From state integrals to $q$-series

STAVROS GAROUFALIDIS AND RINAT KASHAEV

It is well-known to the experts that multi-dimensional state integrals of products of Faddeev’s quantum dilogarithm which arise in Quantum Topology can be written as finite sums of products of basic hypergeometric series in $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i / \tau}$. We illustrate this fact by giving a detailed proof for a family of one-dimensional integrals which includes state-integral invariants of $4_1$ and $5_2$ knots.

1. Introduction

1.1. State-integrals and their $q$-series

Multi-dimensional state integrals of products of Faddeev’s quantum dilogarithm appear in abundance in Quantum Topology, and were studied among others by Hikami [Hik01], Dimofte–Gukov–Lennels–Zagier [DGLZ09], Andersen–Kashaev [AK], and Kashaev–Luo–Vartanov [KLV16]. It is well-known to the experts that such state-integrals can be written as finite sums of products of pairs of $q$-series and $\tilde{q}$-series. The reason for this is a factorized structure of Faddeev’s quantum dilogarithm, the structure of the set of its poles, and the specific form of exponential factors of the integrand of the state-integrals, while its derivation is based on an application of the residue theorem. Instead of formulating a general theorem for multi-dimensional integrals which obscures the principle, we will give a detailed proof for the case of a family of 1-dimensional integrals and illustrate it with some concrete examples taken from [AK, KLV16]. Similar computations appear in mathematical physics [BDP14].
To state our results, recall that Faddeev’s quantum dilogarithm function $\Phi_b(x)$ is given by [Fad95]

$$\Phi_b(x) = \frac{(e^{2\pi b(x+c_b)}; q)_\infty}{(e^{2\pi b^{-1}(x-c_b)}; \tilde{q})_\infty},$$

where

$q = e^{2\pi ib^2}, \quad \tilde{q} = e^{-2\pi ib^2}, \quad c_b = \frac{i}{2}(b + b^{-1}), \quad \Im(b^2) > 0.$

Remarkably, this function admits an extension to all values of $b$ with $b^2 \not\in \mathbb{R}_{\leq 0}$. $\Phi_b(x)$ is a meromorphic function of $x$ with

poles: $c_b + iNb + iNb^{-1}$, \quad zeros: $-c_b - iNb - iNb^{-1}$.

The functional equation

$$\Phi_b(x)\Phi_b(-x) = e^{\pi ix^2}\Phi_b(0)^2, \quad \Phi_b(0) = q^{\frac{1}{24}}\tilde{q}^{-\frac{1}{24}}$$

allows us to move $\Phi_b(x)$ from the denominator to the numerator of the integrand of a state-integral.

For natural numbers $A, B$ with $B > A > 0$, we consider the absolutely convergent integral

$$I_{A,B}(b) = \int_{\mathbb{R}+i\epsilon} \Phi_b(x)^B e^{-A\pi ix^2} dx$$

with small positive $\epsilon$. The condition $B > A > 0$ ensures not only the convergence of $I_{A,B}(b)$ for $\Im(b^2) > 0$, but also the convergence of the $q$-series and the $\tilde{q}$-series (for $|q|, |\tilde{q}| < 1$) that appear in Theorem 1.1 below.

To express the above state-integral in terms of series, consider the generating series

$$F_{A,B}(q, x) = \sum_{m=0}^{\infty} \frac{(-1)^Am q^{A(m+1)/2}}{(q)_m} x^m, \quad \tilde{F}_{A,B}(q, x) = F_{b-A,B}(q, x).$$

Consider the operators $\delta$ and $\delta_k$ (for $k$ a positive natural number) which act on the space of functions of $x$ as follows

$$(\delta F)(x) = x\partial_x F(x), \quad (\delta_k F)(x) = \sum_{s=1}^{\infty} \frac{s^k q^s}{1-q^s} F(q^s x).$$
Likewise, there are operators $\delta$ and $\tilde{\delta}_k$ which act on the space of functions of $\tilde{x}$ and with $q$ replaced by $\tilde{q}$. It is easy to see that any two of the operators $\delta, \delta_k, \tilde{\delta}, \tilde{\delta}_k$ commute and they freely generate over $\mathbb{Q}$ a commutative ring $\mathcal{D} \otimes \tilde{\mathcal{D}}$, where

$$\mathcal{D} = \mathbb{Q}[\delta, \delta_1, \delta_2, \ldots], \quad \tilde{\mathcal{D}} = \mathbb{Q}[\tilde{\delta}, \tilde{\delta}_1, \tilde{\delta}_2, \ldots].$$

Let

$$\mathcal{D}_b = \mathcal{D}[(2\pi i)^{-1}, b^\pm, e_2, e_4, e_6, \ldots], \quad \tilde{\mathcal{D}}_b = \tilde{\mathcal{D}}[(2\pi i)^{-1}, b^\pm, e_2, e_4, e_6, \ldots],$$

where $e_l = e_l(q) = \tilde{\delta}_l(1) \in \mathbb{Z}[[\tilde{q}]]$. Consider the following operator valued polynomial:

$$P_{A,B} = \text{Res}_{w=0} \left( e^{\frac{1}{4} w^2 + A w (b(\delta + \frac{1}{2}) + b^{-1}(\delta + \frac{1}{2}))} \right)^A \times \left( \frac{\phi(bw, \delta\bullet) \tilde{\phi}(b^{-1}w, \tilde{\delta}\bullet)}{b(1 - e^{b^{-1}w})} \right)^B \in \mathcal{D}_b \otimes \tilde{\mathcal{D}}_b,$$

where

$$\phi(w, \delta\bullet) = \exp \left( -\sum_{l=1}^{\infty} \frac{\delta_l}{l!} w^l \right)$$

$$\tilde{\phi}(w, \tilde{\delta}\bullet) = \exp(-\tilde{\delta}w) \exp \left( 2 \sum_{l=even > 0} e_l(\tilde{q}) \frac{w^l}{l!} \right) \times \exp \left( -\sum_{l=1}^{\infty} \frac{\tilde{\delta}_l}{l!} (-w)^l \right).$$

