Math. Res. Lett.
Volume 25, Number 2, 535–559, 2018

Positivity properties for canonical bases of modified quantum affine $\mathfrak{sl}_n$

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The positivity property for canonical bases asserts that the structure constants of the multiplication for the canonical basis are in $\mathbb{N}[v, v^{-1}]$. Let $\mathbf{U}$ be the quantum group over $\mathbb{Q}(v)$ associated with a symmetric Cartan datum. The positivity property for the positive part $\mathbf{U}^+$ of $\mathbf{U}$ was proved by Lusztig. He conjectured that the positivity property holds for the modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$. In this paper, we prove that the structure constants for the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ coincide with certain structure constants for the canonical basis of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. In particular, the positivity property for $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ follows from the positivity property for $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$.

1. Introduction

Let $\mathbf{U}$ be the quantum group over $\mathbb{Q}(v)$ associated with a Cartan datum $(I, \cdot)$, where v is an indeterminate. It is known by Lusztig and Kashiwara that the positive part $\mathbf{U}^+$ of a quantum enveloping algebra $\mathbf{U}$ has a canonical basis with remarkable properties (see Kashiwara [K], Lusztig [L1, L2, L5]). Among them, the deepest one should be the positivity property for the canonical basis of $\mathbf{U}^+$ proved by Lusztig [L1, L2], [L5, 14.4.13], which asserts that the structure constants of the multiplication for the canonical basis of $\mathbf{U}^+$ are in $\mathbb{N}[v, v^{-1}]$ in the case where the Cartan datum $(I, \cdot)$ is symmetric.

Let $\dot{\mathbf{U}}$ be the modified form of $\mathbf{U}$. The algebra $\dot{\mathbf{U}}$ is an associative algebra without unity and the category of $\mathbf{U}$-modules of type 1 is equivalent to the category of unital $\dot{\mathbf{U}}$-modules. The canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$ was constructed by Lusztig [L4, L5]. In [L4, Section 11] and [L5, 25.4.2], he conjectured that the structure constants of the multiplication for $\dot{\mathbf{B}}$ are in $\mathbb{N}[v, v^{-1}]$, i.e., the positivity property holds for $\dot{\mathbf{U}}$, in the case where the Cartan datum $(I, \cdot)$ is symmetric.

Let $\boldsymbol{\mathcal{S}}_{\vartriangle}(n, r)$ be the affine quantum Schur algebra over $\mathbb{Q}(v)$ (see [GV], [G2] and [L6]). An explicit algebra homomorphism ζ_r from $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ to $\boldsymbol{\mathcal{S}}_{\vartriangle}(n, r)$ was constructed by Ginzburg–Vasserot [GV], Lusztig [L6]. According to [L6, 8.2] the map $\zeta_r : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ is not surjective in the case where $n \leqslant r$. In turn, it is proved by Deng–Du–Fu [DDF, 3.8.1] that the map ζ_r can be extended to a surjective algebra homomorphism from $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ to $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$, where $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ is the quantum loop algebra of $\widehat{\mathfrak{gl}}_n$. On the other hand, the quantum Schur algebra $\boldsymbol{\mathcal{S}}(n,r)$ is known to be a quotient of the quantum algebra $\mathbf{U}(\mathfrak{sl}_n)$. The canonical basis of $\boldsymbol{\mathcal{S}}(n,r)$ was defined by Beilinson–Lusztig–MacPherson [BLM] and the positivity property for the canonical basis of $\boldsymbol{\mathcal{S}}(n,r)$ was proved by Green in [G1]. The canonical basis $\mathbf{B}(n,r)$ of the affine quantum Schur algebra $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ was defined in [L6]. Lusztig gave in [L6, 4.5] a sketch of the proof of the positivity property for $\mathbf{B}(n,r)$ based on the property of Kazhdan–Lusztig basis of affine Hecke algebras of type A.

In this paper, we show that there exist good relations among canonical bases of the three algebras $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ and $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$. In Theorem 4.8 we prove that the structure constants for $\mathbf{B}(n,r)$ are determined by the structure constants for the canonical basis $\mathbf{B}(N)^{\mathrm{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. Then the positivity property for $\mathbf{B}(n,r)$ follows from the positivity property for $\mathbf{B}(N)^{\mathrm{ap}}$. This gives an alternate approach for the positivity property of $\mathbf{B}(n,r)$. Using Theorem 4.8, we prove in Theorem 5.4 that the structure constants for the canonical basis $\dot{\mathbf{B}}(n)$ of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ are determined by the structure constants for the canonical basis $\mathbf{B}(N)^{\mathrm{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ for $n < N$. Thus the positivity property for $\dot{\mathbf{B}}(n)$ follows from the positivity property for $\mathbf{B}(N)^{\mathrm{ap}}$. We also discuss in Theorem 6.3 a certain weak positivity property for $\dot{\mathfrak{D}}_\vartriangle(n)$, where $\dot{\mathfrak{D}}_\vartriangle(n)$ is the modified quantum affine $\mathfrak{gl}_n$. We expect that the method of this paper can be used to study Lusztig conjecture [L5, 25.4.2] on the positivity property for other types of modified quantum groups.

Notation: For a positive integer n, let $\Theta_\vartriangle(n)$ (resp., $\widetilde{\Theta}_\vartriangle(n)$) be the set of all matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ (resp. $a_{i,j} \in \mathbb{Z}$, $a_{i,j} \geqslant 0$ for all $i \neq j$) such that

(a) $a_{i,j} = a_{i+n,j+n}$ for $i,j \in \mathbb{Z}$;

(b) for every $i \in \mathbb{Z}$, both sets $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ and $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$ are finite.

Let $\Theta_\vartriangle^+(n) = \{A \in \Theta_\vartriangle(n) \mid a_{i,j} = 0 \text{ for } i \geqslant j\}$. For $r \geqslant 0$, let $\Theta_\vartriangle(n,r) = \{A \in \Theta_\vartriangle(n) \mid \sigma(A) = r\}$, where $\sigma(A) = \sum_{1\leqslant i\leqslant n,\, j\in\mathbb{Z}} a_{i,j}$. For $i,j \in \mathbb{Z}$ let $E^\vartriangle_{i,j} \in \Theta_\vartriangle(n)$ be the matrix $(e^{i,j}_{k,l})_{k,l\in\mathbb{Z}}$ defined by

$$e^{i,j}_{k,l} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{Z}^n_\vartriangle = \{(\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{Z},\ \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$ and $\mathbb{N}^n_\vartriangle = \{(\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^n_\vartriangle \mid \lambda_i \geqslant 0 \text{ for } i \in \mathbb{Z}\}$. $\mathbb{Z}^n_\vartriangle$ has a natural structure of abelian group. For $r \geqslant 0$ let $\Lambda_\vartriangle(n,r) = \{\lambda \in \mathbb{N}^n_\vartriangle \mid \sigma(\lambda) = r\}$, where $\sigma(\lambda) = \sum_{1\leqslant i\leqslant n} \lambda_i$.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate.

2. Preliminaries

2.1. Let $\triangle(n)$ ($n \geqslant 2$) be the cyclic quiver with vertex set $I = \mathbb{Z}/n\mathbb{Z}$ and arrow set $\{i \to i+1 \mid i \in I\}$. We identify I with $\{1, 2, \dots, n\}$. Let $\mathbb{F}$ be a field. For $i \in I$ and $j \in \mathbb{Z}$ with $i < j$, let S_i denote the one-dimensional representation of $\triangle(n)$ with $(S_i)_i = \mathbb{F}$ and $(S_i)_k = 0$ for $i \neq k$ and $M^{i,j}$ the unique indecomposable nilpotent representation of length $j - i$ with top S_i.

For $A \in \Theta^+_\vartriangle(n)$ let $\mathbf{d}(A) \in \mathbb{N}I$ be the dimension vector of $M(A)$, where

$$M(A) = M_{\mathbb{F}}(A) = \bigoplus_{\substack{1\leqslant i\leqslant n\\ i<j,\ j\in\mathbb{Z}}} a_{i,j} M^{i,j}.$$

We will identify naturally $\mathbb{N}I$ with $\mathbb{N}^n_\vartriangle$. The Euler form associated with the cyclic quiver $\triangle(n)$ is the bilinear form $\langle -, - \rangle \colon \mathbb{Z}^n_\vartriangle \times \mathbb{Z}^n_\vartriangle \to \mathbb{Z}$ defined by $\langle \lambda, \mu \rangle = \sum_{1\leqslant i\leqslant n} \lambda_i\mu_i - \sum_{1\leqslant i\leqslant n} \lambda_i\mu_{i+1}$ for $\lambda, \mu \in \mathbb{Z}^n_\vartriangle$.

By Ringel [R], for $A, B, C \in \Theta^+_\vartriangle(n)$, there is a polynomial $\varphi^C_{A,B} \in \mathbb{Z}[v^2]$ such that, for any finite field $\mathbb{F}_q$, $\varphi^C_{A,B}|_{v^2=q}$ is equal to the number of submodules N of $M_{\mathbb{F}_q}(C)$ satisfying $N \cong M_{\mathbb{F}_q}(B)$ and $M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A)$.

Let $\mathfrak{D}_\vartriangle(n)$ be the double Ringel–Hall algebra of the cyclic quiver $\triangle(n)$ introduced in [DDF, (2.1.3.2)] (see also [X]). It was proved in [DDF, 2.5.3] that $\mathfrak{D}_\vartriangle(n)$ is isomorphic to the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. According to [DDF, 2.6.1, 2.6.3(5) and 3.9.2] we have the following result.

Lemma 2.1. *The algebra $\mathfrak{D}_\vartriangle(n)$ is the algebra over $\mathbb{Q}(v)$ generated by u^+_A, $K^{\pm 1}_i$, u^-_A ($A \in \Theta^+_\vartriangle(n)$, $i \in I$) subject to the following relations:*

(1) $K_iK_j = K_jK_i$, $K_iK_i^{-1} = 1$, $u^+_0 = u^-_0 = 1$;

(2) $K^{\mathbf{j}}u^+_A = v^{\langle \mathbf{d}(A),\mathbf{j}\rangle} u^+_A K^{\mathbf{j}}$, $u^-_A K^{\mathbf{j}} = v^{\langle \mathbf{d}(A),\mathbf{j}\rangle} K^{\mathbf{j}} u^-_A$, *where* $K^{\mathbf{j}} = K_1^{j_1} \cdots K_n^{j_n}$ *for* $\mathbf{j} \in \mathbb{Z}^n_\vartriangle$;

(3) $u^+_A u^+_B = \sum_{C\in\Theta^+_\vartriangle(n)} v^{\langle \mathbf{d}(A),\mathbf{d}(B)\rangle} \varphi^C_{A,B} u^+_C$;

(4) $u^-_A u^-_B = \sum_{C\in\Theta^+_\vartriangle(n)} v^{\langle \mathbf{d}(B),\mathbf{d}(A)\rangle} \varphi^C_{B,A} u^-_C$;

(5) commutator relations*: for all* $\lambda,\mu\in\mathbb{N}^n_\vartriangle$,

$$v^{\langle\mu,\mu\rangle}\sum_{\substack{\alpha,\beta\in\mathbb{N}^n_\vartriangle\\ \lambda-\alpha=\mu-\beta\geqslant 0}}\varphi^{\alpha,\beta}_{\lambda,\mu}v^{\langle\beta,\lambda+\mu-\beta\rangle}\widetilde{K}^{\mu-\beta}u^-_{A_\beta}u^+_{A_\alpha}$$
$$=v^{\langle\mu,\lambda\rangle}\sum_{\substack{\alpha,\beta\in\mathbb{N}^n_\vartriangle\\ \lambda-\alpha=\mu-\beta\geqslant 0}}\varphi^{\alpha,\beta}_{\lambda,\mu}v^{\langle\mu-\beta,\alpha\rangle+\langle\mu,\beta\rangle}\widetilde{K}^{\beta-\mu}u^+_{A_\alpha}u^-_{A_\beta},$$

where $\widetilde{K}^\nu:=(\widetilde{K}_1)^{\nu_1}\cdots(\widetilde{K}_n)^{\nu_n}$ *with* $\widetilde{K}_i=K_iK^{-1}_{i+1}$ *for* $\nu\in\mathbb{Z}^n_\vartriangle$*, and*

$$\varphi^{\alpha,\beta}_{\lambda,\mu}=v^{2\sum_{1\leqslant i\leqslant n}(\lambda_i-\alpha_i)(1-\alpha_i-\beta_i)}\prod_{\substack{1\leqslant i\leqslant n\\ 0\leqslant s\leqslant\lambda_i-\alpha_i-1}}\frac{1}{v^{2(\lambda_i-\alpha_i)}-v^{2s}}.$$

Note that the set $\{u^+_AK^{\mathbf{j}}u^-_B \mid A,B\in\Theta^+_\vartriangle(n),\,\mathbf{j}\in\mathbb{Z}^n_\vartriangle\}$ forms a $\mathbb{Q}(v)$-basis of $\mathfrak{D}_\vartriangle(n)$.

