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Artin conjecture for p-adic Galois representations of function fields

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For a global function field K of positive characteristic p, we show that Artin's entireness conjecture for L-functions of geometric p-adic Galois representations of K is true in a non-trivial p-adic disk but is false in the full p-adic plane. In particular, we prove the non-rationality[1] of the geometric unit root L-functions.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements with characteristic p. Let C be a smooth projective geometrically connected curve defined over $\mathbb{F}_q$ with function field K. Let U be a Zariski open dense subset of C with inclusion map $j : U \hookrightarrow C$. Let $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote the absolute Galois group of K. For example, we can take $C = \mathbb{P}^1$, $U = \mathbb{P}^1 - \{0, \infty\}$ and $K = \mathbb{F}_q(t)$.

Let $\pi_1^{\mathrm{arith}}(U)$ denote the arithmetic fundamental group of U. That is,

$$\pi_1^{\mathrm{arith}}(U) = G_K / < I_x >_{x \in |U|},$$

where the denominator denotes the closed normal subgroup generated by the inertial subgroups I_x as x runs over the closed points $|U|$ of U. Let D_x denote the decomposition group of G_K at x. One has the following exact sequence

$$1 \to I_x \to D_x \to \mathrm{Gal}(\bar{k}_x/k_x) \to 1,$$

where k_x denotes the residue field of K at x. The Galois group $\mathrm{Gal}(\bar{k}_x/k_x)$ is topologically generated by the geometric Frobenius element Frob_x which

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is characterized by the property:

$$\mathrm{Frob}_x^{-1} : \alpha \to \alpha^{\# k_x}.$$

Let P_x denote the p-Sylow subgroup of I_x. Then we have the following exact sequence

$$1 \to P_x \to I_x \to I_x^{\mathrm{tame}} = \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 1.$$

Let F_ℓ be a finite extension of $\mathbb{Q}_\ell$, where ℓ is a prime number which may or may not equal to p. Let V be a finite dimensional vector space over F_ℓ. Let

$$\rho : G_K \longrightarrow GL(V)$$

be a continuous ℓ-adic representation of G_K unramified on U. Equivalently,

$$\rho : \pi_1^{\mathrm{arith}}(U) \longrightarrow GL(V)$$

is a continuous representation of $\pi_1^{\mathrm{arith}}(U)$. The representation ρ is called *geometric* if it comes from an ℓ-adic cohomology of a smooth proper variety over U. The geometric representations are the most interesting ones in applications.

Given a representation ρ, its L-function is defined by

$$L(U, \rho, T) = \prod_{x \in |U|} \frac{1}{\det(I - \rho(\mathrm{Frob}_x) T^{\deg(x)} | V)} \in 1 + T R_\ell[[T]],$$

where R_ℓ is the ring of integers in F_ℓ. It is clear that this L-function is trivially ℓ-adic analytic in the open unit disc $|T|_\ell < 1$.

We are interested in further analytic properties of this L-function $L(U, \rho, T)$, especially for those representations which come from geometry [2]. More precisely, we want to know

Question 1.1 (Meromorphic continuation). When and where the L-function $L(U, \rho, T)$ is ℓ-adic meromorphic?

[2] Our definition of a geometric representation depends on ℓ. If $\ell \neq p$, then it forms a compatible system for all $\ell \neq p$ and one allows sub-quotients in the definition of geometric representations. In the case $\ell = p$, our definition does not allow sub-quotients. In fact, this was raised as an open problem in Remark 3.4.

Question 1.2 (Artin's conjecture). Assume that ρ has no geometrically trivial component. When and where the L-function $L(U,\rho,T)$ is ℓ-adic entire (no poles or analytic)?

The answer depends very much on whether ℓ equals to p or not. In the easier case $\ell \neq p$, the Grothendieck trace formula [9] gives the following complete answer.

