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On local holomorphic maps preserving invariant (p, p)-forms between bounded symmetric domains

Yuan Yuan

Let $D, \Omega_1, \ldots, \Omega_m$ be irreducible bounded symmetric domains. We study local holomorphic maps from D into $\Omega_1 \times \cdots \times \Omega_m$ preserving the invariant (p, p)-forms induced from the normalized Bergman metrics up to conformal constants. We show that the local holomorphic maps extends to algebraic maps in the rank one case for any p and in the rank at least two case for certain sufficiently large p. The total geodesy thus follows if $D = \mathbb{B}^n, \Omega_i = \mathbb{B}^{N_i}$ for any p or if $D = \Omega_1 = \cdots = \Omega_m$ with $\mathrm{rank}(D) \geq 2$ and p sufficiently large. As a consequence, the algebraic correspondence between quasi-projective varieties D/Γ preserving invariant (p, p)-forms is modular, where Γ is a torsion free, discrete, finite co-volume subgroup of $\mathrm{Aut}(D)$.

1. Introduction

The study of local holomorphic maps preserving invariant (p, p)-forms was raised by Mok in [Mo4]. The motivations are the geometry of local holomorphic isometric or measure-preserving maps (c.f. [Mo2][Mo3][Mo4][MN][Ng1] [YZ]) and the modularity of algebraic correspondence (c.f. [CU] [MN]). Clozel-Ullmo [CU] considered the problem of the modularity for the algebraic correspondence $Y \subset X \times X$ that commutes with a given Hecke correspondence, where $X = \Omega/\Gamma$ is a quotient of an irreducible bounded symmetric domain Ω by a torsion free, discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$. They reduced the problem to the characterization of algebraic correspondence preserving either the invariant metrics or the invariant volume forms. In the differential geometric formulation, the problem can be further reduced to the local holomorphic isometries or the local holomorphic measure-preserving maps from $U \subset \Omega$ into $\Omega \times \cdots \times \Omega$. Clozel-Ullmo [CU] proved the total geodesy for the local holomorphic isometries arised from the algebraic correspondence for Ω

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being the unit disc $\Delta \subset \mathbb{C}$ or a bounded symmetric domain of rank at least two, by applying the powerful metric rigidity theorem of Mok (see [Mo1]) in the latter case (see also [Mo3]). As a consequence, the modularity of the algebraic correspondences is derived in those cases.

Mok then formulates the more general differential geometric problems studying the local holomorphic isometries or measure-preserving maps between bounded symmetric domains. Let D, Ω be bounded symmetric domains and D irreducible, $U \subset D$ be a connected open set, $F : U \to \Omega$ be a holomorphic isometry up to conformal constants. Assume $\dim_{\mathbb{C}} D \geq 2$. F is shown to be totally geodesic either if $\mathrm{rank}(D) \geq 2$ [Mo4]; or D is the complex unit ball and Ω is the product of complex unit balls [Mo2] [Ng1] [YZ]. On the other hand, Mok constructed the non-totally geodesic embeddings from $\mathbb{B}^n$ into a bounded symmetric domain Ω of rank at least two for n either equal to one [Mo3] or greater than one [Mo5]. By using the reduction of Clozel-Ullmo to holomorphic isometries, Mok deduced the modularity of algebraic correspondence between unit balls of dimension at least two [Mo2].

The original modularity problem of Clozel-Ullmo by reducing to holomorphic measure-preserving maps between the irreducible bounded symmetric domain Ω and its products $\Omega \times \cdots \times \Omega$ was later solved by Mok-Ng [MN]. They showed that, the holomorphic measure-preserving maps, in the case of either $\Omega = \mathbb{B}^n$ with $n \geq 2$ or $\mathrm{rank}(\Omega) \geq 2$, are both totally geodesic.

The major ideas in studying the total geodesy for those local holomorphic maps are quite similar. The first step is to show that these local holomorphic maps can be extended as algebraic maps, i.e. they are defined by polynomials. Along the line of algebraicity, Mok [Mo4] derived a powerful algebraicity theorem for local holomorphic isometries, that played essential roles in the proof the total geodesy. Another quite useful tool is the algebraicity of CR maps derived by Huang [Hu1] (Theorem 2.4), which can be used in both local holomorphic isometries and local holomorphic measure-preserving maps . It is now unclear to the author how the algebraicity theorem of Mok can be used for local holomorphic measure-preserving maps or local holomorphic maps preserving invariant (p, p)-forms. In the proof of total geodesy, if the holomorphic maps is derived from the algebraic correspondence, the Alexander type of theorems (Theorem 2.7 and Theorem 2.8) were used in reducing the interior geometry of the holomorphic maps to the boundary geometry, combining the monodromy argument.

The local holomorphic maps preserving invariant (p, p)-forms between bounded symmetric domains were raised by Mok (c.f. Problem 5.3.1. in [Mo4]). This setting becomes holomorphic isometries if $p = 1$ and holomorphic measure-preserving maps if p reaches the top degree. The total geodesy of such maps are related to the modularity of algebraic correspondence preserving invariant (p, p)-forms. The study of such problems follows the standard ideas by showing firstly the algebraicity and secondly the total geodesy by the monodromy argument. We invoke Huang's algebraicity theorem in the first step, while the difficulty occurs if the bounded symmetric domains are of rank at least two, i.e. the case when the holomorphic bisectional curvature has null directions. However, if p is sufficiently large, more precisely, greater than the null dimensions of all irreducible components, then the induced algebraic real hypersurfaces must have strongly pseudoconvex points. Thus Huang's algebraicity theorem can be applied. To prove the total geodesy, we use the idea developed in [YZ] for the rank one case and the Alexander type theorem derived by [MN] in the higher rank case. The modularity of algebraic correspondence then follows from the standard deduction in [CU].

The geometry of local holomorphic isometries dates back to the celebrated work of Calabi [Ca]. Since then, there has been quite a few works (besides the works mentioned above) on local holomorphic maps that is related to the objects considered in the current setting (e.g. [DL] [Eb] [HY1] [HY2], etc.). The list is by no means to be complete.

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2. Background and main theorems

Write $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$ for the unit ball in $\mathbb{C}^n$. Let

$$\omega_{\mathbb{B}^n} = \sum_{j,k \leq n} \frac{1}{(1-|z|^2)^2} \big((1-|z|^2)\delta_{jk} + \bar{z}_j z_k\big) dz_j \wedge d\bar{z}_k$$

be the invariant Kähler form associated to the normalized Bergman metric. Let $U \subset \mathbb{B}^n$ be a connected open subset. Consider a holomorphic map

$$F = (F_1, \ldots, F_m) : U \to \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m} \tag{1}$$

that preserves invariant (p,p)-forms up to positive conformal constant λ in the sense that

$$\lambda\omega^p_{\mathbb{B}^n} = \sum_{j=1}^{m} F_j^*(\omega^p_{\mathbb{B}^{N_j}}). \tag{2}$$

Let $p \leq n \leq N_i$ for each $1 \leq i \leq m$. More precisely, for each j, F_j is a holomorphic map from U to $\mathbb{B}^{N_j}$. The first main theorem is the following rigidity theorem:

Theorem 2.1. *Suppose $n \geq 2$. Let F be a holomorphic map in a connected open subset U of $\mathbb{B}^n$ into the product of unit balls given in (1) and assume that F satisfies (2). In addition, assume that each F_j is of rank at least p at some point in U for $1 \leq j \leq m$. Then each F_j extends to a totally geodesic holomorphic embedding from $(\mathbb{B}^n, \omega_{\mathbb{B}^n})$ into $(\mathbb{B}^{N_j}, \omega_{\mathbb{B}^{N_j}})$ for all $1 \leq i \leq m$ and thus $\lambda = m$.*

Note that a holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$ is a totally geodesic embedding with respect to the normalized Bergman metric if and only if there are a (holomorphic) automorphism $\sigma \in Aut(\mathbb{B}^n)$ and an automorphism $\tau \in Aut(\mathbb{B}^N)$ such that $\tau \circ F \circ \sigma(z) \equiv (z, 0)$.