2.2. We now recall the definition of affine quantum Schur algebras following [L6]. Let $\mathbb{F}$ be a field and fix an $\mathbb{F}[\varepsilon,\varepsilon^{-1}]$-free module V of rank $r\in\mathbb{N}$, where ε is an indeterminate. A lattice in V is, by definition, a free $\mathbb{F}[\varepsilon]$-submodule L of V satisfying $V=L\otimes_{\mathbb{F}[\varepsilon]}\mathbb{F}[\varepsilon,\varepsilon^{-1}]$. Let $\mathscr{F}_\vartriangle=\mathscr{F}_{\vartriangle,n}$ be the set of all filtrations $\mathbf{L}=(L_i)_{i\in\mathbb{Z}}$ of lattices, where each L_i is a lattice in V such that $L_{i-1}\subseteq L_i$ and $L_{i-n}=\varepsilon L_i$, for all $i\in\mathbb{Z}$. The group G of automorphisms of the $\mathbb{F}[\varepsilon,\varepsilon^{-1}]$-module V acts on $\mathscr{F}_\vartriangle$ by $g\cdot\mathbf{L}=(g(L_i))_{i\in\mathbb{Z}}$ for $g\in G$ and $\mathbf{L}\in\mathscr{F}_\vartriangle$. The group G acts on $\mathscr{F}_\vartriangle\times\mathscr{F}_\vartriangle$ by $g\cdot(\mathbf{L},\mathbf{L}')=(g\cdot\mathbf{L},g\cdot\mathbf{L}')$.

Recall the set $\Theta_\vartriangle(n,r)$ given in §1. By [L6, 1.5] there is a bijection between the set of G-orbits in $\mathscr{F}_\vartriangle\times\mathscr{F}_\vartriangle$ and $\Theta_\vartriangle(n,r)$ by sending $(\mathbf{L},\mathbf{L}')$ to $A=(a_{i,j})_{ij\in\mathbb{Z}}$, where $a_{i,j}=\dim_{\mathbb{F}}L_i\cap L'_j/(L_{i-1}\cap L'_j+L_i\cap L'_{j-1})$. Let $\mathcal{O}_A\subseteq\mathscr{F}_\vartriangle\times\mathscr{F}_\vartriangle$ be the G-orbit corresponding to the matrix $A\in\Theta_\vartriangle(n,r)$.

Let $\mathbb{F}=\mathbb{F}_q$ be the finite field of q elements. For $A,A',A''\in\Theta_\vartriangle(n,r)$ and $(\mathbf{L},\mathbf{L}'')\in\mathcal{O}_{A''}$ let $\nu_{A,A',A'';q}=\#\{\mathbf{L}'\in\mathscr{F}_\vartriangle\mid(\mathbf{L},\mathbf{L}')\in\mathcal{O}_A,\,(\mathbf{L}',\mathbf{L}'')\in\mathcal{O}_{A'}\}$. By [L6, 1.8], there exists a polynomial $\nu_{A,A',A''}\in\mathcal{Z}$ in v^2 such that $\nu_{A,A',A''}|_{v^2=q}=\nu_{A,A',A'';q}$ for any q, a power of a prime number.

Let $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ be the the free $\mathcal{Z}$-module with basis $\{e_A\mid A\in\Theta_\vartriangle(n,r)\}$. According to [L6, 1.9] there is a unique associative $\mathcal{Z}$-algebra structure on $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ with multiplication $e_Ae_{A'}=\sum_{A''\in\Theta_\vartriangle(n,r)}\nu_{A,A',A''}e_{A''}$. Let $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)=\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}\otimes\mathbb{Q}(v)$. The algebras $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ and $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ are called affine quantum Schur algebras.

For $A\in\Theta_\vartriangle(n,r)$ let

$$[A]=v^{-d_A}e_A,\quad\text{where}\quad d_A=\sum_{1\leqslant i\leqslant n,\,i\geqslant k,j<l}a_{i,j}a_{k,l}\tag{2.1}$$

According to [L6, 1.11], the $\mathcal{Z}$-linear map

$$\tau_r : \mathcal{S}_\Delta(n,r)_\mathcal{Z} \longrightarrow \mathcal{S}_\Delta(n,r)_\mathcal{Z}, \quad [A] \longmapsto [{}^tA] \tag{2.2}$$

is an algebra anti-involution, where ${}^tA$ is the transpose of A.

2.3. Let $\mathfrak{S}_{\Delta,r}$ be the group consisting of all permutations $w : \mathbb{Z} \to \mathbb{Z}$ such that $w(i+r) = w(i) + r$ for $i \in \mathbb{Z}$. The extended affine Hecke algebra $\mathcal{H}_\Delta(r)_\mathcal{Z}$ of affine type A over $\mathcal{Z}$ is the (unital) $\mathcal{Z}$-algebra with basis $\{T_w\}_{w\in\mathfrak{S}_{\Delta,r}}$, and multiplication defined by

$$\begin{cases} T_{s_i}^2 = (v^2-1)T_{s_i} + v^2, & \text{for } 1 \leqslant i \leqslant r \\ T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \end{cases}$$

where $s_i \in \mathfrak{S}_{\Delta,r}$ is defined by setting $s_i(j) = j$ for $j \not\equiv i, i+1 \bmod r$, $s_i(j) = j-1$ for $j \equiv i+1 \bmod r$ and $s_i(j) = j+1$ for $j \equiv i \bmod r$, and $\ell(w)$ is the length of w.

Recall the set $\Lambda_\Delta(n,r)$ given in §1. Let $\mathfrak{S}_r$ be the subgroup of $\mathfrak{S}_{\Delta,r}$ generated by s_i for $1 \leqslant i \leqslant r-1$, which is isomorphic to the symmetric group of degree r. For $\lambda \in \Lambda_\Delta(n,r)$, let $\mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1,\ldots,\lambda_n)}$ be the corresponding standard Young subgroup of $\mathfrak{S}_r$ and let $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w \in \mathcal{H}_\Delta(r)_\mathcal{Z}$. For $\lambda, \mu \in \Lambda_\Delta(n,r)$, let $\mathscr{D}_\lambda^\Delta = \{d \mid d \in \mathfrak{S}_{\Delta,r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\lambda\}$ and $\mathscr{D}_{\lambda,\mu}^\Delta = \mathscr{D}_\lambda^\Delta \cap \mathscr{D}_\mu^{\Delta\,-1}$. For $\lambda, \mu \in \Lambda_\Delta(n,r)$ and $d \in \mathscr{D}_{\lambda,\mu}^\Delta$ define

$$\phi_{\lambda,\mu}^d \in \mathrm{End}_{\mathcal{H}_\Delta(r)_\mathcal{Z}}\left(\bigoplus_{\lambda\in\Lambda_\Delta(n,r)} x_\lambda \mathcal{H}_\Delta(r)_\mathcal{Z}\right)$$

by

$$\phi_{\lambda,\mu}^d(x_\nu h) = \delta_{\mu\nu} \sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w h$$

for $\nu \in \Lambda_\Delta(n,r)$ and $h \in \mathcal{H}_\Delta(r)_\mathcal{Z}$.

For $\lambda \in \Lambda_\Delta(n,r)$, $1 \leqslant i \leqslant n$ and $k \in \mathbb{Z}$ let

$$R_{i+kn}^\lambda = \{\lambda_{k,i-1}+1, \lambda_{k,i-1}+2, \ldots, \lambda_{k,i-1}+\lambda_i = \lambda_{k,i}\}, \tag{2.3}$$

where $\lambda_{k,i-1} = kr + \sum_{1\leqslant t\leqslant i-1} \lambda_t$. By Varagnolo–Vasserot [VV, 7.4] (see also [DF1, 9.2]), there is a bijective map

$$\jmath_\Delta : \{(\lambda, d, \mu) \mid d \in \mathscr{D}_{\lambda,\mu}^\Delta, \lambda, \mu \in \Lambda_\Delta(n,r)\} \longrightarrow \Theta_\Delta(n,r) \tag{2.4}$$

sending (λ, d, μ) to the matrix $A = (|R_k^\lambda \cap dR_l^\mu|)_{k,l\in\mathbb{Z}}$. Varagnolo–Vasserot showed in [VV] that there is an algebra isomorphism

$$\mathfrak{h} : \mathrm{End}_{\mathcal{H}_\Delta(r)_\mathcal{Z}}\left(\bigoplus_{\lambda\in\Lambda_\Delta(n,r)} x_\lambda \mathcal{H}_\Delta(r)_\mathcal{Z}\right) \to \mathcal{S}_\Delta(n,r)_\mathcal{Z}$$

such that $\mathfrak{h}(\phi^d_{\lambda,\mu}) = e_A$, where $A = \jmath_\Delta(\lambda, d, \mu)$. We identify

$$\mathrm{End}_{\mathcal{H}_\Delta(r)_\mathcal{Z}}\left(\bigoplus_{\lambda\in\Lambda_\Delta(n,r)} x_\lambda \mathcal{H}_\Delta(r)_\mathcal{Z}\right)$$

with $\mathcal{S}_\Delta(n,r)_\mathcal{Z}$ via $\mathfrak{h}$.

2.4. It was shown in [DDF] that the double Ringel–Hall algebra $\mathfrak{D}_\Delta(n)$ and the affine quantum Schur algebra $\boldsymbol{\mathcal{S}}_\Delta(n,r)$ are related by a surjective algebra homomorphism ζ_r. Let $\Theta^\pm_\Delta(n) := \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i = j\}$. For $A \in \Theta^\pm_\Delta(n)$ and $\mathbf{j} \in \mathbb{Z}^n_\Delta$, define $A(\mathbf{j}, r) \in \boldsymbol{\mathcal{S}}_\Delta(n,r)$ by

$$A(\mathbf{j}, r) = \begin{cases} \sum_{\lambda\in\Lambda_\Delta(n,r-\sigma(A))} v^{\lambda\cdot\mathbf{j}}[A + \mathrm{diag}(\lambda)], & \text{if } \sigma(A) \leqslant r; \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda \cdot \mathbf{j} = \sum_{1\leqslant i\leqslant n} \lambda_i j_i$. For $A \in \Theta^+_\Delta(n)$ let

$$\widetilde{u}^\pm_A = v^{\dim \mathrm{End}(M(A)) - \dim M(A)} u^\pm_A.$$

We have the following result.

Theorem 2.2 ([DDF, 3.6.3, 3.8.1]). *For $r \geqslant 0$, the linear map $\zeta_r : \mathfrak{D}_\Delta(n) \to \boldsymbol{\mathcal{S}}_\Delta(n,r)$ satisfying*

$$\zeta_r(K^{\mathbf{j}}) = 0(\mathbf{j}, r),\ \zeta_r(\widetilde{u}^+_A) = A(\mathbf{0}, r), \ \ \textit{and}\ \ \zeta_r(\widetilde{u}^-_A) = ({}^tA)(\mathbf{0}, r),$$

for all $\mathbf{j} \in \mathbb{Z}^n_\Delta$ and $A \in \Theta^+_\Delta(n)$, is a surjective algebra homomorphism.

