Counting Lattice Points in norm balls on higher rank simple Lie groups

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We establish an error estimate for counting lattice points in Euclidean norm balls (associated to an arbitrary irreducible linear representation) for lattices in simple Lie groups of real rank at least two. Our approach utilizes refined spectral estimates based on the existence of universal pointwise bounds for spherical functions on the groups involved. We focus particularly on the case of the special linear groups where we give a detailed proof of error estimates which constitute the first improvement of the best current bound established by Duke, Rudnick and Sarnak in 1991, and are nearly twice as good in some cases.

1. The lattice point counting problem in higher rank simple groups

Let G denote a connected non-compact simple Lie group with finite center, and Γ a lattice in G, namely discrete subgroup of finite covolume. Let vol denote the Haar measure on G normalized so that Γ has co-volume 1. We denote by K a maximal compact subgroup of G. Let $\tau \colon G \to GL(N, \mathbb{R})$ be a non-trivial irreducible representation of G on N-dimensional Euclidean space. We assume (without lost of generality) that $\tau(K) \subset SO(N)$. Let $\|\cdot\|^2$ denote the Euclidean norm $tr(A^t A)$ on $GL(N, \mathbb{R})$, and let

$$\|g\|_\tau^2 = \|\tau(g)\|^2 = \mathrm{tr}\left(\tau(g)^t \tau(g)\right).$$

We will consider balls of radius T in G with respect to $\|\cdot\|_\tau$, namely:

$$(1.1) \qquad B_T^\tau = \{g \in G \colon \|g\|_\tau \leq T\}.$$

Note that B_T^τ is invariant under left and right translations by K, and we will call sets satisfying this condition bi-K-invariant, or radial sets. In the present paper we will study the lattice point counting problem in the balls B_T^τ, namely we will aim to establish an asymptotic formula for $|B_T^\tau \cap \Gamma|$ in the form:

$$\frac{|B_T^\tau \cap \Gamma|}{\mathrm{vol}\left(B_T^\tau\right)} = 1 + O\left(\mathrm{vol}(B_T^\tau)^{-\kappa}\right),$$

with $\kappa > 0$ as large as possible, and $T \geq T_0 > 0$. Before we state our main results, let us recall what is currently the best exponent known for the error term in the higher rank case, established by Duke, Rudnick and Sarnak. For the group $G = SL\left(n+1, \mathbb{R}\right)$ with $n \geq 2$, and the balls B_T^τ, it is as follows.

Theorem 1. *[2, Thm. 3.1] Let $\Gamma \subset SL\left(n+1, \mathbb{R}\right)$ be any lattice. Then for T sufficiently large, and any $\eta > 0$,*

$$(1.2) \qquad \frac{|B_T^\tau \cap \Gamma|}{vol\left(B_T^\tau\right)} = 1 + O_\eta\left(vol(B_T^\tau)^{-\frac{1}{n(n+1)(n+2)}+\eta}\right).$$

We will denote this exponent by $\kappa_0 = \kappa_0(n) = \frac{1}{n(n+1)(n+2)}$.

Our purpose in the present paper is to improve this exponent for a large collection of families B_T^τ, associated with suitable irreducible representations of $SL\left(n+1, \mathbb{R}\right)$. We note that our method applies in principle to any connected higher-rank simple Lie group with finite center, but for simplicity of exposition we will concentrate below only on the case of $G = SL\left(n+1, \mathbb{R}\right)$ with $n \geq 2$. Let us begin by stating a special case of our main result and comparing it to the exponent cited above.

Theorem 2. *Let $G = SL\left(n+1, \mathbb{R}\right)$, $n \geq 2$ and $\Gamma \subset G$ be any lattice. Consider the adjoint representation of $SL\left(n+1, \mathbb{R}\right)$. Then, for $T \geq T_0$*

1) *For n odd:*

$$\frac{\left|B_T^{Ad} \cap \Gamma\right|}{vol\left(B_T^{Ad}\right)} = 1 + O\left(vol\left(B_T^{Ad}\right)^{-2\frac{n}{n+1}\kappa_0}(\log T)^q\right).$$

2) *For n even:*

$$\frac{\left|B_T^{Ad} \cap \Gamma\right|}{vol\left(B_T^{Ad}\right)} = 1 + O\left(vol\left(B_T^{Ad}\right)^{-2\frac{n}{n+2}\kappa_0}(\log T)^q\right).$$

In both cases q and T_0 are positive numbers which depends on n but not on the lattice, and can be made explicit.

Thus, for large n, the exponent established above is nearly twice as large as the exponent κ_0 established by [2] for the error term. The first case where the estimate is improved is for $G = SL_4(\mathbb{R})$.

Anticipating our results below, let us note that we will establish an error term exponent for Euclidean norm balls associated with any irreducible representation τ of G. These representations are classified by dominant weights, and for a certain non-empty cone of dominant weights we will establish the exponent stated in Theorem 2. For dominant weights belonging to certain other cones, we will establish an exponent which improves on κ_0 but is smaller than the one stated in Theorem 2.

2. Preliminaries and notation

2.1. Roots and weights in the Lie algebra of $SL(n+1, \mathbb{R})$

In the present section we establish notation and record some preliminaries. Our discussion is based on [9].

Let $G = SL(n+1, \mathbb{R})$, $K = SO(n+1, \mathbb{R})$ a maximal compact subgroup and let A be the following $\mathbb{R}$-split Cartan subgroup,

$$A = \left\{ \operatorname{diag}(a_1, \ldots, a_{n+1}) : \prod a_i = 1 \text{ and } a_i > 0, \quad \text{for } 1 \leq i \leq n+1 \right\}.$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{a}$ of A. Let $(X, Y)_{\mathfrak{g}} = \operatorname{tr}(\operatorname{ad}_{\mathfrak{g}}(X) \operatorname{ad}_{\mathfrak{g}}(Y))$ be the Killing form on $\mathfrak{g}$. For $\gamma, \delta \in \mathfrak{a}^* = \operatorname{Hom}(\mathfrak{a}, \mathbb{R})$ we set $\langle \gamma, \delta \rangle = \frac{2(\gamma, \delta)}{(\delta, \delta)}$, where $(\cdot, \cdot)$ is the form induced from $(\cdot, \cdot)_{\mathfrak{g}}$. Let $\Phi \subset \mathfrak{a}^*$ be the root system for the pair $(\mathfrak{g}, \mathfrak{a})$. Fix a choice of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$ and denote the set of positive roots by Φ^+. Denote by $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha_i(H) \geq 0, \ 1 \leq i \leq n\}$ the non-negative Weyl chamber with respect to Δ. Let $\left\{ \tilde{\beta}_j \right\}_{j=1}^{n} \subset \mathfrak{a}^+$ be defined by $\alpha_i(\tilde{\beta}_j) = \delta_{i,j} \ 1 \leq i, j \leq n$, namely the dual basis of the simple roots. An element $\gamma \in \mathfrak{a}^*$ is called a weight if the numbers $\langle \gamma, \alpha \rangle$ are integers for all $\alpha \in \Phi$, and denote by Λ the set of all weights. For $\gamma \in \Lambda$, if the integers $\langle \gamma, \alpha \rangle$ are non-negative for all $\alpha \in \Delta$, then the weight is called dominant. Let Λ^+ denote the set of dominant weights. We denote by λ_i, $1 \leq i \leq n$ the fundamental weights, namely those satisfying the equations $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$ $1 \leq i, j \leq n$. Another example of a dominant weight is half the sum of the positive roots, denoted ρ, which is equal to $\sum_{i=1}^{n} \lambda_i$. We recall that finite