Theorem 1.3. *Assume that $\ell \neq p$. The L-function $L(U,\rho,T)$ is a rational function in $F_\ell(T)$. If ρ has no geometrically trivial component, then $L(U,\rho,T)$ is a polynomial in $F_\ell[T]$.*

In the case $\ell = p$, the situation is much more subtle. A general conjecture of Katz [11] as proved by Emerton-Kisin [8] says that the above two questions still have a complete positive answer if we restrict to the closed unit disc. That is, we have

Theorem 1.4. *Assume that $\ell = p$. The L-function $L(U,\rho,T)$ is p-adic meromorphic on the closed unit disc $|T|_p \leq 1$. If ρ has no geometrically trivial component, then the L-function $L(U,\rho,T)$ is p-adic analytic (no poles) on the closed unit disc $|T|_p \leq 1$.*

The extension of the above results to larger p-adic disc is more subtle. For any given $\epsilon > 0$, there are examples [15] showing that the L-function $L(U,\rho,T)$ is not p-adic meromorphic in the disc $|T|_p < 1+\epsilon$, disproving another conjecture of Katz [11]. However, if ρ comes from geometry, then Dwork's conjecture [6] as proved by the second author [16][17] shows the L-function is indeed a good p-adic function:

Theorem 1.5. *Assume that $\ell = p$. If ρ comes from geometry, then the L-function $L(U,\rho,T)$ is p-adic meromorphic in the whole p-adic plane $|T|_p < \infty$.*

The aim of this paper is to study Artin's entireness conjecture for such L-functions of geometric p-adic representations. Our main result is the following theorem.

Theorem 1.6. *Assume that $\ell = p$ and ρ comes from geometry with no geometrically trivial components. Then, there is a positive constant $c(p,\rho)$ such that the L-function $L(U,\rho,T)$ is p-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p,\rho)$. Furthermore, there are geometrically non-trivial rank* *one geometric p-adic representations ρ such that $L(U,\rho,T)$ is not p-adic analytic (in fact having infinitely many poles) in $|T|_p < \infty$.*

The second part of the theorem shows that Artin's conjecture is false in the entire plane $|T|_p < \infty$. It shows that the first part of the theorem is best one can hope for, and Artin's conjecture is true in a larger disk than the closed unit disk for geometric p-adic representations. An interesting further question is how big the constant $c(p,\rho)$ can be. Our proof gives an explicit positive constant depending only on p and some embedding rank of ρ. If ρ comes from the slope zero part of an ordinary overconvergent F-crystal on U and the uniformizer of R_p is p, one can take $c(p,\rho) = p-1$ which is independent of ρ.

2. ℓ-adic case: $\ell \neq p$

Since $\ell \neq p$, the restriction of the ℓ-adic representation ρ to P_x is of finite order and thus the representation ρ is almost tame. In fact, by class field theory, ρ itself has finite order up to a twist if ρ has rank one. Thus, there are not too many such ℓ-adic representations. The L-function $L(U,\rho,T)$ is always a rational function. This follows from Grothendieck's trace formula [9]:

Theorem 2.1. *Let $\mathcal{F}_\rho$ denote the lisse ℓ-adic sheaf on U associated with ρ. Then, there are finite dimensional vector spaces $H_c^i(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho)$ $(i = 0,1,2)$ over F_ℓ such that*

$$L(U,\rho,T) = \prod_{i=0}^{2} \det(I - \mathrm{Frob}_q T | H_c^i(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho))^{(-1)^{i-1}} \in F_\ell(T).$$

If U is affine, then $H_c^0 = 0$. If ρ does not contain a geometrically trivial component, then $H_c^2 = 0$. Thus, in most cases, it is H_c^1 that is the most interesting.

Corollary 2.2. *Let U be affine. Assume that ρ does not contain a geometrically trivial component. Then, the L-function*

$$L(U,\rho,T) = \det(I - \mathrm{Frob}_q T | H_c^1(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho))$$

is a polynomial.

This is the ℓ-adic function field analogue of Artin's entireness conjecture.