Let $D \subset \mathbb{C}^n$ be the Harish-Chandra realization of an irreducible bounded symmetric domain. Let ω_D be the invariant Kähler form associated to the normalized Bergman metric.

Given any nonzero holomorphic tangent vector X, define N_X to be the null space of the Hermitian bilinear form $R(X, \bar{X}, V, \bar{W})$ on the holomorphic tangent space with respect to X, where R is the Riemannian curvature operator of the bounded symmetric domain D. Define $\mathcal{N} = \max_X \dim_{\mathbb{C}} N_X$ to be the null dimension of D [Mo1]. Note that $\mathcal{N}$ is invariant at each point, as D is symmetric. In particular, $\mathcal{N}_D < n$ for the bounded symmetric domain D in $\mathbb{C}^n$ and $\mathcal{N}_{\mathbb{B}^n} = 0$ for any n.

Assume $\operatorname{rank}(D) \geq 2$. Let $U \subset D$ be a connected open subset. Consider a holomorphic map

$$F = (F_1, \ldots, F_m) : U \to D \times \cdots \times D \tag{3}$$

that preserves invariant (p,p)-forms up to the positive conformal constant λ in the sense that

$$\lambda\omega^p_D = \sum_{j=1}^{m} F_j^*(\omega^p_D). \tag{4}$$

The second main theorem is the following rigidity theorem:

Theorem 2.2. *Let $D \subset \mathbb{C}^n$ be the Harish-Chandra realization of an irreducible bounded symmetric domain of rank at least 2, and let F be a holomorphic map in a connected open subset U of D into $D \times \cdots \times D$ given in (3) satisfying (4). Assume each F_i is of full rank at some point in D and $p > \mathcal{N}_D$. Then each F_i extends to the holomorphic automorphim $F_i \in Aut(D)$ for all $1 \leq i \leq m$ and thus $\lambda = m$.*

Let $X := \Omega/\Gamma$ be the quotient of an irreducible bounded symmetric domain Ω by a torsion free, discrete, finite co-volume subgroups of $\mathrm{Aut}(\Omega)$. Let T_Y be the algebraic correspondence. By definition, $Y \subset X \times X$ is an irreducible subvariety such that the canonical projections $\pi_i|_Y$ are finite surjective morphisms for $i = 1, 2$. Denote the degree of π_i by d_i. For any fixed point $z_0 \in X$, there exists an open set $U_0 \subset X$ of z_0 such that the covering map $\pi : \Omega \to X$ and $\pi_2 : Y \to X$ are local biholomorphisms on $\pi^{-1}(U_0)$ and $\pi_2^{-1}(U_0)$, respectively. Moreover, π_1 is also a local biholomorphism on $\pi_2^{-1}(U_0)$. Write $\pi_2^{-1}(U_0) = V_1 \cup \cdots \cup V_{d_2}$ as disjoint union of open sets in Y, and $\pi_2^{-1}(z_0) = \{y_1, \ldots, y_{d_2}\}$ with $y_j \in V_j$. Let $W_j = \pi_1(V_j)$ be the open set containing $z_j = \pi_1(y_j)$. In this case, $\pi_1 : V_j \to W_j, \pi_2 : V_j \to U_0$ are both biholomorphisms for each $1 \leq j \leq d_2$. Denote $\pi_1|_{V_j} \circ \pi_2^{-1}$ by $\theta_j : U_0 \to W_j$ for each j.

We say that the algebraic correspondence T_Y locally preserves the invariant (p,p)-forms ω_Ω^p on X induced from the universal covering Ω if for any smooth $(n-p, n-p)$-form β with compact support in U_0 such that

$$\frac{1}{d_1} \int_X T_Y^* \beta \wedge \omega_\Omega^p = \int_X \beta \wedge \omega_\Omega^p.$$

The detailed presentation on algebraic correspondence is referred to [CU].

The standard deduction in [CU] yields:

$$\frac{1}{d_1} \sum_{j=1}^{d_2} \theta_j^* \omega_\Omega^p = \omega_\Omega^p \text{ on } U_0. \tag{5}$$

By lifting ω_Ω from X to its universal covering Ω, the holomorphic map $\pi_1 \circ \pi_2^{-1} = (\theta_1, \ldots, \theta_{d_2}) : U_0 \to \Omega \times \cdots \times \Omega$ can be considered as the germs of holomorphic map preserving the invariant (p,p)-forms ω_Ω^p in the sense of (5).

When $\Omega = \mathbb{B}^n$ for $n \geq 2$, suppose (5) holds for any p. Then each $\theta_j \in \mathrm{Aut}(\mathbb{B}^n)$ as the straightforward application of Theorem 2.1. When $\mathrm{rank}(\Omega) \geq$

2, suppose (5) holds for any $p > \mathcal{N}_\Omega$. Then each $\theta_j \in \mathrm{Aut}(\Omega)$ as the straightforward application of Theorem 2.2. In both cases, the subvariety $Y \subset X \times X$ raised from such θ_j is the modular correspondence (see, also, the argument in [CU]). Therefore, we have proved the following theorem:

Theorem 2.3. *Let $X = \Omega/\Gamma$ and $Y \subset X \times X$ be the algebraic correspondence preserving the invariant (p,p)-form ω_Ω^p. Then Y is modular either if $\Omega = \mathbb{B}^n$ for $n \geq 2$ and $1 \leq p \leq n$; or if rank$(\Omega) \geq 2$ for $n \geq p > \mathcal{N}_\Omega$.*

Let us state the main ingredients in the proofs of Theorem 2.1 and Theorem 2.2. The first one is the following algebraicity theorem of Huang [Hu1] for CR maps.

Theorem 2.4. *(Huang [Hu1]) Let $M_1 \subset \mathbb{C}^n$ and $M_2 \subset \mathbb{C}^N$ be real algebraic hypersurfaces with $n > 1$ and $N \geq n$. Let $p \in M_1$ be a strongly pseudoconvex point. Suppose that h is a holomorphic map from an open neighborhood U_p of p to $\mathbb{C}^N$ such that $h(U_p \cap M_1) \subset M_2$ and $h(p)$ is also a strongly pseudoconvex point, then h is Nash algebraic.*

Recall that a function $h(z,\bar{z})$ is called a Nash algebraic function over $\mathbb{C}^n$ if there is an irreducible polynomial $P(z,\xi,X)$ in $(z,\xi,X) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$ with $P(z,\bar{z},h(z,\bar{z})) \equiv 0$ over $\mathbb{C}^n$. A holomorphic map is called Nash algebraic if each component of the map is a Nash algebraic function.