dimensional irreducible representations of $SL\left(n+1,\mathbb{R}\right)$, are in a bijective correspondence with dominant weights, see e.g. [11, Theorem 5.5].

2.2. Volumes of radial balls in $SL\left(n+1,\mathbb{R}\right)$

Every $g \in G = SL\left(n+1,\mathbb{R}\right)$ can be written as $g = k_1 a k_2$ where $k_i \in K$ and $a \in A^+$. This decomposition yield the integration formula [10, p.142] :

Proposition 3. *Given a Haar measure on G, for $f \in C_c\left(SL\left(n+1,\mathbb{R}\right)\right)$ we have,*

$$(2.1) \qquad \int_{SL(n+1,\mathbb{R})} f\left(g\right) dg = \int_{K \times \mathfrak{a}^+ \times K} f\left(k_1 \exp\left(H\right) k_2\right)$$
$$\times \prod_{\alpha \in \Phi^+} \sinh\left(\alpha\left(H\right)\right) dk_1 dH dk_2$$

where dH is a suitable choice of Lebesgue measure on $\mathfrak{a}$, and dk is the Haar probability measure on the maximal compact subgroup K.

This formula was used in [7, 12] to compute the asymptotic volume of general norm balls in connected noncompact semisimple Lie groups. The computation applies in particular to the balls we investigate. Let τ, B_T^τ, Γ and vol be as in §1. Let λ be the highest weight for the representation τ. Applying [7, Thm 2.7] (or equivalently [12, Corollary 1.1]) to $B_T^\tau \subset G$ we get the following asymptotics,

$$\mathrm{vol}\left(B_T^\tau\right) \sim C_1 \left(\log T\right)^l T^{\frac{1}{m_1}},$$

where $l \in \mathbb{N}$ and $C_1 > 0$ is a constant independent of T. The rate of growth, namely m_1, is given by the following expression :

$$(2.2) \qquad m_1 = \min_{j \in \{1, \ldots, n\}} \frac{\lambda\left(\tilde{\beta}_j\right)}{2\rho\left(\tilde{\beta}_j\right)}.$$

Let $I = I(\lambda) = \left\{ 1 \leq i \leq n \colon m_1 = \frac{\lambda(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)} \right\}$ be the set of minimizing indices, namely the set of indices where the minimum is obtained.

3. Averaging operators and counting lattice points

3.1. The spectral method of counting lattice points

Let $G = SL(n+1, \mathbb{R})$, $\Gamma \subset G$ any lattice and B_T^τ be as in §1. Let $\pi_{G/\Gamma}$ be the unitary representation of G on $L^2(G/\Gamma)$, given by

$$\left(\pi_{G/\Gamma}(g)\,\tilde{f}\right)(h\Gamma) = \tilde{f}\left(g^{-1}h\Gamma\right), \quad \forall \tilde{f} \in L^2(G/\Gamma), h \in G/\Gamma.$$

Let b_T^τ denote the normalized indicator function, $\frac{\chi_{B_T^\tau}}{\text{vol}(B_T^\tau)}$. The averaging operators $\pi_{G/\Gamma}(b_T^\tau)$ are defined by,

$$\left(\pi_{G/\Gamma}(b_T^\tau)\left(\tilde{f}\right)\right)(x) = \frac{1}{\text{vol}\left(B_T^\tau\right)} \int_{B_T^\tau} \pi_{G/\Gamma}(g)\,\tilde{f}(x\Gamma)\,dg, \quad \forall \tilde{f} \in L^2(G/\Gamma).$$

We let $L_0^2(G/\Gamma)$ denote the space of L^2-functions on G/Γ with zero integral, and we let $\pi_{G/\Gamma}^0$ denote the restriction of the representation $\pi_{G/\Gamma}$ to $L_0^2(G/\Gamma)$.

We will use [6, Theorem 1.9] to establish our estimate of the error term. To apply this result to our families of balls B_T^τ it is enough to show that the following two conditions are satisfied.

1) The families are Lipschitz admissible in the sense of [5, Theorem 3.15].

2) The averaging operators $\pi_{G/\Gamma}(b_T^\tau)$ satisfy the quantitative mean ergodic theorem, with rate function given as a negative power of the volume.

Condition 1 is explained and established in [5, Theorem 3.15]. The arguments for showing Condition 2 are of spectral nature, and we turn to explain how to exploit the spherical spectrum of $L_0^2(G/\Gamma)$ for this purpose.

3.2. Spectral estimates of spherical functions

We will use concepts from the theory of Gelfand pairs and the theory of Banach $*$-algebras, and for a general exposition of this theory we refer to [3, 14].

Let G be a connected simple Lie group with a finite center, $K \subset G$ a maximal compact subgroup. It is well known that (G, K) is a Gelfand pair, so $L^1(K\backslash G/K)$ is a commutative Banach $*$-algebra. The map $f \mapsto f^*$ defined by, $f^*(x) = \overline{f(x^{-1})}$ is the involution of $L^1(K\backslash G/K)$. Denote by Σ the Gelfand spectrum of $L^1(K\backslash G/K)$. We can identify Σ with the set of bounded (G, K)-spherical functions ω on G [14, Theorem 8.2.7]. Using this

identification the Gelfand transform is given by $\hat{f}(\omega) = \int_G f(g)\,\omega\left(g^{-1}\right)dg$. We denote by $\Sigma^+ \subset \Sigma$ the subset of positive definite spherical functions for (G, K). Consider any unitary representation $\pi\colon G \to \mathcal{U}(\mathcal{H})$ with no invariant unit vectors. Such a representation defines a nondegenrate $*$-representation of $L^1(K\backslash G/K)$ on $\mathcal{H}$. This representation is defined by $f \mapsto \pi(f) = \int_G f(g)\,\pi(g)\,dg$. By the spectral theorem of $*$-representations [3, Theorem 1.54] there is a unique regular projection-valued measure, denoted P^π, on the spectrum of the algebra, such that P^π is supported on Σ^+, and the following formula holds :

