Iitaka's $C_{n,m}$ conjecture for 3-folds in positive characteristic

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In this paper, we prove that for a fibration $f : X \to Z$ from a smooth projective 3-fold to a smooth projective curve, over an algebraically closed field k with $\operatorname{char} k = p > 5$, if the geometric generic fiber $X_{\bar{\eta}}$ is smooth, then subadditivity of Kodaira dimensions holds, i.e.
$$\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(Z).$$

1. Introduction

Throughout this paper, a *fibration* means a projective morphism $f : X \to Y$ between varieties such that the natural morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

The Kodaira dimension is one of the most important birational invariants and plays a key role in the birational classification of algebraic varieties. For a fibration, we have the following conjecture on Kodaira dimensions, which was proposed by Iitaka in characteristic zero.

Conjecture 1.1 ($C_{n,m}$). *Let $f : X \to Z$ be a fibration between smooth projective varieties of dimension n and m respectively over an algebraically closed field k with $\operatorname{char} k = p \geq 0$, whose geometric generic fiber $X_{\bar{\eta}}$ is integral and smooth. Then*

$$\kappa(X) \geq \kappa(X_{\bar{\eta}}) + \kappa(Z).$$

In characteristic zero, many results related to this conjecture are known [4, 9, 10, 12, 17, 18, 22–24, 26–28, 36, 37]. In particular, this conjecture was reduced to problems in the minimal model program by Kawamata [24, Corollary 1.2].

In positive characteristic, Conjecture 1.1 has been proved in some cases recently. Chen and Zhang showed $C_{n,n-1}$ [13, Theorem 1.2]. Under the assumption that $p > 5$, $C_{3,1}$ was shown when $k = \overline{\mathbb{F}_p}$ by Birkar, Chen and

Zhang [6, Theorem 1.2], when $X_{\overline{\eta}}$ is of general type [14, Theorem 1.5], and when the genus of Z is at least two by Zhang [40, Corollary 1.9]. Furthermore, when f has singular geometric generic fiber, its dualizing sheaf, denoted by $\omega_{X_{\overline{\eta}}}$ (the same notation with canonical sheaf because they coincide with each other when $X_{\overline{\eta}}$ is smooth), was considered by [32] and [40], and under some special situations, an analogous inequality $\kappa(X) \geq \kappa(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}) + \kappa(Z)$ was proved.

The aim of this paper is to prove the theorem below.

Theorem 1.2. *Conjecture $C_{3,m}$ holds when* $\mathrm{char} k = p > 5$.

The proof relies on the minimal model program for varieties of dimension at most three in characteristic $p > 5$ developed by several mathematicians including Birkar, Cascini, Hacon, Tanaka, Waldron and Xu. The case when $\kappa(X_{\overline{\eta}}) = 2$ has been proved by the first author [14, Theorem 1.5]. In this paper we only need to consider the cases $\kappa(X_{\overline{\eta}}) = 0, 1$.

This paper is organized as follows. Section 2 includes some basic results to be used in our proof, including minimal model theory of 3-folds, vector bundles on elliptic curves and weak positivity of push-forward of pluri-relative canonical sheaves. Section 3 and 4 are devoted to study the cases $\kappa(X_{\overline{\eta}}) = 0$ and 1 respectively.

Notation and Conventions: In this paper, we fix an algebraically closed field k of characteristic $p > 0$. A *k-scheme* is a separated scheme of finite type over k. A *variety* means an integral k-scheme, and a *curve* (resp. *surface, n-fold*) means a variety of dimension one (resp. two, n).

Let $\varphi : S \to T$ be a morphism of schemes and let T' be a T-scheme. Then we denote by $S_{T'}$ and $\varphi_{T'} : S_{T'} \to T'$ respectively the fiber product $S \times_T T'$ and its second projection. For a prime $p \in \mathbb{Z}$, $\mathbb{F}_p$ and $\mathbb{Z}_{(p)}$ denote respectively $\mathbb{Z}/p\mathbb{Z}$ and the localization of $\mathbb{Z}$ at $p\mathbb{Z}$. For a Cartier, $\mathbb{Z}_{(p)}$-Cartier or $\mathbb{Q}$-Cartier divisor D on S (resp. an $\mathcal{O}_S$-module $\mathcal{G}$), the pullback of D (resp. $\mathcal{G}$) to $S_{T'}$ is denoted by $D_{T'}$ or $D|_{S_{T'}}$ (resp. $\mathcal{G}_{T'}$ or $\mathcal{G}|_{S_{T'}}$) if it is well-defined. Similarly, for a homomorphism of $\mathcal{O}_S$-modules $\alpha : \mathcal{F} \to \mathcal{G}$ the pullback of α to $S_{T'}$ is denoted by $\alpha_{T'} : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$.

