

Weierstrass cycles in moduli spaces and the Krichever map

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We analyze cohomological properties of the Krichever map and use the results to study Weierstrass cycles in moduli spaces and the tautological ring.

Let us consider a point p on a smooth projective connected curve C over \mathbb{C} of genus g . We say that a natural number n is a non-gap if there exists a function that is holomorphic on $C \setminus p$ and has a pole of order n at the point p (in other words $h^0(\mathcal{O}(np)) > h^0(\mathcal{O}((n-1)p))$).

It is obvious that the set of all non-gaps is a semigroup; it is easy to derive from Riemann-Roch theorem that the number of gaps (the cardinality of the complement to the set of non-gaps in \mathbb{N}) is equal to g . We denote by H the set consisting of 0 and of all integers n such that $h^0(\mathcal{O}(np)) > h^0(\mathcal{O}((n-1)p))$ (in other words, we include 0 and all non-gaps into H). One says that H is the Weierstrass semigroup at p .

One says that a subsemigroup H of \mathbb{N}_0 such that $\#(\mathbb{N}_0 \setminus H) = g$ and $0 \in H$ is a numerical semigroup of genus g ; obviously any Weierstrass semigroup belongs to this class. (Here \mathbb{N}_0 stands for the semigroup of non-negative integers). The point p is a Weierstrass point if the first non-gap is $\leq g$ (i.e. $H \neq \{0, g+1, g+2, \dots\}$). There exist only a finite number of Weierstrass points on a curve. Instead of Weierstrass semigroup H , one can consider a decreasing sequence of integers such that s_i is the largest integer with

$$h^0(K_C(-s_i p)) = i.$$

Here K_C denotes the canonical line bundle on C . It follows from the Riemann-Roch theorem that this sequence (the Weierstrass sequence of the point p) has the form $s_i = a_{g-i+1} - 1$ if $1 \leq i \leq g$ and $s_i = g - 1 - i$ if $i \geq g + 1$. Here $1 = a_1 < \dots < a_g$ denotes the increasing sequence of gaps.

Notice that all these statements remain correct if p is a nonsingular point of an irreducible (not necessarily smooth) curve and the canonical line bundle is replaced by the dualizing sheaf ω_C . (Every irreducible curve is a

Cohen-Macaulay curve; hence it is not necessary to consider a complex of sheaves talking about the dualizing sheaf.)¹ Any numerical semigroup of genus g is a Weierstrass semigroup at a point on an irreducible curve of (arithmetic) genus g ; see Section 3.

Let us consider the moduli space $\mathcal{M}_{g,1}$ of non-singular irreducible curves of genus g with one marked point (one can characterize this space as the universal curve).² If H is a numerical semigroup of genus g , we denote by \mathcal{M}_H the subset of $\mathcal{M}_{g,1}$ consisting of curves with marked points having Weierstrass semigroup H . The closure $W_H = \overline{\mathcal{M}_H}$ of the Weierstrass set \mathcal{M}_H in $\mathcal{M}_{g,1}$ is called a Weierstrass cycle. Under some conditions, we calculate the cohomology class $[W_H]$ dual to this cycle. (Our methods can be used also to calculate the element of Chow ring specified by Weierstrass cycle).

Our problem is closely related to the problem of the calculation of the homomorphism induced by the Krichever map $k : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$. Here $\widehat{\mathcal{M}}_g$ stands for the moduli space of triples (C, p, z) , where C is a complex connected smooth projective curve of genus g with a point p and a map $z : D \rightarrow \mathbb{D}$ is an isomorphism from a closed set D onto the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ obeying $z(p) = 0$.³ We use the notation $\text{Gr}(\mathcal{H})$ for the Sato Grassmannian (as defined in [23]) and the notation $\text{Gr}_d(\mathcal{H})$ for index d component of Grassmannian. The Krichever map sends a triple (C, p, z) into the space V , the closure of functions on the boundary of the disk D that can be extended to holomorphic differentials on the complement of D . (A function $f(z)$ on S^1 is considered as a differential $f(z)dz$ restricted to the boundary of D .) The kernel and the cokernel of $\pi_-|_V : V \rightarrow \mathcal{H}_-$ are identified with $H^0(C, \omega_C)$ and $H^1(C, \omega_C)$ respectively (see [18], [23]); hence the index $\pi_-|_V$ is $g - 1$. Here $\pi_-|_V$ stands for the orthogonal projection of V into \mathcal{H}_- ; the projection is defined with respect to Hermitian inner product $\langle f_1, f_2 \rangle = \int_{S^1} f_1(z) \overline{f_2(z)} dz / 2\pi$. Hence the image of the Krichever map lies in the component $\text{Gr}_{g-1}(\mathcal{H})$. The Krichever map commutes with the natural action of S^1 on $\widehat{\mathcal{M}}_g$ and on $\text{Gr}(\mathcal{H})$. Thus it induces a homomorphism of the S^1 -equivariant cohomology of the connected component $\text{Gr}_{g-1}(\mathcal{H})$ of $\text{Gr}(\mathcal{H})$ into the S^1 -equivariant cohomology of $\widehat{\mathcal{M}}_g$. The latter is isomorphic to the conventional cohomology of $\mathcal{M}_{g,1}$ (see [10] for more detail). In [10], we have

¹All curves we consider are reduced irreducible projective curves. (in other words we work with projective integral curves)

²One can consider this space as an orbifold or as a moduli space of a stack. However, we are interested only in cohomology over \mathbb{C} , therefore it is sufficient to consider it as a topological space.

³The embedding into Grassmannian induces topology on $\widehat{\mathcal{M}}_g$. Of course, this topology can be described without the reference to Grassmannian.

calculated the images of a set of multiplicative generators under the homomorphism induced by the Krichever map in the S^1 -equivariant cohomology of Grassmannian; in the present paper, we will give an explicit formula for this homomorphism on additive generators of this cohomology. In the paper [11], we identified the S^1 -equivariant cohomology of Grassmannian with the ring of shifted symmetric functions (see also [9]). We describe the homomorphism induced by the Krichever map on this ring; we specify the answers for various additive generators of S^1 -equivariant cohomology.

Weierstrass cycles W_H are related to intersections of Schubert cycles in the Grassmannian with the image of Krichever map. This allows us to obtain the information about classes $[W_H]$ from the analysis of the homomorphism induced by the Krichever map in the S^1 -equivariant cohomology. The same technique is used to obtain relations in the tautological rings of moduli spaces. We obtain also similar results for the moduli spaces of irreducible (possibly singular) curves with embedded disks.

