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Regularity of envelopes in Kähler classes

Valentino Tosatti

We establish the $C^{1,1}$ regularity of quasi-psh envelopes in a Kähler class, confirming a conjecture of Berman.

1. Introduction

Let (X^n, ω) be a compact Kähler manifold, and $\theta = \omega + \sqrt{-1}\partial\overline{\partial}v$ a closed real $(1,1)$-form cohomologous to ω, where $v \in C^\infty(X, \mathbb{R})$. The envelope (or extremal function) u_θ is defined by

$$\begin{aligned} u_\theta(x) &= \sup\{u(x) \mid u \in PSH(X, \theta), u \leqslant 0\} \\ &= -v + \sup\{u(x) \mid u \in PSH(X, \omega), u \leqslant v\}, \end{aligned}$$

is a θ-psh function with minimal singularities in the class $[\omega]$, and has received much attention recently (see for example [6, 7, 14] and references therein). By Berman-Demailly [8], we know that the complex Hessian (or equivalently the Laplacian) of u_θ belongs to $L^\infty(X)$, and so in particular u_θ is $C^{1,\alpha}(X)$ for all $0 < \alpha < 1$. A direct PDE proof was given by Berman in [4].

Here we establish the optimal regularity result for the envelope, which was previously only known when $[\omega] \in H^2(X, \mathbb{Q})$ by [3] (see also [5]). This resolves affirmatively a conjecture of Berman [3, Conjecture 1.10]:

Theorem 1.1. *The envelope u_θ is in $C^{1,1}(X)$.*

This is in general optimal, see e.g. [3, Example 5.2] for examples on toric manifolds.

In fact, combining our result with the arguments in [12, Proof of Theorem 2.5], we obtain the same $C^{1,1}$ regularity result for the "rooftop envelopes"

$$P(v_1, \dots, v_k)(x) = \sup\left\{u(x) \mid u \in PSH(X, \omega), u \leqslant \min_{j=1,\dots,k} v_j\right\},$$

where the v_j's are $C^{1,1}$ functions, see Theorem 3.1 below.

Also, using Theorem 1.1 together with the arguments in [3, Theorem 3.4], we obtain a shlightly shorter proof of the identity

$$(\theta + \sqrt{-1}\partial\overline{\partial}u_\theta)^n = \chi_{\{u_\theta=0\}}\theta^n, \tag{1.1}$$

which clearly implies

$$\int_{\{u_\theta=0\}} \theta^n = \int_X \omega^n, \tag{1.2}$$

and which was proved (in more generality) in [8, Corollary 2.5]. Indeed it is classical that the Monge-Ampère operator $(\theta + \sqrt{-1}\partial\overline{\partial}u_\theta)^n$ vanishes outside the contact set $\{u_\theta = 0\}$ (see e.g. [3, Proposition 3.1] or [6, Proposition 2.10]), and by Theorem 1.1 we know that $\nabla_i u_\theta$ is Lipschitz (for any $1 \leqslant i \leqslant n$, working in a local coordinate chart) and so $\nabla\nabla_i u_\theta = 0$ a.e. on the set $\{\nabla_i u_\theta = 0\}$ (see e.g. [1, Theorem 3.2.6]), which contains the contact set. Therefore a.e. on the contact set we have $\nabla^2 u_\theta = 0$ and so $\theta + \sqrt{-1}\partial\overline{\partial}u_\theta = \theta$, which proves (1.1).

The proof of the Theorem 1.1, which is given in section 2, is obtained by using Berman's result [4] that the envelope u_θ is in fact the limit of solutions of a 1-parameter family of complex Monge-Ampère equations, together with the technique recently introduced by Chu, Weinkove and the author [9, 10] to obtain uniform $C^{1,1}$ estimates for such equations. A generalization of this result to "rooftop envelopes" (in the sense of [12]) is proved in section 3.

After the first version of this paper was posted on the arXiv, we were informed that J. Chu and B. Zhou independently proved Theorem 1.1 in [11].

2. $C^{1,1}$ regularity of envelopes

In this section we give the proof of Theorem 1.1. Following the approach of [4], we consider the family of complex Monge-Ampère equations

$$(\theta + \sqrt{-1}\partial\overline{\partial}u_\beta)^n = e^{\beta u_\beta}\omega^n, \tag{2.1}$$

where $\beta \in \mathbb{R}_{\geqslant 0}$, the function u_β is smooth and $\theta + \sqrt{-1}\partial\overline{\partial}u_\beta$ is a Kähler metric on X. This is solvable thanks to the work of Aubin [2] and Yau [15]. Recall also that we write $\theta = \omega + \sqrt{-1}\partial\overline{\partial}v$ for a smooth function v.