3. Canonical bases for affine quantum Schur algebras

3.1. Let W_r be the subgroup of $\mathfrak{S}_{\Delta,r}$ generated by s_i for $1 \leqslant i \leqslant r$. For $i, j \in \mathbb{Z}$ such that $i \not\equiv j \bmod r$, define $(i,j) \in \mathfrak{S}_{\Delta,r}$ by setting $(i,j)(k) = k$ for $k \not\equiv i, j \bmod r$, $(i,j)(k) = j + k - i$ for $k \equiv i \bmod r$ and $(i,j)(k) = i + k - j$ for

$k \equiv j \bmod r$. Note that $(i,j) \in W_r$ for all i,j. By definition we have $(i,j) = (i+tr, j+tr)$ for $t \in \mathbb{Z}$ and $(i,i+1) = s_i$. Let

$$T = \bigcup_{w\in W_r, 1\leqslant i\leqslant r} ws_iw^{-1} = \{(i,j) \in W_r | 1 \leqslant i \leqslant r,\ i,j \in \mathbb{Z},\ i<j,\ i \not\equiv j \mod r\}.$$

For $y, w \in W_r$, we write $y \leqslant w$ if there exist $t_i \in T$ $(1 \leqslant i \leqslant m)$ for some $m \in \mathbb{N}$ such that $w = t_1t_2\cdots t_m y$ and $\ell(t_it_{i+1}\cdots t_my) > \ell(t_{i+1}t_{i+2}\cdots t_my)$ for $1 \leqslant i \leqslant m$. The partial ordering $\leqslant$ on W_r is called the Bruhat order. Let ρ be the permutation of $\mathbb{Z}$ sending j to $j+1$ for all $j \in \mathbb{Z}$. Then we have $\mathfrak{S}_{\vartriangle,r} = \langle\rho\rangle \ltimes W_r$, where $\langle\rho\rangle \cong \mathbb{Z}$ is the subgroup of $\mathfrak{S}_{\vartriangle,r}$ generated by ρ. The Bruhat order on W_r can be extended to $\mathfrak{S}_{\vartriangle,r}$ by define $\rho^i y \leqslant \rho^j w$ (for $y, w \in W_r$) if and only if $i = j$ and $y \leqslant w$.

Let $^- : \mathcal{H}_\vartriangle(r)_\mathcal{Z} \to \mathcal{H}_\vartriangle(r)_\mathcal{Z}$ be the ring involution defined by $\bar{v} = v^{-1}$ and $\bar{T}_w = T^{-1}_{w^{-1}}$. Let $\mathcal{H}(W_r)$ be the $\mathcal{Z}$-subalgebra of $\mathcal{H}_\vartriangle(r)_\mathcal{Z}$ generated by T_{s_i} for $1 \leqslant i \leqslant r$. Let $\{C'_w \mid w \in W_r\}$ be the Kazhdan–Lusztig basis of $\mathcal{H}(W_r)$ defined in [KL, 1.1(c)]. For $y, w \in W_r$ and $a, b \in \mathbb{Z}$ let $P_{\rho^a y, \rho^b w} = \delta_{a,b}P_{y,w}$, where $P_{y,w} \in \mathcal{Z}$ is the Kazhdan–Lusztig polynomial. For $w = \rho^a x \in \mathfrak{S}_{\vartriangle,r}$ with $a \in \mathbb{Z}$ and $x \in W_r$, let $C'_w = T^a_\rho C'_x$. Then for $w \in \mathfrak{S}_{\vartriangle,r}$ we have $\overline{C'_w} = C'_w$ and

$$C'_w = \sum_{y\leqslant w,\, y\in\mathfrak{S}_{\vartriangle,r}} v^{\ell(y)-\ell(w)} P_{y,w} \widetilde{T}_y$$

where $\widetilde{T}_y = v^{-\ell(y)}T_y$. The set $\{C'_w \mid w \in \mathfrak{S}_{\vartriangle,r}\}$ is called the canonical basis of $\mathcal{H}_\vartriangle(r)_\mathcal{Z}$.

For $d \in \mathscr{D}^\vartriangle_{\lambda,\mu}$ let $T_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu} = \sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w$ and $\widetilde{T}_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu} = v^{-\ell(d^+)}T_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu}$, where d^+ is the unique longest element in $\mathfrak{S}_\lambda d\mathfrak{S}_\mu$. According to [C, (1.10)] and [DDPW, 4.35] we have the following result.

Lemma 3.1. *For $\lambda, \mu \in \Lambda_\vartriangle(n,r)$ and $d \in \mathscr{D}^\vartriangle_{\lambda,\mu}$ we have*

$$C'_{d^+} = \sum_{\substack{y\in\mathscr{D}^\vartriangle_{\lambda,\mu}\\ y\leqslant d}} v^{\ell(y^+)-\ell(d^+)} P_{y^+,d^+} \widetilde{T}_{\mathfrak{S}_\lambda y\mathfrak{S}_\mu},$$

where y^+ is the unique longest element in $\mathfrak{S}_\lambda y\mathfrak{S}_\mu$.

3.2. We now recall the definition of canonical bases of affine quantum Schur algebras. Note that $C'_{w_{0,\lambda}} = v^{-\ell(w_{0,\lambda})}x_\lambda$, where $w_{0,\lambda}$ is the longest element in $\mathfrak{S}_\lambda$. We define a map $^- : \mathcal{S}_\vartriangle(n,r)_\mathcal{Z} \to \mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ by $v \mapsto \bar{v} = v^{-1}$, $f \mapsto \bar{f}$, where for $f \in \mathrm{Hom}_{\mathcal{H}_\vartriangle(r)_\mathcal{Z}}(x_\mu\mathcal{H}_\vartriangle(r)_\mathcal{Z}, x_\lambda\mathcal{H}_\vartriangle(r)_\mathcal{Z})$, $\bar{f} \in \mathrm{Hom}_{\mathcal{H}_\vartriangle(r)_\mathcal{Z}}(x_\mu\mathcal{H}_\vartriangle(r)_\mathcal{Z}, x_\lambda\mathcal{H}_\vartriangle(r)_\mathcal{Z})$

is defined by $\bar{f}(C'_{w_0,\mu}h) = \overline{f(C'_{w_0,\mu})}h$ for $h \in \mathcal{H}_\vartriangle(r)_\mathcal{Z}$. The map $^- : \mathcal{S}_\vartriangle(n,r)_\mathcal{Z} \to \mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ is a ring involution (cf. [D]).

For $A \in \Theta_\vartriangle(n)$ let $\mathrm{ro}(A) = \big(\sum_{j\in\mathbb{Z}} a_{i,j}\big)_{i\in\mathbb{Z}}$ and $\mathrm{co}(A) = \big(\sum_{i\in\mathbb{Z}} a_{i,j}\big)_{j\in\mathbb{Z}}$. For $A \in \widetilde{\Theta}_\vartriangle(n)$ and $i \neq j \in \mathbb{Z}$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum\limits_{s\leqslant i, t\geqslant j} a_{s,t}, & \text{if } i < j; \\ \sum\limits_{s\geqslant i, t\leqslant j} a_{s,t}, & \text{if } i > j. \end{cases}$$

For $A, B \in \widetilde{\Theta}_\vartriangle(n)$, define $B \preccurlyeq A$ by the condition $\sigma_{i,j}(B) \leqslant \sigma_{i,j}(A)$ for all $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and, for some pair (i,j) with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$. For $A, B \in \widetilde{\Theta}_\vartriangle(n)$ define $B \sqsubseteq A$ if and only if $B \preccurlyeq A$, $\mathrm{co}(B) = \mathrm{co}(A)$ and $\mathrm{ro}(B) = \mathrm{ro}(A)$. Put $B \sqsubset A$ if $B \sqsubseteq A$ and $B \neq A$. According to [DF1, 6.1] we know that the order relation $\sqsubseteq$ is a partial order relation on $\widetilde{\Theta}_\vartriangle(n)$.

Lusztig proved in [L6] that there is a unique $\mathcal{Z}$-basis

$$\mathbf{B}(n,r) := \{\theta_{A,r} \mid A \in \Theta_\vartriangle(n,r)\} \tag{3.1}$$

for $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ such that $\overline{\theta_{A,r}} = \theta_{A,r}$ and

$$\theta_{A,r} - [A] \in \sum_{\substack{B\in\Theta_\vartriangle(n,r)\\ B\sqsubset A}} v^{-1}\mathbb{Z}[v^{-1}][B], \tag{3.2}$$

(see also [DF3, 7.6]). The set $\mathbf{B}(n,r)$ is called the canonical basis of $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$.

3.3. For $w \in \mathfrak{S}_{\vartriangle,r}$ let $\mathscr{L}(w) = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leqslant i \leqslant r,\ i < j,\ w(i) > w(j)\}$ and $\mathscr{R}(w) = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leqslant j \leqslant r,\ i < j,\ w(i) > w(j)\}$. The following result is given in [DDF, (3.2.1.1)] (see also [DF2, 5.2]).

Lemma 3.2. *For $w \in \mathfrak{S}_{\vartriangle,r}$, we have $\ell(w) = |\mathscr{L}(w)| = |\mathscr{R}(w)|$.*

For $i \in \mathbb{Z}$ the image of i in $\mathbb{Z}/r\mathbb{Z}$ will be denoted by $\bar{i}$. The following corollary can be proved by a standard argument by using Lemma 3.2. So we omit the proof.

Corollary 3.3. *Let $x \in \mathfrak{S}_{\vartriangle,r}$ and $i_0, j_0 \in \mathbb{Z}$ such that $i_0 < j_0$ and $\overline{i_0} \neq \overline{j_0}$. Then we have $x < (i_0,j_0)x$ if and only if $x^{-1}(i_0) < x^{-1}(j_0)$, i.e. i_0 occurs in the left of j_0 in the sequence $(x(s))_{s\in\mathbb{Z}}$.*

For $i \in \mathbb{Z}$ let

$$(-\infty, i] = \{\boldsymbol{a} = (a_s)_{s \leqslant i} | a_s \in \mathbb{Z}\} \quad \text{and} \quad [i, +\infty) = \{\boldsymbol{a} = (a_s)_{s \geqslant i} | a_s \in \mathbb{Z}\}.$$

If either $\boldsymbol{a}, \boldsymbol{b} \in (-\infty, i]$ or $\boldsymbol{a}, \boldsymbol{b} \in [i, +\infty)$, we write $\boldsymbol{a} \leqslant \boldsymbol{b}$ if $a_s \leqslant b_s$ for all s. Given $\boldsymbol{a} = (a_s) \in \mathbb{Z}^n_\vartriangle$ and an integer i we let $(a_s)^{\text{sorted}}_{s \leqslant i} = (b_s)_{s \leqslant i}$ such that $\{a_s | s \leqslant i\} = \{b_s | s \leqslant i\}$ and $b_{s-1} \leqslant b_s$ for $s \leqslant i$. Similarly we may define $(a_s)^{\text{sorted}}_{s \geqslant i}$ for $\boldsymbol{a} \in \mathbb{Z}^n_\vartriangle$ and $i \in \mathbb{Z}$.

By Corollay 3.3 we have the following result.