Fix an embedding $\iota : \bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. A representation ρ is called ι-pure of weight $w \in \mathbb{R}$ if each eigenvalue of Frob_x acting on V has absolute value $q^{\deg(x)w/2}$ for all $x \in |U|$. A representation ρ is called ι-mixed of weights at most w if each irreducible subquotient of ρ is ι-pure of weights at most w. If ρ is ι-pure of weight w for every embedding ι, then ρ is called pure of weight w. Similarly, if ρ is ι-mixed of weights at most w for every ι, then ρ is called mixed of weights at most w. The fundamental theorem of Deligne [4] on the Weil conjectures implies

Theorem 2.3. *If ρ is geometric, then ρ is mixed with integral weights. Furthermore, if ρ is mixed of weights at most w, then $H^i_c(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho)$ is mixed of weights at most $w + i$.*

The ℓ-adic function field Langlands conjecture for $\mathrm{GL}(n)$, which was established by Lafforgue [13], implies

Theorem 2.4. *If ρ is irreducible, then ρ is geometric up to a twist and hence pure of some weight.*

Thus, in the ℓ-adic case with $\ell \neq p$, essentially all ℓ-adic representations are geometric from the viewpoint of L-functions.

3. *p*-adic case

In the case $\ell = p$, the restriction of the p-adic representation ρ to P_x can be infinite and thus ρ can be very wildly ramified. The L-function $L(U, \rho, T)$ is naturally more complicated and cannot be rational in general. One can ask for its p-adic meromorphic continuation. The function $L(U, \rho, T)$ is trivially p-adic analytic in the open unit disc $|T|_p < 1$ as the coefficients are in the ring R_p. It was shown in [15] that $L(U, \rho, T)$ is not p-adic meromorphic in general, disproving a conjecture of Katz [11]. However, one can show that $L(U, \rho, T)$ is p-adic meromorphic on the closed unit disc $|T|_p \leq 1$. Its zeros and poles on the closed unit disc are controlled by p-adic étale cohomology of ρ. This was proved by Emerton-Kisin [8], confirming a conjecture of Katz [11]. That is,

Theorem 3.1. *For any p-adic representation ρ of $\pi_1^{\mathrm{arith}}(U)$, the quotient*

$$\frac{L(U, \rho, T)}{\prod_{i=0}^{2} \det(I - \mathrm{Frob}_q T | H^i_c(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho))^{(-1)^{i-1}}}$$

has no zeros and poles on the closed unit disc $|T|_p \leq 1$.

In the case that ρ has rank one, this was first proved by Crew [3]. Note that $H^2_c(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho) = 0$ since U is a curve and $\ell = p$. If U is affine, then $H^0_c(U \otimes \bar{\mathbb{F}}_q, \mathcal{F}_\rho) = 0$. This gives

Corollary 3.2. *Let U be affine. Then, the L-function $L(U, \rho, T)$ is p-adic analytic on the closed unit disc $|T|_p \leq 1$.*

The (compatible) p-adic analogue of a lisse ℓ-adic sheaf (or ℓ-adic representation) on U for $\ell \neq p$ is an overconvergent F-isocrystal over U, which is not a p-adic representation. Its pure slope parts, under the Newton-Hodge decomposition, are p-adic representations up to twists (unit root F-isocrystals, no longer overconvergent in general). P-adic representations arising in this way are also called *geometric*, as they are a natural generalization of the geometric representations we defined before. For geometric p-adic representations, the following meromorphic continuation was conjectured by Dwork [6] and proved by the second author [16] [17].

Theorem 3.3. *If the p-adic representation ρ is geometric, then the L-function $L(U, \rho, T)$ is p-adic meromorphic everywhere.*

Remark 3.4. It would be interesting to know if a sub-quotient of a geometric p-adic representation remains geometric in terms of our general definition.

Unlike the ℓ-adic case, most p-adic representations are not geometric. It seems very difficult to classify geometric p-adic representations, even in the rank one case. This may be viewed as the p-adic Langlands program for function fields of characteristic p, which is still wide open[3].

Our first new result of this paper is to show that the Artin entireness conjecture fails for L-functions $L(U, \rho, T)$ of geometric p-adic representations, even for non-trivial rank one ρ.

