

A note on algebraic rank, matroids, and metrized complexes

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We show that the algebraic rank of divisors on certain graphs is related to the realizability problem of matroids. As a consequence, we produce a series of examples in which the algebraic rank depends on the ground field. We use the theory of metrized complexes to show that equality between the algebraic and combinatorial rank is not a sufficient condition for smoothability of divisors, thus giving a negative answer to a question posed by Caporaso, Melo, and the author.

1. Introduction

In [4], Caporaso defines the *algebraic rank*, an invariant on graphs which reflects the ranks of line bundles with prescribed degrees on all the nodal curves dual to the given graph. This invariant was studied in [5], and was shown to be bounded above by the combinatorial rank of divisors on graphs (in the sense of [2, 3]).

In this note, we further study the algebraic rank. Using a construction of Cartwright from [6], we assign a finite graph G_M and a divisor D_M to any simple matroid of rank 3, where G_M is the Levi graph of M (see Definition 2.2). Our main result is the following:

Theorem (2.4). *Let k be an algebraically closed field. Then D_M has algebraic rank 2 over k if and only if M is realizable over k .*

Since there are matroids that are realizable over a certain field but not another, we conclude:

Corollary (2.5). *There exists a divisor whose algebraic rank depends on the field.*

On a different vein, as the algebraic rank is bounded from above by the combinatorial rank, it is natural to ask how it compares with other

refinements of the combinatorial rank. In Section 3, we examine the relation between the rank of divisors on metrized complexes, and the algebraic rank of divisors obtained by forgetting the metrized complex structure. We show, using two examples, that in general, there is no inequality between one and the other. As an application, we obtain a negative answer to the following question, that originally appeared in [5, Question 2.11]:

Question 1.1. *Assume that the algebraic and combinatorial rank of a divisor D coincide. Is D smoothable?*

1.1. Algebraic rank

In what follows, we assume familiarity with divisor theory on finite vertex-weighted graphs, see [2, 3] for an exposition. Throughout this note, we shall refer to the standard rank of divisors on graphs (defined in *loc. cit.*) as the *combinatorial rank*.

The *algebraic rank* is an invariant of divisor classes, first introduced in [4]. Given a divisor D on a finite graph G , the invariant should reflect the ranks of line bundles on nodal curves dual to G , whose degrees on each component are prescribed by D . That is, the degree of the restriction to each component equals the coefficient of D at the corresponding vertex. However, this involves making several choices:

- 1) The graph G admits many different dual curves.
- 2) The invariant is intended to be well defined on divisor classes, but there is no canonical choice of representative in the divisor class of D .
- 3) There are many different line bundles with degrees prescribed by the divisors in the class.

The algebraic rank is made free of all choices, by defining it as

$$r_k^{\text{alg}}(D) = \max_{X \in M_k^{\text{alg}}(G)} \left[\min_{E \sim D} \left(\max_{L \in \text{Pic}^E(X)} (h^0(L) - 1) \right) \right],$$

where $M_k^{\text{alg}}(G)$ is the set of isomorphism classes of curves defined over k having G as their dual graph, and $\text{Pic}^E(X)$ is the set of line bundles on X whose multidegree is prescribed by E .

As shown in [5], the algebraic rank satisfies the following familiar properties:

Theorem 1.2. *The algebraic rank of a divisor on a finite graph is bounded above by the combinatorial rank, satisfies a Riemann-Roch equality, and a specialization lemma.*

Let R be a discrete valuation ring with fraction field K and algebraically closed residue field k . Let X be a smooth projective curve over K , and \mathcal{X} a regular semistable model for X . That is, \mathcal{X} is a proper, flat, and regular scheme over R , whose generic fiber is isomorphic to X , and whose special fiber is a reduced nodal curve X_0 over k . Combining the theorem above with Baker's Specialization Lemma ([2, 3]), we get:

Corollary 1.3. *Let D be a divisor on X that specializes to a divisor D_G on G . Then*

$$r(D) \leq r_k^{\text{alg}}(D_G) \leq r_G(D_G),$$

where r_G stands for the combinatorial rank.

Definition 1.4. A divisor D as above is said to be a *lifting* of D_G . If D_G admits a lifting to a divisor D with $r(D) = r_G(D_G)$, then we say that D_G is *smoothable*.

2. Dependence of the algebraic rank on the ground field

In this section we show that a divisor can have different algebraic ranks over different (algebraically closed) fields. For that purpose, we appeal to matroid theory. Rather than defining matroids in general, we only define here simple matroids of rank 3. See [11] for a comprehensive treatment of matroids of any rank.