Corollary 3.4. *Let $y, w \in \mathfrak{S}_{\vartriangle,r}$. If $y \leqslant w$ then for any $i \in \mathbb{Z}$ we have $(y(s))^{\text{sorted}}_{s \leqslant i} \leqslant (w(s))^{\text{sorted}}_{s \leqslant i}$ and $(y(s))^{\text{sorted}}_{s \geqslant i} \geqslant (w(s))^{\text{sorted}}_{s \geqslant i}$.*

3.4. Recall the map $\jmath_\vartriangle$ defined in (2.4). Given $A \in \Theta_\vartriangle(n, r)$, write $y_A = w$ if $A = \jmath_\vartriangle(\lambda, w, \mu)$. For $A, B \in \Theta_\vartriangle(n, r)$, define $B \leqslant^{Bo} A$ by the condition $\text{ro}(B) = \text{ro}(A)$, $\text{co}(B) = \text{co}(A)$ and $y_B \leqslant y_A$. Put $B <^{Bo} A$ if $B \leqslant^{Bo} A$ and $B \neq A$. Then $\leqslant^{Bo}$ is a partial order relation on $\Theta_\vartriangle(n, r)$.

Recall that V is a $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$-free module of rank $r \in \mathbb{N}$. Let $\{v_1, v_2, \ldots, v_r\}$ be a fixed $\mathbb{F}[\varepsilon, \varepsilon^{-1}]$-basis of V. We set $v_{i+kr} = \varepsilon^{-k} v_i$ for $1 \leqslant i \leqslant r$ and $k \in \mathbb{Z}$.

Lemma 3.5. *Let $A \in \Theta_\vartriangle(n, r)$, $\lambda = \text{ro}(A)$ and $\mu = \text{co}(A)$. Let $\mathbf{L}(A) = (L_i)_{i \in \mathbb{Z}}$ and $\mathbf{L}'(A) = (L'_i)_{i \in \mathbb{Z}}$ where*

$$L_{i+kn} = \text{span}_\mathbb{F} \left\{ v_a | a \in \bigcup_{t \leqslant i+kn} R^\lambda_t \right\} = \text{span}_\mathbb{F} \left\{ v_a | a \leqslant \sum_{1 \leqslant j \leqslant i} \lambda_j + kr \right\}$$

$$L'_{i+kn} = \text{span}_\mathbb{F} \left\{ v_{y_A(a)} | a \in \bigcup_{t \leqslant i+kn} R^\mu_t \right\} = \text{span}_\mathbb{F} \left\{ v_{y_A(a)} | a \leqslant \sum_{1 \leqslant j \leqslant i} \mu_j + kr \right\}$$

for $1 \leqslant i \leqslant n$ and $k \in \mathbb{Z}$. Then we have $(\mathbf{L}(A), \mathbf{L}'(A)) \in \mathcal{O}_A$.

Proof. By definition we have

$$L_i \cap L'_j = \text{span}_\mathbb{F} \left\{ v_a | a \in \bigcup_{t \leqslant i} R^\lambda_t,\ a \in \bigcup_{t \leqslant j} y_A(R^\mu_t) \right\}$$

for $i, j \in \mathbb{Z}$. Hence for $i, j \in \mathbb{Z}$ we have

$$\frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} = \text{span}_\mathbb{F} \{ \overline{v}_a | a \in R^\lambda_i \cap y_A(R^\mu_j) \}.$$

The assertion follows. □

Lemma 3.6. (1) *If* $A, B \in \Theta_\vartriangle(n,r)$ *and* $B \leqslant^{Bo} A$ *then* $B \sqsubseteq A$.
(2) *If* $A, B \in \Theta_\vartriangle(n,r)$ *and* $B <^{Bo} A$ *then* $B \sqsubset A$.

Proof. If $B \leqslant^{Bo} A$ then $\mathrm{ro}(B) = \mathrm{ro}(A)$, $\mathrm{co}(B) = \mathrm{co}(A)$ and $y_B \leqslant y_A$. We denote $\lambda = \mathrm{ro}(B)$ and $\mu = \mathrm{co}(B)$. Let $\mathbf{L} = \mathbf{L}(A) = \mathbf{L}(B)$, $\mathbf{L}' = \mathbf{L}'(A)$ and $\mathbf{L}'' = \mathbf{L}'(B)$. Then by Lemma 3.5 we have $(\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A$ and $(\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_B$. By definition for $i, j \in \mathbb{Z}$ we have

$$L_i/(L_i \cap L'_{j-1}) = \mathrm{span}_{\mathbb{F}} \left\{ \overline{v}_{y_A(a)} | y_A(a) \in \bigcup_{t \leqslant i} R_t^\lambda,\ a \in \bigcup_{t \geqslant j} R_t^\mu \right\},$$

$$L_i/(L_i \cap L''_{j-1}) = \mathrm{span}_{\mathbb{F}} \left\{ \overline{v}_{y_B(a)} | y_B(a) \in \bigcup_{t \leqslant i} R_t^\lambda,\ a \in \bigcup_{t \geqslant j} R_t^\mu \right\}$$

$$L'_j/(L_{i-1} \cap L'_j) = \mathrm{span}_{\mathbb{F}} \left\{ \overline{v}_{y_A(a)} | a \in \bigcup_{t \leqslant j} R_t^\mu,\ y_A(a) \in \bigcup_{t \geqslant i} R_t^\lambda \right\},$$

$$L''_j/(L_{i-1} \cap L''_j) = \mathrm{span}_{\mathbb{F}} \left\{ \overline{v}_{y_B(a)} | a \in \bigcup_{t \leqslant j} R_t^\mu,\ y_B(a) \in \bigcup_{t \geqslant i} R_t^\lambda \right\}.$$

Since $y_B \leqslant y_A$, by Corollary 3.4 we have

$$\dim_{\mathbb{F}}(L_i/(L_i \cap L''_{j-1})) \leqslant \dim_{\mathbb{F}}(L_i/(L_i \cap L'_{j-1}))$$
$$\dim_{\mathbb{F}}(L''_j/(L_{i-1} \cap L''_j)) \leqslant \dim_{\mathbb{F}}(L'_j/(L_{i-1} \cap L'_j)).$$

Hence by [L6, 1.6(a)] we conclude that $B \sqsubseteq A$. Thus (1) holds. Now we assume $B <^{Bo} A$. Suppose that $B \not\prec A$. Then by (1) we have $B \preccurlyeq A$ and $\mathrm{ro}(B) = \mathrm{ro}(A)$. Hence by [DF1, 6.1] we see that B and A have the same off diagonal entries. Since $\mathrm{ro}(B) = \mathrm{ro}(A)$ we must have $A = B$. This is a contradiction. Hence $B \prec A$. The assertion (2) follows. □

3.5. For $A \in \Theta_\vartriangle(n,r)$ let y_A^+ be the unique longest element in $\mathfrak{S}_\lambda y_A \mathfrak{S}_\mu$, where $\lambda = \mathrm{ro}(A)$ and $\mu = \mathrm{co}(A)$. The following result is given in [DF3, 7.1].

Lemma 3.7. *For* $A \in \Theta_\vartriangle(n,r)$ *we have* $\ell(y_A^+) = d_A + \ell(w_{0,\mu})$ *where* $\mu = \mathrm{co}(A)$ *and* d_A *is given in* (2.1).

For $\lambda, \mu \in \Lambda_\vartriangle(n,r)$ and $d \in \mathscr{D}^\vartriangle_{\lambda,\mu}$, define $\theta^d_{\lambda,\mu} \in \mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ as follows:

$$\theta^d_{\lambda,\mu}(x_\nu h) = \delta_{\mu\nu} v^{\ell(w_{0,\mu})} C'_{d^+} h,$$

where $\nu \in \Lambda_\vartriangle(n,r)$, $h \in \mathcal{H}_\vartriangle(r)_\mathcal{Z}$ and d^+ is the unique longest element in $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$.

Proposition 3.8. *Assume $\lambda, \mu \in \Lambda_\vartriangle(n,r)$, $d \in \mathscr{D}_{\lambda,\mu}$ and $A = \jmath_\vartriangle(\lambda, d, \mu) \in \Theta_\vartriangle(n,r)$. Then we have*

$$\theta^d_{\lambda,\mu} = \theta_{A,r} = \sum_{\substack{B \in \Theta_\vartriangle(n,r) \\ B \leqslant^{B o} A}} v^{\ell(y_B^+) - \ell(y_A^+)} P_{y_B^+, y_A^+}[B],$$

where $P_{y_B^+, y_A^+}$ is the Kazhdan–Lusztig polynomial.

Proof. By definition we have $\overline{\theta^d_{\lambda,\mu}} = \theta^d_{\lambda,\mu}$. Furthermore, by Lemma 3.1 we conclude that

$$\theta^d_{\lambda,\mu} = \sum_{\substack{x \in \mathscr{D}^\vartriangle_{\lambda,\mu} \\ x \leqslant d}} v^{\ell(x^+) - \ell(d^+)} P_{x^+, d^+} \widetilde{\phi^x_{\lambda,\mu}}, \tag{3.3}$$

where $\widetilde{\phi^x_{\lambda,\mu}} = v^{\ell(w_{0,\mu}) - \ell(x^+)} \phi^x_{\lambda,\mu}$. In addition, by Lemma 3.7 we have $[B] = \widetilde{\phi^x_{\lambda,\mu}}$ for $B \in \Theta_\vartriangle(n,r)$ with $B = \jmath_\vartriangle(\lambda, x, \mu)$. Consequently, by Lemma 3.6 and the uniqueness of $\theta_{A,r}$ we conclude that $\theta_{A,r} = \theta^d_{\lambda,\mu}$. The assertion follows. □

4. Connection between $\mathbf{B}(n,r)$ and $\mathbf{B}(N)^{\mathrm{ap}}$

4.1. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ be the $\mathbb{Q}(v)$-subalgebra of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ generated by the elements $u^+_{E^\vartriangle_{i,i+1}}$, $u^-_{E^\vartriangle_{i+1,i}}$ and $\widetilde{K}_i^{\pm 1}$ for $i \in I$. Then $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ is isomorphic to quantum affine $\widehat{\mathfrak{sl}}_n$. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ be the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ generated by the elements $u^+_{E^\vartriangle_{i,i+1}}$ for $i \in I$. Let $U(\widehat{\mathfrak{sl}}_n)^+_\mathcal{Z}$ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ generated by $\widetilde{u}^+_{mE^\vartriangle_{i,i+1}}$ for $i \in I$ and $m \in \mathbb{N}$. The algebra $U(\widehat{\mathfrak{sl}}_n)^+_\mathcal{Z}$ is the $\mathcal{Z}$-form of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$.

Let $\mathfrak{D}_\vartriangle(n)^+_\mathcal{Z} = \mathrm{span}_\mathcal{Z}\{\widetilde{u}^+_A \mid A \in \Theta^+_\vartriangle(n)\}$. Then $\mathfrak{D}_\vartriangle(n)^+_\mathcal{Z}$ is a $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ and $U(\widehat{\mathfrak{sl}}_n)^+_\mathcal{Z}$ is a proper subalgebra of $\mathfrak{D}_\vartriangle(n)^+_\mathcal{Z}$. According to [VV, Prop 7.5], there is a $\mathbb{Q}$-algebra involution on $\mathfrak{D}_\vartriangle(n)^+_\mathcal{Z}$ such that $\bar{v} = v^{-1}$ and

$\overline{\widetilde{u}_A^+} = \widetilde{u}_A^+$ for $A \in \Theta_\vartriangle^+(n)$ with $M(A)$ being semisimple. Furthermore, there is a unique $\mathcal{Z}$-basis

$$\mathbf{B}(n) := \{\theta_A^+ \mid A \in \Theta_\vartriangle^+(n)\} \tag{4.1}$$

for $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^+$ such that $\overline{\theta_A^+} = \theta_A^+$ and

$$\theta_A^+ - \widetilde{u}_A^+ \in \sum_{\substack{B \prec A,\, B \in \Theta_\vartriangle^+(n) \\ \mathbf{d}(B)=\mathbf{d}(A)}} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_B^+. \tag{4.2}$$

The set $\mathbf{B}(n)$ is called the canonical basis of $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^+$. For $A, B \in \Theta_\vartriangle^+(n)$ we write

$$\theta_A^+ \theta_B^+ = \sum_{C \in \Theta_\vartriangle^+(n)} \mathsf{f}_{A,B,C}\theta_C^+, \tag{4.3}$$

where $\mathsf{f}_{A,B,C} \in \mathcal{Z}$. Note that if $\mathsf{f}_{A,B,C} \neq 0$ then $\mathbf{d}(C) = \mathbf{d}(A) + \mathbf{d}(B)$.