The second main ingredient is the following normal form and the deep rigidity result proved by Huang [Hu2] [Hu3]. Let $\mathbb{H}^n = \{(Z,W) \in \mathbb{C}^{n-1} \times \mathbb{C} : \Im W - |Z|^2 > 0\}$ be the Siegel upper half space. Assign the weight of W to be 2 and that of Z to be 1. Denote by $o_{wt}(k)$ terms with weighted degree higher than k and by $P(k)$ a function of weighted degree k. For $p_0 = (Z_0, W_0) \in \partial\mathbb{H}^n$, write $\sigma_0^p : (Z,W) \to (Z + Z_0, W + W_0 + 2iZ \cdot \overline{Z_0})$ for the standard Heisenberg translation. The following Cayley transformation

$$\rho_n(Z,W) = \left(\frac{2Z}{1 - iW}, \frac{1 + iW}{1 - iW}\right) \tag{6}$$

biholomorphically maps $\mathbb{H}^n$ to $\mathbb{B}^n$, and biholomorphically maps $\partial\mathbb{H}^n$, the Heisenberg hypersurface, to $\partial\mathbb{B}^n \backslash \{(0,1)\}$. Let F be a rational proper holomorphic map from $\mathbb{H}^n$ to $\mathbb{H}^N$. By a result of Cima-Suffridge [CS], F is holomorphic in a neighborhood of $\partial\mathbb{H}^n$. The following normalization theorem is proved by Huang:

Theorem 2.5. *(Huang [Hu2-3]) For any $q \in \partial\mathbb{H}^n$, there is an element $\tau \in Aut(\mathbb{H}^{N+1})$ such that the map*

$$F_q^{**} = \tau \circ F \circ \sigma_p^0 = ((f_q^{**})_1, \ldots, (f_q^{**})_{n-1}, \phi_q^{**}, g_q^{**}) = (f_q^{**}, \phi_p^{**}, g_q^{**})$$

takes the following normal form:

$$\begin{aligned} f_q^{**}(Z,W) &= Z + \frac{i}{2} a^{(1)}(Z) W + o_{wt}(3), \\ \phi_q^{**}(Z,W) &= \phi^{(2)}(Z) + o_{wt}(2), \\ g_q^{**}(Z,W) &= W + o_{wt}(4) \end{aligned}$$

with

$$(\bar{Z} \cdot a^{(1)}(Z))|Z|^2 = |\phi^{(2)}(Z)|^2. \tag{7}$$

Writing $(f_q^{**})_l = Z_j + \frac{i}{2}\sum_{k=1}^{n-1} a_{lk} Z_k W + o_{wt}(3)$. In particular, if the $(n-1)\times(n-1)$ Hermitian matrix $(a_{lk})_{1\le l,k\le n-1} \equiv 0$ for all $q \in \partial\mathbb{H}^n$, then F is totally geodesic.

Let $\Xi_j = \omega_{\mathbb{H}^n} - F_j^* \omega_{\mathbb{H}^{N_j}}$ and write

$$\Xi = \Xi_{jk} dZ_j \otimes d\bar{Z}_k + \Xi_{jn} dZ_j \otimes d\bar{W} + \Xi_{nj} dW \otimes d\bar{Z}_j + \Xi_{nn} dW \otimes d\bar{W}.$$

The following proposition is proved in [YZ] connecting the normal form and Ξ.

Proposition 2.6. *([YZ]) Assume that $F = (f_1, \ldots, f_{n-1}, \phi, g) : \mathbb{H}^n \to \mathbb{H}^N$ is a proper rational holomorphic map that satisfies the normalization (at the origin) in Theorem 2.5. Then*

$$\Xi_{jk}(0) = -2i(f_k)_{Z_j W}(0) = a_{kj}.$$

The third main ingredients in the proofs are the following Alexander type of theorems characterizing the automorphim of an irreducible bounded symmetric domain.

Theorem 2.7. *(Mok-Ng [MN]) Let $D \subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Suppose b be a smooth point on ∂D. Let $U_b \subset \mathbb{C}^n$ be an open neighborhood of b in $\mathbb{C}^n$ and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap D) \subset D$ and $f(U_b \cap \partial D) \subset \partial D$. Then, there exists an automorphism $F : D \to D$ such that $F|_{U_b \cap D} = f|_{U_b \cap D}$.*

Theorem 2.8. *(Alexander [Al]) Let* $\mathbb{B}^n$ *be the complex unit ball of complex dimension* $n \geq 2$*. Let* $b \in \partial\mathbb{B}^n$*,* U_b *be a connected open neighborhood of* b *in* $\mathbb{C}^n$*, and* $f : U_b \to \mathbb{C}^n$ *be a nonconstant holomorphic map such that* $f(U_b \cap \partial\mathbb{B}^n) \subset \partial\mathbb{B}^n$*. Then, there exists an automorphism* $F : \mathbb{B}^n \to \mathbb{B}^n$ *such that* $F|_{U_b \cap \mathbb{B}^n} = f|_{U_b \cap \mathbb{B}^n}$*.*

3. Algebraic extension

3.1. Bounded symmetric domains of rank one

Lemma 3.1. *Let the Hermitian holomorphic vector bundle* $E \to X$ *over a complex manifold* S *be Griffiths negative. Then* $\wedge^k E$ *is also Griffiths negative.*

Proof. Firstly, we show that $\otimes^k E$ is Griffiths negative. Inductively, it suffices to show that $E \otimes E$ is Griffiths negative. Use ∇, Θ to denote the connection and curvature operator of a Hermitian vector bundle (following expository in [De]). It follows that

$$\Theta(\nabla_{E\otimes E}) = \Theta(\nabla_E) \otimes I_E + I_E \otimes \Theta(\nabla_E)$$

(see [De] formula (V-4.2') on p.258), where I_E is the identity matrix with rank equal to rank(E). One can easily check that $E \otimes E$ is Griffiths negative by showing that for any nonzero local holomorphic section s,

$$\left(\Theta(\nabla_{E\otimes E}) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \bar{x}^j}, s, \overline{s} \right) \right)$$

is a strictly negative-definite $m^2 \times m^2$ matrix, where $\{x_i\}$ is the holomorphic local coordinate of X, $m = \dim_{\mathbb{C}} S$.