Theorem 3.5. *There are geometrically non-trivial rank one geometric p-adic representations ρ on certain affine curves U over $\mathbb{F}_p$ such that the L-function $L(U, \rho, T)$ is p-adic meromorphic on $|T|_p < \infty$, but having infinitely many poles.*

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[3]The p-adic Langlands correspondence for overconvergent F-isocrystals was recently studied by Tomoyuki Abe. This is compatible with the ℓ-adic situation with $\ell \neq p$. In our framework of convergent geometric unit root F-isocrystals, no one knows how to formulate the correspondence yet, even in the rank one case. In fact, our negative result below on the Artin entireness conjecture suggests something completely new happens in this new situation.

Proof. Let $p > 2$ be an odd prime and $N > 4$ be a positive integer prime to p. Let Y be the component of ordinary non-cuspidal locus of the modulo p reduction of the compactified modular curve $X_1(Np)$. This is an affine curve over the finite field $\mathbb{F}_p$. Let $E_1(Np)$ be the universal elliptic curve over Y. Its relative p-adic étale cohomology is a rank one geometric p-adic representation ρ of $\pi_1^{\mathrm{arith}}(Y)$. For a non-zero integer k, the k-th tensor power $\rho^{\otimes k}$ is again a geometric p-adic representation of $\pi_1^{\mathrm{arith}}(Y)$. The Monsky trace formula gives the following relation

$$L(Y, \rho^{\otimes k}, T) = \frac{D(k+2, T)}{D(k, pT)}, \tag{1}$$

where $D(k,T)$ is the characteristic power series of the U_p-operator acting on the space of overconvergent p-adic cusp forms of weight k and tame level N. The series $D(k,T)$ is a p-adic entire function. Equation (1) implies that the L-function $L(Y, \rho^{\otimes k}, T)$ is p-adic meromorphic in T, which was first proved by Dwork in [5] via Monsky's trace formula, see also [12] and [2].

We want to show that the L-function $L(Y, \rho^{\otimes k}, T)$ is not p-adic entire for infinitely many integers k. For this purpose, we need to describe the coefficients of the L-function in more detail, following Coleman [2, Appendix I]. In the following, we fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

For an order $\mathcal{O}$ in a number field, let $h(\mathcal{O})$ denote the class number of $\mathcal{O}$. If γ is an algebraic integer, let O_γ be the set of orders in $\mathbb{Q}(\gamma)$ containing γ. For a positive integer m, let $W_{p,m}$ denote the finite set of p-adic units $\gamma \in \mathbb{Q}_p$ such that $\mathbb{Q}(\gamma)$ is an imaginary quadratic field, γ is an algebraic integer and

$$\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\gamma)}(\gamma) = p^m.$$

By Coleman [2, Theorem I1], for all integers k, we have

$$D(k,T) = \exp\left(\sum_{m=1}^{\infty} A_m(k) \frac{T^m}{m}\right),$$

where

$$A_m(k) = \sum_{\gamma \in W_{p,m}} \sum_{\mathcal{O} \in O_\gamma} h(\mathcal{O}) B_N(\mathcal{O}, \gamma) \frac{\gamma^k}{\gamma^2 - p^m},$$

and $B_N(\mathcal{O}, \gamma)$ is the number of elements of $\mathcal{O}/N\mathcal{O}$ of order N fixed under multiplication by p^m/γ. This is really another form of the Monsky trace formula. It follows that

$$L(Y, \rho^{\otimes k}, T) = \exp\left(\sum_{m=1}^{\infty} C_m(k)\frac{T^m}{m}\right),$$

where

$$C_m(k) = A_m(k+2) - A_m(k)p^m = \sum_{\gamma \in W_{p,m}} \sum_{\mathcal{O} \in O_\gamma} h(\mathcal{O}) B_N(\mathcal{O}, \gamma)\gamma^k.$$

It is clear that $C_m(k)$ is an algebraic number in $\mathbb{Q}_p$. To proceed, we first recall the following basic fact regarding linear independence of square roots of integers.

Lemma 3.6. Let $1 \le n_1 < n_2 < \cdots < n_l$ be square free integers. Suppose $a_1, \ldots, a_l \in \mathbb{Q}$. If $a_i \neq 0$ for some i, then $a_1\sqrt{n_1} + \cdots + a_l\sqrt{n_l} \neq 0$.