For a series $F(x, \tilde{x})$, we define:

$$\langle F(x, \tilde{x}) \rangle = F(1, 1).$$

**Theorem 1.1.** We have:

$$\mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \left\langle P_{A,B} \left( F_{A,B}(q, x) \tilde{F}_{A,B}(\tilde{q}, \tilde{x}) \right) \right\rangle.$$
Corollary 1.2. Writing $P_{A,B} = \sum_k p_k P_k$ (a finite sum), for $p_k \in \mathcal{D}_b$ and $P_k \in \tilde{\mathcal{D}}_b$, it follows that

$$I_{A,B}(b) = \left( \frac{\bar{q}}{q} \right)^{\frac{B-2A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \sum_k g_k(q) G_k(\bar{q})$$

where

$$g_k(q) = \langle p_k F_{A,B} \rangle, \quad G_k(\bar{q}) = \langle P_k \tilde{F}_{A,B} \rangle.$$

Remark 1.3. The left hand side of Equation (8) has analytic continuation to the cut plane $\mathbb{C} \setminus \{b^2 | b^2 < 0\}$ whereas each of the series $g_k$ and $G_k$ is only well-defined in the upper-half plane $\{b^2 | \Im(b^2) > 0\}$.

Remark 1.4. $P_{A,B}$, as a polynomial in the variables $e_2, e_4, \ldots$ has degree $B - 1$, where the degree of $e_l$ is $l$. $P_{A,B}$ as a Laurent polynomial in $b$ has $b$-monomials of degrees in $\{-B + 1, -B + 3, \ldots, B - 3, B - 1\}$.

1.2. $q$-difference equations

Next we describe a linear $q$-difference equation of $F_{A,B}(q, x)$. Consider the operators $\hat{x}$ and $\hat{E}$ which act on $f(x) \in \mathbb{Q}\langle q \rangle[[x]]$ by:

$$(\hat{E} f)(x) = f(qx), \quad (\hat{x} f)(x) = xf(x).$$

Observe that $\hat{E} \hat{x} = q \hat{x} \hat{E}$.

Lemma 1.5. (a) We have:

$$F_{A,B}(q^{-1}, x) = \tilde{F}_{A,B}(q, x).$$

(b) $F_{A,B}$ satisfies the linear $q$-difference equation

$$(1 - \hat{E})^B - (-1)^A q^A x \hat{E}^A \right) F_{A,B}(q, x) = 0.$$
Corollary 1.6. (a) If we define \( \omega(q, x) = F_{A,B}(q, qx)/F_{A,B}(q, x) \) and \( \omega(q, x)_n = \prod_{j=1}^n \omega(q, q^j x) \), then \( \omega \) satisfies the nonlinear equation

\[
\sum_{j=0}^B (-1)^j \left( \frac{B}{j} \right) \omega(q, x)_j - (-1)^A q^A x \omega(q, x)_A = 0.
\]

(b) \( F \) is an admissible power series in the sense of Kontsevich-Soibelman [KS11, Sec.6], the limit \( \lim_{q \to 1} \omega(q, x) = \omega(x) \in \mathbb{Q}[x] \) exists and satisfies the algebraic equation (also known as the Nahm equation or the gluing equation)

\[
(1 - \omega(x))^B = (-1)^A x \omega(x)^A.
\]

The Nahm equation has been studied by several authors including [Zag07, Sec.3], [Vla, VZ11], [RV, Sec.4].

1.3. The case of the 4_1 knot

We now specialize Corollary 1.2 to the invariant of the 4_1 and 5_2 knots is given by [KLV16, AK]

\[
\mathcal{I}_{1,2} = \mathcal{I}_{4_1}, \quad \mathcal{I}_{2,3} = \mathcal{I}_{5_2}.
\]

In this section, let

\[
F(q, x) = F_{1,2}(q, x) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q)_n^2} x^n.
\]

Corollary 1.7. (a) We have:

\[
\mathcal{I}_{4_1}(b) = -\frac{i}{2} \left( \frac{q}{\tilde{q}} \right)^{1/24} \left( bG(q)g(\tilde{q}) - b^{-1}G(\tilde{q})g(q) \right)
\]

where

\[
g(q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q)_n^2}
\]

\[
G(q) = \sum_{m=0}^{\infty} \left( 1 + 2m - 4 \sum_{s=1}^{\infty} \frac{q^{s(m+1)}}{1 - q^s} \right) (-1)^m \frac{q^{m(m+1)/2}}{(q)_m^2}
\]
(b) The series \( g(q) \) and \( G(q) \) are given in terms of \( F(q, x) \) by:

\[
\begin{align*}
g(q) &= \langle F \rangle \\
G(q) &= \langle (2 + 2\delta - 4\delta_1)F \rangle
\end{align*}
\]

(c) \( F \) satisfies the linear \( q \)-difference equation

\[
F(q, q^{-1}x) + F(q, qx) = (2 - x)F(q, x)
\]

The series \( g(q) \) that appears in Theorem 1.7 was known to the first author and Zagier to be closely related to the 4\(_1\) knot. For a detailed discussion of experimental facts below, see [GZ]. Empirically, it appears that

- the pair \((g(q), G(q))\) is related to the 3D index of the 4\(_1\) knot,
- the radial asymptotics of the pair \((g(q), G(q))\) are related to the asymptotics of the Kashaev invariant of the 4\(_1\) knot,
- the above observations for 4\(_1\) also hold for the case of 5\(_2\) knot discussed below.