2. Preliminaries

In this section, we recall some basic results which will be used in the proof.

2.1. Minimal models of 3-folds

Existence of (log) minimal models of 3-folds in positive characteristic $p > 5$ was first proved for canonical singularities by Hacon and Xu [19], and in general by Birkar [5] (see [39] for the lc case). The result on Mori fiber spaces was proved for terminal singularities by Cascini, Tanaka and Xu [11], and in general by Birkar and Waldron [7]. We collect some results in the following theorem, which will be used in our proof.

Theorem 2.1. *Assume that the base field k has characteristic $p > 5$. Let $f : X \to Z$ be a contraction from a normal 3-fold, and let Δ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(1) If either (X, Δ) is klt and $K_X + \Delta$ is pseudo-effective over Z, or (X, Δ) is lc and $K_X + \Delta$ has a weak Zariski decomposition [1], then (X, Δ) has a log minimal model over Z.

(2) If (X, Δ) is a dlt pair and Z is a smooth projective curve with $g(Z) \geq 1$, then every step of LMMP in [5, Sec. 3.5-3.7] starting from (X, Δ) is over Z.

Proof. For (1) please refer to [5, Theorem 1.2 and Proposition 8.3].

For (2), since (X, Δ) is dlt, every $K_X + \Delta$-extremal ray is a generated by a rational curve by cone theorem [7, Theorem 1.1], which is contracted by f since $g(Z) \geq 1$. So for an extremal contraction $X \to \bar{X}$, if there is a divisorial contraction or a flip $\sigma : X \dashrightarrow X^+$ as in [5, Sec. 3.5-3.7], there exist natural morphism $\bar{f} : \bar{X} \to Z$ and $f^+ : X^+ \to Z$ fitting into the following commutative diagram
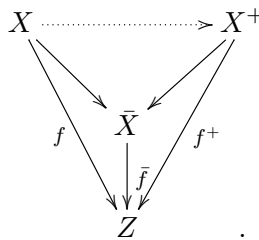
Note that $(X^+, \Delta^+ = \sigma_* \Delta)$ is a dlt pair. We can show this assertion by induction. □

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[1]i.e., there exists a birational projective morphism $\mu : W \to X$ such that $\nu^*(K_X + \Delta) = P + M$ where P is nef over Z and M is effective

2.2. Covering Theorem

The result below is [[21], Theorem 10.5] when X and Y are both smooth, and the proof there also applies when varieties are normal.

Theorem 2.2. ([21, Theorem 10.5]) *Let $f : X \to Y$ be a proper surjective morphism between complete normal varieties. If D is a Cartier divisor on Y and E an effective f-exceptional divisor on X, then*

$$\kappa(X, f^*D + E) = \kappa(Y, D).$$

As a corollary we get the following useful result.

Lemma 2.3. *Let $g : W \to Y$ be surjective projective morphism between projective varieties. Assume Y is normal and let $L_1, L_2 \in \mathrm{Pic}^0(Y)$ be two line bundles on Y. If $g^*L_1 \sim_{\mathbb{Q}} g^*L_2$ then $L_1 \sim_{\mathbb{Q}} L_2$.*

Proof. Let $L = L_1 \otimes L_2^{-1}$. Denote by $\sigma : W' \to W$ the normalization and $g' = g \circ \sigma : W' \to Y$. Then $g'^*L \sim_{\mathbb{Q}} 0$. Applying Theorem 2.2 to $g' : W' \to Y$ gives that $L \sim_{\mathbb{Q}} 0$, which is equivalent to that $L_1 \sim_{\mathbb{Q}} L_2$.   □

2.3. Adjunction

Lemma 2.4. *Assume that the base field k has characteristic $p > 5$. Let (X, Δ) be a normal, $\mathbb{Q}$-factorial, lc 3-fold (not necessarily projective). Let C be a projective lc center of (X, Δ) and $\tilde{C}$ be the normalization of C. If $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial, then $(K_X + \Delta)|_{\tilde{C}}$ is $\mathbb{Q}$-trivial.*