Secondly, as $\wedge^k E$ is a subbundle of $\otimes^k E$, $\wedge^k E$ is also Griffiths negative by Proposition (6.10) in [De] (p.340). In fact, there is an analogue Gauss-Codazzi equation for the vector bundle,

$$\Theta_{\wedge^k E}(u, u) = \Theta_{\otimes^k E}(u, u) - |\beta \cdot u|^2,$$

where $u \in TS \otimes \wedge^k E$ and $\beta \in \wedge^{1,0}(S, Hom(\wedge^k E, \otimes^k E / \wedge^k E))$ is the second fundament form of $\wedge^k E$ in $\otimes^k E$. □

Theorem 3.2. *Let* $F := (F_1, \ldots, F_m) : U \to \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$ *be the holomorphic map defined on* $U \subset \mathbb{B}^n$ *that preserves invariant* (p, p)*-forms up to* *the positive conformal factor* λ *in the sense that*

$$\lambda \omega^p_{\mathbb{B}^n} = \sum_{j=1}^{m} F_j^*(\omega^p_{\mathbb{B}^{N_j}}) \text{ on } U. \tag{8}$$

Let $p \le n \le N_i$ *for each* $1 \le i \le m$. *Assume that each* F_j *is of rank at least* p *at some point in* $\mathbb{B}^n$ *for all* j. *Then* F *is Nash algebraic.*

Proof. Consider $S_1 \subset \wedge^p(TU)$ and $S_2 \subset \wedge^p(T\mathbb{B}^{N_1}) \times \cdots \times \wedge^p(T\mathbb{B}^{N_m})$ as follows:

$$S_1 := \left\{(t,\zeta) \in \wedge^p(TU) : \lambda\omega^p_{\mathbb{B}^n}(t)(\zeta,\bar{\zeta}) = 1\right\}, \tag{9}$$

and

$$\begin{aligned} S_2 := \{(z_1,\xi_1,\ldots,z_m,\xi_m) \in \wedge^p(T\mathbb{B}^{N_1}) \times \cdots \times \wedge^p(T\mathbb{B}^{N_m}) : \\ \omega^p_{\mathbb{B}^{N_1}}(z_1)(\xi_1,\overline{\xi_1}) + \cdots + \omega^p_{\mathbb{B}^{N_m}}(z_m)(\xi_m,\overline{\xi_m}) = 1\}. \end{aligned} \tag{10}$$

The defining functions ρ_1, ρ_2 of S_1, S_2 are, respectively, as follows:

$$\begin{aligned} \rho_1 &= \lambda\omega^p_{\mathbb{B}^n}(t)(\zeta,\zeta) - 1, \\ \rho_2 &= \omega^p_{\mathbb{B}^{N_1}}(z_1)(\xi_1,\xi_1) + \cdots + \omega^p_{\mathbb{B}^{N_m}}(z_m)(\xi_m,\xi_m) - 1. \end{aligned}$$

Here $\{t\}, \{z_i\}$ are the canonical Euclidean coordinates on $\mathbb{C}^n, \mathbb{C}^{N_i}$ respectively. Then one can easily check that the map $(F_1, dF_1, \ldots, F_m, dF_m)$ maps S_1 to S_2 according to the equation (8). It is obvious that S_1, S_2 are both real algebraic hypersurfaces by the expression of the complex hyperbolic metric of the unit ball. To finish the proof the theorem, it suffices to show that $(F_1, dF_1, \ldots, F_m, dF_m)$ maps a strongly pseudoconvex point on S_1 to a strongly pseudoconvex point in S_2. We show the strong pseudoconvexity of S_2 at $Q = (0, \xi_1, \ldots, 0, \xi_m)$ as follows and the strong pseudoconvexity of S_1 follows from the same computation.

Let $\{s_{K_j}\}$ be the basis of $\wedge^p T\mathbb{B}^{N_i}$ and write $\xi_j = \xi_{K_j} s_{K_j}$, where K_j is the multi-index $(k_{j_1}, \ldots, k_{j_p})$. Denote $\Delta_j = \binom{N_j}{p}$ to be the rank of the vector bundle $\wedge^p T\mathbb{B}^{N_j}$. By applying $\partial\bar{\partial}$ to ρ_2 at $Q = (0, \xi_1, \ldots, 0, \xi_m)$, we have the following Hessian matrix

$$\mathscr{H} = \begin{bmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & C_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_m & 0 \\ 0 & 0 & \cdots & 0 & C_m \end{bmatrix} \tag{11}$$

where $B_j, C_j, j = 1, 2, \ldots, m$ are function-valued matrices with the following (in)equalities:

$$B_j := \left(\partial_{z_{jk}}\partial_{\bar{z}_{jl}}\rho_2\right)(Q) = \left(-\sum_{K_j,L_j=1}^{\Delta_j} \Theta_{\wedge^p T\mathbb{B}^{N_i}}\left(\frac{\partial}{\partial z_{jk}}, \frac{\partial}{\partial \bar{z}_{jl}}, s_{K_j}\bar{s}_{L_j}\right)(0)\xi_{K_j}\bar{\xi}_{L_j}\right) \geq \delta|\xi_j|^2 I_{N_j}, \tag{12}$$

$$C_j := \left(\partial_{s_{K_j}}\partial_{\bar{s}_{L_j}}\rho_2\right)(Q) = p!\left(\delta_{K_jL_j}\right) \geq \delta I_{\Delta_j}, \tag{13}$$

at $(0, \xi_1, \ldots, 0, \xi_m)$ for some $\delta > 0$. Here B_j is positive definite (the inequality in (12) holds) because $\wedge^p T\mathbb{B}^{N_j}$ is Griffiths negative by applying Lemma 3.1. This implies that $Q \in S_2$ is a strongly pseudoconvex point.

Without loss of generality, by composing elements from $Aut(\mathbb{B}^n)$ and $Aut(\mathbb{B}^{N_1}) \times \cdots \times Aut(\mathbb{B}^{N_m})$, one can assume that $F(0) = 0$ and the rank of each F_j at 0 is at least p. Therefore, there exists $0 \neq \zeta \in \wedge^p T_0\mathbb{B}^n$, such that $dF_j(\zeta) \neq 0$ for all j. After rescaling, we assume that $(0, \zeta) \in S_1$. Now the theorem follows by applying the algebracity theorem of Huang (Theorem 2.4) to the map $(F_1, dF_1, \ldots, F_m, dF_m)$ from S_1 into S_2. □

3.2. Bounded symmetric domains of rank at least two

Theorem 3.3. *Let $D, \Omega_1, \ldots, \Omega_m$ be the Harish-Chandra realization of irreducible bounded symmetric domains in $\mathbb{C}^n, \mathbb{C}^{N_1}, \ldots, \mathbb{C}^{N_m}$ respectively. Let $F := (F_1, \ldots, F_m) : U \subset D \to \Omega_1 \times \cdots \times \Omega_m$ be the holomorphic map defined on $U \subset D$ that preserves invariant (p, p)-forms in the sense that*

$$\lambda\omega_D^p = \sum_{j=1}^m F_j^*(\omega^p_{\Omega_{N_j}}), \text{ for } \lambda > 0, \tag{14}$$

where each F_i is of rank at least p at some point in D and $p \leq n \leq N_i$ for all $i \in \{1, \ldots, m\}$. Assume $p > \max\{\mathcal{N}_D, \mathcal{N}_{\Omega_1}, \ldots, \mathcal{N}_{\Omega_m}\}$. Then F is Nash algebraic.