Proof. Suppose $p_1, \ldots, p_s$ are the prime factors of all n_i. Then we may write

$$a_1\sqrt{n_1} + \cdots + a_l\sqrt{n_l} = f(\sqrt{p_1}, \ldots, \sqrt{p_s}),$$

where $f(y_1, \ldots, y_s)$ is a polynomial with rational coefficients of degree at most 1 with respect to every y_j. Now the lemma follows from the main result of [1]. $\square$

Corollary 3.7. *Keep notations as in Lemma 3.6. Suppose none of* $a_1, \ldots, a_l$ *are* 0. *Then*

$$\mathbb{Q}(\sqrt{n_1}, \ldots, \sqrt{n_l}) = \mathbb{Q}(a_1\sqrt{n_1} + \cdots + a_l\sqrt{n_l}).$$

Proof. Put $c = a_1\sqrt{n_1} + \cdots + a_l\sqrt{n_l}$. We need to show that $\sqrt{n_i} \in \mathbb{Q}(c)$ for every i. If not, without loss of generality, we may suppose $\sqrt{n_1} \notin \mathbb{Q}(c)$. Then there exists an automorphism $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(c) = c$ and $\sigma(\sqrt{n_1}) = -\sqrt{n_1}$. Putting these together, we will obtain a non-trivial linear relation of $\sqrt{n_1}, \ldots, \sqrt{n_l}$ over $\mathbb{Q}$; this contradicts with the above lemma. $\square$

We need the following key property.

Lemma 3.8. For $k \ge 1$, the field generated by all the algebraic numbers $C_m(k)$ in $\mathbb{Q}_p$ is equal to the compositum of all imaginary quadratic fields in $\mathbb{Q}_p$ in which p splits. In particular, this field is an infinite algebraic extension of $\mathbb{Q}$ in $\mathbb{Q}_p$.

Proof. Let γ be a p-adic unit such that $\mathbb{Q}(\gamma)$ is an imaginary quadratic field and $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\gamma)}(\gamma) = p^m$. It is clear that in this case p splits in $\mathbb{Q}(\gamma)$. Thus $C_m(k)$ is contained in the compositum of all imaginary quadratic fields in $\mathbb{Q}_p$ in which p splits.

Conversely, let K be any imaginary quadratic field in $\mathbb{Q}_p$ in which p splits. Write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Without loss of generality, we may suppose $\bar{\mathfrak{p}} = p\mathbb{Z}_p \cap \mathcal{O}_K$. Write $m = h(\mathcal{O}_K)$, then $\mathfrak{p}^m = (\gamma)$ is a principal ideal, and $\bar{\mathfrak{p}}^m = (\bar{\gamma})$. It follows that $\mathrm{Norm}_{\mathbb{Q}}^{\mathbb{Q}(\gamma)}(\gamma) = \gamma\bar{\gamma} = p^m$ and γ is a p-adic unit. By replacing γ with γ^n and m with mn for some suitable positive integer n, we may further suppose that $\bar{\gamma} \equiv 1 \mod N\mathcal{O}_K$. In particular, we have $B_N(\mathcal{O}, \gamma) > 0$ for any $\mathcal{O} \in O_\gamma$.

We claim that for $\gamma' \in K \cap W_{p,m}$, if $B_N(\mathcal{O}, \gamma') > 0$ for some $\mathcal{O} \in O_{\gamma'}$, then $\gamma' = \gamma$. In fact, since $\mathrm{Norm}_{\mathbb{Q}}^{K}(\gamma') = p^m$, we first have that $(\gamma') = \mathfrak{p}^i\bar{\mathfrak{p}}^j$ with $i + j = m$. Note thtat γ' is a p-adic unit; this yields $j = 0$. That is $(\gamma) = (\gamma')$. Thus we may write $\gamma' = u\gamma$ for some unit u.