Berman shows in [4] that

$$|u_\beta| \leqslant C, \quad |\Delta_g u_\beta| \leqslant C, \quad |u_\beta - u_\theta| \leqslant C\frac{\log\beta}{\beta}, \tag{2.2}$$

for a uniform constant C independent of β (and which depends only on the $C^{1,1}$ norm of v), from which it follows that u_β converges to u_θ in $C^{1,\alpha}(X)$ for any $0 < \alpha < 1$, as $\beta \to \infty$.

Our main result is that for all $\beta \in \mathbb{R}_{\geqslant 0}$ we have

$$|\nabla^2 u_\beta|_g \leqslant C, \tag{2.3}$$

for a uniform C, which immediately implies Theorem 1.1. As will be apparent from the proof, the constant C depends only on the $C^{1,1}$ norm of v. Let $\varphi = u_\beta + v$ and rewrite (2.1) as

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\beta(\varphi - v)}\omega^n, \tag{2.4}$$

where $\tilde{\omega} := \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is a Kähler metric, and the idea is to follow very closely the method introduced by Chu, Weinkove and the author in [9, 10]. We thus let $\lambda_1(\nabla^2\varphi)$ be the largest eigenvalue of $\nabla^2\varphi$ with respect to g, and the goal is to prove that $\lambda_1(\nabla^2\varphi) \leqslant C$ for a uniform constant C. Indeed, once we prove this, since the trace of $\nabla^2\varphi$ is $\Delta_g\varphi$ which is bounded below by $-n$, we will conclude that

$$|\nabla^2\varphi|_g \leqslant C,$$

which implies (2.3). To this end, we apply the maximum principle to

$$Q = \log\lambda_1(\nabla^2\varphi) + h(|\partial\varphi|^2_g) - A\varphi,$$

(defined on the set where $\lambda_1(\nabla^2\varphi) > 0$, which we may assume is nonempty) where $A > 0$ is a uniform constant to be determined and

$$h(s) = -\frac{\lambda}{2}\log\left(1 + \sup_M |\partial\varphi|^2_g - s\right), \tag{2.5}$$

where $\lambda = (1 + 2\sup_X |\partial v|^2_g)^{-1} \leqslant 1$, is a small uniform constant. The only difference between this quantity and the corresponding one in [10] is that there we just took $\lambda = 1$. We have

$$\frac{\lambda}{2} \geqslant h' \geqslant \frac{\lambda}{2 + 2\sup_M |\partial\varphi|^2_g} > 0, \quad \text{and} \quad h'' = \frac{2}{\lambda}(h')^2 \geqslant 2(h')^2, \tag{2.6}$$

where we are evaluating h and its derivatives at $|\partial\varphi|^2_g$. The bounds (2.2) show that the the last two terms in Q are uniformly bounded.

We work at a point x_0 where the maximum is achieved, and as in [10] we choose local normal coordinates for g near x_0, so that $(\tilde{g}_{i\bar{j}})(x_0)$ is diagonal, as well as constant vector fields $\{V_\alpha\}$ near x_0 which at that point form an orthonormal basis of eigenvectors of $\nabla^2\varphi$, with $\nabla^2\varphi(V_1, V_1)(x_0) = \lambda_1$.

We also apply the same perturbation argument as in [10], so that Q gets replaced by the local quantity $\hat{Q}$ defined near x_0 as in [10] by

$$\hat{Q} = \log \lambda_1(\Phi) + h(|\partial\varphi|^2_g) - A\varphi,$$

where Φ is the endomorphism of TX given by

$$\Phi^\mu_\nu = g^{\mu\gamma}(\nabla^2_{\gamma\nu}\varphi - \delta_{\gamma\nu} + V_1^\gamma V_1^\nu),$$

where (V_1^ν) are the components of V_1. The largest eigenvalue $\lambda_1(\Phi)$ now varies smoothly near x_0 and $\hat{Q}$ achieves a local maximum at that point. Writing $\lambda_\alpha = \lambda_\alpha(\Phi)$, the goal is to show that $\lambda_1(x_0) \leqslant C$, for a uniform constant C. We claim that at x_0 we have