Proof. The proof uses the similar argument in Theorem 3.2 of reducing the algebraicity of holomorphic maps to the algebraicity of CR maps. Consider $S_1 \subset \wedge^p(TU)$ and $S_2 \subset \wedge^p(T\Omega_1) \times \cdots \times \wedge^p(T\Omega_m)$ as follows:

$$S_1 := \left\{(t, \zeta) \in \wedge^p(TU) : \lambda\omega_D^p(t)(\zeta, \bar{\zeta}) = 1\right\}, \tag{15}$$

and

$$S_2 := \{(Z^{(1)}, \xi^{(1)}, \dots, Z^{(m)}, \xi^{(m)}) \in \wedge^p(T\Omega_1) \times \cdots \times \wedge^p(T\Omega_m) : \omega^p_{\Omega_1}(Z^{(1)})(\xi^{(1)}, \overline{\xi^{(1)}}) + \cdots + \omega^p_{\Omega_m}(Z^{(m)})(\xi^{(m)}, \overline{\xi^{(m)}}) = 1\}. \tag{16}$$

The defining functions ρ_1, ρ_2 of S_1, S_2 are, respectively, as follows:

$$\rho_1 = \lambda\omega^p_D(t)(\zeta, \overline{\zeta}) - 1,$$
$$\rho_2 = \omega^p_{\Omega_1}(Z^{(1)})(\xi^{(1)}, \overline{\xi^{(1)}}) + \cdots + \omega^p_{\Omega_m}(Z^{(m)})(\xi^{(m)}, \overline{\xi^{(m)}}) - 1.$$

Here $\{t\}, \{Z^{(i)}\}$ are the canonical Euclidean coordinates on $\mathbb{C}^n, \mathbb{C}^{N_i}$ respectively. It is obvious that S_1, S_2 are both real algebraic hypersurfaces by the expression of the Bergman metrics [FK]. Moreover one can easily check that the map $(F_1, dF_1, \dots, F_m, dF_m)$ maps S_1 to S_2 according to the equation (21). To finish the proof of the theorem, it suffices to show that $(F_1, dF_1, \dots, F_m, dF_m)$ maps a strongly pseudoconvex point on S_1 to a strongly pseudoconvex point in S_2. We show the strong pseudoconvexity of S_2 at $Q = (0, \xi^{(1)}, \dots, 0, \xi^{(m)})$ as follows and the strong pseudoconvexity of S_1 follows from the same computation.

Let $\{s^{(j)}\}$ be the basis of $\wedge^p T\Omega_j$ and write $\xi^{(j)} = u^{(j)}_{j_1\cdots j_p} s^{(j)}_{j_1\cdots j_p}$, where $s^{(j)}_{j_1\cdots j_p} = \frac{\partial}{\partial Z^{(j)}_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial Z^{(j)}_{j_p}}$ for $j_1 < \cdots < j_p$. Note that

$$\omega^p_{\Omega_j}(Z^{(j)})(s^{(j)}_{j_1\cdots j_p}, \overline{s^{(j)}_{j'_1\cdots j'_p}}) = \det(\omega^{(j)}_{j_s j'_t})_{(j_1,\dots,j_p; j'_1,\dots,j'_p)},$$

where $\omega^{(j)}_{j_s j'_t} = \omega_{\Omega_j}(\frac{\partial}{\partial Z^{(j)}_{j_s}}, \overline{\frac{\partial}{\partial Z^{(j)}_{j'_t}}})$. It follows by straightforward calculation under the normal coordinates that, at $0 \in \Omega_j$,

$$\begin{aligned} &\partial_{Z^{(j)}_k} \bar{\partial}_{Z^{(j)}_l} \omega^p_{\Omega_j}(Z^{(j)})(s^{(j)}_{j_1\cdots j_p}, \overline{s^{(j)}_{j'_1\cdots j'_p}}) \\ &= \begin{cases} 0 & \text{if } \sharp(\{j_1,\dots,j_p\} \cap \{j'_1,\dots,j'_p\}) \le p-2 \\ \pm R^{(j)}_{k\bar{l}s\bar{t}} & \text{if } \sharp(\{j_1,\dots,j_p\} \cap \{j'_1,\dots,j'_p\}) = p-1 \\ -\sum_{1\le i\le q} R^{(j)}_{k\bar{l}j_i\overline{j_i}} & \text{if } \{j_1,\dots,j_p\} = \{j'_1,\dots,j'_p\}) \end{cases}, \end{aligned}$$

where $R^{(j)}_{i\bar{j}s\bar{t}}$ is the holomorphic bisectional curvature of ω_{Ω_j}, that is seminegative definite; $\{s, t\}$ are the two distinct indices such that $\{j_1, \dots, j_p\} \setminus \{s\} = \{j'_1, \dots, j'_p\} \setminus \{t\}$; and the sign in the second case depends on the positions of s, t in $\{j_1, \dots, j_p\}$ and $\{j'_1, \dots, j'_p\}$, respectively.

Therefore, by applying $\partial\bar{\partial}$ to ρ_2 at $Q=(0,\xi^{(1)},\ldots,0,\xi^{(m)})$, we have the following Hessian matrix

$$\mathscr{H}=\begin{bmatrix} B_1 & 0 & \cdots & 0 & 0 \\ 0 & C_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_m & 0 \\ 0 & 0 & \cdots & 0 & C_m \end{bmatrix}, \tag{17}$$

where $B_j, C_j, D_j, j=1,2,\ldots,m$ are function-valued matrices with the following expressions:

$$\begin{aligned} B_j &:= \left(\partial_{Z_k^{(j)}}\bar{\partial}_{Z_l^{(j)}}\rho_2\right)(Q) \\ &= \Bigg(-\sum_{j_1<\cdots<j_p}\sum_{1\le i\le q} R^{(j)}_{k\bar{l}j_i\overline{j_i}}|u^{(j)}_{j_1\cdots j_p}|^2 \\ &\qquad \pm \sum_{j_1<\cdots<j_{p-1},s\neq t} R^{(j)}_{k\bar{l}s\bar{t}}u^{(j)}_{j_1\cdots s\cdots j_{p-1}}\overline{u^{(j)}_{j_1\cdots t\cdots j_{p-1}}}\Bigg)(Q), \end{aligned} \tag{18}$$

and

$$C_j := \left(\partial_{u^{(j)}_{j_1\cdots j_p}}\bar{\partial}_{u^{(j)}_{j'_1\cdots j'_p}}\rho_2\right)(Q)=\left(\det(\omega^{(j)}_{j_s j'_t})_{(j_1,\ldots,j_p;j'_1,\ldots,j'_p)}\right)(0), \tag{19}$$

where C_j equals the $(\Delta_j\times\Delta_j)$-matrix of $\omega^p_{\Omega_j}$ at 0, that is positive definite, and $\Delta_j=\binom{N_j}{p}$ is the rank of the holomorphic vector bundle $\wedge^p T\Omega_j$.