In a separate paper [14], we show how to use the ideas of present paper to obtain estimates for dimensions of Weierstrass cycles. We perform calculations for moduli spaces of irreducible curves of low genera.

1. Krichever Map

In the introduction, we have described the Krichever map $k : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$ of the moduli space $\widehat{\mathcal{M}}_g$ into Segal-Wilson version of Sato Grassmannian (see [23] for more detail). This construction can be generalized to projective integral curves (the marked point p should be non-singular, the disk D should consist of non-singular points, instead of holomorphic differentials one should consider sections of the dualizing sheaf of C). This follows from the results of [23] and from the remark that the dualizing sheaf of Cohen-Macaulay curve is a torsion-free rank one sheaf. We will denote by $\widehat{\mathcal{C}\mathcal{M}}_g$ the moduli space of triples (C, p, z) where C is a projective integral curve of genus g , and p is a nonsingular point, and z is a local coordinate system around p sending a closed set D containing p onto \mathbb{D} with $z(p) = 0$; the extension of the Krichever map to this space will be also denoted by k . The extended Krichever map is an embedding of $\widehat{\mathcal{C}\mathcal{M}}_g$ into Grassmannian (this follows from the results of [23]); we can define the topology on $\widehat{\mathcal{C}\mathcal{M}}_g$ using

this embedding. The image of this embedding is called the Krichever locus.⁴

Notice that a reasonable (separable) moduli space of singular curves (even of Gorenstein curves) does not exist; see [8]. It is important that we consider curves with embedded disks. Identifying the points of $\widehat{\mathcal{CM}}_g$ corresponding to the same curve C with different embedded disks we obtain a non-separable space.

We have used the dualizing sheaf in the construction of Krichever map; however, as it was shown in [23], one can use any torsion-free rank one sheaf.⁵

Using q -differentials, one can construct a more general Krichever map $k_q : \widehat{\mathcal{M}}_g \rightarrow \text{Gr}(\mathcal{H})$ for each $q \in \mathbb{Z}$; this corresponds to using the q -th power of dualizing sheaf. It is clear that $k_1 = k$. In general the map k_q for $q > 1$ or $q < 0$ cannot be defined for general irreducible curves, but it can be defined for Gorenstein curves where the dualizing sheaf is an invertible sheaf (is a line bundle).

It is easy to check that the images of any triple (C, p, z) under k_q and k_{1-q} are closed subspaces of $L^2(S^1)$ orthogonal with respect to bilinear inner product

$$(1.1) \quad (f_1, f_2) = \frac{1}{2\pi} \int_{S^1} f_1(z) f_2(z) dz;$$

in other words, we have

$$(1.2) \quad k_{1-q}(C, p, z) = (k_q(C, p, z))^\perp$$

where $^\perp$ denotes orthogonal complement (see [22]) with respect to the bilinear inner product. In particular, for $q = 1$,

$$(1.3) \quad k_0(C, p, z) = (k_1(C, p, z))^\perp.$$

One should emphasize that (1.3) is correct for all irreducible curves, but to make sense of (1.2) we should assume that C is a Gorenstein curve. All maps k_q are S^1 -equivariant; one can study the induced homomorphisms on the S^1 -equivariant cohomology. The answers are formulated in terms of

⁴The Krichever map can be defined also for reducible curves, but in this case this map is not an embedding and it is not continuous. In particular, the Krichever map on the space of nodal curves with disks is discontinuous.

⁵Notice that our definition of the Krichever locus is not quite standard. Usually this locus is defined as the image of the general Krichever map.

lambda-classes and psi-classes (see [10] for the analysis of these problems for non-singular curves) .

The Hodge bundle \mathbb{E} on $\widehat{\mathcal{CM}}_g$ is defined as a bundle having the space of holomorphic sections of dualizing sheaf as a fiber. (See a rigorous definition below.) This is an S^1 -equivariant vector bundle whose S^1 -equivariant Chern classes are called lambda-classes and denoted by $\lambda_1, \dots, \lambda_g$. Restricting them to $\widehat{\mathcal{M}}_g$, we obtain conventional lambda-classes. (Recall, that the S^1 -equivariant cohomology of $\widehat{\mathcal{M}}_g$ coincides with cohomology of universal curve $\mathcal{M}_{g,1}$, see [10].) Lambda-classes can be considered as elementary symmetric functions of lambda-roots (of Chern roots of the Hodge bundle).

S^1 -equivariant cohomology can be regarded as an algebra over polynomial ring $\mathbb{C}[u]$, where $u = c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$. The psi-class $\psi \in H_{S^1}(\widehat{\mathcal{CM}}_g)$ will be defined as $-u$. It was shown in [10] that restricting to $\widehat{\mathcal{M}}_g$ we obtain the standard definition of psi-class.

The subring of the ring $H_{S^1}(\widehat{\mathcal{CM}}_g)$ generated by lambda-classes and psi-class will be called tautological ring. It will follow from our results that the tautological ring can be characterized as the image of S^1 -equivariant cohomology of Grassmannian by the homomorphism k^* induced by the Krichever map. We will prove some relations in the tautological ring; these relations can be restricted to relations in the tautological ring of the universal curve.

Let us consider submanifolds Gr_d^l of $\text{Gr}_d(\mathcal{H})$ consisting of points W such that the orthogonal projection $\pi_l : W \rightarrow z^{-l}\mathcal{H}_-$ is surjective. (Here $l \geq 0$.) The action of S^1 on $\text{Gr}_d(\mathcal{H})$ generates an action on Gr_d^l for each $l \geq 0$. The kernels of the projection $\pi_l : W \rightarrow z^{-l}\mathcal{H}_-$ can be considered as fibers of an equivariant vector bundle \mathcal{E}_l over Gr_d^l . This bundle has rank $d + l$. Using the Krichever map, we can embed $\widehat{\mathcal{CM}}_g$ into Gr_{g-1}^1 ; the Hodge bundle is a pullback of the S^1 -equivariant vector bundle \mathcal{E}_1 . (This statement can be considered as a rigorous definition of the Hodge bundle.)

It is proved in [9] and [11] that the S^1 -equivariant cohomology ring of Grassmannian $\text{Gr}_d(\mathcal{H})$ can be identified with the ring $\Lambda^*(z||u)$ of “polynomial” functions of variables $(z_i)_{i \in \mathbb{N}}$ and the variable u that become symmetric with respect to the variables $\{x_i\}$, where $x_i = z_i + (d + 1 - i)u$, for $i \geq 1$. These functions are called shifted symmetric functions [2], [11].