Recall that the index of an ideal triangulation was introduced in [DGG14, DGG13], necessary and sufficient conditions for its convergence was established in [Gar16] and its topological invariance was proven in [GHRS15]. For a detailed discussion of the above experimental facts, see [GZ].

### 1.4. The case of the 5\(_2\) knot

In this section, let

\[
F(q, x) = F_{2,3}(q, x) = \sum_{m=0}^{\infty} t_m(q)x^m, \quad \tilde{F}(q, \tilde{x}) = F_{1,3}(q, \tilde{x}) = \sum_{m=0}^{\infty} T_m(q)\tilde{x}^m
\]

where

\[
t_m(q) = \frac{q^{m(m+1)}}{(q^3)_m^3}, \quad T_n(q) = (-1)^n q^{\frac{1}{2}n(n+1)} \frac{1}{(q^3)_n^3} = t_n(q^{-1}).
\]

Let

\[
R_{m,n}(q, \tilde{q}) = -\frac{b^2}{2} \left(1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)}(q) - 3E_2^{(m)}(q)\right)
\]

\[
- \frac{1}{2\pi i} + \frac{1}{2} \left(1 + 2m - 3E_1^{(m)}(q)\right) \left(1 + 2n - 6E_1^{(n)}(\tilde{q})\right)
\]

\[
+ \frac{b^2}{2} \left(-n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)}(\tilde{q}) - 3E_2^{(n)}(\tilde{q})\right),
\]

\[
q = \exp(2\pi i \frac{1}{b}), \quad \tilde{q} = \exp(2\pi i \frac{1}{2b}).
\]
where $E_l^{(m)}(q)$ are defined in Equation (29a). For $k = 1, \ldots, 4$ let

(18) \quad g_k(q) = \sum_{m=0}^{\infty} p_k(m)t_m(q), \quad G_k(\tilde{q}) = \sum_{n=0}^{\infty} P_k(n)T_n(\tilde{q}),

where

(19a) \quad p_{1,m}(q) = 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) \\
+ 9E_1^{(m)^2}(q) - 3E_2^{(m)}(q)

(19b) \quad p_{2,m}(q) = 1 + 2m - 3E_1^{(m)}(q)

(19c) \quad p_{3,m}(q) = 1

and

(20a) \quad P_{1,m}(q) = 1

(20b) \quad P_{2,m}(q) = 1 + 2n - 6E_1^{(n)}(\tilde{q})

(20c) \quad P_{3,m}(q) = -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) \\
- 9E_1^{(n)^2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}).

**Corollary 1.8.** (a) We have:

(21) \quad I_{2,3}(q) = -e^{\frac{3\pi i}{4}} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{8}} \sum_{m,n=0}^{\infty} R_{m,n}(q, \tilde{q}) t_m(q) T_n(\tilde{q})

\quad = -e^{\frac{3\pi i}{4}} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{8}} \left( - \frac{b^2}{2} g_1(q) G_1(\tilde{q}) - \frac{1}{2\pi i} g_3(q) G_1(\tilde{q}) \\
+ \frac{1}{2} g_2(q) G_2(\tilde{q}) + \frac{b^2}{2} g_3(q) G_3(\tilde{q}) \right)

(b) $F$ and $\tilde{F}$ satisfy the linear $q$-difference equations

\[ F(q, q^3 x) - (3 - q^2 x) F(q, q^2 x) + 3F(q, qx) - F(q, x) = 0 \]

\[ \tilde{F}(q, q^3 x) - 3\tilde{F}(q, q^2 x) + (3 - q^2 x) \tilde{F}(q, qx) - \tilde{F}(q, x) = 0. \]

**Remark 1.9.** A computation gives that $P(A, B) = P(B - A, B)$ for $(A, B) = (1, 2)$ and $(A, B) = (2, 3)$ corresponding to the invariants of the 41 and 52 knots. In all other cases that we tried, we found that $P(A, B)$ is not equal to $P(B - A, B)$.
2. Proofs

2.1. A residue computation

To relate the state-integral $I_{A,B}$ to a sum, we will apply the residue theorem on a semicircle $\gamma_R$ with center 0 and radius $R$, oriented counterclockwise in the upper half-plane:

![\gamma_R]

Then, we will take the limit $R \to \infty$. To compute the residue of the integrand, we need to expand $\Phi_b(x)$ near the pole

$$x_{m,n} = c_b + ibm + ib^{-1}n$$

for natural numbers $m$ and $n$. Let

$$\phi_m(x) = \frac{(q^{m+1}e^x; q)_\infty}{(q^{m+1}; q)_\infty}$$

$$\tilde{\phi}_n(x) = \frac{(\tilde{q}; \tilde{q})_\infty (\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}e^x; \tilde{q})_\infty (\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}$$