$$\begin{aligned} 0 \geqslant \Delta_{\tilde{g}}\hat{Q} \geqslant\ & 2\sum_{\alpha>1} \frac{\tilde{g}^{i\bar{i}}|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}|\partial_i(\varphi_{V_1V_1})|^2}{\lambda_1^2} \\ & + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) + \beta h'|\partial\varphi|^2_g + h''\tilde{g}^{i\bar{i}}|\partial_i|\partial\varphi|^2_g|^2 \\ & + (A - C)\sum_i \tilde{g}^{i\bar{i}} - An + \frac{\beta}{4} \\ \geqslant\ & 2\sum_{\alpha>1} \frac{\tilde{g}^{i\bar{i}}|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1(\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}^{p\bar{p}}\tilde{g}^{q\bar{q}}|V_1(\tilde{g}_{p\bar{q}})|^2}{\lambda_1} - \frac{\tilde{g}^{i\bar{i}}|\partial_i(\varphi_{V_1V_1})|^2}{\lambda_1^2} \\ & + h'\sum_k \tilde{g}^{i\bar{i}}(|\varphi_{ik}|^2 + |\varphi_{i\bar{k}}|^2) + h''\tilde{g}^{i\bar{i}}|\partial_i|\partial\varphi|^2_g|^2 \\ & + (A - C)\sum_i \tilde{g}^{i\bar{i}} - An. \end{aligned} \tag{2.7}$$

Indeed, as in [10, (2.7)] we have

$$\Delta_{\tilde g}\hat Q = \frac{\Delta_{\tilde g}(\lambda_1)}{\lambda_1} - \frac{\tilde g^{i\bar i}|\partial_i(\varphi_{V_1V_1})|^2}{\lambda_1^2} + h'\Delta_{\tilde g}(|\partial\varphi|^2_g) + h''\tilde g^{i\bar i}|\partial_i|\partial\varphi|^2_g|^2 + A\sum_i \tilde g^{i\bar i} - An, \tag{2.8}$$

and as in [10, (2.8)]

$$\Delta_{\tilde g}(\lambda_1) \geqslant 2\sum_{\alpha>1}\tilde g^{i\bar i}\frac{|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1-\lambda_\alpha} + \tilde g^{i\bar i}V_1V_1(\tilde g_{i\bar i}) - C\lambda_1\sum_i \tilde g^{i\bar i}. \tag{2.9}$$

The Monge-Ampère equation (2.4) in local coordinates reads

$$\log\det\tilde g = \log\det g + \beta\varphi - \beta v, \tag{2.10}$$

and so applying V_1V_1 to this and evaluating at x_0 we obtain

$$\begin{aligned}\tilde g^{i\bar i}V_1V_1(\tilde g_{i\bar i}) &= \tilde g^{p\bar p}\tilde g^{q\bar q}|V_1(\tilde g_{p\bar q})|^2 + V_1V_1(\log\det g)\\ &\quad + \beta V_1V_1(\varphi) - \beta V_1V_1(v)\\ &= \tilde g^{p\bar p}\tilde g^{q\bar q}|V_1(\tilde g_{p\bar q})|^2 + V_1V_1(\log\det g) + \beta\lambda_1 - \beta V_1V_1(v)\\ &\geqslant \tilde g^{p\bar p}\tilde g^{q\bar q}|V_1(\tilde g_{p\bar q})|^2 + V_1V_1(\log\det g) + \beta(\lambda_1 - C)\\ &\geqslant \tilde g^{p\bar p}\tilde g^{q\bar q}|V_1(\tilde g_{p\bar q})|^2 + V_1V_1(\log\det g) + \frac{\beta}{2}\lambda_1,\end{aligned} \tag{2.11}$$

since we may assume that at x_0 the largest eigenvalue λ_1 is large. This gives

$$\Delta_{\tilde g}(\lambda_1) \geqslant 2\sum_{\alpha>1}\tilde g^{i\bar i}\frac{|\partial_i(\varphi_{V_\alpha V_1})|^2}{\lambda_1-\lambda_\alpha} + \tilde g^{p\bar p}\tilde g^{q\bar q}|V_1(\tilde g_{p\bar q})|^2 - C\lambda_1\sum_i\tilde g^{i\bar i} + \frac{\beta}{2}\lambda_1. \tag{2.12}$$

Next, at x_0,

$$\begin{aligned}(2.13)\quad \Delta_{\tilde g}(|\partial\varphi|^2_g) &= \sum_k \tilde g^{i\bar i}(|\varphi_{ik}|^2+|\varphi_{i\bar k}|^2)+2\beta\mathrm{Re}\left(\sum_k \varphi_k(\varphi-v)_{\bar k}\right)\\ &\quad + \tilde g^{i\bar i}\partial_i\partial_{\bar i}(g^{k\bar\ell})\varphi_k\varphi_{\bar\ell}\\ &\geqslant \sum_k \tilde g^{i\bar i}(|\varphi_{ik}|^2+|\varphi_{i\bar k}|^2) - C\sum_i \tilde g^{i\bar i}+2\beta|\partial\varphi|^2_g\\ &\quad - 2\beta\mathrm{Re}\left(\sum_k \varphi_k v_{\bar k}\right)\\ &\geqslant \sum_k \tilde g^{i\bar i}(|\varphi_{ik}|^2+|\varphi_{i\bar k}|^2) - C\sum_i \tilde g^{i\bar i}+\beta|\partial\varphi|^2_g-\beta|\partial v|^2_g,\end{aligned}$$