To show that each B_j is positive definite for all j, fix $V^{(j)}=(v_1^{(j)},\ldots,v_{N_j}^{(j)})$ to be a nonzero vector in the holomorphic tangent bundle $T\Omega_j$ at $0\in\Omega_j$. Using a change of coordinate at $0\in\Omega_j$ after a unitary transformation with constant entries, one can assume that $R^{(j)}_{V^{(j)}\overline{V^{(j)}}s\bar{t}}(0)$ is a diagonal matrix. Therefore,

$$\begin{aligned} V^{(j)}B_j\overline{V^{(j)}}^t &= -\sum_{j_1<\cdots<j_p}\sum_{1\le i\le p} R^{(j)}_{V^{(j)}\overline{V^{(j)}}j_i\overline{j_i}}(0)|u^{(j)}_{j_1\cdots j_p}|^2 \\ &\qquad \pm\sum_{j_1<\cdots<j_{p-1},s\neq t} R^{(j)}_{V^{(j)}\overline{V^{(j)}}s\bar{t}}(0)u^{(j)}_{j_1\cdots s\cdots j_{p-1}}\overline{u^{(j)}_{j_1\cdots t\cdots j_{p-1}}} \\ &= -\sum_{j_1<\cdots<j_p}\sum_{1\le i\le p} R^{(j)}_{V^{(j)}\overline{V^{(j)}}j_i\overline{j_i}}(0)|u^{(j)}_{j_1\cdots j_p}|^2 \end{aligned} \tag{20}$$

.

Without loss of generality, by composing with the automorphism groups, one can assume that $F(0)=0$ and also $F_1,\dots,F_m$ are of rank at least p at 0. Therefore, there exists $(0,\zeta)\in S_1$, such that $dF_j(\zeta)\neq 0$ for all j. Moreover, there exists at least one $(j_1,\dots,j_p)$ such that $u^{(j)}_{j_1\cdots j_p}(0,dF_j(\zeta))\neq 0$. It follows that for this particular $(j_1,\dots,j_p)$,

$$V^{(j)}B_j\overline{V^{(j)}}^t \geq -\sum_{1\leq i\leq p} R^{(j)}_{V^{(j)}\overline{V^{(j)}}j_i\overline{j_i}}(0)|u^{(j)}_{j_1\cdots j_p}|^2>0,$$

where the second inequality follows from the assumption $p>\mathcal{N}_{\Omega_j}$, implying $-\sum_{1\leq i\leq p}R^{(j)}_{V^{(j)}\overline{V^{(j)}}j_i\overline{j_i}}(0)>0$. This shows the positivity of B_j and thus the positivity of $\mathscr{H}$. Therefore $Q\in S_2$ is a strongly pseudoconvex point.

Now the theorem follows by applying the algebracity theorem of Huang (Theorem 2.4) to the map $(F_1,dF_1,\dots,F_m,dF_m)$ from S_1 into S_2. $\square$

The algebraicity in the case of unit balls follows also directly from Theorem 3.3.

Corollary 3.4. *Let $F:=(F_1,\dots,F_m):U\subset\mathbb{B}^n\to\mathbb{B}^{N_1}\times\cdots\times\mathbb{B}^{N_m}$ be the holomorphic map defined on $U\subset\mathbb{B}^n$ that preserves invariant (p,p)-forms in the sense that*

$$\lambda\omega^p_{\mathbb{B}^n}=\sum_{j=1}^m F_j^*(\omega^p_{\mathbb{B}^{N_j}}),\ \textit{for}\ \lambda>0, \tag{21}$$

where none of F_i is a constant map and $p\leq n\leq N_i$ for all $i\in\{1,\dots,m\}$. Then F is Nash algebraic.

4. Total geodesy

4.1. Bounded symmetric domains of rank one

Lemma 4.1. *Let $F:\mathbb{B}^n\to\mathbb{B}^N$ be a rational, proper holomorphic map. Let $\Xi=\omega_{\mathbb{B}^n}-F^*\omega_{\mathbb{B}^N}$. Then Ξ is a non-negative $(1,1)$-form in $\mathbb{B}^n$ that is real analytic on an open neighborhood of $\overline{\mathbb{B}^n}$. Moreover, F is a totally geodesic embedding if and only if either $\Xi\equiv 0$ in $\mathbb{B}^n$ or $\Xi\equiv 0$ on an open piece V of $\partial\mathbb{B}^n$ of real dimension $2n-1$.*

Proof. It follows from the Schwarz Lemma that Ξ is non-negative and $\Xi\equiv 0$ if and only if F is a totally geodesic embedding. That Ξ is real analytic on an open neighborhood of $\overline{\mathbb{B}^n}$ is proved in Corollary 2.3 in [YZ]. Moreover,

Ξ is closely related to the first fundamental form of the CR map between unit spheres. By applying Proposition 2.6 to the second normalization $F_z^{**} = \sigma \circ F \circ \tau_z^0$ for each $z \in V$, it follows that $a_{kj} = 0$ for all z. Hence F is totally geodesic by Theorem 2.5. $\square$

The following is a trivial result in calculus.

Lemma 4.2. *Let $V \subset \mathbb{C}^{N_1}$ be a connected open set and $F = (f_1, \ldots, f_{N_2}) : V \to \mathbb{C}^{N_2}$ be a holomorphic map. Let $\{w_i\}_{i=1}^{N_1}$ and $\{z_k\}_{k=1}^{N_2}$ be the coordinates of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Assume $N_1 \geq p, N_2 \geq p$. Then*

$$(22)\qquad \begin{aligned} &F^*(dz_{k_1} \wedge \cdots \wedge dz_{k_p} \wedge d\overline{z_{l_1}} \wedge \cdots \wedge d\overline{z_{l_p}}) \\ &= \sum_{i_1 < \cdots < i_p, j_1 < \cdots < j_p} \det\left(\frac{\partial(f_{k_1}, \ldots, f_{k_p})}{\partial(w_{i_1}, \ldots, w_{i_p})}\right) \det\overline{\left(\frac{\partial(f_{l_1}, \ldots, f_{l_p})}{\partial(w_{j_1}, \ldots, w_{j_p})}\right)} \\ &\qquad \times dw_{i_1} \wedge \cdots \wedge dw_{i_p} \wedge d\overline{w_{j_1}} \wedge \cdots \wedge d\overline{w_{j_p}}. \end{aligned}$$

We are ready to present the proof of Theorem 2.1. Let X be the union of the branch varieties of F_j for $1 \leq j \leq m$. Since $\dim_{\mathbb{C}} X \leq n-1$, for any $Q_0 \in \mathbb{C}^n \setminus X$, there is a real curve γ connecting Q_0 and U such that any branch of F is holomorphically continued along γ to the germ of holomorphic map at Q_0, still denoted by F. Define $E = \cup_{j=1}^m \{z \in \mathbb{B}^n \setminus X \big| |F_j(z)| = 1\}$ and $\dim_{\mathbb{R}} E \leq 2n-1$.

