n V^i(z_i)$ *is cyclic* $\mathfrak{g}[t]$*-module with cyclic vector* $\otimes_{i=1}^n v^i$.

Proof. Let $x \in \mathfrak{g}_{\bar{0}}$. Then the operator $x \otimes t^k$ acts on the tensor product $\bigotimes V^i(z_i)$ by the usual tensor product formula for representations of Lie algebras. Therefore all the operators

$$x(i) = \underbrace{\mathrm{Id} \otimes \cdots \mathrm{Id}}_{i-1} \otimes x \otimes \mathrm{Id} \otimes \cdots \mathrm{Id}$$

on $\bigotimes_{i=1}^n V^i(z_i)$ can be written as linear combinations of the operators $x \otimes t^k$ (via the Vandermonde determinant).

Now assume that $x \in \mathfrak{g}_{\bar{1}}$. Then one has

$$(x \otimes t^k)(u_1 \otimes \cdots \otimes u_n) = (z_1^k x(1) + z_2^k (-1)^{\deg u_1} x(2) + \cdots \\ + z_n^k (-1)^{\deg u_1 + \cdots + \deg u_{n-1}} z_n^k x(n))(u_1 \otimes \cdots \otimes u_n).$$

Therefore for any $i = 1, \dots, n$ the operator $(-1)^{\deg u_1 + \cdots + \deg u_{i-1}} x(i)$ can be expressed as linear combination of the operators $x \otimes t^j$, $0 \le j \le n-1$ (the case $i = 1$ corresponds to just $x(1)$). Therefore the operators $x \otimes t^j$, $0 \le j \le n-1$ do generate the whole tensor product $\bigotimes_{i=1}^n V^i(z_i)$ acting on the tensor product of cyclic vectors. □

The universal enveloping algebra $\mathrm{U}(\mathfrak{g}[t])$ has natural grading coming for the degree of t, $\mathrm{U}(\mathfrak{g}[t]) = \bigoplus_{s \ge 0} \mathrm{U}(\mathfrak{g}[t])_s$ (for example, $\mathrm{U}(\mathfrak{g}[t])_0 = \mathrm{U}(\mathfrak{g})$). Let us introduce the increasing filtration F_s on the tensor product $\bigotimes_{i=1}^n V^i(z_i)$ as follows:

$$F_s = \mathrm{U}(\mathfrak{g}[t])_s(v^1 \otimes \cdots \otimes v^n).$$

The associated graded space is cyclic $\mathrm{U}(\mathfrak{g}[t])$ module. An important feature is that it is now equipped with the additional grading. We denote the graded module by $V^1(z_1) * \cdots * V^n(z_n)$.

Let V be the irreducible 3-dimensional representation of $\mathfrak{osp}(1,2)$.

Theorem 1.6. *For any pairwise distinct $z_1, \dots, z_n$ the graded tensor product $V(z_1) * \cdots * V(z_n)$ is isomorphic to the Weyl module W_{-n} as $\mathfrak{osp}(1,2)[t]$ modules.*

Proof. It is easy to see that all the defining relations of the Weyl module do hold in the graded tensor product. Therefore, we have a surjection $W_{-n} \to V(z_1) * \cdots * V(z_n)$. Since the dimension of the right hand side is 3^n and this is the upper estimate for the dimension of the left hand side (see Lemma 1.2), the surjection is the isomorphism. □

Corollary 1.7. *The graded tensor product $V(z_1) * \cdots * V(z_n)$ does not depend (as $\mathfrak{osp}(1,2)[t]$-module) on the (pairwise distinct) parameters z_i.*

Corollary 1.8. *Vectors* (1.1) *form a basis of the Weyl module W_{-n}.*

Now let us consider the twisted algebra $\mathfrak{osp}(1,2)[t]^\sigma$. For a complex number z one defines the 3-dimensional evaluation $\mathfrak{osp}(1,2)[t]^\sigma$-module $V^\sigma(z)$ via the same formula as above. We have the following theorem.

Theorem 1.9. *Assume that the numbers $z_1, \dots, z_n$ satisfy the conditions $z_i^2 \ne z_j^2$ for $i \ne j$. Then the graded tensor product $V^\sigma(z_1) * \cdots * V\sigma(z_n)$ is well defined and is isomorphic to the Weyl module W_{-n}^σ as $\mathfrak{osp}(1,2)[t]$ modules.*

Proof. The only difference with the untwisted case is the condition $z_i^2 \ne z_j^2$, which guaranties the cyclicity of the tensor product of evaluation modules in the twisted case. □

Corollary 1.10. *The graded tensor product $V^\sigma(z_1) * \cdots * V^\sigma(z_n)$ does not depend (as $\mathfrak{osp}(1,2)[t]^\sigma$-module) on the parameters z_i, satisfying $z_i^2 \neq z_j^2$ for all $i \neq j$. Vectors* (1.3) *form a basis of the Weyl module W^σ_{-n}.*

1.4. The positive n case

In this subsection we define the modules W_n for $n > 0$. We first consider the untwisted case.

We define vector $w_n = e_0^n w_{-n} \subset W_{-n}$. We note that there is a symmetry (the A_1 Weyl group action) on W_{-n} interchanging e with f and g^+ with g^-. This symmetry sends w_{-n} to w_n and vice versa. Therefore, the module W_{-n} enjoys the basis of the form

$$\begin{gathered} f_{a_1} \cdots f_{a_s} g^-_{b_1} \cdots g^-_{b_k} w_n, \\ 0 \le b_1 < \cdots < b_k \le n-1, \quad 0 \le a_1 \le \cdots \le a_s \le n-k-s. \end{gathered} \tag{1.4}$$

Let $W_n = \mathrm{U}(\mathfrak{n}^- \otimes t\mathbb{C}[t]).w_n \subset W_{-n}$, so W_n is a module for the shifted Borel subalgebra, generated by g_1^- and g_0^+. In other words, the module W_n is generated from the vector w_n by the action of the operators f_i and g_i^-, $i > 0$.