Lemma 2.1. We have:

$$\Phi_b(x + x_{m,n}) = \Phi_b(x + c_b) \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{1}{(q; q)_\infty} \frac{1}{(q^m; q)_\infty} \frac{\phi_m(2\pi bx)}{(q; q)_\infty} \frac{\tilde{\phi}_n(2\pi b^{-1}x)}{(\tilde{q}; \tilde{q})_\infty}. \tag{25}$$

Proof. Equation (1) implies the functional equations

$$\frac{\Phi_b(x + c_b + ib)}{\Phi_b(x + c_b)} = \frac{1}{1 - qe^{2\pi bx}}$$

$$\frac{\Phi_b(x + c_b + ib^{-1})}{\Phi_b(x + c_b)} = \frac{1}{1 - \tilde{q}^{-1}e^{2\pi b^{-1}x}}$$

which give

$$\Phi_b(x + x_{m,n}) = \Phi_b(x + c_b) \frac{1}{(qe^{2\pi bx}; q)_\infty} \frac{1}{(q^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_\infty}$$

$$\Phi_b(x + c_b) = \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(qe^{2\pi bx}; q)_\infty}{(\tilde{q}e^{2\pi b^{-1}x}; \tilde{q})_\infty}. \tag{23}$$
Thus,

\[
\Phi_b(x + x_{m,n}) = \frac{(q; q)_\infty}{(\tilde{q}; q)_\infty} \frac{1}{(q_{e^{2\pi bx}}; q)_\infty} \frac{1}{(\tilde{q}; q)^{-1}_\infty} \frac{(q; q)_m}{(q_{e^{2\pi bx}}; q)_m} \frac{(q_{e^{2\pi b^{-1}x}}; \tilde{q}^{-1})_n}{(q; q)^{-1}_n}
\]

\[
\times \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(q; q)^{-1}_\infty}{(q^{m+1}e^{2\pi bx}; q)^{-1}_\infty} \frac{(q; q)^{-1}_\infty}{(q_{e^{2\pi b^{-1}x}}; \tilde{q})^{-1}_\infty} \frac{(q; q)^{-1}_\infty}{(q_{e^{2\pi b^{-1}x}}; \tilde{q}^{-1})_n}
\]

\[
= \frac{(q; q)_\infty}{(q; q)^{-1}_\infty} \frac{1}{(q^{m+1}e^{2\pi bx}; q)^{-1}_\infty} \frac{(q; q)^{-1}_\infty}{(q_{e^{2\pi b^{-1}x}}; \tilde{q})^{-1}_\infty} \frac{(q; q)^{-1}_\infty}{(q_{e^{2\pi b^{-1}x}}; \tilde{q}^{-1})_n}
\]

The result follows. □

The decoupling of \((m, n)\) in the quadratic form comes as follows: since \(A, m, n\) are integers, \(e^{A\pi imn} = 1\) and a computation gives

\[
e^{-A\pi i(x + x_{m,n})^2} = i^A \left( \frac{q}{\tilde{q}} \right)^{\frac{A}{2}} t_m^A(q) \tilde{t}_n^A(\tilde{q}) e^{-A\pi i x^2 + 2A\pi x(b(m + \frac{1}{2}) + b^{-1}(n + \frac{1}{2}))}
\]

where

\[
t_m^A(q) = (-1)^Am q^{-\frac{A(m+1)}{2}}, \quad \tilde{t}_n^A(\tilde{q}) = (-1)^An \tilde{q}^{-\frac{A(n+1)}{2}}.
\]

The Dedekind function \(\eta(\tau) = q^{1/24}(q; q)_\infty\) (with \(q = e^{2\pi i \tau}\)) satisfies the modular equation \(\eta(-\tau^{-1}) = \sqrt{-i}\eta(\tau)\) [And76]. It follows that

\[
(26) \quad \frac{(q; q)_\infty}{(q; q)^{-1}_\infty} = e^{\frac{\pi i}{24}} \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{24}} b^{-1}.
\]

After we set \(w = x/(2\pi)\), the above discussion implies that

\[
(27) \quad \mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \times \sum_{m,n=0}^{\infty} (\text{Res}_{w=0} F_{A,B,m,n}(w)) t_m^A(q) \tilde{t}_n^A(\tilde{q}) \frac{(q; q)_m}{(\tilde{q}; q^B)_n} \frac{(\tilde{q}^{-1}; q^{-1})^B_n}{(q^{-1}; \tilde{q}^{-1})^B_n},
\]

where

\[
(28) \quad F_{A,B,m,n}(w) = e^{\frac{A}{4\pi i}w^2 + Aw(b(m + \frac{1}{2}) + b^{-1}(n + \frac{1}{2}))} \left( \frac{\phi_m(bw) \tilde{\phi}_n(b^{-1}w)}{b(1 - e^{b^{-1}w})} \right)^B.
\]
2.2. The Taylor series of $\phi_m(x)$ and $\tilde{\phi}_n(x)$

In this section we express the Taylor series of $\phi_m(x)$ and $\tilde{\phi}_n(x)$ in terms of the $q$-series $E_l^{(m)}(q)$ and $\tilde{E}_l^{(m)}(\tilde{q})$ defined by:

\begin{align}
E_l^{(m)}(q) &= \sum_{s=1}^{\infty} \frac{s^{l-1} q^s (m+1)}{1 - q^s} = \langle \delta_l(x^m) \rangle \\
\tilde{E}_l^{(m)}(\tilde{q}) &= \begin{cases} 
-n + E_1^{(n)}(\tilde{q}) & \text{if } l = 1 \\
E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\
2E_l^{(0)}(\tilde{q}) - E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even}
\end{cases}
\end{align}