At the first step, we are going to show $\dim_{\mathbb{R}} E \leq 2n-2$. Suppose not. Then there is a curve γ connecting U and a point $Q_0 \in E$ such that $\dim_{\mathbb{R}} O = 2n-1$, where $O \subset E$ is an open neighborhood of Q_0. Moreover, assume $\{Q_0\} = \gamma \cap E$. Without loss of generality, assume $Q_0 \in O \subset \{z \in E \big| |F_1(z)| = 1\}$. Since equation (2) holds in a small open neighborhood of γ for any branch of F by the holomorphic continuation, one has the following equation as points Q_s on γ approach Q_0:

$$\lambda \omega_{\mathbb{B}^n}^p(Q_s) = \sum_{j=1}^m F_j^*(\omega_{\mathbb{B}^{N_j}}^p)(Q_s).$$

Denote the coordinates of $\mathbb{B}^n$ and $\mathbb{B}^{N_1}$ by $\{z_i\}_{i=1}^n$ and $\{w_k\}_{k=1}^{N_1}$ respectively and $F_1 = (f_1, \ldots, f_{N_1})$. It follows that

$$(23)\qquad \begin{aligned} \lambda \omega_{\mathbb{B}^n}^p(Q_s) &\geq F_1^*(\omega_{\mathbb{B}^{N_1}}^p)(Q_s) \geq F_1^*\left(\frac{\sum_{k=1}^{N_1} dw_k \wedge d\overline{w_k}}{1-|w|^2}\right)^p(Q_s) \\ &= \frac{1}{(1-|F_1(Q_s)|^2)^p} \sum_{k_1 < \cdots < k_p} C_{k_1,\ldots,k_p} \Theta_{k_1,\ldots,k_p}(Q_s), \end{aligned}$$

where each $C_{k_1,\ldots,k_p}$ is the nonnegative constant coefficient in front of $dw_{k_1} \wedge \cdots \wedge dw_{k_p} \wedge d\overline{w_{k_1}} \wedge \cdots \wedge d\overline{w_{k_p}}$ of $\left(\sum_{k=1}^{N_1} dw_k \wedge d\overline{w_k}\right)^p$ and

$$\Theta_{k_1,\ldots,k_p} = \sum_{i_1<\cdots<i_p, j_1<\cdots<j_p} \det\left(\frac{\partial(f_{k_1},\ldots,f_{k_p})}{\partial(z_{i_1},\ldots,z_{i_p})}\right) \det\left(\overline{\frac{\partial(f_{k_1},\ldots,f_{k_p})}{\partial(z_{j_1},\ldots,z_{j_p})}}\right) \times dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \cdots \wedge d\overline{z_{j_p}},$$

by Lemma 4.2. Here two (p, p)-forms α_1, α_2 satisfy $\alpha_1 \geq \alpha_2$ if and only if $\alpha_1 - \alpha_2$ is a nonnegative (p, p)-form.

By comparing the coefficients of $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z_{i_1}} \wedge \cdots \wedge d\overline{z_{i_p}}$ in the equation (23), as $Q_s \to Q_0$, $\omega^p_{\mathbb{B}^n}(Q_s)$ is bounded and hence

$$\frac{1}{(1 - |F_1(Q_s)|^2)^p} \left| \frac{\partial(f_{k_1},\ldots,f_{k_p})}{\partial(z_{i_1},\ldots,z_{i_p})}(Q_s) \right|^2$$

is bounded. If follows that for any $k_1 < \cdots < k_p, i_1 < \cdots < i_p$ and any $Q_0 \in O$,

$$\frac{\partial(f_{k_1},\ldots,f_{k_p})}{\partial(z_{i_1},\ldots,z_{i_p})}(Q_0) = 0.$$

By the uniqueness of holomorphic functions, it follows that

$$\frac{\partial(f_{k_1},\ldots,f_{k_p})}{\partial(z_{i_1},\ldots,z_{i_p})} \equiv 0$$

on U. This contradicts to the assumption that F_1 is of rank at least p.

Given any point in $\partial\mathbb{B}^n \setminus X$, still denoted by Q_0, it follows from the previous step that there is a curve, still denoted by γ, connecting U and Q_0, such that $\gamma \cap E = \emptyset$. Since the equation (2) holds on a small open neighborhood O' of γ and as $Q_s \in O'$ approaches Q_0, the coefficients of $\omega^p_{\mathbb{B}^n}(Q_s)$ go to $+\infty$, then the coefficients of $F_j^*(\omega^p_{\mathbb{B}^{N_j}})(Q_s)$ go to $+\infty$ for a certain j. This implies $|F_j(Q_0)| = 1$. Hence F_j maps an open piece of $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^{N_j}$. Assume that F_j maps an open subset of $\partial\mathbb{B}^n$ into $\partial\mathbb{B}^{N_j}$ exactly for $1 \leq j \leq m_0$ after reordering $\{1, \ldots, j\}$. By the theorems of Forstneric [Fo] and Cima-Suffridge [CS], each F_j extends to the unique proper holomorphic map between $\mathbb{B}^n$ and $\mathbb{B}^{n_j}$, which is rational, for $1 \leq j \leq m_0$. Therefore, there exists an open subset of $\partial\mathbb{B}^n$, still denoted by O such that $F_j(O) \subset \partial\mathbb{B}^{N_j}$ for $1 \leq j \leq m_0$ and $F_j(O) \subset \mathbb{B}^{N_j}$ for $m_0 + 1 \leq j \leq m$. Rewrite the equation (2):

$$(24)\qquad \sum_{j=1}^{m_0}\left(\omega^p_{\mathbb{B}^n}-(F_j^*\omega_{\mathbb{B}^{N_j}})^p\right)+(\lambda-m_0)\omega^p_{\mathbb{B}^n}=\sum_{j=m_0+1}^{m}F_j^*(\omega^p_{\mathbb{B}^{N_j}})$$

Letting $\Xi_j=\omega_{\mathbb{B}^n}-F_j^*\omega_{\mathbb{B}^{N_j}}$ and applying difference formula, it follows that

$$(25)\qquad \begin{aligned}&\sum_{j=1}^{m_0}\Xi_j\wedge\left(\sum_{t=0}^{p-1}\omega^t_{\mathbb{B}^n}\wedge(F_j^*\omega_{\mathbb{B}^{N_j}})^{p-1-t}\right)+(\lambda-m_0)\omega^p_{\mathbb{B}^n}\\&=\sum_{j=m_0+1}^{m}F_j^*(\omega^p_{\mathbb{B}^{N_j}}).\end{aligned}$$

For each j, it follows from $\Xi_j\geq 0$ that

$$(26)\qquad \Xi_j\wedge\omega^{p-1}_{\mathbb{B}^n}\leq\Xi_j\wedge\left(\sum_{t=0}^{p-1}\omega^t_{\mathbb{B}^n}\wedge(F_j^*\omega_{\mathbb{B}^{N_j}})^{p-1-t}\right)\leq p\Xi_j\wedge\omega^{p-1}_{\mathbb{B}^n}.$$

Now we rewrite equation (25) in the coordinate of the Siegel upper half space $\mathbb{H}^n=\{(Z,W)\in\mathbb{C}^{n-1}\times\mathbb{C}:\Im W-|Z|^2>0\}$. Applying the Cayley transformation $\rho_n(Z,W)$, one can compute the normalized Bergman metric on $\mathbb{H}^n$, denoted by $\omega_{\mathbb{H}^n}$, by pulling back the normalized Bergman metricon $\mathbb{B}^n$, as follows:

$$(27)\qquad \begin{aligned}\omega_{\mathbb{H}^n}=&\sum_{j,k<n}\frac{\delta_{jk}(\Im W-|Z|^2)+\bar{Z}_jZ_k}{(\Im W-|Z|^2)^2}dZ_j\wedge d\bar{Z}_k+\frac{dW\wedge d\overline{W}}{4(\Im W-|Z|^2)^2}\\&+\sum_{j<n}\frac{\bar{Z}_jdZ_j\wedge d\overline{W}}{2i(\Im W-|Z|^2)^2}-\sum_{j<n}\frac{Z_jdW\wedge d\bar{Z}_j}{2i(\Im W-|Z|^2)^2}.\end{aligned}$$