where to derive the first line we have applied $\partial_{\bar k}$ to (2.10). But then

$$\beta h'|\partial v|^2_g \leqslant \frac{\beta\lambda}{2}|\partial v|^2_g \leqslant \frac{\beta\sup_X|\partial v|^2_g}{2+4\sup_X|\partial v|^2_g}\leqslant\frac{\beta}{4},$$

and so combining this with (2.8), (2.12) and (2.13), we see that (2.7) holds.

Now the rest of the proof proceeds exactly as in [10], since (2.7) is the same as [10, (2.6)], and the specific form of the PDE (2.4) is not used anymore in [10] after that point. At a couple of places we used that $h''=2(h')^2$, but in fact the inequality $h''\geqslant 2(h')^2$ is enough, and this holds in our case. The constant A is chosen at the end of the argument of [10, Proof of Theorem 1.2], and it equals $A=C+3$, where C is the uniform constant in (2.7). This completes the proof of Theorem 1.1.

3. Rooftop envelopes

In this section we consider a generalization of Theorem 1.1, as follows.

Suppose we are now given $C^{1,1}$ functions $v_j, j=1,\dots,k$ on a compact Kähler manifold (X,ω), and we consider the "rooftop envelope"

$$P(v_1,\dots,v_k)(x)=\sup\{u(x)\mid u\in PSH(X,\omega), u\leqslant \min_{j=1,\dots,k} v_j\}.$$

When $k=1$ this is essentially the same as the envelope we considered in Theorem 1.1, but with a weaker regularity assumption. Darvas-Rubinstein proved in [12] that $P(v_1,\dots,v_k)$ has bounded Laplacian on X, in particular it is in $C^{1,\alpha}(X)$ for all $0<\alpha<1$, and that if $[\omega]\in H^2(X,\mathbb{Q})$ then $P(v_1,\dots,v_k)$ is in $C^{1,1}(X)$. This last point used the regularity results of

Berman and Demailly [3, 8], and another proof was also given by Berman [5]. Using Theorem 1.1, we can prove the $C^{1,1}$ regularity of $P(v_1, \dots, v_k)$ in general Kähler classes:

Theorem 3.1. *The rooftop envelope $P(v_1, \dots, v_k)$ is in $C^{1,1}(X)$.*

Proof. The argument in [12, Proof of Theorem 2.5] reduces this result to proving the case when $k = 1$. So we have a function $v \in C^{1,1}(X)$, and consider the envelope

$$P(v)(x) = \sup\{u(x) \mid u \in PSH(X, \omega), u \leqslant v\},$$

and the goal is to show that $P(v)$ is also in $C^{1,1}(X)$.

By using convolution in local charts and gluing them with a partition of unity (see e.g. the appendix in [13]) can choose a sequence v_j of smooth functions which converge to v in $C^{1,\alpha}(X)$ for some fixed $0 < \alpha < 1$, and such that $\|v_j\|_{C^{1,1}(X,g)} \leqslant C$ for all j. For each j and $\beta \geqslant 0$ solve

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\beta(\varphi - v_j)}\omega^n, \tag{3.1}$$

where $\varphi = \varphi_{j,\beta}$ and $\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$. As mentioned earlier, Berman [4] proved that

$$|\varphi| \leqslant C, \quad |\Delta_g \varphi| \leqslant C, \quad |\varphi - P(v_j)| \leqslant C\frac{\log \beta}{\beta}, \tag{3.2}$$

for a uniform constant C independent of j, β, from which it follows that for any j fixed φ converges to $P(v_j)$ in $C^{1,\alpha}(X)$ for any $0 < \alpha < 1$, as $\beta \to \infty$. From Theorem 1.1 and its proof, we also have that

$$|\nabla^2 \varphi|_g \leqslant C,$$

independent of j, β. Therefore $\|P(v_j)\|_{C^{1,1}(X,g)} \leqslant C$ for all j. On the other hand we have that $P(v_j) \to P(v)$ uniformly as $j \to \infty$, which follows easily from the definition, and so we conclude that $P(v) \in C^{1,1}(X)$ as well. □

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Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208, USA
E-mail address: tosatti@math.northwestern.edu

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