A matrix $A = (a_{i,j}) \in \Theta_\vartriangle(n)$ is said to be aperiodic if for every integer $l \neq 0$ there exists $1 \leqslant i \leqslant n$ such that $a_{i,i+l} = 0$. Let $\Theta_\vartriangle(n)^{\mathrm{ap}}$ be the set of all aperiodic matrices in $\Theta_\vartriangle(n)$. Let $\Theta_\vartriangle^+(n)^{\mathrm{ap}} = \Theta_\vartriangle^+(n) \cap \Theta_\vartriangle(n)^{\mathrm{ap}}$.

By Lusztig [L3] we know that the set

$$\mathbf{B}(n)^{\mathrm{ap}} := \{\theta_A^+ \mid A \in \Theta_\vartriangle^+(n)^{\mathrm{ap}}\} \tag{4.4}$$

forms a $\mathcal{Z}$-basis for $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$ and is called the canonical basis of $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$. The following positivity result for $U(\widehat{\mathfrak{sl}}_n)_\mathcal{Z}^+$ was proved by Lusztig.

Theorem 4.1 ([L5, 14.4.13]). *For $A, B, C \in \Theta_\vartriangle^+(n)^{\mathrm{ap}}$ we have $\mathsf{f}_{A,B,C} \in \mathbb{N}[v, v^{-1}]$.*

4.2. Let $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^0$ be the $\mathcal{Z}$-subalgebra of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ generated by $K_i^{\pm 1}$ and $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$ for $1 \leqslant i \leqslant n$ and $t > 0$, where $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right] = \prod_{s=1}^t \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}$. Let $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^{\geqslant 0} = \mathfrak{D}_\vartriangle(n)_\mathcal{Z}^+ \mathfrak{D}_\vartriangle(n)_\mathcal{Z}^0$. Then $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^{\geqslant 0}$ is a $\mathcal{Z}$-subalgebra of $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}$.

Recall the map ζ_r defined in Theorem 2.2. Let $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}^{\geqslant 0}$ be the $\mathcal{Z}$-submodule of $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$ spanned by the elements $A(\mathbf{0}, r)[\mathrm{diag}(\lambda)]$ for $A \in \Theta_\vartriangle^+(n)$ and $\lambda \in \Lambda_\vartriangle(n,r)$. Since $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}^{\geqslant 0} = \zeta_r(\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^{\geqslant 0})$, we conclude that $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}^{\geqslant 0}$ is a $\mathcal{Z}$-subalgebra of $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$. The algebra $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}^{\geqslant 0}$ is called a Borel subalgebra of $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}$.

Lemma 4.2. *The set $\{\theta_{A+\mathrm{diag}(\lambda),r} \mid A \in \Theta_\vartriangle^+(n),\ \lambda \in \Lambda_\vartriangle(n, r - \sigma(A))\}$ forms a $\mathcal{Z}$-basis of $\mathcal{S}_\vartriangle(n,r)_\mathcal{Z}^{\geqslant 0}$.*

Proof. By definition the set

$$\{[A+\mathrm{diag}(\lambda)] \mid |A \in \Theta_\vartriangle^+(n),\ \lambda \in \Lambda_\vartriangle(n, r-\sigma(A))\}$$

forms a $\mathcal{Z}$-basis of $\mathcal{S}_\vartriangle(n,r)_{\mathcal{Z}}^{\geqslant 0}$. Furthermore, by (3.2), for $A \in \Theta_\vartriangle^+(n)$ and $\lambda \in \Lambda_\vartriangle(n, r-\sigma(A))$, we have

$$\theta_{A+\mathrm{diag}(\lambda),r} - [A+\mathrm{diag}(\lambda)] \in \sum_{\substack{B\in\Theta_\vartriangle^+(n),\ \mu\in\Lambda_\vartriangle(n,r-\sigma(B))\\ B+\mathrm{diag}(\mu)\sqsubset A+\mathrm{diag}(\lambda)}} \mathcal{Z}[B+\mathrm{diag}(\mu)].$$

The assertion follows. □

According to [DF3, 7.7(2) and 7.9] we have the following result (see also [F2, 3.7]).

Lemma 4.3. *For $A \in \Theta_\vartriangle^+(n)$ we have $\zeta_r(\theta_A^+) = \sum_{\mu\in\Lambda_\vartriangle(n,r-\sigma(A))} \theta_{A+\mathrm{diag}(\mu),r}$. In particular we have*

$$[\mathrm{diag}(\lambda)]\zeta_r(\theta_A^+) = \begin{cases} \theta_{A+\mathrm{diag}(\lambda-\mathrm{ro}(A)),r} & \text{if } \lambda-\mathrm{ro}(A)\in\mathbb{N}_\vartriangle^n \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\zeta_r(\theta_A^+)[\mathrm{diag}(\lambda)] = \begin{cases} \theta_{A+\mathrm{diag}(\lambda-\mathrm{co}(A)),r} & \text{if } \lambda-\mathrm{co}(A)\in\mathbb{N}_\vartriangle^n \\ 0 & \text{otherwise.} \end{cases}$$

for $\lambda \in \Lambda_\vartriangle(n,r)$.

For $A, B \in \Theta_\vartriangle(n,r)$ we write

$$\theta_{A,r}\theta_{B,r} = \sum_{C\in\Theta_\vartriangle(n,r)} \mathsf{g}_{A,B,C,r}\theta_{C,r} \tag{4.5}$$

where $\mathsf{g}_{A,B,C,r} \in \mathcal{Z}$. If $\mathsf{g}_{A,B,C,r} \neq 0$ then we have $\mathrm{co}(A) = \mathrm{ro}(B)$, $\mathrm{ro}(A) = \mathrm{ro}(C)$ and $\mathrm{co}(B) = \mathrm{co}(C)$.

Lemma 4.4. *Let $A, B\in\Theta_\vartriangle^+(n)$, $\lambda\in\Lambda_\vartriangle(n, r-\sigma(A))$ and $\mu\in\Lambda_\vartriangle(n,r-\sigma(B))$. If $\mathrm{co}(A)+\lambda = \mathrm{ro}(B)+\mu$ then we have*

$$\mathsf{g}_{A+\mathrm{diag}(\lambda),B+\mathrm{diag}(\mu),C',r} = \begin{cases} \mathsf{f}_{A,B,C} & \text{if } C' = C+\mathrm{diag}(\lambda+\mathrm{ro}(A-C)) \\ & \text{for some } C\in\Theta_\vartriangle^+(n), \\ 0 & \text{otherwise.} \end{cases}$$

for $C' \in \Theta_\vartriangle(n,r)$, where $\mathsf{f}_{A,B,C}$ is as given in (4.3).

Proof. By Lemma 4.3 we have

$$\begin{aligned}&\theta_{A+\mathrm{diag}(\lambda),r}\theta_{B+\mathrm{diag}(\mu),r}\\&=[\mathrm{diag}(\lambda+\mathrm{ro}(A))]\zeta_r(\theta_A^+)\zeta_r(\theta_B^+)[\mathrm{diag}(\mu+\mathrm{co}(B))]\\&=\sum_{\substack{C\in\Theta_\Delta^+(n),\,\mathbf{d}(C)=\mathbf{d}(A)+\mathbf{d}(B)\\ \lambda+\mathrm{ro}(A)-\mathrm{ro}(C)\in\mathbb{N}_\Delta^n}}\mathsf{f}_{A,B,C}\theta_{C+\mathrm{diag}(\lambda+\mathrm{ro}(A)-\mathrm{ro}(C)),r}[\mathrm{diag}(\mu+\mathrm{co}(B))].\end{aligned}$$

If $\mathbf{d}(C)=\mathbf{d}(A)+\mathbf{d}(B)$ then we have $\mathrm{ro}(C)-\mathrm{co}(C)=\mathrm{ro}(A+B)-\mathrm{co}(A+B)$ and hence $\mathrm{co}(C)+\lambda+\mathrm{ro}(A)-\mathrm{ro}(C)=\lambda+\mathrm{co}(A+B)-\mathrm{ro}(B)=\mu+\mathrm{co}(B)$. Thus we have

$$\theta_{A+\mathrm{diag}(\lambda),r}\theta_{B+\mathrm{diag}(\mu),r}=\sum_{\substack{C\in\Theta_\Delta^+(n),\,\mathbf{d}(C)=\mathbf{d}(A)+\mathbf{d}(B)\\ \lambda+\mathrm{ro}(A)-\mathrm{ro}(C)\in\mathbb{N}_\Delta^n}}\mathsf{f}_{A,B,C}\theta_{C+\mathrm{diag}(\lambda+\mathrm{ro}(A)-\mathrm{ro}(C)),r}.$$

The assertion follows. □

4.3. For $m\in\mathbb{Z}$ there is a map

$$\eta_m:\Theta_\Delta(n)\to\Theta_\Delta(n) \tag{4.6}$$

defined by sending $A=(a_{i,j})_{i,j\in\mathbb{Z}}$ to $(a_{i,mn+j})_{i,j\in\mathbb{Z}}$. Note that if $A=\jmath_\Delta(\lambda,d,\mu)\in\Theta_\Delta(n,r)$ then $\eta_m(A)=\jmath_\Delta(\lambda,d\rho^{mr},\mu)\in\Theta_\Delta(n,r)$.

Lemma 4.5. *Let $A\in\Theta_\Delta(n)$ and $m\in\mathbb{Z}$. If $a_{i,j}=0$ for $1\leqslant i\leqslant n$ and $j\leqslant mn$, then $\eta_k(A)\in\Theta_\Delta^+(n)$ for $k\leqslant m-1$.*

Proof. Let $B^{(k)}=\eta_k(A)$. If $k\leqslant m-1$, $1\leqslant i\leqslant n$ and $i\geqslant j$, then $kn+j\leqslant(m-1)n+j\leqslant(m-1)n+i\leqslant mn$ and hence $b_{i,j}^{(k)}=a_{i,kn+j}=0$. Thus $B^{(k)}\in\Theta_\Delta^+(n)$ for $k\leqslant m-1$. □

Lemma 4.6. *Let $A\in\Theta_\Delta(n,r)$ with $\lambda=\mathrm{ro}(A)$ and $\mu\in\mathrm{co}(A)$. Then we have $\theta_{A,r}\cdot\theta_{\mu,\mu}^{\rho^{mr}}=\theta_{\eta_m(A),r}=\theta_{\lambda,\lambda}^{\rho^{mr}}\cdot\theta_{A,r}$ for $m\in\mathbb{Z}$.*

Proof. Note that $C'_{w_{0,\mu}}=v^{-\ell(w_{0,\mu})}x_\mu$. Since $\rho^rx=x\rho^r$ for $x\in\mathfrak{S}_{\Delta,r}$ we have $\mathfrak{S}_\mu\rho^{mr}\mathfrak{S}_\mu=\mathfrak{S}_\mu\mathfrak{S}_\mu\rho^{mr}=\mathfrak{S}_\mu\rho^{mr}$. It follows that $w_{0,\mu}\rho^{mr}$ is the longest element in $\mathfrak{S}_\mu\rho^{mr}\mathfrak{S}_\mu$. This together with Proposition 3.8 implies that

$$\theta_{A,r}\theta_{\mu,\mu}^{\rho^{mr}}(C'_{w_{0,\mu}})=\theta_{A,r}(C'_{w_{0,\mu}\cdot\rho^{mr}})=\theta_{A,r}(C'_{w_{0,\mu}}T_\rho^{mr})=C'_{d^+\rho^{mr}}, \tag{4.7}$$

where $d\in\mathscr{D}_{\lambda,\mu}^\Delta$ is such that $\jmath_\Delta(\lambda,d,\mu)=A$ and d^+ is the unique longest element in $\mathfrak{S}_\lambda d\mathfrak{S}_\mu$. Furthermore since $\mathfrak{S}_\lambda d\rho^{mr}\mathfrak{S}_\mu=\mathfrak{S}_\lambda d\mathfrak{S}_\mu\rho^{mr}$, we see that