**Proposition 2.2.** We have:

\begin{align}
\phi_m(x) &= \exp \left( -\sum_{l=1}^{\infty} \frac{1}{l!} E_l^{(m)}(q) x^l \right) \\
\tilde{\phi}_n(x) &= \exp \left( \sum_{l=1}^{\infty} \frac{1}{l!} \tilde{E}_l^{(m)}(\tilde{q}) x^l \right).
\end{align}

The proof of this proposition is given in Section 2.6. The first few terms in Equations (30a)–(30a) are given by:

\begin{align}
\phi_m(x) &= \exp \left( -E_1^{(m)} x - \frac{1}{2} E_2^{(m)} x^2 - \frac{1}{6} E_3^{(m)} x^3 - \frac{1}{24} E_4^{(m)} x^4 - \cdots \right) \\
&= 1 - E_1^{(m)} x + \frac{1}{2} (E_1^{(m)2} - E_2^{(m)}) x^2 \\
&\quad + \frac{1}{6} (-E_1^{(m)3} + 3E_1^{(m)} E_2^{(m)} - E_3^{(m)}) x^3 \\
&\quad + \frac{1}{24} (E_1^{(m)4} - 6E_1^{(m)2} E_2^{(m)} + 3E_2^{(m)2} + 4E_1^{(m)} E_3^{(m)} - E_4^{(m)}) x^4 + \cdots \\
\tilde{\phi}_n(x) &= \exp \left( \tilde{E}_1^{(n)} x + \frac{1}{2} \tilde{E}_2^{(n)} x^2 + \frac{1}{6} \tilde{E}_3^{(n)} x^3 + \frac{1}{24} \tilde{E}_4^{(n)} x^4 - \cdots \right) \\
&= 1 + \tilde{E}_1^{(n)} x + \frac{1}{2} (\tilde{E}_1^{(n)2} + \tilde{E}_2^{(n)}) x^2 \\
&\quad + \frac{1}{6} (\tilde{E}_1^{(n)3} + 3\tilde{E}_1^{(n)} \tilde{E}_2^{(n)} + \tilde{E}_3^{(n)}) x^3 \\
&\quad + \frac{1}{24} (\tilde{E}_1^{(n)4} + 6\tilde{E}_1^{(n)2} \tilde{E}_2^{(n)} + 3\tilde{E}_2^{(n)2} + 4\tilde{E}_1^{(n)} \tilde{E}_3^{(n)} + \tilde{E}_4^{(n)}) x^4 + \cdots
\end{align}
where $E_l^{(m)} = E_l^{(m)}(q)$ and $\tilde{E}_l^{(m)} = \tilde{E}_l^{(m)}(\tilde{q})$.

2.3. The connection with the differential operators $\delta_l$ and $\tilde{\delta}_l$

In this section we connect the series $E_l^{(m)}(q)$ and $\tilde{E}_l^{(m)}(\tilde{q})$ with the action of the differential operators $\delta_l$ and $\tilde{\delta}_l$ on a series $F(x)$ and $\tilde{F}(\tilde{x})$ respectively. Consider formal power series

$$F(x) = \sum_{m=0}^{\infty} t(m)x^m \quad \tilde{F}(\tilde{x}) = \sum_{m=0}^{\infty} \tilde{t}(m)\tilde{x}^m.$$  

**Lemma 2.3.** We have:

\begin{align*}
\sum_{m=0}^{\infty} \left( \prod_{j=1}^{r} E_{l_j}^{(m)}(q) \right) t(m) &= \left\langle \prod_{j=1}^{r} \delta_{l_j} F \right\rangle \\
\sum_{m=0}^{\infty} m^r t(m) &= \langle \delta^r F \rangle \\
\sum_{n=0}^{\infty} \left( \prod_{j=1}^{r} \tilde{E}_{l_j}^{(n)}(\tilde{q}) \right) \tilde{t}(n) &= \left\langle \prod_{j=1}^{r} \tilde{\delta}_{l_j} \tilde{F} \right\rangle \\
\sum_{n=0}^{\infty} n^r \tilde{t}(n) &= \langle \tilde{\delta}^r \tilde{F} \rangle.
\end{align*}

**Proof.** For a positive natural number $l$ we have:

$$\sum_{m=0}^{\infty} E_l^{(m)}(q)t(m) = \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle t(m) = \left\langle \delta_l \left( \sum_{m=0}^{\infty} t(m)x^m \right) \right\rangle = \langle \delta_l F \rangle.$$

Moreover, for positive natural numbers $l, l'$ we have:

$$\sum_{m=0}^{\infty} E_l^{(m)}(q)E_{l'}^{(m)}(q)t(m) = \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle \langle \delta_{l'}(x^m) \rangle t(m)$$

$$= \left\langle \delta_l \left( \sum_{m=0}^{\infty} \langle \delta_{l'}(x^m) \rangle t(m)x^m \right) \right\rangle.$$
Now,
\[ \langle \delta \nu (x^m) \rangle t(m)x^m = \sum_{s=1}^{\infty} \frac{s' - 1}{q^s} q^{s' m} t(m) x^m = \delta \nu (x^m) t(m) \]
and summing up over \( m \), we obtain that
\[ \sum_{m=0}^{\infty} \langle \delta \nu (x^m) \rangle t(m)x^m = \delta \nu F(q, x). \]
It follows that
\[ \sum_{m=0}^{\infty} E_{l}^{(m)}(q) \tilde{E}_{l}^{(m)}(q) t(m) = \langle \delta_{l} \delta_{l} F \rangle. \]
The general case of Equation (32) follows by induction on \( r \). Equation (33) is obvious. □