Note that $\omega_{\mathbb{H}^n}$ is also an invariant metric under the action of the holomorphic automorphism group of $\mathbb{H}^n$. Still denote $\rho^{-1}(Q_0)$ by Q_0. Without loss of generality, one may assume that $Z(Q_0)=0$ by composing the holomorphic automorphism of $\mathbb{H}^n$. One chooses Q_s such that $Z(Q_s)=0$. Hence one has:

$$\omega_{\mathbb{H}^n}(Q_s)=\sum_{k<n}\frac{1}{\Im W(Q_s)}dZ_k\wedge d\bar{Z}_k+\frac{1}{4(\Im W(Q_s))^2}dW\wedge d\overline{W}.$$

The right hand side of equation (25) is bounded. However, because the blown-up rate for $\omega^{p-1}_{\mathbb{H}^n}(Q_s)(\frac{\partial}{\partial Z_1}\wedge\cdots\frac{\partial}{\partial Z_p}\wedge\frac{\partial}{\partial W}\wedge\frac{\partial}{\partial \bar{Z}_1}\wedge\cdots\frac{\partial}{\partial \bar{Z}_p}\wedge\frac{\partial}{\partial \overline{W}})$, and thus by equation (26), that for $\Xi_j \wedge \big(\sum_{t=0}^{p-1} \omega_{\mathbb{H}^n}^t \wedge (F_j^*\omega_{\mathbb{H}^{N_j}})^{p-1-t}\big)(Q_s)(\frac{\partial}{\partial Z_1} \wedge \cdots \frac{\partial}{\partial Z_p} \wedge \frac{\partial}{\partial W} \wedge \frac{\partial}{\partial \bar{Z}_1} \wedge \cdots \frac{\partial}{\partial \bar{Z}_p} \wedge \frac{\partial}{\partial \overline{W}})$ are both $\frac{1}{(\Im W(Q_s))^p}$, while that of $\omega_{\mathbb{H}^n}^p(Q_s)(\frac{\partial}{\partial Z_1} \wedge \cdots \frac{\partial}{\partial Z_p} \wedge \frac{\partial}{\partial W} \wedge \frac{\partial}{\partial \bar{Z}_1} \wedge \cdots \frac{\partial}{\partial \bar{Z}_p} \wedge \frac{\partial}{\partial \overline{W}})$ is $\frac{1}{(\Im W(Q_s))^{p+1}}$, which is higher. Therefore, $m_0 = \lambda$. It follows that

$$
\begin{aligned}
(28)\qquad \sum_{j=m_0+1}^{m} F_j^*(\omega_{\mathbb{B}^{N_j}}^p) &= \sum_{j=1}^{m_0} \Xi_j \wedge \left(\sum_{t=0}^{p-1} \omega_{\mathbb{B}^n}^t \wedge (F_j^*\omega_{\mathbb{B}^{N_j}})^{p-1-t} \right) \\
&\geq \sum_{j=1}^{m_0} \Xi_j \wedge \omega_{\mathbb{B}^n}^{p-1}.
\end{aligned}
$$

We only need consider the case $p \geq 2$ (the case $p = 1$ is solved in [YZ]). By the similar argument as above, it follows that $\Xi_j \equiv 0$ on an open piece of $\partial\mathbb{B}^n$ of real dimension $2n-1$ for each $1 \leq j \leq m_0$. Hence F_j is a totally geodesic embedding for each j by Lemma 4.1, and therefore $\Xi_j \equiv 0$ on $\mathbb{B}^n$. Now the equation (28) reads

$$
(29)\qquad \sum_{j=m_0+1}^{m} F_j^*(\omega_{\mathbb{B}^{N_j}}^p) \equiv 0 \text{ on } U,
$$

implying $F_j^*(\omega_{\mathbb{B}^{N_j}}^p) \equiv 0$ on U for $m_0 + 1 \leq j \leq m$. It follows from Lemma 4.2 that for any $1 \leq k_1 < \cdots < k_p \leq N_j, 1 \leq i_1 < \cdots < i_p \leq n$,

$$
\frac{\partial(f_{jk_1}, \ldots, f_{jk_p})}{\partial(z_{i_1}, \ldots, z_{i_p})} \equiv 0,
$$

where $F_j = (f_{j1}, \ldots, f_{jN_j})$ for each $m_0 + 1 \leq j \leq m$. This contradicts to the assumption that each F_j is of rank at least p. Hence $m_0 = m$ and the theorem is proved.

4.2. Bounded symmetric domains of rank at least two

We now prove Theorem 2.2. We follow the similar argument as in the proof of Theorem 2.1 (see also [Mo2] [MN]). By Theorem 3.3, F is algebraic. Let X be the union of the branch varieties of F_j and subvarieties where F_i is not of full rank for $1 \leq j \leq m$. Since $\dim_{\mathbb{C}} X \leq n-1$, for any point $Q_0 \in \mathbb{C}^n \setminus X$, there is a real curve γ connecting Q_0 and U such that any branch of F is holomorphically continued along γ to the germ of holomorphic map at Q_0. Fix one branch of such map, still denoted by F. Fix a smooth boundary point $q \in \partial D \setminus X$ and an open neighborhood U_q of q outside X, and choose a smooth curve $\gamma(t)$ such that $\gamma(0) \in U \setminus X, \gamma(1) = q$. Moreover, the equation (4) is preserved along $\gamma(t)$ unless $\gamma(t)$ or some $F_i(\gamma(t))$ touches ∂D.

By the expression of the Bergman metric ω_D of D (c.f. [FK]), $\omega_D^p(z)$ blows up if and only if $z \to \partial D$. When $\gamma(t) \in D$, $F_j^* \omega_D^p(\gamma(t))$ does not blow up by equation (4), implying $F_j(\gamma(t)) \in D$. Hence $F_j(U_q \cap D) \subset D$ for all j. As $\gamma(t) \to q' \in \partial D \cap U_q$, $F_j^* \omega_D^p(\gamma(t))$ blows up for some j, implying that $F_j(q') \in \partial D$. By possibly shrinking U_q, there exists some j such that $F_j(\partial D \cap U_q) \subset \partial D$. By Theorem 2.7, $F_j \in Aut(D)$. The theorem then follows by the induction.

Remark 4.3. *When $p = m$, Theorem 2.2 is due to Mok-Ng [MN].*

Remark 4.4. *One can also follow exactly the same argument in [MN] by using Henkin-Tumanov's characterization [TK] of the automorphism of D. For the simplicity of argument, here we invoke the Alexander type theorem by Mok-Ng.*

Remark 4.5. *When $D = \mathbb{B}^n$, then Theorem 2.2 follows also by the similar argument using Alexander's Theorem (Theorem 2.8), and the theorem in this case is due to Mok [Mo2] and Mok-Ng [MN], when $p = 1$ and $p = n$, respectively.*

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Department of Mathematics, Syracuse University
215 Carnegie building, Syracuse, NY 13244, USA
E-mail address: yyuan05@syr.edu

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