$d^+\rho^{mr}$ is the longest element in $\mathfrak{S}_\lambda d\rho^{mr}\mathfrak{S}_\mu$. It follows from (4.7) that

$$\theta_{\eta_m(A),r}(C'_{w_{0,\mu}}) = \theta^{d\rho^{mr}}_{\lambda,\mu}(C'_{w_{0,\mu}}) = C'_{d^+\rho^{mr}} = \theta_{A,r}\theta^{\rho^{mr}}_{\mu,\mu}(C'_{w_{0,\mu}}).$$

Thus we have $\theta_{A,r}\cdot\theta^{\rho^{mr}}_{\mu,\mu} = \theta_{\eta_m(A),r}$. This implies that $\theta^{d^{-1}}_{\mu,\lambda}\cdot\theta^{\rho^{-mr}}_{\lambda,\lambda} = \theta^{d^{-1}\rho^{-mr}}_{\mu,\lambda}$. Applying the map τ_r given in (2.2), we get $\theta^{\rho^{mr}}_{\lambda,\lambda}\cdot\theta_{A,r} = \tau_r(\theta^{d^{-1}}_{\mu,\lambda}\cdot\theta^{\rho^{-mr}}_{\lambda,\lambda}) = \tau_r(\theta^{d^{-1}\rho^{-mr}}_{\mu,\lambda}) = \theta_{\eta_m(A),r}$. □

Assume $N \geqslant n$. There is a natural injective map

$$\widetilde{\ }: \Theta_\vartriangle(n) \longrightarrow \Theta_\vartriangle(N), \quad A = (a_{i,j}) \longmapsto \widetilde{A} = (\widetilde{a}_{i,j}),$$

where $\widetilde{A} = (\widetilde{a}_{i,j})$ is defined by

$$\widetilde{a}_{k,l+mN} = \begin{cases} a_{k,l+mn}, & \text{if } 1 \leqslant k,l \leqslant n;\\ 0, & \text{if either } n < k \leqslant N \text{ or } n < l \leqslant N \end{cases}$$

for $m \in \mathbb{Z}$. Note that the map $\widetilde{\ }: \Theta_\vartriangle(n) \longrightarrow \Theta_\vartriangle(N)$ induces a map from $\Theta^+_\vartriangle(n)$ to $\Theta^+_\vartriangle(N)$. Similarly, there is an injective map

$$\widetilde{\ }: \mathbb{Z}^n_\vartriangle \longrightarrow \mathbb{Z}^N_\vartriangle, \quad \lambda \longmapsto \widetilde{\lambda},$$

where $\widetilde{\lambda}_i = \lambda_i$ for $1 \leqslant i \leqslant n$ and $\widetilde{\lambda}_i = 0$ for $n+1 \leqslant i \leqslant N$.

It is easy to see that there is an injective algebra homomorphism (not sending 1 to 1)

$$\iota_{n,N} : \boldsymbol{\mathcal{S}}_\vartriangle(n,r) \longrightarrow \boldsymbol{\mathcal{S}}_\vartriangle(N,r), \quad [A] \longmapsto [\widetilde{A}] \ \text{ for } A \in \Theta_\vartriangle(n,r)$$

(see [DDF, §4.1]).

Let $\Theta_\vartriangle(n,r)^{\mathrm{ap}} = \Theta_\vartriangle(n)^{\mathrm{ap}} \cap \Theta_\vartriangle(n,r)$.

Lemma 4.7. *Assume $N > n$. Then for $A \in \Theta_\vartriangle(n,r)$ we have $\widetilde{A} \in \Theta_\vartriangle(N,r)^{\mathrm{ap}}$ and $\iota_{n,N}(\theta_{A,r}) = \theta_{\widetilde{A},r}$. In particular we have $\mathrm{g}_{A,B,C,r} = \mathrm{g}_{\widetilde{A},\widetilde{B},\widetilde{C},r}$ for $A,B,C \in \Theta_\vartriangle(n,r)$, where $\mathrm{g}_{A,B,C,r}$ is as given in* (4.5).

Proof. The first assertion follows from the definition of $\widetilde{A}$. The second assertion follows from Proposition 3.8 and (3.3). □

Recall the map η_m defined in (4.6). The structure constants for the canonical basis $\mathbf{B}(n,r) = \{\theta_{A,r} \mid A \in \Theta_\vartriangle(n,r)\}$ of the affine quantum Schur algebra $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ and the canonical basis $\mathbf{B}(N)^{\mathrm{ap}} = \{\theta^+_A \mid A \in \Theta^+_\vartriangle(N)^{\mathrm{ap}}\}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ are related as follows.

Theorem 4.8. *Assume $N \geqslant n$. Let $A, B, C \in \Theta_\vartriangle(n,r)$ and $C' \in \Theta_\vartriangle(N,r)$.*

(1) *We have*

$$\mathsf{g}_{\widetilde{\eta_k(A)},\widetilde{\eta_k(B)},C',r} = \begin{cases} \mathsf{g}_{A,B,X,r} & \text{if } C' = \eta_{2k}(X) \text{ for some } X \in \Theta_\vartriangle(n,r) \\ 0 & \text{otherwise} \end{cases}$$

for $k \in \mathbb{Z}$, where $\mathsf{g}_{A,B,X,r}$ is as given in (4.5).

(2) *If $N > n$ and $\mathrm{co}(A) = \mathrm{ro}(B)$, then there exist $k_0 \in \mathbb{Z}$ such that for $k \leqslant k_0$, $\widetilde{\eta_k(A)}$, $\widetilde{\eta_k(B)}$, $\widetilde{\eta_{2k}(C)} \in \Theta^+_\vartriangle(N) \cap \Theta_\vartriangle(N,r)^{\mathrm{ap}}$ and $\mathsf{g}_{A,B,C,r} = \mathsf{f}_{\widetilde{\eta_k(A)},\widetilde{\eta_k(B)},\widetilde{\eta_{2k}(C)}}$, where $\mathsf{f}_{\widetilde{\eta_k(A)},\widetilde{\eta_k(B)},\widetilde{\eta_{2k}(C)}}$ is as given in* (4.3).

Proof. If $\mathrm{co}(A) \neq \mathrm{ro}(B)$ then $\theta_{A,r}\theta_{B,r} = \theta_{\eta_k(A)}\theta_{\eta_k(B)} = 0$ for any $k \in \mathbb{Z}$. Now we assume $\mathrm{co}(A) = \mathrm{ro}(B)$. Let $\lambda = \mathrm{ro}(A)$ and $\nu = \mathrm{co}(B)$. Then by Lemma 4.6 we have

$$\begin{aligned} \theta_{\eta_k(A),r}\theta_{\eta_k(B),r} &= \theta^{\rho^{kr}}_{\lambda,\lambda}\theta_{A,r}\theta_{B,r}\theta^{\rho^{kr}}_{\nu,\nu} \\ &= \sum_{X \in \Theta_\vartriangle(n,r)} \mathsf{g}_{A,B,X,r}\theta_{\eta_k(X),r}\theta^{\rho^{kr}}_{\nu,\nu} \\ &= \sum_{X \in \Theta_\vartriangle(n,r)} \mathsf{g}_{A,B,X,r}\theta_{\eta_{2k}(X),r} \end{aligned} \tag{4.8}$$

for $k \in \mathbb{Z}$. Applying $\iota_{n,N}$ to (4.8) gives that

$$\iota_{n,N}(\theta_{\eta_k(A),r})\iota_{n,N}(\theta_{\eta_k(B),r}) = \sum_{X \in \Theta_\vartriangle(n,r)} \mathsf{g}_{A,B,X,r}\iota_{n,N}(\theta_{\eta_{2k}(X),r})$$

for $k \in \mathbb{Z}$. Thus by Lemma 4.7 we have

$$\theta_{\widetilde{\eta_k(A)},r}\theta_{\widetilde{\eta_k(B)},r} = \sum_{X \in \Theta_\vartriangle(n,r)} \mathsf{g}_{A,B,X,r}\theta_{\widetilde{\eta_{2k}(X)},r} \tag{4.9}$$

for $k \in \mathbb{Z}$. The assertion (1) follows. The assertion (2) follows from the assertion (1), Lemma 4.4, Lemma 4.5 and Lemma 4.7. □

As a corollary to Theorem 4.8, together with Theorem 4.1 we have the following positivity property for $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$. This gives an alternate approach to Lusztig's result on positivity property for $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ in [L6, 4.5].

Corollary 4.9. *For $A, B, C \in \Theta_\vartriangle(n,r)$ we have $\mathsf{g}_{A,B,C,r} \in \mathbb{N}[v, v^{-1}]$.*

4.4. There is an injective map from $\mathbf{B}(n)$ to $\mathbf{B}(N)^{\mathrm{ap}}$ defined by sending θ_A^+ to $\theta_{\widetilde{A}}^+$ for $A \in \Theta_\vartriangle^+(n)$. The structure constants for the canonical basis $\mathbf{B}(n)$ of $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^+$ and the canonical basis $\mathbf{B}(N)^{\mathrm{ap}}$ of $U(\widehat{\mathfrak{sl}}_N)_\mathcal{Z}^+$ are related as follows.

Theorem 4.10. *Assume $N > n$. For $A, B, C \in \Theta_\vartriangle^+(n)$ we have $\mathsf{f}_{A,B,C} = \mathsf{f}_{\widetilde{A},\widetilde{B},\widetilde{C}}$, where $\mathsf{f}_{A,B,C}$ is as given in* (4.3)*.*

Proof. There exist $\lambda, \mu \in \mathbb{N}_\vartriangle^n$ such that $\lambda + \mathrm{co}(A) = \mu + \mathrm{ro}(B)$ and $\lambda + \mathrm{ro}(A) - \mathrm{ro}(C) \in \mathbb{N}_\vartriangle^n$. Let $r = \sigma(\lambda) + \sigma(A)$. Then by Lemmas 4.4 and 4.7 we have

$$\begin{aligned}\mathsf{f}_{A,B,C} &= \mathsf{g}_{A+\mathrm{diag}(\lambda),B+\mathrm{diag}(\mu),C+\mathrm{diag}(\lambda+\mathrm{ro}(A-C))}\\ &= \mathsf{g}_{\widetilde{A}+\mathrm{diag}(\widetilde{\lambda}),\widetilde{B}+\mathrm{diag}(\widetilde{\mu}),\widetilde{C}+\mathrm{diag}(\widetilde{\lambda}+\mathrm{ro}(\widetilde{A}-\widetilde{C}))} = \mathsf{f}_{\widetilde{A},\widetilde{B},\widetilde{C}}.\end{aligned}$$

□

The following result is a generalization of Theorem 4.1, which gives the positivity property for $\mathfrak{D}_\vartriangle(n)_\mathcal{Z}^+$.

Corollary 4.11. *For $A, B, C \in \Theta_\vartriangle^+(n)$ we have $\mathsf{f}_{A,B,C} \in \mathbb{N}[v, v^{-1}]$.*

Proof. The assertion follows from Theorem 4.1 and Theorem 4.10. □

5. Positivity properties for $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$

5.1. Recall that $I = \mathbb{Z}/n\mathbb{Z}$ and I is identified with $\{1, 2, \ldots, n\}$. There is an algebra grading over $\mathbb{Z}[I]$

$$\mathbf{U}(\widehat{\mathfrak{sl}}_n) = \bigoplus_{\nu \in \mathbb{Z}[I]} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_\nu$$

defined by the condition $\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'}\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu''} \subseteq \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'+\nu''}$, $\widetilde{K}_i \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_0$, $u^+_{E^\vartriangle_{i,i+1}} \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_i$, $u^-_{E^\vartriangle_{i+1,i}} \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{-i}$ for all $\nu', \nu'' \in \mathbb{Z}[I]$, $i \in I$.