$$(3.1) \qquad \forall f \in L^1(K\backslash G/K) \quad \langle \pi(f)\,u, v\rangle = \int_{\Sigma^+} \hat{f}(\omega)\,dP^\pi_{u,v}(\omega)\,,$$

where $P^\pi_{u,v}$ is the scalar complex bounded Borel measure on the spectrum Σ^+ determined by the pair of vectors u, v and the projection valued measure P^π. Therefore

$$
\begin{aligned}
(3.2) \qquad \|\pi(f)\|^2 &= \sup_{\|v\|=1}\ \langle \pi(f)\,v, \pi(f)\,v\rangle \\
&= \sup_{\|v\|=1}\ \langle \pi(f^* * f)\,v, v\rangle \\
&= \sup_{\|v\|=1} \int_{\Sigma^+} \widehat{f^* * f}(\omega)\,dP^\pi_{v,v}(\omega) \\
&= \sup_{\|v\|=1} \int_{\Sigma^+} \widehat{f^*}(\omega) \cdot \hat{f}(\omega)\,dP^\pi_{v,v}(\omega) \\
&= \sup_{\|v\|=1} \int_{\Sigma^+} \left|\hat{f}(\omega)\right|^2 dP^\pi_{v,v}(\omega)\,.
\end{aligned}
$$

A positive definite spherical function for (G, K) arise as the matrix coefficient associated with the unique K-invariant unit vector of a uniquely determined irreducible unitary representation [14, Theorem 8.4.8]. Our approach is based on a remarkable uniform spectral estimate, which is a special feature of the spherical unitary representation theory of simple Lie groups of real rank at least 2. Namely, we will use the fact that all the non-constant positive definite spherical functions can be *bounded by one and the same positive function* on the group. Any such bounding function F, which is refered to as a universal pointwise bound, gives a norm bound on all bi-K-invariant averaging operators on the group, as follows.

Theorem 4. *Let F be an upper bound for all matrix coefficients associated with K-invariant unit vectors of irreducible non-trivial unitary representations. Let π be any unitary representation without invariant unit vectors, as above. Then, for any bi-K-invariant function $f \in L^1(G)$*

$$(3.3) \qquad \|\pi(f)\| \le \int_G |f(g)| F(g)\, dg.$$

Proof. Let ω be a non-constant positive definite spherical function. Let π_ω be the unique irreducible unitary representation of G and v_ω the unique (up to scalar) K-fixed cyclic unit vector (see [14, Theorem 8.4.8]) satisfying $\omega(g) = \langle v_\omega, \pi_\omega(g) v_\omega \rangle$.

Since F is a universal pointwise bound, for every $\omega \in \Sigma^+$ we have

$$\left| \omega\left(g^{-1}\right) \right| = \left| \left\langle v_\omega, \pi_\omega\left(g^{-1}\right) v_\omega \right\rangle \right| = \left| \left\langle \pi_\omega(g) v_\omega, v_\omega \right\rangle \right| \le F(g).$$

Thus $\left| \hat{f}(\omega) \right| = \left| \int_G f(g) \omega\left(g^{-1}\right) dg \right| \le \int_G |f(g)| F(g)\, dg$ for every $\omega \in \Sigma^+$. Substituting this into equation (3.2) gives,

$$\begin{aligned}
\|\pi(f)\|^2 &= \sup_{\|v\|=1} \int_{\Sigma^+} \left| \hat{f}(\omega) \right|^2 dP^\pi_{v,v}(\omega) \\
&\le \sup_{\|v\|=1} \int_{\Sigma^+} \left(\int_G |f(g)| F(g)\, dg \right)^2 dP^\pi_{v,v}(\omega) \\
&= \left(\int_G |f(g)| F(g)\, dg \right)^2.
\end{aligned}$$

$\square$

3.3. The integrability bound

The error estimate in the lattice point counting problem provided by Duke, Rudnick and Sarnak can be derived using the following argument. Suppose that each non-trivial positive definite spherical function ω on G belong to $L^{p+\eta}(G)$ for any $\eta > 0$, and that the $L^{p+\eta}(G)$-norm of ω is uniformly bounded. This is certainly the case if there exists a universal pointwise bound F which is in $L^{p+\eta}(G)$ for every $\eta > 0$. Then, applying (3.3) to $\pi^0_{G/\Gamma}$ and b^τ_T, we get the following inequality:

$$\begin{aligned}
\left\| \pi^0_{G/\Gamma}\left(b^\tau_T\right)\left(\tilde{f}\right) \right\|_{L^2(G/\Gamma)} &\le \left\| \pi^0_{G/\Gamma}\left(b^\tau_T\right) \right\| \left\| \tilde{f} \right\|_{L^2(G/\Gamma)} \\
&\le \left(\frac{1}{\mathrm{vol}\left(B^\tau_T\right)} \int_{B^\tau_T} F(g)\, dg \right) \left\| \tilde{f} \right\|_{L^2(G/\Gamma)}.
\end{aligned}$$

This inequality gives us a quantitative mean ergodic theorem. Hence the conditions mentioned in §3.1 are met, and we can apply [6, Theorem 1.9] to write, for $T \geq T_0 > 0$:

$$(3.4) \qquad \left| \frac{|B_T^\tau \cap \Gamma|}{\text{vol}\left(B_T^\tau\right)} - 1 \right| < C \left(\frac{1}{\text{vol}\left(B_T^\tau\right)} \int_{B_T^\tau} F\left(g\right) dg \right)^{\frac{1}{1+d}} .$$

with $d = \dim\left(G/K\right) = \frac{(n+1)(n+2)}{2} - 1$ (see e.g. [6, remark 1.10]).

Thus the quality of the error estimate depends on the upper bound for the integral. One possibility is to use the $L^{p+\eta}(G)$ integrability condition for F, so that using Hölder's inequality :

$$\leq C_\eta \left(\frac{1}{\text{vol}\left(B_T^\tau\right)} \right)^{\frac{1}{p(d+1)} - \eta} \|F\|_{L^{p+\eta}(G)}^{1/(d+1)} ,$$

where C_η is a computable positive constant. This estimate, namely $\kappa_0 = \frac{1}{p(d+1)}$ is the one established in [2], using the fact that for $G = SL(n+1, \mathbb{R})$ the exponent of integrability is $p = 2n$.

In the next section we will focus on $SL(n+1, \mathbb{R})$ and show how to derive a better upper estimate for the integral $\int_{B_T^\tau} F\left(g\right) dg$, for certain functions F, thus improving the error estimate arising from the exponent of integrability of F.