The ring $\Lambda^*(z||u)$ can be identified with the ring $\Lambda(x||u)$ of functions $\alpha(u, x_1, x_2, \dots)$ in variables $(x_i)_{i \in \mathbb{N}}$ and in u that are symmetric with respect to the variables $(x_i)_{i \in \mathbb{N}}$, and can be obtained from a polynomial $\tilde{\alpha}(u, z_1, \dots, z_N) \in \mathbb{C}[u, z_1, \dots, z_N]$, by means of substitution $z_i = x_i - (d +$

$1 - i)u$ for $i \geq 1$. (Hence $\alpha(u, x_1, x_2, \dots) = \tilde{\alpha}(u, z_1, \dots, z_N)$. The function α is defined on sequences $(x_i)_{i \in \mathbb{N}}$ obeying $x_i = (d + 1 - i)u$ for $i \gg 0$)

Let $k = k_1 : \widehat{\mathcal{CM}}_g \rightarrow \text{Gr}_{g-1}(\mathcal{H})$ be the Krichever map. Let α be a S^1 -equivariant cohomology class of $\text{Gr}_{g-1}(\mathcal{H})$ represented by a function $\alpha(u, x_1, \dots, x_i, \dots)$ symmetric with respect to (x_i) that becomes a polynomial $\tilde{\alpha}(u, z_1, \dots, z_N) \in \mathbb{C}[u, z_1, \dots, z_N]$ with $z_i = x_i - (g - i)u$ for $i \geq 1$ and for some $N \gg 0$. We will prove the following statements:

Theorem 1.1. The classes $\{-k^*x_i : 1 \leq i \leq g\}$ are Chern roots of the bundle \mathbb{E}^V dual to the Hodge bundle \mathbb{E} and $k^*x_i = -(g - i)\psi$ for $i > g$ (or equivalently, $k^*z_i = 0$ for all $i > g$.) It follows that $\alpha(-\psi, k^*(x_1), \dots, k^*(x_i), \dots)$ is well defined. We prove that

$$k^*\alpha = \alpha(-\psi, k^*(x_1), \dots, k^*(x_i), \dots).$$

In other words

$$k^*\alpha = \alpha_g(\psi, k^*x_1, \dots, k^*x_g).$$

Here we obtain α_g from $\tilde{\alpha}$ by setting

$$\alpha_g(\psi, k^*x_1, \dots, k^*x_g) = \tilde{\alpha}(k^*u, k^*z_1, \dots, k^*z_N)$$

where $z_i = x_i - (g - i)u$ for $i \geq 1$.

Proof. Denote $d = g - 1$. Let $\mathcal{H}_{i,j}$ be the linear subspace of \mathcal{H} spanned by $\{z^s : i \leq s \leq j\}$ and denote $\underline{\mathcal{H}}_{i,j}$ the product bundle $\mathcal{H}_{i,j} \times \text{Gr}_d^l$. We consider the action of S^1 on $\underline{\mathcal{H}}_{i,j}$ defined by

$$(1.4) \quad (t, (f, V)) \mapsto (t^{-1}f(t^{-1}z), t(V)).$$

Here V is a point in $\text{Gr}_d(\mathcal{H})$, f is vector in $\mathcal{H}_{i,j}$ and $t \in S^1$; Here we define $t(V)$ as the space of functions $t^{-1}f(t^{-1}z)$ for $f(z) \in V$. Then $\underline{\mathcal{H}}_{i,j}$ is a S^1 -equivariant vector bundle over Gr_d^l . Then the total S^1 -equivariant Chern classes of the bundle $\underline{\mathcal{H}}_{i,j}$ is given by the formula

$$c^T(\underline{\mathcal{H}}_{i,j}) = \prod_{m=i}^j (1 - (m + 1)u).$$

Let f_{ln} be the inclusion maps $\text{Gr}_d^l \hookrightarrow \text{Gr}_d^n$ and $\text{Gr}_d^l \hookrightarrow \text{Gr}_d(\mathcal{H})$ respectively. The induced map of f_{ln} and f_l on the equivariant cohomology are denoted by f_{ln}^* and f_l^* respectively.

Let $\{x_1, \dots, x_{d+l}\}$ be the S^1 -equivariant Chern roots of \mathcal{E}_l^\vee . The S^1 -equivariant cohomology $H_{S^1}^*(\text{Gr}_d^l)$ of Gr_d^l can be identified with the algebra $\Lambda(x_1, \dots, x_{d+l} \| u)$ of polynomials in x_1, \dots, x_{d+l}, u over \mathbb{C} symmetric with respect to $\{x_1, \dots, x_{d+l}\}$. The inclusion map $f_{ln} : \text{Gr}_d^l \hookrightarrow \text{Gr}_d^n$ induces an algebra homomorphism $f_{ln}^* : \Lambda(x_1, \dots, x_{d+n} \| u) \rightarrow \Lambda(x_1, \dots, x_{d+l} \| u)$. The projective limit of the projective system $(\Lambda(x_1, \dots, x_{d+l} \| u), f_{ln}^*)$ is the S^1 -equivariant cohomology $H_{S^1}^*(\text{Gr}_d(\mathcal{H}))$. It can be identified with the ring $\Lambda(x \| u)$ defined above. Let α be an S^1 -equivariant cohomology class in $H_{S^1}^*(\text{Gr}_d(\mathcal{H}))$ represented by a function $\alpha(u, x_1, \dots)$ in $\Lambda(x \| u)$. To compute $k^*\alpha$, we only need to compute $\alpha(k^*u, k^*x_1, \dots)$.