Proof. By [5, Lemma 6.5], we can take a crepant partial resolution $\mu : X' \to X$ such that

$$K_{X'} + D + \Delta' \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) \cdots (\clubsuit)$$

where D is a reduced irreducible divisor dominant over C and $(X', D + \Delta')$ is dlt. Then considering the restriction of the relation $\clubsuit$ on D, by the adjunction formula [25, 5.3], we have

$$K_D + \Delta_D \sim_{\mathbb{Q}} \mu^*(K_X + \Delta)|_D.$$

Then D is a normal projective surface hence granted a natural morphism $D \to \tilde{C}$, and (D, Δ_D) is log canonical by [5, Lemma 5.2]. Applying [33, Theorem 1.2], we have that $K_D + \Delta_D$ is semi-ample, thus $\mu^*(K_X + \Delta)|_D$

is $\mathbb{Q}$-trivial since $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial. We can conclude that $(K_X + \Delta)|_{\tilde{C}}$ is $\mathbb{Q}$-trivial by Lemma 2.3. $\qquad\square$

2.4. Vector bundles on elliptic curves

In this subsection, we recall some facts about vector bundles on elliptic curves, which are used in the proof of Theorem 3.2.

Theorem 2.5. *Let C be an elliptic curve, and let $\mathcal{E}_C(r, d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d.*

(1) *([2, Theorem 5])For each $r > 0$, there exists a unique element $\mathcal{E}_{r,0}$ of $\mathcal{E}_C(r, 0)$ with $H^0(C, \mathcal{E}_{r,0}) \neq 0$. Moreover, for every $\mathcal{E} \in \mathcal{E}_C(r, 0)$ there exists an $\mathcal{L} \in \mathrm{Pic}^0(C)$ such that $\mathcal{E} \cong \mathcal{E}_{r,0} \otimes \mathcal{L}$.*

(2) *([30, Proposition 2.10]) When the Hasse invariant $\mathrm{Hasse}(C)$ is nonzero, $F_C^*\mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$. When $\mathrm{Hasse}(C)$ is zero,*

$$F_C^*\mathcal{E}_{r,0} \cong \bigoplus_{1 \leq i \leq \min\{r,p\}} \mathcal{E}_{\lfloor (r-i)/p \rfloor + 1, 0},$$

where $\lfloor r \rfloor$ denotes the round down of r.

Theorem 2.6 ([29, 1.4. Satz]). *Let $\mathcal{E}$ be a vector bundle on a smooth projective curve C. If $F_C^{e*}\mathcal{E} \cong \mathcal{E}$ for some $e > 0$, then there exists an étale morphism $\pi : C' \to C$ from a smooth projective curve C' such that $\pi^*\mathcal{E} \cong \bigoplus \mathcal{O}_{C'}$.*

Proposition 2.7. *Let $\mathcal{E}$ be a vector bundle on an elliptic curve C. Then there exists a finite morphism $\pi : C' \to C$ from an elliptic curve C' such that $\pi^*\mathcal{E}$ is a direct sum of line bundles.*

Proof. We may assume that for every finite morphism $\varphi : B \to C$ from an elliptic curve B, $\varphi^*\mathcal{E}$ is indecomposable. Set $d := \deg \mathcal{E}$ and $r := \mathrm{rank}\mathcal{E}$. We show that $r = 1$. Let $Q \in C$ be a closed point. Replacing $\mathcal{E}$ by $((r_C)^*\mathcal{E})(-dQ)$, we may assume that $d = 0$. Here $r_C : C \to C$ is the morphism given by multiplication by r. Hence Theorems 2.5 and 2.6 imply that when the Hasse invariant of C is nonzero (resp. zero), there exists an étale morphism $\pi : C' \to C$ (resp. an $e > 0$) such that $\pi^*\mathcal{E}$ (resp. $F_C^{e*}\mathcal{E}$) is a direct sum of line bundles. This implies that $r = 1$. $\qquad\square$

2.5. Weak positivity

The following positivity result will be used in the proof of the case when the geometric generic fiber has Kodaira dimension one.

Theorem 2.8. *Assume that* char$k = p > 5$. *Let* $f : X \to Z$ *be a fibration from a smooth projective 3-fold to a smooth projective curve. Suppose that the geometric generic fiber* $X_{\overline{\eta}}$ *has at most rational double points as singularities. If* $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}) = 1$, *then there exists a real number* $c > 0$ *such that* $f_* \omega_{X/Z}^m$ *contains a nef subbundle of rank at least* cm *for sufficiently divisible* $m > 0$.

Before proving Theorem 2.8, we recall some results.

Theorem 2.9 ([13, 3.2]). *Let* $f : X \to Z$ *be a surjective morphism between smooth projective varieties, over an algebraically closed field of positive characteristic, whose geometric generic fiber is a smooth elliptic curve. Then* $\kappa(X, K_{X/Z}) \geq 0$.