Definition 2.1. A rank 3 *simple matroid* M consists of a finite set \mathcal{E} and a collection of subsets \mathcal{F} satisfying:

- 1) Every pair of distinct elements E_1 and E_2 in \mathcal{E} are contained in exactly one element of \mathcal{F} .
- 2) \mathcal{F} contains at least two elements.

We refer to every E in \mathcal{E} as an *element* of the matroid, and to F in \mathcal{F} as a *flat*. The unique flat containing a pair E_i and E_j of elements will be denoted F_{ij} .

In [6], Cartwright assigns a graph and a divisor to every simple matroid of rank 3 as follows:

Definition 2.2. Let M be a simple matroid of rank 3. Let G_M be the incidence graph of M . Namely, G_M has a vertex v_E for each element E , and a vertex v_F for each flat F of M . There is an edge between v_E and v_F whenever E is contained in F . We define D_M to be the divisor with a single chip on every vertex v_E corresponding to an element of M .

As shown in [6, Proposition 2.2], the combinatorial rank of D_M is 2. A similar construction was used independently by Castravet and Tevelev to study planar realization of hypertrees [7].

Definition 2.3. A rank 3 simple matroid is *realizable* over k if there is a bijection between its elements and a collection $\{\ell_1, \dots, \ell_m\}$ of lines in \mathbb{P}_k^2 , such that $\ell_{i_1}, \dots, \ell_{i_k}$ have non-empty intersection if and only if the corresponding elements of M are contained in a flat.

The rest of this section is devoted to proving the following result:

Theorem 2.4. *Let M be a simple matroid of rank 3. Then D_M has algebraic rank 2 over k if and only if M is realizable over k .*

Corollary 2.5. *There exists a divisor whose algebraic rank depends on the field.*

Proof. Let M be the Fano matroid, consisting of seven elements, such that every three of them are contained in a unique flat (Figure 1). Then it is known that M is realizable over k if and only if the characteristic of k is 2 [9, Example 4.5]. Therefore, Theorem 2.4 implies that $r_{\mathbb{C}}^{\text{alg}}(D_M)$ is at most 1, but $r_{\mathbb{F}_2}^{\text{alg}}(D_M) = 2$.

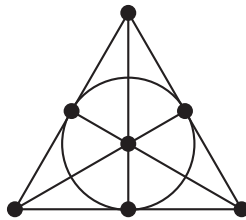


Figure 1: The Fano Plane. The lines correspond to the elements of the Fano matroid, and the marked points correspond to the flats.

□

The graph G_M from the corollary is known as the *Heawood graph*. A similar graph but with generic edge lengths was used by Jensen to show that the locus of Brill–Noether general graphs is not dense in the moduli space of tropical curves [8].

Let (L, H) be a linear series on a nodal curve X . Then H has a *basepoint* at a point p if all the sections of H vanish at p .

Lemma 2.6. *Let H be a linear series of rank $r \geq 0$ on a nodal curve X , and let Y be the union of the components where not all the global sections of H vanish identically. Then H induces a non-degenerate rational map from the smooth locus of Y to \mathbb{P}^r .*

Proof. Let $\pi : \tilde{Y} \rightarrow Y$ be the normalization of Y . Then the pullback to \tilde{Y} of the sections of H induce a rational map $\tilde{\phi} : \tilde{Y} \dashrightarrow \mathbb{P}^r$. For every smooth point p of Y such that $\tilde{\phi}$ is defined on $\pi^{-1}p$, let $\phi(p) = \tilde{\phi}(\pi^{-1}p)$. To see that the map is non-degenerate, suppose by contradiction that the image of ϕ is contained in a hyperplane whose defining equation is $a_0 \cdot x_0 + \cdots + a_r \cdot x_r = 0$. Then $a_0 \cdot \sigma_0 + \cdots + a_r \cdot \sigma_r = 0$, contradicting the fact that $\{\sigma_i\}$ is a basis. \square

Remark 2.7. If H has no basepoint at a node of X , then ϕ may be extended to that point.

Example 2.8. Let X be a union of three rational curves A, B and C , such that each pair meets at a single point. Let L be a line bundle of degree 1 on each component, and suppose that it has a single basepoint at the intersection of A and B . Then the map ϕ which corresponds to the complete linear system of L is defined everywhere away from the basepoint, and maps C isomorphically onto a line. Since there is a basepoint on A and B , their images $\phi(A)$ and $\phi(B)$ are single points. Those points must be distinct, as ϕ is an isomorphism on C .