The Krichever map $k : \widehat{\mathcal{CM}}_g \rightarrow \text{Gr}_{g-1}(\mathcal{H})$ are composition of maps:

$$\widehat{\mathcal{CM}}_g \xrightarrow{\quad {}^l k \quad} \text{Gr}_{g-1}^l \xrightarrow{\quad f_l \quad} \text{Gr}_{g-1}(\mathcal{H}),$$

where ${}^l k : \widehat{\mathcal{CM}}_g \rightarrow \text{Gr}_{g-1}^l$ is the modified Krichever map. Notice that the pull back bundle ${}^l k^*\mathcal{E}_l$ on $\widehat{\mathcal{CM}}_g$ has a orthogonal direct sum decomposition:

$${}^l k^*\mathcal{E}_l = {}^1 k^*\mathcal{E}_1 \oplus k^*\underline{\mathcal{H}}_{-l, -1}.$$

This implies that $\{-kx_i : 1 \leq i \leq d+l\}$ forms equivariant Chern roots of the direct sum bundles ${}^1 k^*\mathcal{E}_1 \oplus k^*\underline{\mathcal{H}}_{-l, -1}$. Hence we can set $k^*x_i = -(g-i)\psi$ for $i \geq g+1$ and $\{kx_i : 1 \leq i \leq g\}$ the equivariant Chern roots of the bundle ${}^1 k^*\mathcal{E}_1$. This proves our assertion. □

Schubert cycles $\overline{\Sigma}_\mu$ specify S^1 -equivariant cohomology classes Ω_μ^T corresponding to Okounkov-Olshanski shifted Schur functions s_μ^* . Let us recall the definition of the shifted Schur functions following [2]. The factorial Schur polynomial depending on partition μ and variables $\{z_1, \dots, z_n\}$ is given by the formula:

$${}^n t_\mu(z_1, \dots, z_n) = \frac{\det[(z_i \downarrow \mu_j + n - j)]_{i,j=1}^n}{\det[(z_i \downarrow n - j)]_{i,j=1}^n},$$

where the symbol $(z \downarrow i)$ stands for the i -th falling factorial power of the variable z :

$$(z \downarrow i) = \begin{cases} z(z-1) \cdots (z-i+1), & i = 1, 2, \dots; \\ 1, & i = 0. \end{cases}$$

After the change of variables $z'_i = z_i - n + i$ for $1 \leq i \leq n$, we obtain the shifted Schur polynomials ${}^n s_\mu^*(z'_1, \dots, z'_n) = {}^n t_\mu(z_1, \dots, z_n)$. The shifted

Schur polynomials satisfy the stability conditions ${}^{n+1}s_\mu^*(z_1, \dots, z_n, 0) = {}^n s_\mu^*(z_1, \dots, z_n)$ which allows us to define the shifted Schur functions $s_\mu^*(z_1, z_2, \dots)$ in the sequence of variables $\{z_1, z_2, \dots\}$. The stability condition expressed in terms of factorial Schur functions looks as follows: given any $g \in \mathbb{Z}$,

$$\begin{aligned} & {}^{n+1}t_\mu(X_1 + (n - g) + 1, \dots, X_n + (n - g) + 1, 0) \\ &= {}^n t_\mu(X_1 + (n - g), \dots, X_n + (n - g)) \end{aligned}$$

for $n > l(\mu)$. Here $X_i = z'_i - i + g$ for any $i \geq 1$. For more details, see [2]. It follows from the results of [10] that the equivariant Schubert class in $H_{S^1}^*(\text{Gr}_{g-1}(\mathcal{H}))$ corresponding to the partition μ is given by the formula:

$$\Omega_\mu^T = s_\mu^*(z_1, z_2, \dots)u^{|\mu|} = {}^n t_\mu \left(\frac{x_1 + (n - g)u}{u}, \dots, \frac{x_n + (n - g)u}{u} \right) u^{|\mu|},$$

for all $n > l(\mu)$. Here (z_i) is the sequence of variables defined by $z_i = (x_i + (i - g)u)/u$ for all i , and $|\mu|$, the weight of a partition μ , is defined to be $\sum_i \mu_i$. Note that $x_i + (i - g)u = 0$ for all i sufficiently large in $H_{S^1}^*(\text{Gr}_{g-1}(\mathcal{H}))$ and thus the sequence of variables (z_i) defined by $z_i = (x_i + (i - g)u)/u$ makes sense in s_μ^* . Using this statement and the Theorem 1.1, we obtain

Corollary 1.1.

$$k^* \Omega_\mu^T = {}^g s_\mu^*(z_1, \dots, z_g)(-\psi)^{|\mu|} = {}^g t_\mu(z'_1, \dots, z'_g)(-\psi)^{|\mu|}$$

where $\{z_1, \dots, z_g\}$ is the set of variables defined by $z_i = (k^*x_i - (i - g)\psi)/(-\psi)$ for $1 \leq i \leq g$ and $\{z'_1, \dots, z'_g\}$ is the set of variables defined by $z'_i = (k^*x_i - (n - g)\psi)/(-\psi)$ for $1 \leq i \leq g$. and n is a positive integer such that $n > l(\mu)$.

The factorial Schur function ${}^g t_\mu(X_1 + (n - g), \dots, X_g + (n - g))$ is an inhomogeneous symmetric function; we will represent it as a sum of homogeneous polynomials:

$${}^g t_\mu(X_1 + (n - g), \dots, X_g + (n - g)) = \sum_i t_\mu^i(X_1, \dots, X_g),$$

where $t_\mu^i(X_1, \dots, X_g)$ is a homogeneous symmetric polynomial of degree i in X_1, \dots, X_g . Hence we can write

$$(1.5) \quad k^* \Omega_\mu^T = \sum_i t_\mu^i(k^*x_1, \dots, k^*x_g)(-\psi)^{|\mu|-i}.$$

Shifted Schur functions form a basis in the space of all shifted symmetric functions, and therefore we can say that conversely Theorem 1.1 follows from Corollary 1.1.

Denote Ψ_μ the $l(\mu) \times l(\mu)$ matrix whose ij -th entry is given by

$$(\Psi_\mu)_{ij} = \begin{cases} \sum_{a+b=\mu_i+j-i} h_a(k^*x_1, \dots, k^*x_g) e_b(0, 1, 2, \dots, \mu_i - i + g - 1) \psi^b, & \text{if } \mu_i - i + g \geq 1; \\ \sum_{a+b=\mu_i+j-i} e_a(k^*x_1, \dots, k^*x_g) h_b(0, 1, 2, \dots, i - \mu_i - g) \psi^b, & \text{if } \mu_i - i + g \leq 0. \end{cases}$$

We can also consider another matrix (of the size $l(\mu') \times l(\mu')$) defined by

$$(\Psi'_\mu)_{ij} = \begin{cases} \sum_{a+b=\mu'_i+j-i} e_a(k^*x_1, \dots, k^*x_g) h_b(0, 1, 2, \dots, \mu'_i - i + g - 1) \psi^b, & \text{if } \mu'_i - i + g \geq 1; \\ \sum_{a+b=\mu'_i+j-i} h_a(k^*x_1, \dots, k^*x_g) e_b(0, 1, 2, \dots, i - \mu'_i - g) \psi^b, & \text{if } \mu'_i - i + g \leq 0. \end{cases}$$