The following lemma will be frequently used.

Lemma 2.10 ([38, Lemma 3.2]). *Let* $f : X \to Z$ *be a fibration between normal quasi-projective varieties. Let* L *be a* f-*nef* $\mathbb{Q}$-*Cartier divisor on* X *such that* $L_\eta \sim_{\mathbb{Q}} 0$ *where* η *is the generic point of* Z. *Assume* $\dim Z \leq 3$. *Then there exist a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ f' \downarrow & & \downarrow f \\ Z' & \xrightarrow{\psi} & Z \end{array}$$

with ϕ, ψ *projective birational, and an* $\mathbb{Q}$-*Cartier divisor* D *on* Z' *such that* $\phi^* L \sim_{\mathbb{Q}} f'^* D$. *Furthermore, if* f *is flat and* Z *is* $\mathbb{Q}$-*factorial, then we can take* $X' = X$ *and* $Z' = Z$.

The next lemma is a consequence of Tanaka's vanishing theorem for surfaces [35].

Lemma 2.11. *Let* $g : Y \to Z$ *be a generically smooth surjective morphism from a smooth projective surface to a smooth projective curve. Let* H *be a nef and* g-*big divisor on* Y. *Then* $g_* \mathcal{O}_Y(K_{Y/Z} + lH)$ *is a nef vector bundle for every* $l \gg 0$.

Proof. Let A be an ample divisor on Z with $\deg A \geq \deg K_Z + 2$. Then $A - K_Z - z$ is ample for a closed point $z \in Z$ where z is seen as a divisor on Z. Note that $\nu(H) \geq 1$ and $H + g^*(A - K_Z - z)$ is nef and big. Denote by Y_z the fiber of g over z. By [35, Theorem 2.6] we see that

$$H^1(Y, K_Y + H + g^*(A - K_Z) + (l-1)H - Y_z)$$
$$= H^1(Y, K_Y + H + g^*(A - K_Z - z) + (l-1)H) = 0$$

for $l \gg 0$. Thus for a closed point $z \in Z$, by the long exact sequence arising from taking cohomology of the exact sequence below

$$0 \to \mathcal{O}_Y(K_{Y/Z} + g^*A + lH - Y_z) \to \mathcal{O}_Y(K_{Y/Z} + g^*A + lH)$$
$$\to \mathcal{O}_Y(K_{Y/Z} + g^*A + lH)|_{Y_z} \to 0$$

we conclude that the restriction

$$H^0(Y, K_{Y/Z} + g^*A + lH) \to H^0(Y_z, (K_{Y/Z} + g^*A + lH)|_{Y_z})$$

is surjective. This implies that $(g_*\mathcal{O}_Y(K_{Y/Z} + lH))(A)$ is generically globally generated. On the other hand, if z is general then Y_z is smooth, applying [31, Corollary 2.23], since $H|_{Y_z}$ is ample, we have that for $l \gg 0$ the morphism

$$H^0(Y_z, \phi_{Y_z}^{(e)} \otimes \mathcal{O}_{Y_z}(K_{Y_z} + lH_z)) : H^0(Y_z, K_{Y_z} + lp^eH_z) \to H^0(Y_z, K_{Y_z} + lH_z)$$

is surjective. This implies that the homomorphism ([14, Section 2])

$$g_{Z^e*}(\phi_{Y/Z}^{(e)} \otimes \mathcal{O}_{Y_{Z^e}}((K_{Y/Z} + lH)_{Z^e})) \otimes \mathcal{O}_{Z^e}(A) :$$
$$g_*\mathcal{O}_Y(K_{Y/Z} + lp^eH + g^*(A + z)) \cong g_*\mathcal{O}_Y(K_{Y/Z} + lp^eH) \otimes \mathcal{O}_{Z^e}(A)$$
$$\to g_{Z^e*}\mathcal{O}_{Y_{Z^e}}((K_{Y/Z} + lH)_{Z^e}) \otimes \mathcal{O}_{Z^e}(A + z)$$
$$\cong F_Z^{e*}g_*\mathcal{O}_Y(K_{Y/Z} + lH) \otimes \mathcal{O}_{Z^e}(A)$$

is generically surjective. Thus for every $e > 0$, $F_Z^{e*}(g_*\mathcal{O}_Y(K_{Y/Z} + lH)) \otimes \mathcal{O}_{Z^e}(A)$ is generically globally generated, and hence is nef. We conclude that $g_*\mathcal{O}_Y(K_Y + lH)$ is nef by applying [14, Proposition 4.7].   □