Proposition 1.11. $\dim W_n = 3^{n-1}$. *The vectors*

$$\begin{gathered} f_{a_1} \cdots f_{a_s} g^-_{b_1} \cdots g^-_{b_k} w_{-n}, \\ 1 \le b_1 < \cdots < b_k \le n-1, \quad 1 \le a_1 \le \cdots \le a_s \le n-s-k \end{gathered} \tag{1.5}$$

form a basis of W_n.

Proof. The vectors (1.5) belong to the basis of the module W_{-n} and hence are linear independent. Now we know that for the 2^{n-k} dimensional Weyl module for $\mathfrak{sl}_2$ the part generated by $f_1, f_2, \dots$ from the lowest weight vector has basis of the form $f_{a_1} \cdots f_{a_s}$, where $1 \le a_1 \le \cdots \le a_s \le n-s-k$. Now using filtration from the proof of Lemma 1.1 we obtain that (1.5) is indeed a basis. □

Corollary 1.12. *The character of W_n is equal to*

$$\sum_{k=0}^{n-1} q^{k(k+1)/2} \binom{n-1}{k}_q \sum_{s=0}^{n-k-1} x^{n-k-2s} q^s \binom{n-k-1}{s}_q.$$

Now let us work out the twisted case.

We define vector $w_n^\sigma = e_0^n w_{-n}^\sigma \subset W_{-n}^\sigma$. Let

$$W_n^\sigma = \mathrm{U}((\mathfrak{osp}(1,2)\otimes t\mathbb{C}[t])^\sigma).w_n^\sigma \subset W_{-n},$$

i.e. W_n^σ is a cyclic module for the shifted Borel subalgebra, generated by g_1^- and e_0. The module W_n^σ is generated from the vector w_n by the action of the operators f_i, $i = 2,4,\dots,2n-2$ and g_i^-, $i = 1,3,\dots,2n-1$.

Proposition 1.13. $\dim W_n = 2\cdot 3^{n-1}$. *The vectors*

$$f_{a_1}\cdots f_{a_s} g_{b_1}^- \cdots g_{b_k}^- w_n^\sigma, \tag{1.6}$$
$$1 \le b_1 < \dots < b_k \le 2n-3, \quad 2 \le a_1 \le \dots \le a_s \le 2(n-s-k)$$

and

$$f_{a_1}\cdots f_{a_s} g_{b_1}^- \cdots g_{b_{k-1}}^- g_{2n-1}^- w_n^\sigma, \tag{1.7}$$
$$1 \le b_1 < \dots < b_{k-1} \le 2n-3, \quad 0 \le a_1 \le \dots \le a_s \le 2(n-s-k).$$

form a basis of W_n^σ.

Proof. We first prove that the elements (1.7) belong to W_n^σ (this is obvious for (1.6), but not for (1.7)). The only problem is the operator f_0 popping up in (1.7). However, it always comes multiplied by g_{2n-1}^-. Therefore, we only need to check that $f_0^m g_{2n-1}^- w_n \in W_n^\sigma$ for all $m > 0$. First, we note that by the weight reason $f_0^n g_{2n-1}^- w_n = 0$. Second, it suffices to prove the claim for $m = n-1$, since W_n^σ is e_0-invariant. Third, the vectors $f_0^{n-1} g_{2n-1}^- w_n$ is proportional to $f_2^{n-1} g_1^- w_n$. Indeed, since the weight space containing both vectors is one-dimensional, we only need to check that $f_2^{n-1} g_1^- w_n \ne 0$. Assume that this vector vanishes. Then $e_0^{n-1} f_2^{n-1} g_1^- w_n = 0$. However, up to a nonzero constant, this vector is equal to $g_{2n-1}^- w_n$, which does not vanish.

So we know that all the vectors (1.6) and (1.7) belong to W_n^σ. We also know that they are linearly independent, since they belong to the basis (1.3) of the whole space W_{-n}^σ. Since the sets (1.6) and (1.7) contain $2\cdot 3^{n-1}$ elements, we are left to show that the dimension of $W_{-n}^\sigma / W_n^\sigma \ge 3^{n-1}$. We know that relations in W_{-n}^σ are generated by $e_0^{n+1} w_{-n}$. Since the vector $e_0^n w_{-n}$ is trivial in the quotient, we conclude that

$$\dim W_{-n}^\sigma / W_n^\sigma \ge \dim W_{-n+1}^\sigma = 3^{n-1}.$$

□

Corollary 1.14. *The character of* W_n^σ *is equal to*

$$\sum_{k=0}^{n-1} q^{k^2} \binom{n-1}{k}_{q^2} \sum_{s=0}^{n-k-1} x^{n-k-2s} q^{2s} \binom{n-k-1}{s}_{q^2} + x^{-1} q^{2n-1} \sum_{k=0}^{n-1} q^{k^2} \binom{n-1}{k}_{q^2} \sum_{s=0}^{n-k-1} x^{n-k-2s} \binom{n-k-1}{s}_{q^2}.$$

1.5. The limit procedure

In this subsection we consider the $n \to \infty$ limits of the modules W_{-n} and W_{-n}^σ.

Lemma 1.15. *For $n > 0$ there exists an embedding of $\mathfrak{osp}(1,2)[t]$ modules $W_{-n} \to W_{-n-1}$, defined by $w_{-n} \mapsto g_n^+ w_{-n-1}$.*

Proof. First, we show that the vector $g_n^+ w_{-n-1}$ satisfies all the annihilation conditions for the cyclic vector of W_{-n}. Clearly, $h_0 g_n^+ w_{-n-1} = -n g_n^+ w_{-n-1}$. Now, for $x \in \mathfrak{n}^-$ and $k \geq 0$ one has $x_k g_n^+ w_{-n-1} = 0$ (because $x_k w_{-n-1} = 0$) and $[x_k, g_n^+] w_{-n-1} = 0$ because

$$[x_k, g_n^+] \subset (\mathfrak{n}^- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathbb{C}g^+ \otimes t^n \mathbb{C}[t].$$

Finally, for $k > 0$ $h_k g_n^+ w_{-n-1} = 0$, because $[h_k, g_n^+] = g_{n+k}^+$. We conclude that there exists a surjective homomorphism of $\mathfrak{osp}(1,2)[t]$-modules $W_{-n} \to \mathrm{U}(\mathfrak{osp}(1,2)[t]) g_n^+ w_{-n-1}$.