Here μ' denotes the conjugate partition of μ . Using the determinant formula for double Schur functions, we obtain

$$k^* \Omega_\mu^T = \det \Psi_\mu = \det \Psi'_\mu.$$

If $l(\mu) \leq g$, $\mu_i - i + g > 1$ for $1 \leq i \leq g$. Thus

$$k^* \Omega_\mu^T = \det \left[\sum_{a+b=\mu_i+j-i} h_a(k^*x_1, \dots, k^*x_g) e_a(1, 2, \dots, \mu_i - i + g - 1) \psi^b \right]_{1 \leq i, j \leq l(\mu)}.$$

Similarly, if $l(\mu) \leq g$ and $\mu'_i - i + g > 1$, we can also obtain the dual formula

$$k^* \Omega_\mu^T = \det \left[\sum_{a+b=\mu'_i+j-i} e_a(k^*x_1, \dots, k^*x_g) h_b(1, 2, \dots, \mu'_i - i + g - 1) \psi^b \right]_{1 \leq i, j \leq l(\mu')}.$$

These two formulas are useful when we compute the cohomology classes of the Weierstrass cycles.

We can consider also cohomology classes p_s corresponding to symmetric functions

$$p_s(u, x_1, \dots, x_n, \dots) = \sum_{i=1}^{\infty} \{x_i^s - (-1)^s (i - d - 1)^s u^s\}$$

(these classes constitute a multiplicative system of generators of equivariant cohomology). Applying Theorem 1.1, we obtain

Corollary 1.2.

$$k^* p_s = \text{ch}_s(\mathbb{E}) - \sum_{i=1}^g (i - g)^s \psi^s,$$

where $\text{ch}_s(\mathbb{E})$ stands for the s -th component of the Chern character of Hodge bundle \mathbb{E} .

All statements proved above are valid not only for the space $\widehat{\mathcal{CM}}_g$, but also for its S^1 -invariant subspaces, in particular, for the subspace $\widehat{\mathcal{M}}_g$ consisting of smooth curves. For $\widehat{\mathcal{M}}_g$, some of our statements can be simplified.

For the moduli space of pointed smooth curves [20], the Mumford formula

$$(1.6) \quad c(\mathbb{E})c(\mathbb{E}^*) = 1$$

implies that $h_a(x_1, \dots, x_g) = (-1)^a \lambda_a$. Hence the Ψ -matrix can be expressed in the form:

$$(1.7) \quad (\Psi_{\mu})_{ij} = \begin{cases} \sum_{a+b=\mu_i+j-i} (-1)^a e_b(0, 1, 2, \dots, \mu_i - i + g - 1) \lambda_a \psi^b, & \text{if } \mu_i - i + g \geq 1; \\ \sum_{a+b=\mu_i+j-i} (-1)^a h_b(0, 1, 2, \dots, i - \mu_i - g) \lambda_a \psi^b, & \text{if } \mu_i - i + g \leq 0. \end{cases}$$

If we are working with the moduli space $\widehat{\mathcal{M}}_g$, the Chern character of the Hodge bundle can be expressed in terms of kappa-classes [20]. Therefore we obtain:

Corollary 1.3.

$$k^* p_s = \begin{cases} \sum_{i=1}^g (i - g)^{2r} \psi^{2r}, & \text{if } s = 2r; \\ B_{2r} \kappa_{2r} / 2r - \sum_{i=1}^g (i - g)^{2r-1} \psi^{2r-1}, & \text{if } s = 2r - 1. \end{cases}$$

2. Weierstrass cycles

The Schubert cells Σ_S on $\text{Gr}(\mathcal{H})$ are labeled by decreasing sequences of integers $S : s_1 > s_2 > \dots$ such that the sets $S_+ = \{s_i : i \geq 1\} \cap \mathbb{Z}_+$ and $S_- = \mathbb{Z}_- \setminus \{s_i : i \geq 1\}$ are both finite sets⁶. The virtual cardinality of a sequence S is defined as $d = \#S_+ - \#S_-$. The closure of Σ_S is the Schubert cycles $\overline{\Sigma}_S$. Given a sequence S , we define its corresponding partition μ by $\mu_i = s_i + i - d$, for all i , where d is the virtual cardinality of S . The equivariant Schubert class of $\overline{\Sigma}_S$ in $H_{S^1}^*(\text{Gr}(\mathcal{H}))$ is Ω_μ^T where μ is the partition corresponding to S . For more details, see [10] and [11].

Theorem 2.1. A point $k(C, p, z)$ of the Krichever locus belongs to the Schubert cell Σ_S defined by the Weierstrass sequence S at the point p .

(See [1] where this statement is attributed to Mumford.)

Assume that H is a numerical semigroup of genus g . Let A_H^{alg} be the linear subspace of \mathcal{H} generated by elements of the form $\{z^{-h} : h \in H\}$ whose closure is denoted by A_H . Suppose that $\{h_1, \dots, h_l\}$ is a generating set of H . Then $A_H^{\text{alg}} = \mathbb{C}[[z^{-h_1}, \dots, z^{-h_l}]]$. The affine curve $\text{Spec } A_H^{\text{alg}}$ is called a monomial curve. Let us consider the filtration in $\mathbb{C}((z))$ by $\{z^{-n}\mathbb{C}[[z]] : n \in \mathbb{Z}\}$. There is a natural filtration of A_H^{alg} from the filtration of $\mathbb{C}((z))$. Then we obtain the associated graded algebra $\text{gr}(A_H^{\text{alg}})$ from the filtration of A_H^{alg} . The complete irreducible curve C_H also called a monomial curve is given by $\text{Proj}(\text{gr } A_H^{\text{alg}})$ and is the one point completion of $\text{Spec } A_H^{\text{alg}}$. In other words, $C_H = \text{Spec } A_H^{\text{alg}} \cup \{p\}$, where p is a smooth point so that $z(p) = 0$. We can check that $A_H = k_0(C_H, p, z)$ and thus A_H^{alg} is the space of meromorphic functions on C with the only possible pole at p . Since $z^{-h} \in A_H^{\text{alg}}$, we see that H is the Weierstrass semigroup at p . Hence every numerical semigroup of genus g is a Weierstrass semigroup of a smooth point on an irreducible curve of genus g .