4. Universal pointwise bounds for $SL\left(n+1, \mathbb{R}\right)$

4.1. Universal pointwise bounds

Let $G = SL\left(n+1, \mathbb{R}\right)$, $K = SO\left(n+1, \mathbb{R}\right)$ and $F \in C_c\left(G\right)$ a bi-K-invariant function. By the integration formula (2.1) we have,

$$\int_G F\left(g\right) dg = \int_{K \times \mathfrak{a}^+ \times K} F\left(k_1 \exp\left(H\right) k_2\right) \prod_{\alpha \in \Phi^+} \sinh\left(\alpha\left(H\right)\right) dk_1 dH dk_2$$

$$= \int_{\mathfrak{a}^+} F\left(\exp\left(H\right)\right) \prod_{\alpha \in \Phi^+} \sinh\left(\alpha\left(H\right)\right) dH.$$

Hence we can consider F to be a function on $\mathfrak{a}^+$. The functions we will use as universal bounds for the positive definite spherical functions for (G, K),

will all have the following form. For all $H \in \mathfrak{a}^+$

(4.1) $\quad F_\theta(H) = P(H) e^{-\theta(H)}$ for some $\theta \in \mathfrak{a}^*$ where $\theta(H) > 0 \ \forall H \in \mathfrak{a}^+$.

Here P is a positive function which can be bounded by a polynomial function in $\|H\|$.

Three distinct functions which can serve as bounds for the positive definite spherical functions of (G, K) are, first, a suitable root of the Harish Chandra Ξ_G-function, (see e.g. [1]), second, a function constructed by Howe and Tan (see [8, theorem 3.3.12]), and third, a sharper version of it constructed by Oh (see [13]). Let us turn to describe them in greater detail.

4.1.1. Harish-Chandra's Ξ_G-function.

It is a well-known fact that for a suitable $n = n_G$ the function $\Xi_G^{\frac{1}{n}}$ is a bound for the non-constant positive definite spherical functions of (G, K), see the discussion in [1]. Furthermore the Harish-Chandra function satisfies (see [4, Theorem 4.6.4])

$$\Xi_G\left(e^H\right) \leq C\left(1 + \|H\|\right)^{|\Phi^+|} e^{-\rho(H)}, \quad \forall H \in \mathfrak{a}^+.$$

For $H \in \mathfrak{a}^+$, write $H = \text{diag}(h_1, \ldots, h_{n+1})$, where $h_i \geq h_{i+1}$ and $\sum_{i=1}^{n+1} h_i = 0$. Then the first universal pointwise bound is given by

$$F_{\rho/n}(H) = C\left(1 + \|H\|\right)^{|\Phi^+|} e^{-\frac{\rho(H)}{n_G}}$$

so that the linear function θ in this case is ρ/n_G. The constant n_G has been computed explicitly for all simple Lie groups of real rank at least two, and can be taken at the least integer k such that all non-constant positive-definite spherical functions on G are in $L^{2k+\eta}(G)$ for every $\eta > 0$. For $G = SL(n + 1, \mathbb{R})$ it is equal to n.

4.1.2. Howe-Tan's function..

Howe and Tan [8, theorem 3.3.12] showed that the bi-K-invariant function :

$$k_1 \exp(H) k_2 \ \mapsto \ \min_{i \neq j} \Xi_{SL(2,\mathbb{R})}\left(\exp\begin{pmatrix} \frac{h_i - h_j}{2} & 0 \\ 0 & \frac{h_j - h_i}{2} \end{pmatrix}\right),$$

is a bound for all the non-constant positive definite spherical functions of (G, K). We recall that $\Xi_{SL(2,\mathbb{R})}\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \sim \frac{\log a}{a}, \quad a > 0$. Using this asymptotic of $\Xi_{SL(2,\mathbb{R})}$, and the fact that $h_i \geq h_{i+1}$ we conclude that a second

universal pointwise bound is provided by the function $F_{\beta/2}$ given by

$$F_{\beta/2}(H) = C \cdot \beta(H) \cdot e^{-\frac{1}{2}\beta(H)},$$

where $\beta = \sum_{i=1}^{n} \alpha_i$ is the highest root, $\beta(H) = h_1 - h_{n+1}$.

4.1.3. Oh's function. Using the same spectral approach more efficiently, by utilizing strongly orthogonal systems, Oh [13] showed that in (4.1) we can take the linear functional $\theta = \gamma$ given explicitly as follows.

$$\gamma = \begin{cases} \frac{1}{2}\left(\sum_{i=1}^{(n-1)/2} i\alpha_i + \sum_{i=(n+1)/2}^{n}(n+1-i)\alpha_i\right) & n \text{ odd} \\ \frac{1}{2}\left(\sum_{i=1}^{n/2} i\alpha_i + \frac{n}{2}\alpha_{n/2+1} + \sum_{i=n/2+2}^{n}(n+1-i)\alpha_i\right) & n \text{ even.} \end{cases}$$

Thus a third universal pointwise bound is $F_\gamma(H) = P(H)e^{-\gamma(H)}$, where $P(H)$ is bounded by an explicit polynomial in $\|H\|$.

Remark 5.　　1) We note that for $n = 2$ namely for $G = SL(3, \mathbb{R})$, the functions constructed by Howe-Tan and by Oh are the same. As we shall see below, it follows that this case is the only one for which we will not achieve an improvement of the error estimate in the lattice point counting problem.

2) As will become apparent in the next section, all three functions discussed above on $G = SL(n+1, \mathbb{R})$ satisfy that they are in $L^p(G)$ if and only if $p > 2n$.

4.2. Estimating integrals of universal pointwise bounds

Our task now is to estimate the integral of the universal pointwise bound on norm balls B_T^τ. This amounts to bounding the integral of a polynomial times an exponential function on suitable regions of Euclidean space.

Therefore consider, in Euclidean space $\mathbb{R}^k$, the region :

$$D = \left\{(t_1, \ldots, t_k) : \forall i, \quad t_i \geq 0, \quad \sum_{i=1}^{k} m_i t_i \leq S\right\}.$$

We begin with the following

Lemma 6. *Given $k \in \mathbb{N}$ and $0 < m_1 \leq \cdots \leq m_k$, for all $S > 0$,*

$$\int_D P(t_1, \ldots, t_k) e^{\sum_{i=1}^{k} t_i} dt_1 \cdots dt_k \leq C e^{\frac{S}{m_1}} S^{\deg(P)+k-1},$$

for suitable C, where P is any polynomial function.