Second, we check that $\dim \mathrm{U}(\mathfrak{osp}(1,2)[t]) g_n^+ w_{-n-1} \geq 3^n$. Indeed, the vectors

$$e_{a_1} \cdots e_{a_s} g_{b_1}^+ \cdots g_{b_k}^+ g_n^+ w_{-n-1}$$

with $0 \leq b_1 < \cdots < b_k \leq n-1$ and $0 \leq a_1 \leq \cdots \leq a_s \leq n-k-s$ are linearly independent, since they belong to the set of basis vectors (1.1) (with $n+1$ instead of n). The number of these elements is 3^n. □

Let $L = \lim_{n \to \infty} W_{-n}$ be the $\mathfrak{osp}(1,2)[t]$-module, obtained via the embeddings from Lemma 1.15. We define the character of L as follows:

$$\mathrm{ch}L(x,q) = \lim_{n \to \infty} q^{n(n-1)/2} \mathrm{ch}W_{-n}(x, q^{-1}).$$

Our goal is to find a formula for the character of L. The following lemma is well known (see e.g. [A]).

Lemma 1.16. $\sum_{k\geq 0}\frac{q^{k(k-1)/2}}{(q)_k}x^k=\prod_{i=0}^{\infty}(1+q^ix)$.

Theorem 1.17. $\mathrm{ch}L(x,q)=\prod_{i=0}^{\infty}(1+q^ix)\prod_{i=0}^{\infty}(1+q^ix^{-1})$.

Proof. Recall the basis (1.1) of the space W_{-n}. Let $l_0(n)\in W_{-n}$ be the vector $g_0^+g_1^+\cdots g_{n-1}^+w_{-n}$. We note that the embedding $W_{-n}\to W_{-n-1}$ sends $l_0(n)$ to $l_0(n+1)$. Therefore, in the limit we obtain the vector $l_0=\lim_{n\to\infty}l_0(n)\in L$. We note that the summand in the character of L corresponding to l_0, is just $1(=x^0q^0)$. It is convenient to parametrize the images of the elements $g_{b_1}^+\cdots g_{b_k}^+w_{-n}$, $0\leq b_1\leq\cdots\leq b_k\leq n-1$ in the limit space L by the set

$$g_{-c_1}^-\cdots g_{-c_k}^-l_0,\quad 0\leq c_1<\cdots<c_k$$

(although we do not have the action of the operators with negative powers of t). Then we obtain the basis of L of the form

(1.8)
$$e_{a_1}\cdots e_{a_s}g_{-c_1}^-\cdots g_{-c_k}^-l_0,\quad 0\leq c_1<\cdots<c_k,\quad 0\leq a_1\leq\cdots\leq a_s\leq k-s.$$

The character of the basis vectors (1.8) with fixed k and s is equal to

$$x^{-k+2s}\frac{q^{k(k-1)/2}}{(q)_k}\binom{k}{s}_{q^{-1}}=\frac{x^{-k+2s}q^{k(k-1)/2-k(s-k)}}{(q)_s(q)_{k-s}}$$
$$=\frac{q^{s(s-1)/2}x^s}{(q)_s}\times\frac{q^{(k-s)(k-s-1)/2}x^{s-k}}{(q)_{k-s}}.$$

Therefore

$$\mathrm{ch}L(x,q)=\sum_{s\leq k}\frac{q^{s(s-1)/2}x^s}{(q)_s}\times\frac{q^{(k-s)(k-s-1)/2}x^{s-k}}{(q)_{k-s}}$$
$$=\prod_{i=0}^{\infty}(1+q^ix)\prod_{i=0}^{\infty}(1+q^ix^{-1})$$

(the last equality comes form Lemma 1.16). □

Remark 1.18. We note that the character $\mathrm{ch}L(x,q)$ is equal to the (normalized) theta-function divided by the η-function $\prod_{i\geq 1}(1-q^i)$.

Now let us turn to the twisted case. Here we state the twisted analogues of the results from the previous subsection.

Lemma 1.19. *For $n > 0$ there exists an embedding of $\mathfrak{osp}(1,2)[t]^\sigma$ modules $W^\sigma_{-n} \to W^\sigma_{-n-1}$, defined by $w_{-n} \mapsto g^+_{2n-1} w_{-n-1}$.*

Let $L^\sigma = \lim_{n\to\infty} W^\sigma_{-n}$ be the $\mathfrak{osp}(1,2)[t]^\sigma$-module. We define the character of L as follows:

$$\mathrm{ch} L(x,q) = \lim_{n\to\infty} q^{n^2} \mathrm{ch} W^\sigma_{-n}(x, q^{-1}).$$

Let $l_0(n) = g_1^+ \cdots g^+_{2n-1} w^\sigma_{-n} \in W^\sigma_{-n}$ and let $l_0 = \lim_{n\to\infty} l_0(n)$.

Theorem 1.20. $\mathrm{ch} L^\sigma(x,q) = \prod_{i=0}^\infty (1 + q^{2i+1}x) \prod_{i=0}^\infty (1 + q^{2i+1}x^{-1})$.

2. Nonsymmetric Macdonald polynomials of types $A_2^{(2)}$ and $A_2^{(2)\dagger}$

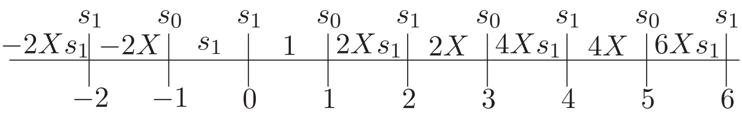
Nonsymmetric Koornwinder polynomials of types $A_{2n}^{(2)}$ and $A_{2n}^{(2)\dagger}$ (see e.g. [OS, H, RY]) are rational functions depending on a parameter q and five independent Hecke parameters v_s, v_l, v_2, v_0, v_z. Nondegenerate nonsymmetric Macdonald polynomials of types A_2^{2n} ($A_2^{2n\dagger}$ correspondingly) are defined as specializations of Koornwinder polynomials at $v_2 \mapsto 1$ ($v_z \mapsto 1$ correspondingly) and equal all other Hecke parameters v_i's (we denote them by v). Functions thus obtained are rational functions in variables $x, q, t = v^2$, more precisely they belong to $\mathbb{Z}(q,t)[x]$. We study the limits $v \to 0$ and $v \to \infty$ for $n = 1$ via the Ram-Yip formula (see also section 6.6, [M3]).