The Weierstrass sequence S of (C, p, z) in $\widehat{\mathcal{CM}}_g$ is closely related to the Weierstrass semigroup H of (C, p) . Let $\varsigma : \mathbb{Z} \rightarrow \mathbb{Z}$ be the translation operator: $\varsigma(n) = n + 1$, for $n \in \mathbb{Z}$. Then $H = \mathbb{Z} - \varsigma(S)$ or equivalently $S = \varsigma^{-1}(\mathbb{Z} - H)$.

Given a numerical semigroup H of genus g let S be a sequence defined by $S = \varsigma^{-1}(\mathbb{Z} - H)$. By (1.3), we have $\mathcal{H}_S = k_0(C_H, p, z)^\perp = k(C_H, p, z)$, where \mathcal{H}_S is the closed subspace of \mathcal{H} generated by $\{z^s : s \in S\}$. Since \mathcal{H}_S belongs

⁶Here \mathbb{Z}_+ and \mathbb{Z}_- are subsets consisting of nonnegative integers and of negative integers respectively.

to Σ_S and $\mathcal{H}_S = k(C_H, p, z)$, \mathcal{H}_S belongs to the intersection of $k(\widehat{\mathcal{CM}}_g)$ and the Schubert cell Σ_S . We conclude that:

Theorem 2.2. The intersection of $k(\widehat{\mathcal{CM}}_g)$ and Σ_S is nonempty if and only if the set $H = \mathbb{Z} - \zeta(S)$ is a numerical semigroup of genus g .

Let us consider the closure $\overline{\Sigma}_S$ of a Schubert cell Σ_S . A point $k(C, p, z)$ belongs to $\overline{\Sigma}_S$ if and only if the Weierstrass sequence $(s_i(p))$ at p obeys the relation $s_i(p) \geq s_i$ for all i .

Lemma 2.1. Let H be a numerical semigroup of genus g and $S = \zeta(\mathbb{Z} - H)$. Then $s_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s_i = g - i - 1$ for $i \geq g + 1$.

Proof. This statement follows from [15], Lemma 3.2.. □

Let Z be the sequence defined by $z_i = 2g - 2i$ for $1 \leq i \leq g$ and $z_i = g - i - 1$ for $i \geq g + 1$. Then $\mathbb{Z} - \zeta(Z)$ is the numerical semigroup of genus g generated by 2. Hence $\mathcal{H}_Z \in k(\widehat{\mathcal{CM}}_g) \cap \Sigma_Z$. If S is any sequence so that $s_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s_i \leq g - i - 1$, then $z_i \geq s_i$ for all i and thus $\mathcal{H}_Z \in k(\widehat{\mathcal{CM}}_g) \cap \overline{\Sigma}_S$.

Proposition 2.1. The set $k(\widehat{\mathcal{CM}}_g) \cap \overline{\Sigma}_S$ is nonempty if and only if the sequence S obeys $s_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s_i \leq g - i - 1$.

Proof. We have seen that if S obeys the relations, $\mathcal{H}_Z \in k(\widehat{\mathcal{CM}}_g) \cap \overline{\Sigma}_S$. Thus $k(\widehat{\mathcal{CM}}_g) \cap \overline{\Sigma}_S$ is nonempty. Conversely, assume that $k(\widehat{\mathcal{CM}}_g) \cap \overline{\Sigma}_S$ is nonempty. Then there exists a sequence $S' = (s'_i)$ such that $s'_i \geq s_i$ and $k(\widehat{\mathcal{CM}}_g) \cap \Sigma_{S'} \neq \emptyset$. By the theorem 2.2, $H' = \mathbb{Z} - \zeta(S')$ is a numerical semigroup of genus g . By the Lemma 2.1, $s'_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s'_i = g - i - 1$ for $i \geq g + 1$. Hence $s_i \leq s'_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s_i \leq s'_i = g - i - 1$ for $i \geq g + 1$ which completes the proof. □

Let us say that a set $\Gamma \subset X$ is a support of a cohomology class $\xi \in H^*(X)$ if the restriction of this class to $X \setminus \Gamma$ is trivial (i.e. $\iota^*(\xi) = 0$ where ι^* stands for the homomorphism of cohomology groups induced by the embedding $X \setminus \Gamma \rightarrow X$). If we consider the G -equivariant cohomology of G -space X , we can apply this notion to the G -invariant subset Γ . If X is a manifold and Γ is a submanifold then Γ is a support of the cohomology class that is dual to Γ ; this cohomology class is denoted by $[\Gamma]$. This statement remains correct in the framework of equivariant cohomology and in the case when Γ is a subvariety of a complex manifold X . Conversely, if a submanifold Γ having

codimension k is a support of a k -dimensional cohomology class ξ of the manifold X , then this cohomology class is proportional to the cohomology class dual to Γ :

$$(2.1) \quad \xi = \text{const}[\Gamma]$$

The same is true if X is a complex manifold and Γ is an irreducible subvariety. If Γ is a reducible subvariety and is a support of a cohomology class having the dimension equal to the real codimension of Γ , then this class is a linear combination of classes dual to irreducible components of Γ . Similar statements are true in equivariant cases.

These results are well known but we were not able to find an appropriate reference. See, however, [12] and [5]. Under some conditions one can apply these statements in the case when X is infinite-dimensional and Γ has finite codimension.

In particular, Schubert cycle $\overline{\Sigma}_\mu$ is a support of the S^1 -equivariant cohomology class Ω_μ^T corresponding to Okounkov-Olshanski shifted Schur functions s_μ^* .

Let us consider a G -equivariant map $f : Y \rightarrow X$ between G -manifolds. If an equivariant cohomology class $\xi \in H_G^*(X)$ has a support that does not intersect $f(Y)$, then $f^*(\xi) = 0$. Applying this remark to the Krichever map, we obtain the following statement.