Proof. Since P is a polynomial function, there exists $c_1 > 0$ such that for all $S > 0$, $P|_D \leq c_1 S^{\deg(P)}$. Next, consider the following re-parametrization of D. Set $0 \leq t_k \leq S/m_k$ and for $1 \leq j \leq k-1$:

$$0 \leq t_j \leq \frac{1}{m_j} \left(S - \sum_{i=j+1}^{k} m_i t_i \right).$$

This allows us to write,

$$\int_D P((t_1, \ldots, t_k)) e^{\sum_{i=1}^{k} t_i} dt_1 \cdots dt_k$$

$$= \int_{t_k=0}^{\frac{S}{m_k}} \cdots \int_{t_1=0}^{\frac{1}{m_1}\left(S-\sum_{i=2}^{k} m_i t_i\right)} P(t_1, \ldots, t_k) e^{\sum_{i=1}^{k} t_i} dt_1 \cdots dt_k$$

$$\leq c_1 S^{\deg P} \int_{t_k=0}^{\frac{S}{m_k}} \cdots \int_{t_1=0}^{\frac{1}{m_1}\left(S-\sum_{i=2}^{k} m_i t_i\right)} e^{\sum_{i=1}^{k} t_i} dt_1 \cdots dt_k$$

$$\leq c_1 S^{\deg P} \int_{t_k=0}^{\frac{S}{m_k}} \cdots \int_{t_2=0}^{\frac{1}{m_2}\left(S-\sum_{i=3}^{k} m_i t_i\right)} \left(e^{\frac{1}{m_1}\left(S-\sum_{i=2}^{k} m_i t_i\right)} - 1 \right) e^{\sum_{i=2}^{k} t_i} dt_2 \cdots dt_k$$

$$\leq c_1 S^{\deg P} e^{\frac{S}{m_1}} \int_{t_k=0}^{\frac{S}{m_k}} \cdots \int_{t_2=0}^{\frac{1}{m_2}\left(S-\sum_{i=3}^{k} m_i t_i\right)} e^{\sum_{i=2}^{k} t_i \left(1-\frac{m_i}{m_1}\right)} dt_2 \cdots dt_k$$

$$\leq c_1 S^{\deg P} e^{\frac{S}{m_1}} \int_{t_k=0}^{\frac{S}{m_k}} \cdots \int_{t_2=0}^{\frac{1}{m_2}\left(S-\sum_{i=3}^{k} m_i t_i\right)} 1 dt_2 \cdots dt_k \leq C e^{\frac{S}{m_1}} S^{\deg P+k-1}.$$

$\square$

Let us now apply this fact to the integrals we are interested in bounding. As in §1, we let τ denote an irreducible representation associated with a dominant weight λ, which is its highest weight.

Corollary 7. *Let $G = SL(n+1, \mathbb{R})$, and B_T^τ be as in §1. If F is a bi-K-invariant function and $F|_{\mathfrak{a}^+}$ is of the form $F_\theta(H) = P(H) e^{-\theta(H)}$, then*

$$\int_{B_T^\tau} F(g)\, dg \leq C_1' (\log T)^{l'} T^{\frac{1}{m_1'}},$$

where $l' \in \mathbb{N}, C'_1 > 0$. Furthermore, setting $\psi = 2\rho - \theta$, the following formula for m'_1 holds:

$$(4.2) \qquad m'_1 = \min_{1 \leq i \leq n} \left(\frac{\lambda\left(\tilde{\beta}_i\right)}{\psi\left(\tilde{\beta}_i\right)} \right).$$

Extending the definition of $I(\lambda)$ in §2.2 above, let us denote by $I'(\lambda)$ the set of minimizing indices in the equation above. This set depends on the functional θ chosen to define the universal estimate, but we will suppress this dependence in the notation.

Proof. By equation (2.1) we have,

$$\int_{B_T^\tau} F(g)\, dg = \int_{\mathfrak{a}^+(T,\tau)} F\mid_{\mathfrak{a}^+}(H) \prod_{\alpha \in \Phi^+} \sinh(\alpha(H))\, dH,$$

where $\mathfrak{a}^+(T,\tau) = \{H \in \mathfrak{a}^+ : \|\tau(\exp(H))\| \leq T\}$. Let λ be the highest weight of τ. Comparing with the max-norm, we find that there is $\tilde{C}$ such that for all $T > 0$,

$$\mathfrak{a}^+(T,\tau) \subset \left\{ H \in \mathfrak{a}^+ : e^{\lambda(H)} \leq \frac{T}{\tilde{C}} \right\}$$
$$= \left\{ H \in \mathfrak{a}^+ : \lambda(H) \leq \log T - \log \tilde{C} \right\}.$$

We denote $D_{T,\tau} = \left\{ H \in \mathfrak{a}^+ : \lambda(H) \leq \log T - \log \tilde{C} \right\}$. Since $\sinh(x) \leq e^x$, we have the following inequality,

$$\int_{B_T^\tau} F_\theta(g)\, dg = \int_{\mathfrak{a}^+(T,\tau)} P(H)\, e^{-\theta(H)} \prod_{\alpha \in \Phi^+} \sinh(\alpha(H))\, dH$$
$$\leq C_1 \int_{D_{T,\tau}} P(H)\, e^{(2\rho-\theta)(H)}\, dH$$

Let $H = \sum_{j=1}^n t_j \frac{\tilde{\beta}_j}{\psi(\tilde{\beta}_j)}$. In terms of these coordinates we have (recall $\psi = 2\rho - \theta$):

$$D_{T,\tau} = \left\{ (t_1,\ldots,t_n) : \forall i, \quad t_i \geq 0, \quad \sum_{j=1}^n \frac{\lambda\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} t_j \leq \log T - \log \tilde{C} \right\},$$

and

$$\int_{B_T^\tau} F_\theta\left(g\right) dg \le C_2 \int_{D_{T,\tau}} P\left(t_1,\ldots,t_n\right) e^{\sum_{j=1}^n t_j} \prod_{j=1}^n dt_j.$$

Thus using Lemma 6 we find that,

$$\int_{B_T^\tau} F_\theta\left(g\right) dg \le C_1'\left(\log T\right)^{l'} T^{\frac{1}{m_1'}}.$$

With $m_1' = \min_{1\le j \le n} \frac{\lambda(\tilde\beta_j)}{\psi(\tilde\beta_j)}$, $C_1' > 0$ and $l' \in \mathbb{N}$. □

Recall the parameter m_1 defined in equation (2.2). Given m_1 and m_1' (which depends also on the choice of the functional θ defining the universal pointwise bound), and the parameter $\kappa_0 = \kappa_0(n)$, and using the foregoing formula we can rewrite equation (3.4) and state our main error estimate as follows.

Corollary 8.

$$(4.3) \qquad \frac{|B_T^\tau \cap \Gamma|}{vol\left(B_T^\tau\right)} = 1 + O\left(vol(B_T^\tau)^{-2n\left(1-\frac{m_1}{m_1'}\right)\kappa_0}\left(\log T\right)^q\right),$$

for some $q \in \mathbb{N}$ and $T \ge T_0$ that can be computed explicitly.