2.1. Ram-Yip formula for type $A_2^{(2)}(A_2^{(2)\dagger})$

We compute specializations of nonsymmetric Macdonald polynomials using methods from the papers [RY, OS]. We use the so called alcove walks, which are certain paths on the set of alcoves. We don't give the general definition of an alcove walk: one can find it in papers [RY, OS]. However we give an explicit construction in our case.

We consider the real line $\mathbb{R}$ and the set of alcoves $(i, i+1), i \in \mathbb{Z}$ and label the walls of alcoves by simple reflections such that the wall i is labeled by s_1 if i is even and by s_0 if i is odd.

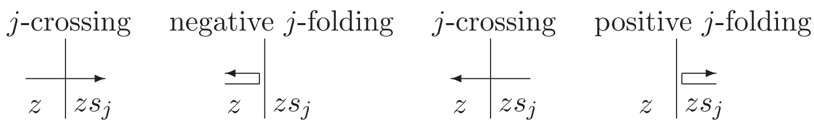
The Weyl group $W = \langle s_1 \rangle \star \langle s_0 \rangle$ (the free product of two cyclic groups of order 2) acts on the set of alcoves simply transitively. We identify W with the set of alcoves: s_1 acts as a reflection in the wall 0 and s_0 acts as a reflection in the wall 1. Hence $2X = s_1 s_0$ acts as a shift by 2. We use additive notation for the group generated by $2X$ (i. e. we write elements of this group as $2nX, n \in \mathbb{Z}$). Any element of W can be written in the form $2nXs_1^b, b \in \{0,1\}$. The following picture illustrates the procedure:



For any alcove a denote the even wall of a by $2\mathrm{wt}(a)$; if it is the left wall then put $\mathrm{d}(a) = 0$, if it is the right wall, then $\mathrm{d}(a) = 1$. In terms of W, for $w = 2nXs_1^b$ one has $\mathrm{wt}(a) = n$, $\mathrm{d}(a) = b$.

An alcove walk is a sequence of simple reflections with some addition information. This sequence is called the type of walk. Take an integer number n and consider an element $2nX \in W$. For $n \geq 0$ the Ram-Yip formula says that $E_{-n}^{A_2^{(2)}}(x;q,t)$ ($E_{-n}^{A_2^{(2)\dagger}}(x;q,t)$) can be constructed from alcove walks of type $(s_1, s_0, \ldots, s_1, s_0)$, $2n$ elements. Analogously, $E_n^{A_2^{(2)}}(x;q,t)$ ($E_n^{A_2^{(2)\dagger}}(x;q,t)$) can be constructed from alcove paths of type $(s_0, \ldots, s_1, s_0)$, $2n-1$ elements. Let us give definition of alcove walks in our case.

Let w be an element of the affine Weyl group. Consider a sequence $(s_{k_1}, \ldots, s_{k_l})$ of simple reflections, such that $w = s_{k_1} \cdots s_{k_l}$ is a reduced decomposition, and a binary word $b_1, \ldots, b_l$, $b_i \in \{0,1\}$. Alcove walk of type w is the path on the set of alcoves starting at the alcove $[0,1]$ (corresponding to $1 \in W$) and consisting of the following steps:



We have folding on the i-th step if $b_i = 0$ and crossing if $b_i = 1$. We call a folding positive if it is "from left to right" and negative if it is "from right to left". Put $J = \{i | b_i = 0\}$. Denote by p_J the final alcove of the walk with such set of foldings. Of course, p_J also depends on w but we omit w to simplify the notation.

Denote by $\mathcal{B}(w)$ the set of alcove walks of type w, i. e. pairs $p = (w, b)$. Let $J_0 = \{i \in J | w_i = s_0\}$ and let J_+ (J_-) be the subset of positive (correspondingly, negative) foldings of $J \backslash J_0$.

Put $\beta_i = s_{k_l} \cdots s_{k_{i+1}} \alpha_{k_i}$, $1 \leq i \leq l$, where α_{k_i} is a simple root. For the walks of types $(s_1, s_0, \ldots, s_1, s_0)$ and $(s_0, \ldots, s_1, s_0)$ one has:

$$\beta_{l-2i} = (s_0 s_1)^i \alpha_0 = -\alpha_1 + (2i+1)\delta,$$
$$\beta_{l-2i-1} = (s_0 s_1)^i s_0 \alpha_1 = -\alpha_1 + (2i+2)\delta.$$

Any β from the affine root lattice of type $A_2^{(2)}$ (aka $A_1^{(1)}$) can be written in the following form:

$$\beta = \beta' + \deg(\beta)\delta,$$

where β' is a root of finite dimensional root system. So in our case we have that $\deg(\beta_{l-i}) = i + 1$.

2.2. Explicit formula in type $A_2^{(2)}$

Theorem 2.1. *(Ram, Yip, $A_2^{(2)}$-case) Put $v = t^{1/2}$. Let $w = (s_1, s_0, \dots, s_1, s_0)$ (with $-2n$ elements for $n < 0$) and $w = (s_0, \dots, s_1, s_0)$ (with $2n-1$ elements for $n \geq 0$). Then:*

$$E_n^{A_2^{(2)}}(x,q,t) = \sum_{p \in \mathcal{B}(w)} v^{(sign(n)-1)/2 + d(p_J) - |J|} \left(1 - v^2\right)^{|J|} \times \prod_{j \in J_0} \frac{\xi_j}{1-\xi_j^2} \prod_{j \in J_+} \frac{1}{1-\xi_j} \prod_{j \in J_-} \frac{\xi_j}{1-\xi_j} x^{wt(p_J)}, \tag{2.1}$$

where $\xi_j = q^{\deg(\beta_j)} v^{-\langle \alpha_1^\vee, \beta_j \rangle} = q^{\deg(\beta_j)} v^2$.