2.4. Proof of Theorem 1.1

Fix natural numbers \( A \) and \( B \) with \( B > A \geq 1 \), and let
\[ t(m) = \frac{(-1)^A q^{\frac{A(m+1)}{2}}}{(q)_m^B}, \quad F(q, x) = \sum_{m=0}^{\infty} t(m)x^m \]
and
\[ \tilde{t}(n) = \frac{(-1)^{(B-A)n} q^{\frac{(B-A)(n+1)}{2}}}{(\tilde{q})_n^B}, \quad \tilde{F}(\tilde{q}, \tilde{x}) = \sum_{n=0}^{\infty} \tilde{t}(n)x^n. \]
Use Equations (27) and (28) and Proposition 2.2 to expand \( F_{A,B,m,n}(w) \) as a power series with coefficients polynomials in the variables \( m, E_{l}^{(m)}(q) \) and \( n, \tilde{E}_{l}^{(n)}(\tilde{q}) \) and \( b^{\pm 1} \) and \((2\pi i)^{-1}\). Now apply Lemma 2.3 to convert the variables \( m, E_{l}^{(m)}(q), n, \tilde{E}_{l}^{(n)}(\tilde{q}) \) in terms of the action of the operators \( \delta, \delta_{l}, \tilde{\delta}, \tilde{\delta}_{l} \) respectively. This concludes the proof of Theorem 1.1. □

2.5. Some auxiliary power series

Consider the auxiliary series
From state integrals to $q$-series

(36) $\frac{1}{ae^x - 1} = \sum_{n=0}^{\infty} p_n(a)x^n$

where

$$p_n(a) = -\frac{a}{n!(1-a)^{n+1}} \sum_{m=0}^{n-1} A_{n,m}a^m \quad p_0(a) = -\frac{1}{1-a}$$

and $A_{n,m}$ are Euler triangular numbers (sequence A008292 in the online encyclopedia of integer sequences [Slo]) that satisfy the recursion

$$A_{n,m} = (n - m)A_{n-1,m-1} + (m + 1)A_{n-1,m}$$

and also given by the sum

$$A_{n,m} = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m + 1 - k)^n.$$

For a detailed discussion on this subject, see [FS70]. A table of the first few numbers $A_{n,m}$ is given by

<table>
<thead>
<tr>
<th>$n \setminus m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td>1</td>
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<tr>
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<td>1</td>
<td>4</td>
<td>1</td>
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</tr>
<tr>
<td>4</td>
<td>1</td>
<td>11</td>
<td>11</td>
<td>1</td>
<td></td>
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</tr>
<tr>
<td>5</td>
<td>1</td>
<td>26</td>
<td>66</td>
<td>26</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>6</td>
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<td>57</td>
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<td>302</td>
<td>57</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>120</td>
<td>1191</td>
<td>2416</td>
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<td>120</td>
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<td></td>
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</tr>
<tr>
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<td>1</td>
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<td>4293</td>
<td>15619</td>
<td>15619</td>
<td>4293</td>
<td>247</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>502</td>
<td>14608</td>
<td>88234</td>
<td>156190</td>
<td>88234</td>
<td>14608</td>
<td>502</td>
<td>1</td>
</tr>
</tbody>
</table>

**Lemma 2.4.** For $l \geq 1$, we have:

(37) $\frac{d^l}{dx^l} \log(1 - q^k e^{bx})|_{x=0} = b^l p_{l-1}(q^k) + b \delta_{l,1}$
Proof. It follows from
\[ \frac{d}{dx} \log(1 - q^k e^{bx}) = b \left( 1 + \frac{1}{q^k e^{bx} - 1} \right) \]
and Equation (36).
\[ \Box \]

For positive natural numbers \( l, r \) with \( l \geq r \) and \( m \) consider the \( q \)-series \( E_{l,r}^{(m)}(q) \) defined by

\[ E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \frac{q^{kr}}{(1 - q^k)^l} \]  

Lemma 2.5. (a) We have

\[ E_{l,r}^{(m)}(q) = \sum_{s=r}^{\infty} \frac{a_{l,s} q^{s(m+1)}}{1 - q^s} \]

where
\[ \frac{x^r}{(1-x)^l} = \sum_{s=r}^{\infty} a_{l,s} x^s \]

(b) It follows that

\[ \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q) = E_{l}^{(m)}(q) \]

Proof. For (a), interchange \( k \) and \( s \) summation:

\[ E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \sum_{s=r}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} \sum_{k=m+1}^{\infty} a_{l,s} q^{sk} \]

\[ = \sum_{s=r}^{\infty} q^{(m+1)s} \sum_{k=0}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} a_{l,s} \frac{q^{(m+1)s}}{1 - q^s} \]

(b) follows from (a) and the fact that

\[ \sum_{r=0}^{l-1} A_{l-1,r} x^r \frac{x^r}{(1-x)^l} = \sum_{s=1}^{\infty} s^{l-1} x^s. \]

\[ \Box \]
Lemma 2.6. We have:

\[
\phi_m(x) = \exp \left( - \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q)x^l \right)
\]

**Proof.** It follows from Lemma 2.4 combined with

\[
\log(\phi_m(x)) = \log \left( \frac{(q^{m+1}e^x;q)_{\infty}}{(q^m;q)_{\infty}} \right)
= \sum_{l=m+1}^{\infty} \left( \log(1 - q^l e^x) - \log(1 - q^l) \right)
\]

\[
\square
\]