This asymptotic formula gives a solution to the lattice point counting problem in $|B_T^\tau \cap \Gamma|$, with exponent $\kappa = \kappa(\tau) = 2n\left(1 - \frac{m_1}{m_1'}\right)\kappa_0$. The exponent $\kappa(\tau)$ is determined by the underlying representation τ and the functional θ defining the universal pointwise estimate F_θ. Our next task therefore is to estimate the exponent just described as τ ranges over the set of irreducible finite-dimensional representations, namely over the space of dominant weights, and establish when does $\kappa(\tau)$ constitute an improvement over κ_0.

5. Improving the error estimates

5.1. Comparing exponents

In order to estimate the exponent from equation (4.3) for various irreducible representations, let us note the following. The largest exponent is achieved when using the functional γ described above, and we will presently give conditions on the irreducible representations for which this largest exponent can be established.

First, given a dominant weight λ recall the set $I = I(\lambda)$ of minimizing indices defined in §2.2.

Proposition 9. *Let $G = SL(n+1, \mathbb{R})$, $\lambda \in \Lambda^+$ and $\theta \in \mathfrak{a}^*$. For every $i \in I = I(\lambda)$ we have $\frac{m_1}{m_1'} \geq \frac{\psi(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)}$, so in particular $\frac{m_1}{m_1'} \geq \min_{1 \leq j \leq n} \frac{\psi(\tilde{\beta}_j)}{2\rho(\tilde{\beta}_j)}$.*

Proof. By definition, for any $1 \leq i \leq n$

$$m_1' = \min_{1 \leq j \leq n} \left(\frac{\lambda\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} \right) \leq \frac{\lambda\left(\tilde{\beta}_i\right)}{\psi\left(\tilde{\beta}_i\right)}.$$

Thus for every $i \in I = I(\lambda)$

$$\frac{m_1}{m_1'} = \frac{\lambda\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)} \left(m_1'\right)^{-1} \geq \frac{\lambda\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)} \frac{\psi\left(\tilde{\beta}_i\right)}{\lambda\left(\tilde{\beta}_i\right)} = \frac{\psi\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)}.$$

$\square$

We can now easily deduce that using the functional ρ/n_G as a universal pointwise bound does not improve the error estimate.

Proposition 10. *Let $\lambda \in \Lambda^+$, and let τ denote the irreducible representation of $SL(n+1, \mathbb{R})$ associated with λ. Then the exponent associated with τ and the linear functional $\frac{\rho}{n_G}$ is equal to κ_0.*

Proof. Indeed in this case $\psi = 2\rho - \frac{\rho}{n} = 2\rho\left(1 - \frac{1}{2n}\right)$. Thus for any $\lambda \in \Lambda^+$,

$$m_1' = \min_{1 \leq j \leq n} \left(\frac{\lambda\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} \right) = \frac{1}{1 - \frac{1}{2n}} \min_{1 \leq j \leq n} \left(\frac{\lambda\left(\tilde{\beta}_j\right)}{2\rho\left(\tilde{\beta}_j\right)} \right) = \frac{1}{1 - \frac{1}{2n}} m_1.$$

This means that $\kappa = 2n\left(1 - \frac{m_1}{m_1'}\right)\kappa_0 = 2n\left(1 - \left(1 - \frac{1}{2n}\right)\right)\kappa_0 = \kappa_0.$ $\square$

The same phenomenon arises with the universal bound defined by the linear functional $\beta/2$, β the highest root.

Proposition 11. *Let $\lambda \in \Lambda^+$, and let τ denote the irreducible representation of $SL(n+1, \mathbb{R})$ associated with λ. Then the exponent associated with τ and the linear functional $\frac{1}{2}\beta$ is less then or equals to κ_0.*

Proof. Recall that $2\rho = \sum_{k=1}^{n} \alpha_k k (n + 1 - k)$, thus $2\rho \left(\tilde{\beta}_j \right) = j (n + 1 - j)$. We also have $\beta = \sum_{k=1}^{n} \alpha_k$, so that $\beta \left(\tilde{\beta}_j \right) = 1$. Now using proposition 9:

$$
\begin{aligned}
\frac{m_1}{m_1'} &\geq \min_{1 \leq j \leq n} \frac{\psi \left(\tilde{\beta}_j \right)}{2\rho \left(\tilde{\beta}_j \right)} = \min_{1 \leq j \leq n} \left(\frac{2\rho \left(\tilde{\beta}_j \right) - \frac{1}{2} \beta \left(\tilde{\beta}_j \right)}{2\rho \left(\tilde{\beta}_j \right)} \right) \\
&= \min_{1 \leq j \leq n} \left(1 - \frac{1}{4\rho \left(\tilde{\beta}_j \right)} \right) = 1 - \left(\max_{1 \leq j \leq n} \frac{1}{4\rho \left(\tilde{\beta}_j \right)} \right) \\
&= 1 - \frac{1}{4 \min_{1 \leq j \leq n} \rho \left(\tilde{\beta}_j \right)} = 1 - \frac{1}{2n}.
\end{aligned}
$$

This means that $\kappa = 2n \left(1 - \frac{m_1}{m_1'} \right) \kappa_0 \leq 2n \left(1 - \left(1 - \frac{1}{2n} \right) \right) \kappa_0 = \kappa_0$. $\square$

Next we consider the universal pointwise bound defined by the linear functional γ described above, and show that as we vary $\lambda \in \Lambda^+$ a better error estimate can be established in many cases.

5.2. Main result : improving the error estimate

As usual, let τ be an irreducible representation of $G = SL(n + 1, \mathbb{R})$ for $n \geq 2$, and let λ be the highest weight. Let $B_T^\tau = \{ g \in G : \| \tau(g) \| \leq T \}$. γ be the functional defined in §4 and let F_γ be the universal pointwise bound defined there. Let m_1, m_1' be defined by equations (2.2) and (4.2). Let $I = I(\lambda)$ and $I' = I'(\lambda)$ be the sets of minimizing indices defining them, with $I'(\lambda)$ defined by the choice $\theta = \gamma$. These choices give rise to the main result of the paper.

Theorem 12. *Let $G = SL(n + 1, \mathbb{R})$ for $n \geq 2$, let Γ any lattice subgroup, and $T \geq T_0$.*

1) *If n is odd and $\frac{n+1}{2} \in I(\lambda) \cap I'(\lambda)$, then*

$$
\frac{|B_T^\tau \cap \Gamma|}{vol\left(B_T^\tau \right)} = 1 + O\left(vol(B_T^\tau)^{-2 \frac{n}{n+1} \kappa_0} (\log T)^q \right),
$$

2) *If n is even and $\frac{n}{2} + 1 \in I(\lambda) \cap I'(\lambda)$ or $\frac{n}{2} \in I(\lambda) \cap I'(\lambda)$, then*

$$
\frac{|B_T^\tau \cap \Gamma|}{vol\left(B_T^\tau \right)} = 1 + O\left(vol(B_T^\tau)^{-2 \frac{n}{n+2} \kappa_0} (\log T)^q \right),
$$

In both cases q and T_0 are positive numbers which depends on n but not on the lattice, and can be made explicit.