Now choose $a \in \mathcal{O}$ such that its image in $\mathcal{O}/N\mathcal{O}$ is of order N and fixed by $\bar{\gamma'}$. Since $\bar{\gamma} \equiv 1 \mod N\mathcal{O}_K$, we get that the product $(\bar{u} - 1)a$ is zero in the quotient group $\mathcal{O}/N\mathcal{O}$. From the assumption that a has order exactly N, one deduces that for every prime l dividing N, the l-adic valuation $v_l(\bar{u} - 1) \geq v_l(N)$. It follows that if u is not 1, then $(u - 1)(\bar{u} - 1) \geq N$. Since $u\bar{u} = 1$, this gives the inequality $4 < N \leq (u - 1)(\bar{u} - 1) \leq 4$ as u has complex norm 1, which is a contradiction if u is not 1!

Consequently, we may write

$$C_m(k) = \left(\sum_{\mathcal{O} \in O_\gamma} h(\mathcal{O}) B_N(\mathcal{O}, \gamma) \right) \gamma^k + \alpha$$

where α is a sum of elements contained in quadratic fields different from K. Since γ^k and $\bar{\gamma}^k$ has different p-adic valuation, we deduce that $\gamma^k \notin \mathbb{Q}$. Thus $\mathbb{Q}(\gamma) = \mathbb{Q}(\gamma^k)$. By the above corollary, we therefore conclude that $K = \mathbb{Q}(\gamma^k)$ is contained in the field generated by $C_m(k)$. This yields the lemma. □

We now return to the proof of the theorem. Let $k \geq 1$ be a positive integer. Let $\mathcal{F}$ denote the relative rigid cohomology of $E_1(Np)$ over Y, which is an ordinary overconvergent F-isocrystal over Y of rank two, sef-dual and pure of weight 1. The rank one p-adic representation ρ is precisely the unit root part of $\mathcal{F}$. It follows that the L-function of the k-th Adams operation of $\mathcal{F}$ is

$$L(Y, \rho^{\otimes k}, T)L(Y, \rho^{\otimes(-k)}, p^k T) = \frac{L(Y, \mathrm{Sym}^k \mathcal{F}, T)}{L(Y, \mathrm{Sym}^{k-2} \mathcal{F}, pT)}.$$

The right side is a rational function with integer coefficients. If both $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes(-k)}, T)$ had a finite number of poles, then the above left side would be a p-adic meromorphic function with a finite number of poles, and it is at the same time a rational function. It would then follow that both $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes(-k)}, T)$ have a finite number of zeros and thus both would be rational functions. This implies that the coefficients of $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes(-k)}, T)$ generate a finite algebraic extension of $\mathbb{Q}$ in $\mathbb{Q}_p$, contradicting to the lemma. We conclude that at least one of the two functions $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes(-k)}, T)$ has infinitely many poles. The theorem is proved. □

Remark 3.9. For any positive integer $k \geq 1$, we believe that both functions $L(Y, \rho^{\otimes k}, T)$ and $L(Y, \rho^{\otimes(-k)}, T)$ have infinitely many poles. But we do not know how to prove it.

Remark 3.10. In the analogous setting of the family of Kloosterman sums, the unit root L-function is again expected to be non-rational, but this remains unknown at present, see page 4 in [10].

Our second result of this paper is to show that for a geometric p-adic representation ρ on a smooth affine curve U over $\mathbb{F}_p$, the L-function $L(U, \rho, T)$ is p-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$ for some positive constant $c(p, \rho)$. In fact, we shall prove a more general theorem in the context of σ-modules as in [16][17]. For simplicity of notation, we use $L(\rho, T)$ to denote $L(U, \rho, T)$.

Theorem 3.11. *Let U be a smooth affine curve over $\mathbb{F}_q$. Let ρ be a unit root σ-module which arises as a pure slope part of an overconvergent σ-module on U. Then, there is a positive constant $c(p, \rho)$ such that the L-function $L(\rho, T)$ is p-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$.*

Proof. Let ϕ be an overconvergent σ-module on U with coefficients in R_p with uniformizer π. Since ϕ is overconvergent, Corollary 3.2 in [17] shows that its L-function $L(\phi, T)$ is p-adic meromorphic everywhere. As U is a smooth affine curve, Corollary 3.3 in [17] further shows that $L(\phi, T)$ is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. Note that $|\pi^{-1}|_p$ is a constant greater than 1. For example, in the case $\pi = p$, we have $|\pi^{-1}|_p = p$.