### 2.6. Proof of Proposition 2.2

Part (a) of Proposition 2.2 follows from Lemma 2.5 and Lemma 2.6. For part (b), we will use the series

\[
E_{l}^{[m]}(q) = \sum_{s=1}^{\infty} \frac{s^{k-1}q^{s(m+1)}}{1 - q^s}
\]

Using

\[
\log(\tilde{\phi}_n(x)) = \log \left( \frac{(\tilde{q}; \tilde{q})_{\infty}}{q e^x; \tilde{q})_{\infty}} \right) + \log \left( \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_{n}}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_{n}} \right)
\]

and the proof of part (a) of Proposition 2.2, it follows that

\[
\log(\tilde{\phi}_n(x)) = \log \left( \frac{(\tilde{q}; \tilde{q})_{\infty}}{q e^x; \tilde{q})_{\infty}} \right) + \log \left( \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_{n}}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_{n}} \right)
= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(0)}(\tilde{q})x^l + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{[n]}(\tilde{q})x^l
= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \left( E_{l,r+1}^{(0)}(\tilde{q}) + E_{l,r+1}^{[n]}(\tilde{q}^{-1}) \right)x^l
\]

where

\[
E_{l,r}^{[n]}(q) = \sum_{k=1}^{n} \frac{q^{kr}}{(1 - q^k)^{l+r}}.
\]
Let

\[
\widetilde{E}_{l,r}^{(n)}(\tilde{q}) = \begin{cases} 
-n + E_{1,1}^{(n)}(\tilde{q}) & \text{if } l = r = 1 \\
E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\
2E_{l,r}^{(0)}(\tilde{q}) - E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even}
\end{cases}
\]

We claim that

\[
E_{l,r}^{(0)}(\tilde{q}) + E_{l,l-r}^{[n]}(\tilde{q}^{-1}) = \widetilde{E}_{l,r}^{(n)}(\tilde{q})
\]

for \(l > r \geq 1\) and

\[
E_{1,1}^{(0)}(\tilde{q}) + E_{1,1}^{[n]}(\tilde{q}^{-1}) = \widetilde{E}_{1,1}^{(n)}(\tilde{q})
\]

Assuming Equations (44) and (45), it follows that

\[
\log(\tilde{\phi}_n(x)) = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \widetilde{E}_{l,r+1}^{(n)}(\tilde{q}) x^l
\]

where the last step follows from part (b) of Lemma 2.5.

It remains to prove Equations (44) and (45). Equation (44) follows from the definition of \(\widetilde{E}_{1,1}^{(n)}(\tilde{q})\) and

\[
E_{l,r}^{(0)}(\tilde{q}) + E_{l,l-r}^{[n]}(\tilde{q}^{-1}) = \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1 - \tilde{q}^k)^l} + \sum_{k=1}^{n} \frac{\tilde{q}^{-k(l-r)}}{(1 - \tilde{q}^{-k})^l}
\]

\[
= \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1 - \tilde{q}^k)^l} + (-1)^l \sum_{k=1}^{n} \frac{\tilde{q}^{kr}}{(1 - \tilde{q}^{-k})^l}
\]

\[
= (1 + (-1)^l) \sum_{k=1}^{n} \frac{\tilde{q}^{kr}}{(1 - \tilde{q}^k)^l} + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^{kr}}{(1 - \tilde{q}^k)^l}
\]

Equation (45) follows from

\[
E_{1,1}^{(0)}(\tilde{q}) + E_{1,1}^{[n]}(\tilde{q}^{-1}) = \sum_{k=1}^{\infty} \frac{\tilde{q}^k}{1 - \tilde{q}^k} + \sum_{k=1}^{n} \frac{\tilde{q}^{-k}}{1 - \tilde{q}^{-k}}
\]

\[
= \sum_{k=1}^{\infty} \frac{1 - \tilde{q}^k}{1 - \tilde{q}^k} - \sum_{k=1}^{n} \frac{1}{1 - \tilde{q}^k} = -n + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^k}{1 - \tilde{q}^k}
\]
This completes the proof of Proposition 2.2. □

2.7. Proof of Lemma 1.5

Part (a) of Lemma 1.5 follows from the definition of $F_{A,B}$ and $\tilde{F}_{A,B}$.

Part (b) follows from an application of Zeilberger’s creative telescoping [Zei91]. To apply the method, define

$$t(m, x) = \frac{(-1)^m A^n_{m+1}}{(q)_m^B} x^m$$

Then, observe that $t$ satisfies the recursions with respect to $m$ and $x$:

$$(1 - q^{m+1}) B t(m + 1, x) = (-1)^A q^{A(m+1)} t(m, x)$$

$$(m, qx) = q^m t(m, x).$$

Now, we eliminate $q^m$ from the above equations as follows. The second equation implies that $t(m + 1, q^j x) = q^{j(m+1)} t(m + 1, x)$. Expanding the first equation, it follows that

$$\sum_{j=0}^{B} (-1)^j \binom{B}{j} t(m + 1, q^j x) = (-1)^A q^{A x} t(m, q^A x)$$

Summing for $m \geq 0$ implies (b). □

Proof. (of Corollary 1.6) The admissibility of $F$ in the sense of Kontsevich-Soibelman, follows from [KS11, Sec.6.1] and [KS11, Thm.9]. Given this, the Nahm Equation (12) for $\omega$ follows easily from part (b) of Lemma 1.5. □