Definition 2.2. Define the elements $c_r(k_{22}, k_{12}, k_{11}) \in \mathbb{Z}[q]$, $r = 1, 2$ by the following recurrent relations:

$$c_1(k_{22}, k_{12}, k_{11}) = q^{2n} c_2(k_{22}-1, k_{12}, k_{11}) + q^{2n-1} c_2(k_{22}, k_{12}-1, k_{11}) + c_1(k_{22}, k_{12}, k_{11}-1), \tag{2.2}$$

$$c_2(k_{22}, k_{12}, k_{11}) = c_2(k_{22}-1, k_{12}, k_{11}) + q^{2n-1} c_2(k_{22}, k_{12}-1, k_{11}) + c_1(k_{22}, k_{12}, k_{11}-1), \tag{2.3}$$

where $n = k_{11} + k_{12} + k_{22}$ in both formulas. The initial values are fixed by $c_r(k_{22}, k_{12}, k_{11}) = 0$, if some k_{ij} is negative, and $c_r(0,0,0) = 1$.

Proposition 2.3. *The specializations of nonsymmetric Macdonald polynomials of the type $A_2^{(2)}$ can be written in the following way ($n \geq 0$):*

$$E_{-n}^{A_2^{(2)}}(x,q,0) = \sum_{k_{22}+k_{12}+k_{11}=n} c_2(k_{22}, k_{12}, k_{11}) x^{k_{22}-k_{11}}. \tag{2.4}$$

$$E_{n+1}^{A_2^{(2)}}(x,q,0) = \sum_{k_{22}+k_{12}+k_{11}=n+1} \left(q^{2n+1} c_2(k_{22}-1, k_{12}, k_{11}) + c_1(k_{22}, k_{12}-1, k_{11})\right) x^{k_{11}-k_{22}+1}. \tag{2.5}$$

$$E_{-n}^{A_2^{(2)}}(x,q^{-1},\infty)=\sum_{k_{22}+k_{12}+k_{11}=n}c_1(k_{22},k_{12},k_{11})x^{k_{11}-k_{22}}. \tag{2.6}$$

$$E_{n+1}^{A_2^{(2)}}(x,q^{-1},\infty)=\sum_{k_{22}+k_{12}+k_{11}=n}c_2(k_{22},k_{12},k_{11})x^{k_{11}-k_{22}+1}. \tag{2.7}$$

In particular, $E_{n+1}^{A_2^{(2)}}(x,q^{-1},\infty)=xE_{-n}^{A_2^{(2)}}(x,q,0)$.

Proof. We first prove (2.4), (2.5) using Theorem 2.1. Let l be a length of an element w. We know that $\beta_{l+1-j}=-\alpha_1+j\delta$. Therefore $\xi_{l+1-j}=q^jv^2=q^jt$. Hence if we study specialization at $t=0$ we can put all denominators to be equal to 1:

$$\begin{aligned}E_n^{A_2^{(2)}}(x,q,0)&=\lim_{v\to 0}\sum_{p\in\mathcal{B}(w)}v^{(sign(n)-1)/2+d(p_J)-|J|}\prod_{j\in J_0}\xi_j\prod_{j\in J_-}\xi_jx^{wt(p_J)}\\&=\lim_{v\to 0}\sum_{p\in\mathcal{B}(w)}v^{(sign(n)-1)/2+d(p_J)-|J|+2|J_0\cup J_-|}q^{\sum_{i\in J_0\cup J_-}i}x^{wt(p_J)}.\end{aligned}$$

We claim that the exponent $(sign(n)-1)/2+d(p_J)-|J|+2|J_0\cup J_-|$ vanishes iff there are no positive 0-foldings and it is positive if such foldings exist. In fact, let J_{0+} and J_{0-} be the sets of positive and negative zero foldings. Then:

$$\begin{aligned}&(sign(n)-1)/2+d(p_J)-|J|+2|J_0\cup J_-|-2|J_{0+}|\\&=(sign(n)-1)/2+d(p_J)-|J_+|-|J_{0+}|+|J_{0-}\cup J_-|=0.\end{aligned} \tag{2.8}$$

Therefore, $(sign(n)-1)/2+d(p_J)-|J|+2|J_0\cup J_-|=2|J_{0+}|$. Denote the set of paths with $|J_{0+}|=0$ by $\mathcal{QB}$. Thus we have:

$$E_n^{A_2^{(2)}}(x,q,0)=\sum_{p\in\mathcal{QB}(w)}q^{\sum_{i\in J_0\cup J_-}i}x^{wt(p_J)}.$$

Now let us write an alcove path in the following form. We encode it by a sequence $\mathbf{h}=(h_0,\dots,h_l)$ (see [HHL]) of 1's and 2's. We put $h_0=1$ if $n>0$ and $h_0=2$ when $n<0$. If i-th step of the path is finished by right arrow (i.e. it is a crossing from left to right or a positive folding) then $h_i=1$. If the i-th step is finished by the left arrow then $h_i=2$. Then subsequences 12 correspond to negative foldings and subsequences 21 correspond to positive. We consider the sequence $(h_0,\dots,h_l)$ as a sequence of pairs 11, 12, 21, 22 and possibly the left most element without pair. Then the set of sequences of pairs with no pair 21 inside corresponds to $\mathcal{QB}(w)$. Denote the set of such sequences by $\mathcal{QS}(n)$.

For any sequence $\mathbf{h}$ of length $l+1$ denote $\mathrm{leg}(\mathbf{h}) = \sum_{h_{l-j}=1, h_{l-j+1}=2} j$. Then:

$$E_n^{A_2^{(2)}}(x,q,0) = \sum_{\mathbf{h}\in\mathcal{QS}(n)} q^{\mathrm{leg}(\mathbf{h})} x^{[(|\{i>0|h_i=1\}|-|\{i>0|h_i=2\}|+1)/2]},$$

where $[y]$ is the integer part of y.