Elements E_1, E_2, E_3 of M are a *basis* if F_{12}, F_{23}, F_{13} are distinct. A basic fact from matroid theory is:

Lemma 2.9. *Every pair of distinct elements can be completed to a basis. If E_1, E_2, E_3 is a basis, then every element E_0 forms a basis with at least two of them.*

Proof. As the matroid is simple, every distinct pair of elements E_1, E_2 is independent [9, Definition 3.8]. The size of any maximally independent set

equals the rank which, by assumption, is three, so E_1, E_2 are contained in a basis. The second part follows immediately by extending E_0 to a basis, and using the matroid exchange property [9, Definition 3.6]. \square

We now turn to prove Theorem 2.4.

Proof. Let G_M and D_M be as in Definition 2.2. If M is realizable over k , then by [6, Theorem 1.5], D_M is smoothable over $k[[t]]$ in the sense of Definition 1.4. That is, there is a curve over $k[[t]]$ whose generic fiber is a smooth $k((t))$ -curve X , whose special fiber has dual graph G_M , and there is a divisor D of rank 2 on X that specializes to D_M . By Corollary 1.3,

$$2 = r(D) \leq r_k^{\text{alg}}(D_M) \leq r_{G_M}(D_M) = 2,$$

so the algebraic rank of D_M over k is 2.

Conversely, suppose that $r_k^{\text{alg}}(D_M) = 2$. From the definition of the algebraic rank, there exists a curve X_M dual to G_M and a line bundle L of rank at least 2 whose degrees are prescribed by D_M . Let (L, H) be a linear series such that $\dim(H) = 3$, and ϕ_M the non-degenerate rational map to \mathbb{P}^2 defined in Lemma 2.6. For an element E of M , we denote by v_E the corresponding vertex of G_M , and C^E the corresponding component of X_M . Similarly, for every flat F , we denote by v_F and C^F the corresponding vertex and component. In what follows, we will show that ϕ_M has no basepoints, maps each C^E isomorphically onto a line, and those lines intersect as prescribed by M .

If H has a basepoint on a component C^F for some flat F then, as D_M has degree zero on v_f , all the sections of H vanish identically on it, and ϕ_M is not defined on that component. If H has no basepoints on C^F , then ϕ_M maps it to a point. Similarly, if H has more than one basepoint on a component C^E then the sections of H vanish identically on it, and if it has a single basepoint then C^E is mapped by ϕ_M to a point. Therefore, since ϕ_M is non-degenerate, there must be an element E_1 such that H has no basepoints on C^{E_1} , and ϕ_M maps it isomorphically onto a line ℓ_1 . We claim, moreover, that ϕ_M is defined on every component. Otherwise, let E_2 be such that all the global sections of H vanish on C^{E_2} . Then they also vanish on C^{E_1} , so H has a basepoint on C^{E_1} , a contradiction. We conclude that ϕ_M is defined on every component, and either maps it to a point on ℓ_1 or a line that intersects ℓ_1 .

Next, we wish to show that, in fact, ϕ_M has not basepoints at all. By the non-degeneracy of the map, there is an element E_2 such that $\phi_M(C^{E_2})$ is another line ℓ_2 meeting ℓ_1 at a point p . Now, suppose for the sake of

contradiction that E_3 is such that H has a basepoint on C^{E_3} . Then $\phi_M(C^{E_3})$ is a point q which is both on ℓ_1 and ℓ_2 . The point q must be different from p since ϕ_M is an isomorphism on C^{E_1} and C^{E_2} . But then, p and q are two different points lying on ℓ_1 and ℓ_2 , contradicting the fact that they are different lines. Therefore, ϕ_M maps every C^E isomorphically onto a line.

It remains to show that each of these components maps to a distinct line. Suppose for contradiction that there are two components $C^{E'}$, $C^{E''}$ that map to the same line, and choose E''' such that E', E'', E''' is a basis for M . As ϕ_M is one-to-one on each component, it follows that $C^{E'''}$ is mapped onto the same line as well. Now, let E be any element of M . By Lemma 2.9, E forms a basis with two of the elements in $\{E', E'', E'''\}$, and therefore, C^E is mapped to the same line. But this contradicts the non-degeneracy of ϕ_M .