We first assume that ϕ is ordinary. For an integer $i \geq 0$, let ϕ_i denote the unit root σ-module on U coming from the slope i-part in the Hodge-Newton decomposition of ϕ. It is no longer overconvergent in general. We need to show that the unit root σ-module L-function $L(\phi_i, T)$ is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. By the definition of ϕ_i and our ordinariness assumption, we have the decomposition

$$L(\phi, T) = \prod_{i \geq 0} L(\phi_i, \pi^i T) = L(\phi_0, T) \prod_{i \geq 1} L(\phi_i, \pi^i T).$$

As mentioned above, the left side is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. For each $i \geq 1$, the right side factor $L(\phi_i, \pi^i T)$ is trivially p-adic analytic with no zeros and poles in the disk $|T|_p < |\pi^{-1}|_p$. We deduce that the first right side factor $L(\phi_0, T)$ is also p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. This proves the theorem in the case $i = 0$.

For $i > 0$, we need to use the proof of Dwork's conjecture in [16][17]. Let $\psi = \phi_i$. We need to prove that $L(\psi, T)$ is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. Let r_i denote the rank of ϕ_i. Define

$$\tau = \wedge^{r_0} \phi_0 \otimes \wedge^{r_1} \phi_1 \otimes \cdots \otimes \wedge^{r_{i-1}} \phi_{i-1}.$$

This is a rank one unit root σ-module on U, not overconvergent in general. Define

$$\varphi = \pi^{-r_1 - \cdots - (i-1) r_{i-1} - i} \wedge^{r_0 + \cdots + r_{i-1} + 1} \phi.$$

Since ϕ is ordinary and overconvergent, it follows that φ is also ordinary and overconvergent. For an integer $j \geq 0$, let φ_j denote the unit root σ-module on U coming from the slope j-part in the Hodge-Newton decomposition of φ. Then, it is easy to check that we have the following decomposition (see equation (5.1) in [17]).

$$L(\varphi \otimes \tau^{-1}, T) = L(\psi, T) \prod_{j \geq 1} L(\varphi_j \otimes \tau^{-1}, \pi^j T).$$

For each $j \geq 1$, the factor $L(\varphi_j \otimes \tau^{-1}, \pi^j T)$ is trivially p-adic analytic with no zeros and poles in the disk $|T|_p < |\pi^{-1}|_p$. To prove that $L(\psi, T)$ is also p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$, it suffices to prove that the left side factor $L(\varphi \otimes \tau^{-1}, T)$ is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$.

Now, the rank one unit root σ-module τ is the slope zero part of the following ordinary and overconvergent σ-module

$$\Phi = \pi^{-r_1-\cdots-(i-1)r_{i-1}} \wedge^{r_0+\cdots+r_{i-1}} \phi.$$

By Theorem 7.8 in [17], we deduce that there is a sequence of nuclear overconvergent σ-modules $\Phi_{\infty,-k}$ $(k \geq 2)$ such that

$$L(\varphi \otimes \tau^{-1}, T) = \prod_{k \geq 1} L(\varphi \otimes \Phi_{\infty,-1-k} \otimes \wedge^k \Phi, T)^{(-1)^{k-1}k}.$$

Since Φ is ordinary and its slope zero part has rank one, $\wedge^k \Phi$ is divisible by π^{k-1}. It follows that for $k \geq 2$, the L-function $L(\varphi \otimes \Phi_{\infty,-1-k} \otimes \wedge^k \Phi, T)$ is trivially p-adic analytic with no zeros and poles in the disk $|T|_p < |\pi^{-1}|_p$. For the remaining case $k = 1$, we apply the one dimensional case of the following n-dimensional integrality result and deduce that $L(\varphi \otimes \Phi_{\infty,-2} \otimes \Phi, T)$ is p-adic analytic in the disk $|T|_p < |\pi^{-1}|_p$. The theorem is proved in the ordinary case.