Let us consider the polynomial $E_{-n}^{A_2^{(2)}}(x,q,0)$. Denote by

$$\mathcal{QS}(i,k_{22},k_{12},k_{11}) \subset \mathcal{QS}(-n)$$

the set of sequences of pairs and the element h_0 such that $h_0 = i$ and there are k_{ij} pairs ij. Put

$$c_i(k_{22},k_{12},k_{11}) = \sum_{\mathbf{h}\in\mathcal{QS}(i,k_{22},k_{21},k_{11})} q^{\mathrm{leg}(\mathbf{h})}.$$

Let us subdivide the sets $\mathcal{QS}(i,k_{22},k_{12},k_{11})$ into three subsets of sequences according to the value of the first pair h_1, h_2 ($(2,2)$, $(1,2)$ or $(1,1)$). Consider the case $i=2$. If $(h_1,h_2)=(2,2)$, then the leg of this sequence will not be changed if we cut (h_0,h_1). It is easy to see that all elements of $\mathcal{QS}(2,k_{22}-1,k_{12},k_{11})$ can be obtained by such a procedure. If $(h_1,h_2)=(1,2)$ then if we cut first two elements then the leg decreases on $2l-1$. If $(h_1,h_2)=(1,1)$ then if we cut first two elements then the leg will not be changed and we obtain $i=1$ instead of $i=2$. So we obtain a recurrent relation (2.3). Recurrent relations (2.2) can be obtained in the same way. Moreover by definition we have that they satisfy (2.4).

Analogously we obtain equation (2.5).

Now let us consider the polynomials $E_n^{A_2^{(2)}}(x,q^{-1},\infty)$. At $t\to\infty$ $\frac{\xi_i}{1-\xi_i^2} \sim \frac{1}{1-\xi_i} \sim -\xi_i^{-1}$. After interchanging $q \to q^{-1}$ we have:

$$\begin{aligned} E_{-n}^{A_2^{(2)}}(x,q^{-1},\infty) &= \lim_{t\to\infty} \sum_{p\in\mathcal{B}(w)} v^{(sign(-n)-1)/2+d(p_J)+|J|} \\ &\quad\times \prod_{j\in J_0\cup J_+} (-q^j v^{-2}) x^{\mathrm{wt}(p_J)} \\ &= \sum_{p\in\mathcal{B}(w)} v^{(-sign(n)-1)/2+d(p_J)+|J|-2|J_0|-2|J_+|} q^{\sum_{j\in J_0\cup J_+} j} x^{\mathrm{wt}(p_J)}. \end{aligned}$$

Similar to (2.8) we have that $(sign(n)-1)/2+d(p_J)+|J|-2|J_0|-2|J_+|=0$ iff $J_{0-}=\emptyset$. So we obtain:

$$E_{-n}^{A_2^{(2)}}(x,q^{-1},\infty)=\sum_{p\in\mathcal{B}(w),|J_{0-}|=0} q^{\sum_{j\in J_0\cup J_+} j} x^{[(|\{i>0|s_1=1\}|-|\{i>0|s_1=2\}|+1)/2]}.$$

So in terms of sequences we have that a walk giving nonzero summand correspond to a sequence of pairs 22, 21, 11 with $h_0=2$. Denote the set of such sequences by $\mathcal{QS}'(n)$ and put $\mathrm{leg}'(\mathbf{h})=\sum_{h_{l-j}=1,h_{l-j-1}=2} j$. Then:

$$E_{-n}^{A_2^{(2)}}(x,q^{-1},\infty)=\sum_{\mathbf{h}\in\mathcal{QS}'(n)} q^{\mathrm{leg}'(\mathbf{h})} x^{[(|\{i>0|h_i=1\}|-|\{i>0|h_i=2\}|+1)/2]}.$$

Now to obtain (2.6) we interchange 1 and 2 in all definitions of the previous paragraph.

Finally:

$$\begin{aligned}E_{n}^{A_2^{(2)}}(x,q^{-1},\infty)&=\lim_{t\to\infty}\sum_{p\in\mathcal{B}(w)} v^{(d(p_J)+|J|}\prod_{j\in J_0\cup J_+}(-q^j v^{-2})x^{\mathrm{wt}(p_J)}\\&=\sum_{p\in\mathcal{B}(w)} v^{d(p_J)+|J|-2|J_0|-2|J_+|}q^{\sum_{j\in J_0\cup J_+} j}x^{\mathrm{wt}(p_J)},\end{aligned}$$

and we obtain (2.7). □

Lemma 2.4. *The unique solution for the quantities $c_r(k_{22},k_{12},k_{11})$ from Definition 2.2 is given by the formulas:*

$$c_1(k_{22},k_{12},k_{11})=q^{k_{12}^2+2k_{22}}\binom{k_{22}+k_{12}+k_{11}}{k_{22},k_{12},k_{11}}_{q^2},\tag{2.9}$$

$$c_2(k_{22},k_{12},k_{11})=q^{k_{12}^2}\binom{k_{22}+k_{12}+k_{11}}{k_{22},k_{12},k_{11}}_{q^2}\tag{2.10}$$

Proof. Direct computation. □

2.3. Dual Macdonald polynomials

In this section we work with Macdonald polynomials of type $A_2^{(2)\dagger}$. We keep the notation from Subsection 2.1 and 2.2.

Theorem 2.5. *(Ram, Yip, $A_2^{(2)\dagger}$-case) Put $v = t^{1/2}$. Then:*

$$E_n^{A_2^{(2)\dagger}}(x,q,t) = \sum_{p\in\mathcal{B}(w)} v^{(sign(n)-1)/2+d(p_J)-|J|}\left(1-v^2\right)^{|J|}$$
$$\times \prod_{j\in J_{0+}} \frac{1}{1-\xi_j^2} \prod_{j\in J_{0-}} \frac{\xi_j^2}{1-\xi_j^2} \prod_{j\in J_+} \frac{1}{1-\xi_j} \prod_{j\in J_-} \frac{\xi_j}{1-\xi_j} x^{wt(p_J)}.$$

Definition 2.6. We define elements $c_r^\dagger(k_{22},k_{21},k_{11}) \in \mathbb{Z}[q], r=1,2$ by the following recurrent relations:

$$\begin{aligned} (2.11)\quad c_1^\dagger(k_{22},k_{21},k_{11}) &= q^{2n}c_2^\dagger(k_{22}-1,k_{21},k_{11}) + q^{2n}c_1^\dagger(k_{22},k_{21}-1,k_{11}) \\ &\quad + c_1^\dagger(k_{22},k_{21},k_{11}-1), \\ (2.12)\quad c_2^\dagger(k_{22},k_{21},k_{11}) &= c_2^\dagger(k_{22}-1,k_{21},k_{11}) + c_1^\dagger(k_{22},k_{21}-1,k_{11}) \\ &\quad + c_1^\dagger(k_{22},k_{21},k_{11}-1), \end{aligned}$$

where $n = k_{11}+k_{21}+k_{22}$. The initial conditions are $c_r^\dagger(k_{22},k_{21},k_{11}) = 0$ if any $k_{ij} < 0$ and $c_r^\dagger(0,0,0) = 1$.