If the intersection of $\overline{\Sigma}_S$ and the Krichever locus $k(\widehat{\mathcal{CM}}_g)$ is empty, then the homomorphism k^* determined by the Krichever map sends the equivariant cohomology class Ω_μ^T into the trivial equivariant cohomology class. Using Theorem 1.1, we obtain a relation in the tautological ring of $\widehat{\mathcal{CM}}_g$:

$$(2.2) \quad \det \Psi_\mu = 0.$$

Here μ stands for a partition corresponding to the sequence S . In particular, the above relation is satisfied if the sequence violates the relations $s_i \leq 2g - 2i$ for $1 \leq i \leq g$ and $s_i \leq g - i - 1$ for $i \geq g + 1$. This relation can be expressed also in terms of shifted Schur functions or factorial Schur functions

$$(2.3) \quad {}^g s_\mu^*(z_1, \dots, z_g) = {}^g t_\mu \left(-\frac{k^*x_1 - l\psi}{\psi}, \dots, -\frac{k^*x_g - l\psi}{\psi} \right) = 0,$$

where $z_i = (k^*x_i - (i - g)\psi)/(-\psi)$ for $1 \leq i \leq g$ and $l \geq l(\mu) - g + 1$. Probably, the most convenient way to express the relations we found is to use the functions t_μ^i (homogeneous components of factorial Schur functions) as in (1.5):

Theorem 2.3. If μ is a partition corresponding to such a sequence S that one cannot find a Weierstrass sequence S' obeying $S' \geq S$ then

$$\sum_i (-\psi)^{|\mu|-i} t_\mu^i(k^*x_1, \dots, k^*x_g) = 0.$$

Of course, these relations are valid also in the case when we restrict ourselves to smooth curves; we obtain relations in the tautological ring of the universal curve $\mathcal{M}_{g,1}$. Using pull-push formula we get relations in \mathcal{M}_g :

$$(2.4) \quad \sum_i (-1)^{|\mu|-i} \kappa_{|\mu|-i-1} t_\mu^i(k^*x_1, \dots, k^*x_g) = 0.$$

However, in the tautological rings of $\mathcal{M}_{g,1}$ and \mathcal{M}_g there exist other relations, in particular, the relations following from the Mumford formula (1.6). Notice that using (1.6) one can get the relations (2.2) on $\mathcal{M}_{g,1}$ with the Ψ -matrix defined by (1.7).

The theorem 2.3 gives an estimate of the tautological ring of the space $\widehat{\mathcal{CM}}_g$ from above (Precise statements can be found below). To obtain an estimate of this ring from below, one can consider the restriction of this ring to the fixed points of the S^1 -action. Since the fixed points of the S^1 -action on $\text{Gr}(\mathcal{H})$ are of the form \mathcal{H}_S , the fixed points on $\widehat{\mathcal{CM}}_g$ correspond to the monomial curves. For each Weierstrass sequence S , the inclusion map $\{\mathcal{H}_S\} \rightarrow \widehat{\mathcal{CM}}_g$ induces a homomorphism on the equivariant cohomology:

$$\text{ev}_S : H_{S^1}^*(\widehat{\mathcal{CM}}_g) \rightarrow H_{S^1}^*(\{\mathcal{H}_S\}) \cong \mathbb{C}[\psi].$$

The ring homomorphism ev_S obeys $\text{ev}_S(\psi) = \psi$ and $\text{ev}_S(\lambda_i) = e_i(s_1 + 1, \dots, s_g + 1)\psi^i$ for all $1 \leq i \leq g$. Taking the direct sum of all ev_S we obtain a ring homomorphism $\text{ev} = \bigoplus_S \text{ev}_S$, where S runs over all the Weierstrass sequences.

The tautological ring of $\widehat{\mathcal{CM}}_g$ denoted by $R = R(\widehat{\mathcal{CM}}_g)$ is the \mathbb{Q} -subalgebra of $H_{S^1}^*(\widehat{\mathcal{CM}}_g)$ generated by $\lambda_1, \dots, \lambda_g$ and ψ . Consider the free polynomial algebra $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]$ generated by commuting variables $\Lambda_1, \dots, \Lambda_g, \Psi$. The ring homomorphism $\epsilon : \mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi] \rightarrow R$ sending $\Lambda_i \rightarrow \lambda_i$ and $\Psi \rightarrow \psi$ induces an isomorphism $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]/\ker \epsilon \cong R$. The tautological ring R is the quotient ring of $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]$ by the ideal of tautological relations $I_{\text{tau}} = \ker \epsilon$. Restricting ev to R , we obtain a ring homomorphism from R to $\bigoplus_S \mathbb{C}[\psi]$. We obtain a homomorphism $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi] \rightarrow \bigoplus_S \mathbb{C}[\psi]$ whose kernel is denoted by I_{ev} . It is obvious

that the ideal $I_{\tau u}$ is contained in I_{ev} . We obtain a surjective homomorphism

$$(2.5) \quad R \rightarrow \mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]/I_{\text{ev}}.$$

Let I be the ideal of $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]$ generated by $\tilde{k}^* \Omega_\mu^T$, where $\tilde{k}^* \Omega_\mu^T$ is the polynomial in $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]$ whose image in R is $k^* \Omega_\mu^T$ with the property that Ω_μ^T is the equivariant Schubert class of the Schubert cycles $\bar{\Sigma}_\mu$ such that the intersection of $\bar{\Sigma}_\mu$ and $k(\widehat{\mathcal{CM}}_g)$ is empty. We also know that I is contained in $I_{\tau u}$ and thus we have a surjective homomorphism

$$(2.6) \quad \mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]/I \rightarrow R.$$

Recall that the Hilbert-Poincaré series $P(A, t)$ of a graded algebra A is the generating function of $h_i(A) = \dim A_i$:

$$P(A, t) = \sum_{i=0}^{\infty} h_i(A) t^i.$$

By (2.6) and (2.5), we have the following estimates

$$h_i(A/I_{\text{ev}}) \leq h_i(R) \leq h_i(A/I).$$

In [14], we present the Hilbert-Poincaré series of $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]/I$ and of $\mathbb{Q}[\Lambda_1, \dots, \Lambda_g, \Psi]/I_{\text{ev}}$ and estimate $h_i(R)$ for curves of genus $g \leq 6$.

Every point V in the Krichever locus is contained in the closed subspace \mathcal{H}' of \mathcal{H} spanned by $\{z^i : i \neq -1\}$. The space \mathcal{H}' has a natural polarized structure coming from the polarized structure of \mathcal{H} . This means that the Krichever map k sends $\widehat{\mathcal{CM}}_g$ to $\text{Gr}_g(\mathcal{H}')$. Schubert cells in $\text{Gr}_g(\mathcal{H}')$ are labeled by sequences S obeying $s_i = g - 1 - i$ for $i \gg 0$; we will use the notation Σ'_S for these cells. It is easy to check that $\Sigma'_S = \Sigma_S \cap \text{Gr}_g(\mathcal{H}')$.