Let us recall the definition of the modified quantum affine algebra $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$. Let X be the quotient of $\mathbb{Z}_\vartriangle^n$ by the subgroup generated by the element $\mathbf{1}$, where $\mathbf{1}_i = 1$ for all i. For $\lambda \in \mathbb{Z}_\vartriangle^n$ let $\bar{\lambda} \in X$ be the image of λ in X. Let $Y = \{\mu \in \mathbb{Z}_\vartriangle^n \mid \sum_{1 \leqslant i \leqslant n} \mu_i = 0\}$. For $\bar{\lambda} \in X$ and $\mu \in Y$ we set $\mu \cdot \bar{\lambda} = \sum_{1 \leqslant i \leqslant n} \lambda_i \mu_i$.

For $i \in I$ let $\boldsymbol{e}_i^\vartriangle \in \mathbb{N}_\vartriangle^n$ be the element satisfying $(\boldsymbol{e}_i^\vartriangle)_j = \delta_{i,j}$ for $j \in I$. There is a natural map $I \to X$ defined by sending i to $\boldsymbol{\alpha}_i^\vartriangle$, where $\boldsymbol{\alpha}_i^\vartriangle = \boldsymbol{e}_i^\vartriangle - \boldsymbol{e}_{i+1}^\vartriangle$. The imbedding $I \to X$ induce a homomorphism $\iota : \mathbb{Z}[I] \to X$.

For $\bar\lambda, \bar\mu \in X$ we set

$$ {}_{\bar\lambda}\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar\mu} = \mathbf{U}(\widehat{\mathfrak{sl}}_n) \Big/ \left(\sum_{\mathbf{j}\in Y} (K^{\mathbf{j}} - v^{\mathbf{j}\cdot\bar\lambda})\mathbf{U}(\widehat{\mathfrak{sl}}_n) + \sum_{\mathbf{j}\in Y} \mathbf{U}(\widehat{\mathfrak{sl}}_n)(K^{\mathbf{j}} - v^{\mathbf{j}\cdot\bar\mu}) \right). $$

Let $\pi_{\bar\lambda,\bar\mu} : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \to {}_{\bar\lambda}\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar\mu}$ be the canonical projection. Let

$$ \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n) := \bigoplus_{\bar\lambda,\bar\mu\in X} {}_{\bar\lambda}\mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar\mu}. $$

We define the product in $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ as follows. Let $\lambda', \mu', \lambda'', \mu'' \in X$ and $\nu', \nu'' \in \mathbb{Z}[I]$ with $\lambda' - \mu' = \iota(\nu')$ and $\lambda'' - \mu'' = \iota(\nu'')$. For $t \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu'}$, $s \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\nu''}$, define

$$ \pi_{\lambda',\mu'}(t)\pi_{\lambda'',\mu''}(s) = \begin{cases} \pi_{\lambda',\mu''}(ts), & \text{if } \mu' = \lambda'' \\ 0 & \text{otherwise.} \end{cases} $$

Then $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ becomes an associative $\mathbb{Q}(v)$-algebra structure with respect to the above product.

5.2. Let $\mathfrak{D}'_\triangle(n)$ be the subalgebra of $\mathfrak{D}_\triangle(n)$ generated by the elements u_A^+, u_A^- and $\widetilde{K}_i^{\pm1}$ for $A \in \Theta_\triangle^+(n)$ and $i \in I$. The algebra $\mathfrak{D}'_\triangle(n)$ is a $\mathbb{Z}[I]$-graded algebra with

$$ \deg(u_A^+) = \sum_{1\leqslant i\leqslant n} d_i i,\ \deg(u_A^-) = -\sum_{1\leqslant i\leqslant n} d_i i \text{ and } \deg(\widetilde{K}_i^{\pm1}) = 0 $$

for $A \in \Theta_\triangle^+(n)$ and $1 \leqslant i \leqslant n$, where $(d_i)_{i\in\mathbb{Z}} = \mathbf{d}(A)$.

Let

$$ \dot{\mathfrak{D}}'_\triangle(n) := \bigoplus_{\bar\lambda,\bar\mu\in X} {}_{\bar\lambda}\mathfrak{D}'_\triangle(n)_{\bar\mu}, $$

where ${}_{\bar\lambda}\mathfrak{D}'_\triangle(n)_{\bar\mu} = \mathfrak{D}'_\triangle(n)/\big(\sum_{\mathbf{j}\in Y}(K^{\mathbf{j}} - v^{\mathbf{j}\cdot\bar\lambda})\mathfrak{D}'_\triangle(n) + \sum_{\mathbf{j}\in Y}\mathfrak{D}'_\triangle(n)(K^{\mathbf{j}} - v^{\mathbf{j}\cdot\bar\mu})\big)$. As in the case of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, there is a natural associative $\mathbb{Q}(v)$-algebra structure on $\dot{\mathfrak{D}}'_\triangle(n)$ inherited from that of $\mathfrak{D}'_\triangle(n)$. We will naturally regard $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ as a subalgebra of $\dot{\mathfrak{D}}'_\triangle(n)$.

For $\bar\lambda, \bar\mu \in X$, let $\pi_{\bar\lambda,\bar\mu} : \mathfrak{D}'_\triangle(n) \to {}_{\bar\lambda}\mathfrak{D}'_\triangle(n)_{\bar\mu}$ be the canonical projection. The algebra $\dot{\mathfrak{D}}'_\triangle(n)$ is naturally a $\mathfrak{D}'_\triangle(n)$-bimodule defined by

$$ t'\pi_{\lambda',\lambda''}(s)t'' = \pi_{\lambda'+\iota(\nu'),\lambda''-\iota(\nu'')}(t'st'') $$

for $t' \in \mathfrak{D}'_\triangle(n)_{\nu'}$, $s \in \mathfrak{D}'_\triangle(n)$, $t'' \in \mathfrak{D}'_\triangle(n)_{\nu''}$ and $\lambda', \lambda'' \in X$.

For $\bar{\lambda} \in X$ let $1_{\bar{\lambda}} = \pi_{\bar{\lambda},\bar{\lambda}}(1)$. The map ζ_r defined in Theorem 2.2 induces an surjective algebra homomorphism

$$\dot{\zeta}_r : \dot{\mathfrak{D}}'_\Delta(n) \to \boldsymbol{\mathcal{S}}_\Delta(n,r)$$

such that for $A \in \Theta^+_\Delta(n)$ and $\bar{\lambda} \in X$, $\dot{\zeta}_r(u^\pm_A 1_{\bar{\lambda}}) = \zeta_r(u^\pm_A)[\mathrm{diag}(\mu)]$, if $\bar{\lambda} = \bar{\mu}$ for some $\mu \in \Lambda_\Delta(n,r)$, and $\dot{\zeta}_r(u^\pm_A 1_{\bar{\lambda}}) = 0$ otherwise (cf. [F1, 3.6]).

The maps $\dot{\zeta}_r$ induce an algebra homomorphism

$$\dot{\zeta} : \dot{\mathfrak{D}}'_\Delta(n) \to \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\Delta(n,r)$$

such that $\dot{\zeta}(x) = (\dot{\zeta}_r(x))_{r\geqslant 0}$ for $x \in \dot{\mathfrak{D}}'_\Delta(n)$. The following result is a generalization of Lusztig [L7, 3.5].

Theorem 5.1. *The map $\dot{\zeta} : \dot{\mathfrak{D}}'_\Delta(n) \to \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\Delta(n,r)$ is injective.*

Proof. Note that the set $\{1_{\bar{\lambda}}\widetilde{u}^+_A\widetilde{u}^-_B \mid A, B \in \Theta^+_\Delta(n),\ \bar{\lambda} \in X\}$ forms a $\mathbb{Q}(v)$-basis for $\dot{\mathfrak{D}}'_\Delta(n)$. We use reduction to absurdity. Assume

$$x = \sum_{A\in\Theta^\pm_\Delta(n),\,\bar{\lambda}\in X} \beta_{A,\bar{\lambda}} 1_{\bar{\lambda}}\widetilde{u}^+_{A^+}\widetilde{u}^-_{{}^t(A^-)} \neq 0 \in \dot{\mathfrak{D}}'_\Delta(n)$$

is such that $\dot{\zeta}(x) = 0$. Then there exist $\boldsymbol{a} \in X$ such that $1_{\boldsymbol{a}}x \neq 0$. Since the set

$$\mathcal{T} := \{A \mid A \in \Theta^\pm_\Delta(n),\ \beta_{A,\boldsymbol{a}} \neq 0\}$$

is finite we may choose a maximal element B in $\mathcal{T}$ with respect to $\preccurlyeq$. We choose $\mu \in \mathbb{N}^n_\Delta$ such that $\bar{\mu} = \boldsymbol{a}$ and $\mu \geqslant \mathrm{ro}(B)$. Let $r_0 = \sigma(\mu)$. Then we have

$$0 = [\mathrm{diag}(\mu)]\dot{\zeta}_{r_0}(x) = \sum_{A\in\mathcal{T}} \beta_{A,\boldsymbol{a}}[\mathrm{diag}(\mu)]A^+(\mathbf{0},r)A^-(\mathbf{0},r).$$

By [DDF, 3.7.3] we have

$$A^+(\mathbf{0},r)A^-(\mathbf{0},r) = A(\mathbf{0},r) + \sum_{C\in\Theta_\Delta(n,r),\, C\prec A} \gamma_{A,C}[C]$$

where $\gamma_{A,C} \in \mathbb{Q}(v)$. This implies that

$$\begin{aligned}
&\sum_{A\in\mathcal{T}} \beta_{A,\boldsymbol{a}}[\mathrm{diag}(\mu)]A^+(\mathbf{0},r)A^-(\mathbf{0},r) \\
&= \beta_{B,\boldsymbol{a}}[\mathrm{diag}(\mu)]\Bigg(B(\mathbf{0},r) + \sum_{\substack{C\in\Theta_\vartriangle(n,r)\\ C\prec B}} \gamma_{B,C}[C]\Bigg) \\
&\quad + \sum_{\substack{A\in\mathcal{T}\\ B\not\preccurlyeq A}} \beta_{A,\boldsymbol{a}}[\mathrm{diag}(\mu)]\Bigg(A(\mathbf{0},r) + \sum_{\substack{C\in\Theta_\vartriangle(n,r)\\ C\prec A}} \gamma_{A,C}[C]\Bigg) \\
&= \beta_{B,\boldsymbol{a}}[B + \mathrm{diag}(\mu - \mathrm{ro}(B))] + f
\end{aligned}$$

where f is a linear combination of $[C' + \mathrm{diag}(\nu)]$ such that $C' \neq B$, $C' \in \Theta^{\pm}_\vartriangle(n)$ and $\nu \in \Theta_\vartriangle(n, r - \sigma(C'))$. Thus we have $\beta_{B,\boldsymbol{a}} = 0$. This is a contradiction. □

5.3. Let $\dot{\mathbf{B}}(n)$ be the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ defined in [L5]. Let $\phi_{r+n,r}$: $\boldsymbol{\mathcal{S}}_\vartriangle(n, r+n) \to \boldsymbol{\mathcal{S}}_\vartriangle(n, r)$ be the algebra homomorphism defined in [L7, 1.11]. According to [L7, 3.4(a)] we have

$$\phi_{r+n,r} \circ \dot{\zeta}_{r+n}(x) = \dot{\zeta}_r(x) \tag{5.1}$$

for all $r \in \mathbb{N}$ and $x \in \dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$. The following result was proved by Schiffmann–Vasserot [SV] (see also Lusztig [L7, 4.1] and Mcgerty [M, 7.10]).

Theorem 5.2. (1) *We have* $\dot{\zeta}_r(\dot{\mathbf{B}}(n)) \subseteq \{0\} \cup \{\theta_{A,r} \mid A \in \Theta_\vartriangle(n,r)\}$.