Proposition 2.7. *We have the following equations ($n \geq 0$):*

$$(2.13)\qquad E_{-n}^{A_2^{(2)\dagger}}(x,q,0) = \sum_{k_{22}+k_{21}+k_{11}=n} c_2^\dagger(k_{22},k_{21},k_{11})x^{k_{11}-k_{22}}.$$

$$(2.14)\qquad E_{n+1}^{A_2^{(2)\dagger}}(x,q,0) = \sum_{k_{22}+k_{21}+k_{11}=n} c_1^\dagger(k_{22},k_{21},k_{11})x^{k_{11}-k_{22}+1}.$$

$$(2.15)\qquad E_{-n}^{A_2^{(2)\dagger}}(x,q^{-1},\infty) = \sum_{k_{22}+k_{21}+k_{11}=n} c_1^\dagger(k_{22},k_{21},k_{11})x^{k_{11}-k_{22}}.$$

$$(2.16)\qquad E_{n+1}^{A_2^{(2)\dagger}}(x,q^{-1},\infty) = \sum_{k_{22}+k_{21}+k_{11}=n} \left(c_2^\dagger(k_{22},k_{21},k_{11}) + c_1^\dagger(k_{22},k_{21},k_{11})\right) \times x^{k_{11}-k_{22}+1}.$$

In particular $E_{n+1}^{A_2^{(2)\dagger}}(x,q,0) = xE_{-n}^{A_2^{(2)\dagger}}(x,q^{-1},\infty)$.

Proof. The proof is completely analogous to the proof of Proposition 2.3. We have the same elements $\xi_j = q^j v^2 = q^j t$. Hence if we study the specialization at $t = 0$ we can put all denominators to be equal to 1:

$$E_n^{A_2^{(2)\dagger}}(x,q,0) = \lim_{v\to 0} \sum_{p\in\mathcal{B}(w)} x^{wt(p_J)} v^{(sign(n)-1)/2+d(p_J)-|J|} \prod_{j\in J_0} \xi_j \prod_{j\in J_-} \xi_j$$
$$= \lim_{v\to 0} \sum_{p\in\mathcal{B}(w)} x^{wt(p_J)} v^{(sign(n)-1)/2+d(p_J)-|J|+2|J_+|+4|J_{0+}|} q^{\sum_{i\in J_0\cup J_-} i}.$$

The exponent $(sign(n)-1)/2 + d(p_J) - |J| + 2|J_+| + 4|J_{0+}|$ vanishes iff there are no negative 0-foldings. Encode alcove paths by binary words in the same way as in the proof of Proposition 2.3. So the nonzero summands correspond to words with no pairs 12. This is the only difference with the previous proof. □

Lemma 2.8. *The unique solution for the quantities $c_r^\dagger(k_{22}, k_{12}, k_{11})$ from Definition 2.6 is given by the formulas:*

$$c_1^\dagger(k_{22},k_{21},k_{11}) = q^{k_{21}(k_{21}-1)+2k_{22}+2k_{21}} \binom{k_{22}+k_{21}+k_{11}}{k_{22},k_{21},k_{11}}_{q^2}, \tag{2.17}$$

$$c_2^\dagger(k_{22},k_{21},k_{11}) = q^{k_{21}(k_{21}-1)} \binom{k_{22}+k_{21}+k_{11}}{k_{22},k_{21},k_{11}}_{q^2} \tag{2.18}$$

Proof. Direct computation. □

3. Comparison

In this section we establish a link between the characters of the Weyl modules and the specialized Macdonald polynomials.

Theorem 3.1. *For any $n \in \mathbb{Z}$ one has*

$$\mathrm{ch}W_n(x,q^2) = E_n^{A_2^{(2)\dagger}}(x,q,0), \quad \mathrm{ch}W_n^\sigma(x,q) = E_n^{A_2^{(2)}}(x,q,0).$$

Proof. Lemma 1.2 and Lemma 1.4 give for $n \geq 0$

$$\mathrm{ch}W_{-n}(x,q^2) = \sum_{k+s\leq n} q^{k(k-1)} x^{-n+k+2s} \binom{n}{k,s,n-k-s}_{q^2}, \tag{3.1}$$

$$\mathrm{ch}W_{-n}^\sigma(x,q) = \sum_{k+s\leq n} q^{k^2} x^{-n+k+2s} \binom{n}{k,s,n-k-s}_{q^2}. \tag{3.2}$$

Formula (3.1) agrees with formulas (2.13), (2.18). Formula (3.2) agrees with formulas (2.4), (2.10).

Using Corollary 1.12 and Corollary 1.14 we obtain the following formulas, $n \geq 0$

$$\text{(3.3)} \quad \mathrm{ch} W_{n+1}(x,q^2) = \sum_{k+s\leq n} q^{k(k+1)+2s} x^{n+1-k-2s} \binom{n}{k,s,n-k-s}_{q^2},$$

$$\text{(3.4)} \quad \mathrm{ch} W^{\sigma}_{n+1}(x,q) = \sum_{k+s\leq n} x^{n+1-k-2s}\left(q^{k^2+2s} + x^{-1}q^{k^2+2n+1}\right) \times \binom{n}{k,s,n-k-s}_{q^2}.$$

Formula (3.3) agrees with formulas (2.14), (2.17).