Thus when n is odd the exponent associated with a dominant weight λ is $\kappa = 2\frac{n}{n+1}\kappa_0$, and when n is even the exponent is $\kappa = 2\frac{n}{n+2}\kappa_0$, provided that λ satisfies the conditions stated above. In both cases, these values are the largest exponent that our method provides, and the exponents are nearly twice as large as κ_0.

Proof. The largest exponent is achieved when the ratio $\frac{m_1}{m'_1}$ is minimized. For the universal pointwise bound associated with γ, setting $\psi = 2\rho - \gamma$ by Proposition 9 we have $\frac{m_1}{m'_1} \geq \min_{1 \leq i \leq n} \frac{\psi(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)}$. To find conditions for which $\frac{m_1}{m'_1} = \min_{1 \leq i \leq n} \frac{\psi(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)}$, we will use the following.

Proposition 13. $\min_{1 \leq i \leq n} \frac{\psi(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)} = \begin{cases} \frac{n}{n+1} & n \text{ is odd} \\ \frac{n+1}{n+2} & n \text{ is even} \end{cases}$. *Moreover for n odd the minimum is attained at $\frac{n+1}{2}$, and for n even at $\left\{\frac{n}{2}, \frac{n}{2}+1\right\}$.*

Proof. We demonstrate this for n odd, and a similar argument applies to the case when n is even. First note that $\psi\left(\tilde{\beta}_j\right)$ and $\rho\left(\tilde{\beta}_j\right)$ have the symmetry, $j \mapsto n+1-j$. Therefore

$$
\min_{1 \leq i \leq n} \frac{\psi\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)} = \min_{1 \leq i \leq \frac{n+1}{2}} \frac{\psi\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)}
$$
$$
= 1 + \min_{1 \leq i \leq \frac{n+1}{2}} \frac{-\frac{1}{2}i}{i(n+1-i)}
$$
$$
= 1 - \frac{1}{2} \max_{1 \leq i \leq \frac{n+1}{2}} \frac{1}{n+1-i}
$$
$$
= 1 - \frac{1}{n+1} = \frac{n}{n+1},
$$

and it is can be easily seen that the minimum is attained at $\frac{n+1}{2}$. $\qquad\square$

Resuming the proof of Theorem 12, we conclude that in the event that n is odd, if $s = \frac{n+1}{2} \in I \cap I'$ then,

$$
\frac{m_1}{m'_1} = \frac{\lambda\left(\tilde{\beta}_s\right)}{2\rho\left(\tilde{\beta}_s\right)} \frac{\psi\left(\tilde{\beta}_s\right)}{\lambda\left(\tilde{\beta}_s\right)} = \frac{\psi\left(\tilde{\beta}_s\right)}{2\rho\left(\tilde{\beta}_s\right)} = \min_{1 \leq i \leq n} \frac{\psi\left(\tilde{\beta}_i\right)}{2\rho\left(\tilde{\beta}_i\right)} = \frac{n}{n+1}.
$$

This implies that the associated exponent will be $\kappa = 2n\left(1 - \frac{m_1}{m_1'}\right)\kappa_0 = \frac{2n}{n+1}\kappa_0$. This exponent is greater then κ_0 if n is odd and $n \geq 3$. Similarly we handle the case that n is even. $\qquad\square$

5.3. Examples of admissible dominant weights

Let us now show that there exist dominant weights for which the improved error estimate is achieved. Namely we give some examples for dominant weights that meet the conditions of theorem 12.

Theorem 14. *Let n be odd. Then every dominant weight μ belonging to the set*

$$W_n = \left\{ \lambda_i + \lambda_{n+1-i} : i \in \left\{ 1, \ldots, \lfloor \frac{n+1}{4} \rfloor \right\} \right\}$$

satisfies the condition of Theorem 12.

Proof. We need to show that for weights $\mu \in W_n$, $\frac{n+1}{2} \in I \cap I'$. Setting $s = \frac{n+1}{2}$ we have to show,

$$\frac{\mu\left(\tilde{\beta}_s\right)}{2\rho\left(\tilde{\beta}_s\right)} \leq \frac{\mu\left(\tilde{\beta}_j\right)}{2\rho\left(\tilde{\beta}_j\right)} \text{ and } \frac{\mu\left(\tilde{\beta}_s\right)}{\psi\left(\tilde{\beta}_s\right)} \leq \frac{\mu\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)}, \forall j \neq s.$$

as noted above

$$\frac{\psi\left(\tilde{\beta}_s\right)}{2\rho\left(\tilde{\beta}_s\right)} \leq \frac{\psi\left(\tilde{\beta}_j\right)}{2\rho\left(\tilde{\beta}_j\right)} \quad \forall j \neq s \implies \frac{\psi\left(\tilde{\beta}_s\right)}{\psi\left(\tilde{\beta}_j\right)} \leq \frac{2\rho\left(\tilde{\beta}_s\right)}{2\rho\left(\tilde{\beta}_j\right)} \quad \forall j \neq s.$$

Hence to show that $\mu \in W_n$ satisfies the property stated in Theorem 12 it is enough to show,

$$(5.1) \qquad\qquad \frac{\mu\left(\tilde{\beta}_s\right)}{\psi\left(\tilde{\beta}_s\right)} \leq \frac{\mu\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} \quad \forall j \neq s$$

Recall that the fundamental weights satisfy $\lambda_i\left(\tilde{\beta}_j\right) = \left(C_n^{-1}\right)_{i,j}$ where C_n is the Cartan matrix of the root system. Hence if $\mu = \sum_{k=1}^{n} q_k \lambda_k$, then

$$C_n^{-1} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} \mu\left(\tilde{\beta}_1\right) \\ \vdots \\ \mu\left(\tilde{\beta}_n\right) \end{pmatrix}.$$

Using the above equation it can be calculated that

$$(\lambda_1 + \lambda_n)\left(\tilde{\beta}_j\right) = (1, 1, \ldots, 1),$$
$$(\lambda_2 + \lambda_{n-1})\left(\tilde{\beta}_j\right) = (1, 2, 2, \ldots, 2, 1),$$

and in general

$$(\lambda_i + \lambda_{n+1-i})\left(\tilde{\beta}_j\right) = \left(1, 2, 3, \ldots, \underbrace{i}_{i\text{'th entry}}, i, \ldots, \underbrace{i}_{n+1-i\text{'th entry}}, \ldots, 2, 1\right).$$