Lemma 3.12. *Let U be a smooth affine variety of equi-dimension n over $\mathbb{F}_q$. Let ϕ be an overconvergent nuclear σ-module on U. Then, the L-function $L(\phi, T)^{(-1)^{n-1}}$ is p-adic analytic (no poles) in the disc $|T|_p < |\pi^{-1}|_p$.*

Proof. In the case that ϕ has finite rank, this integrality is already proved in Corollary 3.3 in [17]. In this case, the finite rank Monsky trace formula (Theorem 3.1 in [17]) states that

$$L(\phi, T)^{(-1)^{n-1}} = \prod_{i=0}^{n} \det(I - \phi_i^* T | M_i^* \otimes_R K)^{(-1)^i},$$

where $R = R_p$ in our current notation and $\det(I - \phi_i^* T | M_i^* \otimes_R K) \in 1 + TR[[T]]$ is a p-adic entire function. Now, the divisibility $\phi_i^* \equiv 0 \pmod{\pi^i}$ (equation (3.4) in [17]) shows that

$$\det(I - \phi_i^* T | M_i^* \otimes_R K) \in 1 + \pi^i TR[[\pi^i T]].$$

This implies the integrality in the finite rank case. For infinite rank nuclear overconvergent σ-modules, the proof is the same. One simply uses the infinite rank nuclear overconvergent trace formula (Theorem 5.8 in [16]):

$$L(\phi, T)^{(-1)^{n-1}} = \prod_{i=0}^{n} \det(I - \Theta_i T | M_i^* \otimes_R K)^{(-1)^i}.$$

Note that there is a misprint of indices in Theorem 5.8 in [16]: $\det(I - \Theta_i T | M_i^* \otimes_R K)$ there should be $\det(I - \Theta_{n-i} T | M_{n-i}^* \otimes_R K)$, compare the finite rank case (Theorem 3.1 in [17]). Now, one uses the same divisibility $\Theta_i \equiv 0 \pmod{\pi^i}$, which follows from the definition of Θ_i (Definition 5.5 in [16]) and the fact that $\phi_i = \phi \otimes \sigma_i$ is divisible by π^i.

□

In the general non-ordinary case, by a similar argument, we may apply the methods in [16][17] for non-ordinary case to give an explicit positive constant $c(p, \rho)$ depending on π and the rank of ϕ such that $L(\rho, T)$ is p-adic analytic in the disk $|T|_p < 1 + c(p, \rho)$. The constant $c(p, \rho)$ depends very badly on the rank of ϕ, and so we would not bother to write it down explicitly. □

The above theorem has a higher dimensional generalization. We state this generalization below.

Remark 3.13. Let U be a smooth affine variety of equi-dimension n over $\mathbb{F}_q$. Let ρ be a unit root σ-module on U. Then, the L-function $L(\rho, T)^{(-1)^{n-1}}$ is p-adic analytic on the closed unit disk $|T|_p \leq 1$. If ρ arises as a pure slope part of an overconvergent σ-module on U. Then, there is a positive constant $c(p, \rho)$ such that the L-function $L(\rho, T)^{(-1)^{n-1}}$ is p-adic analytic (no poles) in the larger disc $|T|_p < 1 + c(p, \rho)$.

The first part follows from Emerton-Kisin's theorem on the Katz conjecture and standard properties of p-adic étale cohomology. The proof of the second part is the same as the above theorem, and use the results in [16][17]. We expect that both parts remain true if U is an equi-dimensional complete intersection (possibly singular) in a smooth affine variety X over $\mathbb{F}_q$. This possible generalization is motivated by the characteristic p entireness result in [14].

Remark 3.14. In the special case that U is the compliment of a hypersurface and the Frobenius lifting is the q-th power lifting of the coordinates, the result of Dwork-Sperber [7] can be used to prove the p-adic analytic continuation of $L(\rho, T)^{(-1)^{n-1}}$ for geometric ordinary ρ in the open disc $|T|_p < p^{(p-1)/(p+1)}$. This result is weaker than our result since the disc $|T|_p < p^{(p-1)/(p+1)}$ is smaller than the disc $|T|_p < p$ obtained in our approach.

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