The nontrivial part is formula (3.4), which we want to compare with formulas (2.5), (2.9) and (2.10). We have conditions (1.6) and (1.7) on elements of basis of W_{n+1}. Elements that satisfy condition (1.6) are parametrized by the data very similar to the parametrization data of (1.3). More precisely we obtain (1.6) if we increase the t-degree of the elements e_a in (1.3) by 2. Thus, under the identification $s = k_{11}$, $k = k_{12}$, $k_{22}+k_{12}+k_{11} = n+1$, the character of the elements (1.6) is equal to

$$\begin{aligned} &\sum_{k_{22}+k_{12}+k_{11}=n+1} c_2(k_{22}-1,k_{12},k_{11}) q^{2k_{22}} x^{k_{11}-k_{22}+1} \\ = &\sum_{k_{22}+k_{12}+k_{11}=n+1} c_1(k_{22}-1,k_{12},k_{11}) x^{k_{11}-k_{22}+1}. \end{aligned}$$

Analogously the elements satisfying condition (1.7) are elements that satisfy conditions (1.3) multiplied by g^-_{2n+1}. So their character is equal to

$$\sum_{k_{22}+k_{12}+k_{11}=n+1} c_2(k_{22},k_{12}-1,k_{11}) q^{2n+1} x^{k_{11}-k_{22}}.$$

□

Let us introduce the PBW filtration on the Weyl module W_{-n}. Namely, we define $F_0 = \mathbb{C}w_{-n}$, $F_{s+1} = F_s + \mathfrak{n}^-[t].F_s$. The associated graded space is a cyclic module for the algebra $\mathbb{C}[e_0,e_1,\dots] \otimes \bigwedge(g^+_0, g^+_1,\dots)$. Let us attach degree 1 to the variables e_i and g^+_i. We thus obtain an additional grading on the module $\mathrm{gr}W_{-n}$. We denote the character by $\mathrm{ch}(\mathrm{gr}W_{-n})(x,q,t)$.

For the twisted Weyl modules we make a similar procedure. First we pass to the graded $\mathbb{C}[e_0,e_1,\dots] \otimes \bigwedge(g^+_0, g^+_1,\dots)$ module $\mathrm{gr}W^{\sigma}_{-n}$. Then we attach degree 1 to the variables e_i and degree 0 to the variables g^+_i. Using this new grading we define the new character depending on x, q, t and denote it by $\mathrm{ch}(\mathrm{gr}W^{\sigma}_{-n})(x,q,t)$.

Theorem 3.2. *Let $n \geq 0$. Then*

$$\mathrm{chgr} W_{-n}(x, q^2, q^2) = E_{-n}^{A_2^{(2)\dagger}}(x, q^{-1}, \infty),$$
$$\mathrm{ch}(\mathrm{gr} W_{-n}^{\sigma})(x, q, q) = E_{-n}^{A_2^{(2)}}(x, q^{-1}, \infty).$$

Proof. It is easy to see that the set of vectors (1.1) forms basis of $\mathrm{gr}W_{-n}$ and the set of vectors (1.3) forms basis of $\mathrm{gr}W_{-n}^{\sigma}$. Hence, we derive the following formulas for the graded characters:

$$(3.5) \quad \mathrm{ch}(\mathrm{gr}W_{-n})(x, q^2, q^2) = \sum_{k+s \leq n} q^{k(k-1)+2k+2s} x^{-n+k+2s} \times \begin{pmatrix} n \\ k, s, n-k-s \end{pmatrix}_{q^2},$$

$$(3.6) \quad \mathrm{ch}(\mathrm{gr}W_{-n}^{\sigma})(x, q, q) = \sum_{k+s \leq n} q^{k^2+s} x^{-n+k+2s} \begin{pmatrix} n \\ k, s, n-k-s \end{pmatrix}_{q^2}.$$

Now formula (3.5) agrees with formulas (2.15), (2.17). Formula (3.6) agrees with formulas (2.6), (2.9). □

Appendix A. Quantum Bruhat graph

Here we briefly describe the methods of a paper [OS]. Although we don't use their techniques, the ideas of [OS] are very important for the content of our paper.

Consider the Weyl group $W = \langle s_0 \rangle \star \langle s_1 \rangle$ of a root system $A_2^{(2)}$.

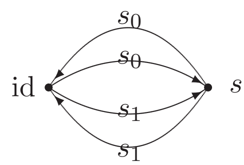
Definition A.1. Let $W(Y)$ be the Coxeter group of the root system Y, s_α be a reflection in the root α, l be the length function on $W(Y)$. Then the quantum Bruhat graph is the following ordered labeled graph:

· the set of vertices is $W(Y)$;

· we have a Bruhat arrow from g to gs_α, if $l(gs_\alpha) = l(g) + 1$;

· we have a quantum arrow from g to gs_α, if $l(gs_\alpha) = l(g) - \langle 2\rho, \alpha \rangle + 1$.

Consider the quantum Bruhat graph of type $\widehat{A_1}$. We want to make a difference between $s_{\alpha_1} = s_1$ and $s_{\alpha_0} = s_0$. So we have the following labeled graph on two vertices:



where arrows from id to s are Bruhat and arrows from s to id are quantum.

Put $\beta_i = s_{k_n} \cdots s_{k_{i+1}} \alpha_{k_i}$, where α_{k_i} is a simple root. We write β in the following form $\beta = \beta' + deg(\beta)\delta$, where $\beta' \in \mathbb{Z}\alpha$.

For any alcove walk (w, b) let $J = \{i | b_i = 0\}$, i. e. the set of foldings of a walk. Then we consider the following path on the quantum Bruhat graph started at element id:

$$\xrightarrow{dir(\beta_{i_1})} \cdots \xrightarrow{dir(\beta_{i_r})}$$

It is easy to see that any odd arrow of this path is quantum and any even is Bruhat. The Bruhat arrows correspond to negative foldings and quantum arrows correspond to positive ones. Define quantum Bruhat paths as paths such that they have no Bruhat (quantum for $A_2^{(2)\dagger}$) edges labeled by s_0. It is proved in [OS] that $E_n^{A_2^{(2)}}(x, q, 0)$ is obtained as a sum of some summands which are in one-to-one correspondence with the quantum Bruhat paths on the quantum Bruhat graph.

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