(2) *For* $A \in \Theta_\vartriangle(n, r+n)^{\mathrm{ap}}$ *we have*

$$\phi_{r+n,r}(\theta_{A,r+n}) = \begin{cases} \theta_{A-E,r} & \text{if } a_{i,i} \geqslant 1 \text{ for } 1 \leqslant i \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

where $E = (\delta_{i,j})_{i,j\in\mathbb{Z}} \in \Theta_\vartriangle(n)$.

For $A \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $A - E \notin \Theta_\vartriangle(n)$ let $\mathfrak{b}_A = (a_r)_{r\geqslant 0} \in \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$, where $a_r = \theta_{A+mE,r}$ if $r = \sigma(A) + mn$ for some $m \geqslant 0$, and $a_r = 0$ otherwise.

Lemma 5.3. *We have* $\dot{\zeta}(\dot{\mathbf{B}}(n)) = \{\mathfrak{b}_A \mid A \in \Theta_\vartriangle(n)^{\mathrm{ap}},\ A - E \notin \Theta_\vartriangle(n)\}$.

Proof. Let $b \in \dot{\mathbf{B}}(n)$. By Theorem 5.1 we have $\dot{\zeta}(b) \neq 0$. Let $r_0 = \min\{r \in \mathbb{N} \mid \dot{\zeta}_r(b) \neq 0\}$. Then by Theorem 5.2(1) and [L6, 8.2] we have $\dot{\zeta}_{r_0}(b) = \theta_{A,r_0}$

for some $A \in \Theta_\vartriangle(n, r_0)^{\mathrm{ap}}$. From (5.1) we see that $\phi_{r_0,r_0-n}(\theta_{A,r_0}) = \phi_{r_0,r_0-n} \circ \dot\zeta_{r_0}(b) = \dot\zeta_{r_0-n}(b) = 0$. Thus by Theorem 5.2(2) we have $A - E \notin \Theta_\vartriangle(n)$. By the proof of [L7, 4.3], we know that if $\dot\zeta_r(b) \neq 0$ for some $r > r_0$, then $r \equiv r_0$ mod n. Furthermore, if $m > 0$ then by (5.1) we have

$$\theta_{A,r_0} = \dot\zeta_{r_0}(b) = \phi_{r_0+n,r_0} \circ \phi_{r_0+2n,r_0+n} \circ \cdots \circ \phi_{r_0+mn,r_0+(m-1)n} \circ \dot\zeta_{r_0+mn}(b).$$

This together with Theorem 5.2 implies that $\dot\zeta_{r_0+mn}(b) = \theta_{A+mE,r_0+mn}$. Thus we have $\dot\zeta(b) = \mathfrak{b}_A$.

On the other hand, if $A' \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $A' - E \notin \Theta_\vartriangle(n)$, by [L6, 8.2] we conclude that there exists $b' \in \dot{\mathbf{B}}(n)$ such that $\dot\zeta_{r_0'}(b') = \theta_{A',r_0'}$, where $r_0' = \sigma(A')$. By the proof above we conclude that $\dot\zeta(b') = \mathfrak{b}_{A'}$. The assertion follows. □

By Theorem 5.1 and Lemma 5.3 we conclude that for each $A \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $A - E \notin \Theta_\vartriangle(n)$, there exists a unique $\mathfrak{c}_A \in \dot{\mathbf{B}}(n)$ such that $\dot\zeta(\mathfrak{c}_A) = \mathfrak{b}_A$. Furthermore we have

$$\dot{\mathbf{B}}(n) = \{\mathfrak{c}_A \mid A \in \Theta_\vartriangle(n)^{\mathrm{ap}},\ A - E \notin \Theta_\vartriangle(n)\}.$$

Thus $\dot{\mathbf{B}}(n)$ is indexed by the set $\{A \in \Theta_\vartriangle(n)^{\mathrm{ap}} \mid A - E \notin \Theta_\vartriangle(n)\}$. For $A, B \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $A - E, B - E \notin \Theta_\vartriangle(n)$ we write

$$\mathfrak{c}_A\mathfrak{c}_B = \sum_{\substack{C\in\Theta_\vartriangle(n)^{\mathrm{ap}}\\ C-E\notin\Theta_\vartriangle(n)}} \mathsf{h}_{A,B,C}\mathfrak{c}_C, \tag{5.2}$$

where $\mathsf{h}_{A,B,C} \in \mathcal{Z}$.

Recall the map η_m defined in (4.6). The structure constants for the canonical basis $\dot{\mathbf{B}}(n)$ of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ and the structure constants for the canonical basis $\mathbf{B}(N)^{\mathrm{ap}} = \{\theta_A^+ \mid A \in \Theta_\vartriangle^+(N)^{\mathrm{ap}}\}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ are related in the following way.

Theorem 5.4. *Assume $N > n$. Let $A, B \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $A - E, B - E \notin \Theta_\vartriangle(n)$. If $C \in \Theta_\vartriangle(n)^{\mathrm{ap}}$ with $C - E \notin \Theta_\vartriangle(n)$ is such that $\mathsf{h}_{A,B,C} \neq 0$, then there exist $m_1, m_2, m_C \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$ such that $\sigma(A) + nm_1 = \sigma(B) + nm_2 = \sigma(C) + nm_C$, $\widetilde{A_k}, \widetilde{B_k}, \widetilde{C_k} \in \Theta_\vartriangle^+(N)^{\mathrm{ap}}$ and*

$$\mathsf{h}_{A,B,C} = \mathsf{f}_{\widetilde{A_k},\widetilde{B_k},\widetilde{C_k}}$$

for $k \leqslant k_0$, where $A_k = \eta_k(A + m_1E)$, $B_k = \eta_k(B + m_2E)$, $C_k = \eta_{2k}(C + m_CE)$ and $\mathsf{f}_{\widetilde{A_k},\widetilde{B_k},\widetilde{C_k}}$ is as given in (4.3).

Proof. By (5.2) we have

$$\mathfrak{b}_A \mathfrak{b}_B = \sum_{\substack{C\in\Theta_\vartriangle(n)^{\mathrm{ap}}\\ C-E\notin\Theta_\vartriangle(n)}} \mathsf{h}_{A,B,C}\mathfrak{b}_C, \tag{5.3}$$

where $\mathsf{h}_{A,B,C} \in \mathcal{Z}$. If $\sigma(A) \not\equiv \sigma(B) \bmod n$ then by definition we have $\mathfrak{b}_A\mathfrak{b}_B = 0$. Now we assume $\sigma(A) \equiv \sigma(B) \bmod n$. Let $\mathcal{X} = \{C \in \Theta_\vartriangle(n)^{\mathrm{ap}} \mid C - E \notin \Theta_\vartriangle(n),\ \mathsf{h}_{A,B,C} \neq 0\}$. We choose $r_0 \in \mathbb{N}$ such that $r_0 \equiv \sigma(A) \bmod n$, $r_0 \geqslant \sigma(A)$, $r_0 \geqslant \sigma(B)$ and $r_0 \geqslant \sigma(C)$ for $C \in \mathcal{X}$. Note that $\sigma(C) \equiv \sigma(A) \bmod n$ for $C \in \mathcal{X}$. Assume $r_0 = \sigma(A) + nm_1 = \sigma(B) + nm_2 = \sigma(C) + nm_C$ for $C \in \mathcal{X}$. Then by (5.3) we have

$$\theta_{A+m_1E,r_0}\theta_{B+m_2E,r_0} = \sum_{C\in\mathcal{X}} \mathsf{h}_{A,B,C}\theta_{C+m_CE,r_0}.$$

This implies that $\mathsf{h}_{A,B,C} = \mathsf{g}_{A+m_1E,B+m_2E,C+m_CE,r_0}$. Now the assertion follows from Theorem 4.8. □

The following theorem gives the positivity property for $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$.

Theorem 5.5. *For $b, b' \in \dot{\mathbf{B}}(n)$ we have $bb' \in \sum_{b''\in\dot{\mathbf{B}}(n)} \mathbb{N}[v, v^{-1}]b''$.*

Proof. The assertion follows from Theorem 4.1 and Theorem 5.4. □

6. A weak positivity property for $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$

For $\lambda, \mu \in \mathbb{Z}^n_\vartriangle$ we set ${}_\lambda\boldsymbol{\mathfrak{D}}_\vartriangle(n)_\mu = \boldsymbol{\mathfrak{D}}_\vartriangle(n)/{}_\lambda I_\mu$, where

$${}_\lambda I_\mu = \sum_{\mathbf{j}\in\mathbb{Z}^n_\vartriangle} (K^{\mathbf{j}} - v^{\lambda\cdot\mathbf{j}})\boldsymbol{\mathfrak{D}}_\vartriangle(n) + \sum_{\mathbf{j}\in\mathbb{Z}^n_\vartriangle} \boldsymbol{\mathfrak{D}}_\vartriangle(n)(K^{\mathbf{j}} - v^{\mu\cdot\mathbf{j}}).$$

Let $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n) := \bigoplus_{\lambda,\mu\in\mathbb{Z}^n_\vartriangle} {}_\lambda\boldsymbol{\mathfrak{D}}_\vartriangle(n)_\mu$. As in the case of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, there is a natural associative $\mathbb{Q}(v)$-algebra structure on $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$ inherited from that of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$ (see [F1]). The algebra $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$ is the modified form of $\boldsymbol{\mathfrak{D}}_\vartriangle(n)$. Let $\{\theta_A \mid A \in \widetilde{\Theta}_\vartriangle(n)\}$ be the canonical basis of $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$ defined in [DF3], where $\widetilde{\Theta}_\vartriangle(n)$ is given in §1.

Proposition 6.1 ([DF3, 7.7]). *There is a surjective algebra homomorphism $\dot{\xi}_r : \dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n) \to \boldsymbol{\mathcal{S}}_\vartriangle(n, r)$ such that*

$$\dot{\xi}_r(\theta_A) = \begin{cases} \theta_{A,r}, & \text{if } A \in \Theta_\vartriangle(n, r); \\ 0, & \text{otherwise.} \end{cases}$$

The maps $\dot{\xi}_r$ induce an algebra homomorphism

$$\dot{\xi} : \dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n) \to \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$$

such that $\dot{\xi}(x) = (\dot{\xi}_r(x))_{r\geqslant 0}$ for $x \in \dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$. Contrast to Theorem 5.1, the map $\dot{\xi}$ is not injective. For $A \in \widetilde{\Theta}_\vartriangle(n)$ let $\overline{\theta_A} = \theta_A + \ker(\dot{\xi}) \in \dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)/\ker(\dot{\xi})$.

Lemma 6.2. *We have $\overline{\theta_A} = 0$ for $A \notin \Theta_\vartriangle(n)$ and the set $\{\overline{\theta_A} \mid A \in \Theta_\vartriangle(n)\}$ forms a $\mathbb{Q}(v)$-basis for $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)/\ker\dot{\xi}$.*

Proof. From Proposition 6.1 we see that $\ker\dot{\xi} = \mathrm{span}_{\mathbb{Q}(v)}\{\theta_A \mid A \in \widetilde{\Theta}_\vartriangle(n), A \notin \Theta_\vartriangle(n)\}$. The assertion follows. □

The following result gives a weak version of the positivity property for $\dot{\boldsymbol{\mathfrak{D}}}_\vartriangle(n)$.

Theorem 6.3. *For $A, B \in \Theta_\vartriangle(n)$ we have $\overline{\theta_A} \cdot \overline{\theta_B} \in \sum_{C\in\Theta_\vartriangle(n)} \mathbb{N}[v, v^{-1}]\overline{\theta_C}$.*

Proof. The assertion follows from Corollary 4.9, Proposition 6.1, and Lemma 6.2. □

Acknowledgements

The first author was supported by the National Natural Science Foundation of China (11671297), Fok Ying Tung Education Foundation (131004), Shanghai Education Development Foundation and Shanghai Municipal Education Commission (Shuguang Program 16SG16).

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Received June 12, 2016