Assume that $k^{-1}\Sigma'_S$ is nonempty. Then a point $(C, p, z) \in k^{-1}\Sigma'_S$ if and only if p has the Weierstrass sequence S . The Weierstrass cycle $\widehat{W}_S = k^{-1}\Sigma'_S \cap \widehat{\mathcal{CM}}_g$ in $\widehat{\mathcal{CM}}_g$ is a support of the cohomology class $k^* \Omega_\mu^T$, where Ω_μ^T is the equivariant cohomology class corresponding to the Schubert cycle $\bar{\Sigma}'_S$ in the equivariant cohomology of Grassmannian $\text{Gr}(\mathcal{H}')$. Of course this statement can be applied also to the Weierstrass cycle $\widehat{W}_S \cap \widehat{\mathcal{M}}_g = k^{-1}\Sigma'_S \cap \widehat{\mathcal{M}}_g$ in $\widehat{\mathcal{M}}_g$. We have mentioned that the equivariant cohomology of $\widehat{\mathcal{M}}_g$ can be identified with cohomology of $\mathcal{M}_{g,1}$ by means of the forgetful map π . We obtain that the Weierstrass cycle W_S in $\mathcal{M}_{g,1}$ is a support of the cohomology

class $(\pi^*)^{-1}k^*\Omega_\mu^T$. In the case when the codimension of the Weierstrass cycle W_S equals to the dimension of the equivariant cohomology class Ω_μ^T (in this case, one says that the Weierstrass cycle has expected dimension), we obtain information about the cohomology class dual to W_S . (We can use the relation between the notion of support and the notion of its dual class; see (2.1).) Namely, if we assume that the Weierstrass cycle W_S is irreducible, then the dual cohomology class $[W_S]$ is proportional to $(\pi^*)^{-1}k^*\Omega_\mu^T$. If we impose stronger condition that the intersection of the Krichever locus $k(\widehat{\mathcal{M}}_g)$ and the Schubert cycle $\overline{\Sigma}'_S$ is in general position, we can say that the coefficient of proportionality is equal to 1. If the Weierstrass cycle W_S is reducible, we can say that $(\pi^*)^{-1}k^*\Omega_\mu^T$ is a linear combination of cohomology classes dual to irreducible components of W_S .

Using the calculation of $k^*\Omega_\mu^T$ in Section 2, we obtain the information about $[W_S]$ in terms of shifted Schur functions or factorial Schur functions:

Theorem 2.4. If the complex codimension of W_S is equal to $|\mu| = \sum \mu_i$ (i.e. the cycle has expected dimension), then

$$(2.7) \quad \begin{aligned} [W_S] &= \text{const } {}^g s_\mu^*(z_1, \dots, z_g)(-\psi)^{|\mu|} \\ &= \text{const } {}^g t_\mu \left(-\frac{k^*x_1}{\psi}, \dots, -\frac{k^*x_g}{\psi} \right) (-\psi)^{|\mu|}, \end{aligned}$$

where const is a non-zero constant, μ is the partition corresponding to the sequence S and z_1, \dots, z_g are the formal variables defined by $z_i = (k^*x_i - (i - g - 1)\psi)/(-\psi)$ for $1 \leq i \leq g$.

In the paper [14] we have analyzed the Weierstrass cycles in the case of low genera $g \leq 6$. We have found that the majority of Weierstrass cycles satisfy the conditions of the theorem (all Weierstrass cycles for $g \leq 4$ have expected dimension and 9 out of 11 Weierstrass cycles have expected dimension for $g = 5$). In all these cases we calculated corresponding cohomology classes using (2.7).

To prove the theorem we notice that the complex codimension of $\overline{\Sigma}'_S$ in $\text{Gr}_g(\mathcal{H}^l)$ is equal to $|S| = \sum_{i=1}^{i_0} (s_i + i - g) + \sum_{i=i_0+1}^\infty (s_i + i - g + 1)$, where i_0 is the index so that $s_{i_0} \geq 0$ and $s_{i_0+1} < 0$. (If $s_i < 0$ for all i , we set $i_0 = 0$.) We associate to S a partition $\mu = (\mu_i)$ by $\mu_i = s_i + i - g$ for $1 \leq i \leq i_0$ and $\mu_i = s_i + i - g + 1$ if $i \geq i_0 + 1$; then the codimension is equal to $|\mu|$. The partition corresponding to a Weierstrass sequence has length at most g by the Riemann-Roch theorem. Therefore the factorial Schur function ${}^{g+1}t_\mu(X_1 + l, \dots, X_g + l, 0) = {}^g t_\mu(X_1 + l, \dots, X_g + l)$ is already in stable

range for $l = 0$. To check that the constant in (2.7) does not vanish we use Serre's theorem [21].

Again it is more convenient to use homogeneous components of factorial Schur functions. Then

$$(2.8) \quad [W_S] = \text{const} \sum_i (-\psi)^{|\mu|-i} t_\mu^i(k^*x_1, \dots, k^*x_g),$$

Notice that in the case when the codimension of W_S is not equal to $|\mu|$ the RHS of (2.7) makes sense, but is not related to $[W_S]$. One can define "virtual Weierstrass cycle" as a homology class dual to the RHS of (2.7). It is interesting to notice that the multiplication rule of Schubert classes in the equivariant cohomology of Grassmannian (see [11],[17]) gives a multiplication rule for "virtual Weierstrass cycles".

One can consider Weierstrass cycles in \mathcal{M}_g defined as images of Weierstrass cycles in $\mathcal{M}_{g,1}$ by the forgetful map. In other words, we define W'_S as a subvariety consisting of curves $C \in \mathcal{M}_g$ containing at least one point with Weierstrass sequence S . Using the pull-push formula, we obtain the following expression for the corresponding cohomology classes

$$(2.9) \quad [W'_S] = \text{const} \sum_i (-1)^{|\mu|-i} \kappa_{|\mu|-i-1} t_\mu^i(k^*x_1, \dots, k^*x_g).$$

Here μ stands for the partition corresponding to S and $\kappa_b = \pi_* \psi^{b+1}$ are the kappa-classes. This expression is valid if W'_S has the expected dimension, i.e. the expression holds if the complex codimension of W'_S in \mathcal{M}_g equals to $|\mu| - 1$.

In a separate paper [14], we have applied the results of the present paper to the moduli space of irreducible curves of low genera. We estimate the dimension of Weierstrass cycles from below; using the calculations of [15] and [16], we show that for $g \leq 6$, this estimate either coincides with the exact dimension or differs by one. If our estimate coincides with the exact dimension, we are able to calculate the homology class of a Weierstrass cycle up to a constant factor; we performed this calculation for $g \leq 6$. We compare the relations in the tautological ring obtained in the present paper with the description of the tautological ring of \mathcal{M}_g obtained by Faber [4].

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