We have seen that ϕ_M maps each C^F to a point, and each C^E isomorphically onto a line. Therefore, every pair E_1 and E_2 of elements of M correspond to distinct lines $\phi_M(C^{E_1})$ and $\phi_M(C^{E_2})$ meeting at $\phi_M(C^{F_{12}})$. Since ϕ_M is one-to-one on each of them, their images cannot meet anywhere else. Therefore, the image of ϕ_M is a realization of M , and the proof is complete. \square

Remark 2.10. Our theorem implies that when a simple matroid M of rank 3 is not realizable over k , the corresponding divisor D_M cannot be smoothed over the fraction field of any discrete valuation ring whose residue field is k . This is a slight strengthening of [6, Theorem 1.5], which implies that D_M is not smoothable over $k[[t]]$.

3. Relation with metrized complexes

In [1], Amini and Baker develop the theory of metrized complexes, a generalization of tropical curves. Roughly speaking, a metrized complex is obtained from a tropical curve by placing smooth curves at the vertices of the graph, and defining linear equivalence in a way that combines chip firing on the graph with linear equivalence on the curves. We refer the reader to [1] for an exposition of metrized complexes and their divisor theory.

The graphs in this section are finite vertex-weighted graphs, and the edges of all the metrized complexes are assumed to have length 1. For a divisor \mathcal{D} , we denote its restriction to a component C by $\mathcal{D}|_C$. For the rest of the section, fix an algebraically closed field k .

Definition 3.1. Let \mathcal{D} be a divisor on a metrized complex C with underlying graph G . If D is the divisor obtained from \mathcal{D} by forgetting the metrized complex structure, we say that \mathcal{D} is an *extension* of D .

Suppose that \mathcal{D} is an extension of D . Then both $r_k^{\text{alg}}(D)$ and $r(\mathcal{D})$ are refinements of the combinatorial rank, obtained by associating algebraic curves to the vertices, and considering line bundles on them. Therefore, we ask whether they are related. Our main result in this section is that, in general, there is no inequality between the ranks in either direction.

In the following example, the algebraic rank of a divisor is strictly greater than the rank of **every** extension to **any** metrized complex.

Example 3.2. Let G be a graph with three vertices u, v, w , such that there is a single edge between u and v , and a single edge between v and w . The weight of each vertex is set to be 1. Let D be the divisor $u + w$ (see Figure 2).

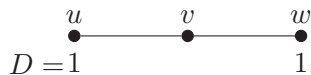


Figure 2: The graph G .

The combinatorial rank of D is 1. Since G is a hyperelliptic graph, it follows from [10, Theorem 1.1] that $r_k^{\text{alg}}(D) = 1$ as well. Now, let \mathcal{D} be any extension of D to a metrized complex \mathcal{C} . That is, \mathcal{C} consists of three components $C_u, C_v,$ and C_w of genus 1. There is single edge connecting C_u and C_v , and a single edge connecting C_v and C_w . \mathcal{D} contains a single chip on C_u and a single chip on C_w . Choose a point x on C_u where \mathcal{D} does not have a chip. Then $\mathcal{D} - x$ is not linearly equivalent to an effective divisor. Therefore, $r_{\mathcal{C}}(\mathcal{D}) = 0$.

We conclude that the answer to Question 1.1 is negative:

Corollary 3.3. *There exists a non-smoothable divisor D whose algebraic and combinatorial ranks are equal.*

Proof. Let G and D be as in the example above, so that $r_k^{\text{alg}}(D) = r_G(D) = 1$. Assume for contradiction that D is smoothable. That is, there exists a semistable model with smooth generic fiber X and special fiber X_0 whose weighted dual graph is G , and a divisor D_X on X of rank 1 whose specialization is D . Let \mathcal{C} be the regularization of X_0 (see Section 1 of [1]), and \mathcal{D} the specialization of D to \mathcal{C} . By the Specialization Lemma for metrized

complexes ([1, Theorem 4.6]), $r_{\mathcal{C}}(\mathcal{D}) \geq r(D) = 1$. But as shown above, \mathcal{D} has rank zero since it is an extension of D . \square

Next, we deal with the converse inequality by finding a divisor D whose algebraic rank is strictly smaller than the rank of **any** of its extensions to metrized complexes. We will, in fact, prove a stronger result: every extension of D can be chosen to be a limit \mathfrak{g}_d^2 . The idea originated from a question posed to the author by Matt Baker at the Joint Mathematical Meetings in Baltimore. We remind the reader of the definition of a limit \mathfrak{g}_d^r on a metrized complex.