3. An application: state-integrals of the 41 and 52 knots

3.1. Proof of Corollary 1.7

Assume now that $(A, B) = (1, 2)$. Then,

$$\frac{1}{(b(1 - e^{b^{-1}w}))^2} = \frac{1}{w^2} - \frac{b^{-1}}{w} + O(1)$$

$$\phi_m(bw) = 1 - 2E_1^{(m)}(q)bw + O(w^2)$$

$$\tilde{\phi}_n(b^{-1}w) = 1 + 2E_1^{(n)}(\tilde{q})b^{-1}w + O(w^2)$$

$$e^{\frac{1}{4\pi}w^2 + w(b(m+1/2) + b^{-1}(n+1/2))} = 1 + \left(\frac{1}{2} + m\right) bw + \left(\frac{1}{2} + n\right) b^{-1}w + O(w^2)$$
Combined with $\tilde{E}_1^{(n)}(\tilde{q}) = -n + E_1^{(n)}(\tilde{q})$, it follows that the residue $R = \text{Res}_{w=0}(F_{1,2,m,n}(w))$ is given by

$$R = \left( b \left( \frac{1}{2} + m - 2E_1^{(m)}(q) \right) - b^{-1} \left( \frac{1}{2} + n - 2E_1^{(n)}(\tilde{q}) \right) \right).$$

The above, together with the fact that $t_n(q) = (-1)^n q^{\frac{1}{2}n(n+1)}$ satisfies $t_n(q^{-1}) = t_n(q)$ implies Equation (14). Equation (17) follows from Equation (11) for $(A, B) = (1, 2)$.

### 3.2. Proof of Corollary 1.8

Assume now that $(A, B) = (2, 3)$. Then,

$$\frac{1}{(b(1 - e^{b^{-1}w}))^3} = -\frac{1}{w^3} + \frac{3b^{-1}}{2w^2} - \frac{b^{-2}}{w} + O(1),$$

$$(\phi_m(bw))^3 = 1 - 3E_1^{(m)}(q) bw + \frac{3}{2} \left( 3E_1^{(m)^2}(q) - E_2^{(m)}(q) \right) b^2 w^2$$
$$+ O(w^3),$$

$$(\tilde{\phi}_n(b^{-1}w))^3 = 1 + 3E_1^{(n)}(\tilde{q}) b^{-1} w + \frac{3}{2} \left( 3\tilde{E}_1^{(n)^2}(\tilde{q}) + \tilde{E}_2^{(n)}(\tilde{q}) \right) b^{-2} w^2$$
$$+ O(w^3),$$

$$e^{\frac{2}{4\pi i} w^2 + 2w(b(m+1/2) + b^{-1}(n+1/2))}$$
$$= 1 + ((1 + 2m)b + (1 + 2n)b^{-1}) w$$
$$+ \left( 1 + \frac{b^2 + b^{-2}}{2} + \frac{1}{2\pi i} + 2b^2 m^2 + 2b^{-2} n^2 + 4mn \right) w^2 + O(w^3).$$

If $R = \text{Res}_{w=0}(F_{2,3,m,n}(w))$, then

$$R_{m,n} = -\frac{b^2}{2} \left( 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12m E_1^{(m)}(q) + 9E_1^{(m)^2}(q) - 3E_2^{(m)}(q) \right)$$
$$- \frac{1}{2\pi i} + \frac{1}{2} \left( 1 + 2m - 3E_1^{(m)}(q) \right) \left( 1 + 2n - 6E_1^{(n)}(\tilde{q}) \right)$$
$$+ \frac{b^{-2}}{2} \left( -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6n E_1^{(n)}(\tilde{q}) - 9E_1^{(n)^2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}) \right).$$
This proves part (a) of Corollary 1.8. Part (b) follows from Equation (11) for $(A, B) = (2, 3)$ and $(A, B) = (1, 3)$. Note that Theorem 1.1 states that

\begin{equation}
I_{2, 3}(q) = -e^{\frac{3\pi i}{4}} \langle P_{2, 3}(F \tilde{F}) \rangle
\end{equation}

where

\begin{align*}
P_{2, 3} &= \frac{-b^2}{2} \left( 1 + 4 \delta + 4 \delta^2 - 6 \delta_1 - 12 \delta_1 \delta_1 + 9 \delta_1^2 - 3 \tilde{\delta}_2 \right) \\
&\quad + \frac{1}{2} \left( 1 + 2 \delta + \frac{i}{\pi} + 2 \tilde{\delta} + 4 \delta \tilde{\delta} - 3 \delta_1 - 6 \tilde{\delta} \delta_1 - 6 e_2(\tilde{q}) - 6 \tilde{\delta}_1 - 12 \delta_1 \delta_1 + 18 \delta_1 \tilde{\delta}_1 \right) \\
&\quad + \frac{b^{-2}}{2} \left( -\tilde{\delta} - \delta^2 + 3 \tilde{\delta}_1 + 6 \tilde{\delta} \delta_1 - 9 \delta_1^2 + 3 \delta_2 \right) .
\end{align*}

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References


[DGLZ09] Tudor Dimofte, Sergei Gukov, Jonatan Lenells, and Don Zagier, Exact results for perturbative Chern-Simons theory with complex


School of Mathematics, Georgia Institute of Technology  
Atlanta, GA 30332-0160, USA  
*E-mail address*: stavros@math.gatech.edu

Section de Mathématiques, Université de Genève  
2-4 rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland  
*E-mail address*: Rinat.Kashaev@unige.ch

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