Next let us recall that by definition of γ

$$\psi = 2\rho - \gamma = \sum_{i=1}^{\frac{n-1}{2}} i\left(n + \frac{1}{2} - i\right)\alpha_i + \sum_{i=\frac{n+1}{2}}^{n} (n+1-i)\left(i - \frac{1}{2}\right)\alpha_i.$$

Recall that ψ has the symmetry given by $\psi\left(\tilde{\beta}_j\right) = \psi\left(\tilde{\beta}_{n+1-j}\right)$ and note that the same is true for every weight in W_n. Hence it is enough to show that $\frac{\mu(\tilde{\beta}_s)}{\psi(\tilde{\beta}_s)} \leq \frac{\mu(\tilde{\beta}_j)}{\psi(\tilde{\beta}_j)}$ $\forall j < s$. Let $\mu = \lambda_i + \lambda_{n+1-i}$. Then

$$\frac{\mu\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} = \begin{cases} \frac{j}{j\left(n+\frac{1}{2}-j\right)} & j < i \\ \frac{i}{j\left(n+\frac{1}{2}-j\right)} & j \geq i \end{cases} = \begin{cases} \frac{1}{\left(n+\frac{1}{2}-j\right)} & j < i \\ \frac{i}{j\left(n+\frac{1}{2}-j\right)} & j \geq i \end{cases}.$$

Thus

$$\min_{1 \le j \le s} \frac{\mu\left(\tilde{\beta}_j\right)}{\psi\left(\tilde{\beta}_j\right)} = \min\left(\min_{1 \le j < i}\left(\frac{1}{(n + \frac{1}{2} - j)}\right), \min_{i \le j \le s}\left(\frac{i}{j\left(n + \frac{1}{2} - j\right)}\right)\right)$$

$$= \min\left(\frac{1}{n - \frac{1}{2}}, \frac{i}{s\left(s - \frac{1}{2}\right)}\right).$$

We would like the minimum to be achieved at $j = s$, and this occurs when

$$\frac{i}{s\left(s - \frac{1}{2}\right)} \le \frac{1}{n - \frac{1}{2}} \iff i \le \frac{n}{n - \frac{1}{2}} \frac{n+1}{4}.$$

This obviously holds for $i < \frac{n+1}{4}$, and so we have shown that for any μ in W_n the condition stated in Theorem 12 (for n odd) holds. Therefore the balls defined by the representations associated with these dominant weights give rise to the an error exponent in the lattice point counting problem stated in Theorem 12. □

Remark 15. A similar result holds for n even, namely for all dominant weights

$$\mu \in \left\{\lambda_i + \lambda_{n+1-i} : i \in \left\{1, \dots, \left\lfloor\frac{n}{4}\right\rfloor\right\}\right\} = W_n$$

condition 2 of theorem 12 is satisfied with $\frac{n}{2} + 1 \in I \cap I'$.

Finally, let us note that $\lambda_1 + \lambda_n = \beta$ is the highest weight of the adjoint representation, and this completes the proof of Theorem 2. □

5.4. Euclidean norm balls associated with an arbitrary irreducible representation

The error estimate given by Theorem 12 is the best that our method can provide. As stated in Theorem 2, it arises for the balls associated with the adjoint representation, among others. Let us now note that for other irreducible representations it is still possible to improve the error estimate beyond the exponent κ_0 established in [2]. We will prove such an improvement for irreducible representations τ when the highest weight $\lambda \in \Lambda^+$ belongs to the

following set:

$$\Lambda_*^+ = \left\{ \lambda \in \Lambda^+ : \exists i \neq 1, n : \frac{\lambda(\tilde{\beta}_i)}{\psi(\tilde{\beta}_i)} \leq \frac{\lambda(\tilde{\beta}_j)}{\psi(\tilde{\beta}_j)} \text{ for } j = 1, \ldots, n \right\}.$$

We note that this set is a union of cones

$$\Lambda_*^+ = \bigcup_{i=2}^{n-1} \Lambda_i^+$$

where

$$\Lambda_i^+ = \left\{ \lambda \in \Lambda^+ : \frac{\lambda(\tilde{\beta}_i)}{\psi(\tilde{\beta}_i)} \leq \frac{\lambda(\tilde{\beta}_j)}{\psi(\tilde{\beta}_j)} \text{ for } j = 1, \ldots, n \right\}.$$

Theorem 16. *Let $G = \mathrm{SL}(n+1, \mathbb{R})$ with $n \geq 2$, let Γ be any lattice subgroup, and $T \geq T_0$. Then for irreducible represenations τ with highest weight $\lambda \in \Lambda_i^+$,*

$$\frac{|B_T^\tau \cap \Gamma|}{\mathrm{vol}(B_T^\tau)} = 1 + O(\mathrm{vol}(B_T^\tau)^{\sigma_i \kappa_0} (\log T)^q)$$

where $\sigma_i = \min(\frac{n}{i}, \frac{n}{n+1-i})$. Here q and T_0 are positive numbers which depends on n but not on the lattice, and can be made explicit.

Proof. We recall that

$$\frac{|B_T^\tau \cap \Gamma|}{\mathrm{vol}(B_T^\tau)} = 1 + O\left(\mathrm{vol}(B_T^\tau)^{-2n\left(1 - \frac{m_1}{m_1'}\right)\kappa_0} (\log T)^q \right)$$

where

$$m_1 = \min_j \frac{\lambda(\tilde{\beta}_j)}{2\rho(\tilde{\beta}_j)} \quad \text{and} \quad m_1' = \min_j \frac{\lambda(\tilde{\beta}_j)}{\psi(\tilde{\beta}_j)}.$$

Since $\lambda \in \Lambda_i^+$,

$$\frac{m_1}{m_1'} \leq \frac{\lambda(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)} \cdot \left(\frac{\lambda(\tilde{\beta}_i)}{\psi(\tilde{\beta}_i)} \right)^{-1} = \frac{\psi(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)} = 1 - \frac{\gamma(\tilde{\beta}_i)}{2\rho(\tilde{\beta}_i)}.$$

The last expression is symmetric with respect to $i \mapsto n+1-i$. so that we may assume that $i \leq (n+1)/2$. Then for $i \leq (n+1)/2$,

$$\frac{m_1}{m_1'} \leq 1 - \frac{i/2}{i(n+1-i)} = 1 - \frac{1}{2(n+1-i)}.$$

This implies the theorem. $\square$

Remark 17.   1) Note that the theorem gives a non-trivial improvement over the bound κ_0 provided $i \neq 1, n$.

2) The best improvement is achieved when $i = (n+1)/2$ for odd n and when $i = n/2$ or $i = n/2 + 1$ for even n.

3) It follows from Theorem 10 that the cones $\Lambda^+_{(n+1)/2}$ for odd n, and $\Lambda^+_{n/2}$ and $\Lambda^+_{n/2+1}$ for even n are not empty.

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