Definition 3.4 ([1]). Let \mathcal{C} be a metrized complex. For each vertex v , choose a k -linear subspace F_v of $k(C_v)$, and let \mathcal{F} be the collection of these spaces. The \mathcal{F} -rank of a divisor \mathcal{D} is the largest integer r such that for any effective divisor \mathcal{E} of degree r , there exists a rational function f on \mathcal{C} whose C_v -part belongs to F_v for all v , such that $\mathcal{D} - \mathcal{E} + \text{div}(f)$ is effective.

If \mathcal{F} can be chosen so that each F_v is a subspace of dimension $r + 1$ (where r is the \mathcal{F} -rank of \mathcal{D}) of the global sections of some line bundle L_v of degree d , then \mathcal{D} is said to be a limit \mathfrak{g}_d^r .

Clearly, if a divisor is a limit \mathfrak{g}_d^r , then its rank is at least r .

Example 3.5. Let G and D be the graph and divisor in Figure 3 with all weights equal to zero.

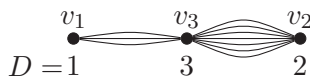


Figure 3: The graph G .

In [5], it was shown that D has combinatorial rank 2, but algebraic rank 1. Here, we strengthen that result, and show that every extension of D to a divisor on a metrized complex is a limit \mathfrak{g}_6^2 .

Indeed, let \mathcal{C} be a metrized complex with underlying finite graph G as in Figure 4, where C_1, C_2, C_3 are smooth rational curves over k .

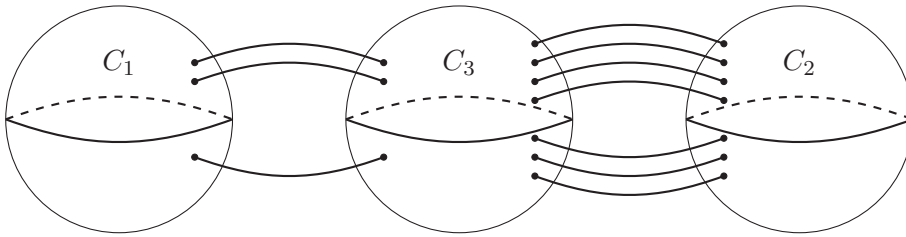


Figure 4: The metrized complex \mathcal{C} .

Let \mathcal{D} be a divisor extending D . Then \mathcal{D} contains a single chip on C_1 , three on C_3 , and two on C_2 . To show that \mathcal{D} is a limit \mathfrak{g}_6^2 , we first need to choose a subspace $H_i \subseteq k(C_i)$ of dimension 3 for every component C_i of \mathcal{D} . Let x^1, y^1, z^1 be the points of C_1 corresponding to the edges coming from C_3 , and x^3, y^3, z^3 the corresponding points on C_3 . Note that the property of being a limit \mathfrak{g}_6^2 is preserved under linear equivalence of divisors. Therefore, we may assume that $\mathcal{D}|_{C_1} = x^1 + y^1$, and $\mathcal{D}|_{C_3} = x^3 + y^3 + z^3$. Now, choose

$$\begin{aligned} : & & H_1 &= H^0(C_1, \mathcal{O}_{C_1}(x^1 + y^1)), \\ : & & H_2 &= H^0(C_2, \mathcal{O}_{C_2}(\mathcal{D}|_{C_2})), \\ : & & H_3 &= H^0(C_3, \mathcal{O}_{C_3}(x^3 + y^3)). \end{aligned}$$

Let \mathcal{E} be an effective divisor of degree 2. By [1, Corollary A.5], we may assume that \mathcal{E} is supported away from the edges. If the degree of $\mathcal{E}|_{C_1}$ is at most 1, then one easily finds functions f_1, f_2, f_3 in H_1, H_2, H_3 such that $\mathcal{D}_{C_i} + \text{div}(f_i)$ is effective and contains E_i for all i . Let f be the rational function on \mathcal{C} whose G -part is identically zero, and C_i -part f_i for each i . Then $\mathcal{D} + \text{div}(f)$ is effective and contains \mathcal{E} .

Otherwise, \mathcal{E} consists of two chips p, q on C_1 . Let f be the rational function on G satisfying $\text{div}(f) = 3(v_1) - 3(v_3)$. Choose f_1 to be the rational function on C_1 such that $\text{div}(f_1) = p + q - x^1 - y^1$, and choose f_2 and f_3 as constant functions on C_2, C_3 respectively. Then $\mathcal{D} + \text{div}(f)$ is effective and contains \mathcal{E} .

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