Proof of Theorem 2.8. Let W be a minimal model of X over Z. Let $\rho : X_{\bar\eta} \to W_{\bar\eta}$ be the induced morphism. Since $\rho_*\mathcal{O}_{X_{\bar\eta}} \cong \mathcal{O}_{W_{\bar\eta}}$, $W_{\bar\eta}$ is normal. Furthermore, since W is terminal, we have $K_{X_{\bar\eta}} \geq \rho^*K_{W_{\bar\eta}}$, and hence $W_{\bar\eta}$ has at most canonical singularities. In particular, replacing X with a minimal model, with the loss of smoothness we may assume that $K_{X/Z}$ is f-nef.

Then by [33, Theorem 1.2], $K_{X_{\overline{\eta}}}$ is semi-ample, and since $p > 5$, the geometric generic fiber of the Iitaka fibration $I_{\overline{\eta}} : X_{\overline{\eta}} \to C_{\overline{\eta}}$ is a smooth elliptic curve over $k(\overline{\eta})$ by [8, Theorem 7.18]. For the generic fiber X_{η} and sufficiently divisible positive integer n, since $H^0(X_{\overline{\eta}}, nK_{X_{\overline{\eta}}}) \cong H^0(X_{\eta}, nK_{X_{\eta}}) \otimes_{k(\eta)} k(\overline{\eta})$, we see that the Iitaka fibration $I_{\overline{\eta}} : X_{\overline{\eta}} \to C_{\overline{\eta}}$ coincides with the Iitaka fibration $I_{\eta} : X_{\eta} \to C_{\eta}$ tensoring with $k(\overline{\eta})$. Thus the geometric generic fiber of I_{η} is a smooth elliptic curve.

Considering the relative Iitaka fibration of $f : X \to Z$, whose geometric generic fiber is a smooth elliptic curve, we get a birational morphism $u : X' \to X$, a fibration $g : Y \to Z$ with Y smooth, and an elliptic fibration $h : X' \to Y$ fitting into the following commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{\ u\ } & X \\
{\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y & \xrightarrow{\ g\ } & Z.
\end{array}
$$

Note that the geometric generic fiber $C_{\overline{\eta}}$ of $g : Y \to Z$ is normal, and hence smooth. By Lemma 2.10, we may assume that $u^* K_{X/Z} \sim_{\mathbb{Q}} h^* H$ for a nef g-big $\mathbb{Q}$-Cartier divisor on Y. By Theorem 2.9, we have $\kappa(X', K_{X'/Y}) \geq 0$, and hence there exists an injective homomorphism $h^* \omega_{Y/Z}^m \to \omega_{X'/Z}^m$ for sufficiently divisible $m > 0$. Let $l \gg 0$ be an integer such that lH is Cartier and $u^* l K_{X/Z} \sim h^* lH$. Then we have natural homomorphisms

$$
\begin{aligned}
(g_* \mathcal{O}_Y(K_{Y/Z} + lH))^{\otimes m} &\to g_* \mathcal{O}_Y(m(K_{Y/Z} + lH)) \\
&\cong g_* h_* \mathcal{O}_{X'}(mh^*(K_{Y/Z} + lH)) \\
&\hookrightarrow f_* u_* \mathcal{O}_{X'}(mK_{X'/Z} + u^* lmK_{X/Z}) \\
&\cong f_* \mathcal{O}_X(m(l+1)K_{X/Z}).
\end{aligned}
$$

Replacing l if necessary, we may assume that the first homomorphism is generically surjective. By Lemma 2.11, $g_* \mathcal{O}_Y(K_{Y/Z} + lH)$ is nef, and hence so is $g_* \mathcal{O}_Y(m(K_{Y/Z} + lH))$. This completes the proof. $\qquad\square$

3. The case $\kappa(X_{\overline{\eta}}) = 0$

In this section, we prove Theorem 1.2 in the case when the Kodaira dimension of the geometric generic fiber is equal to zero. It is proved as a consequence of Theorems 3.1 and 3.2.

Theorem 3.1 ([15]). *Let $f : X \to Z$ be a surjective morphism from a normal projective variety X over an algebraically closed field of characteristic $p > 0$ to a smooth projective variety Z, and let Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some $a > 0$ not divisible by p. Assume that $(X_{\bar\eta}, \Delta_{\bar\eta})$ is F-pure, where $\bar\eta$ is the geometric generic point of Z. If $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + L)$ for some $\mathbb{Q}$-divisor L on Z, then L is pseudo-effective.*

Theorem 3.1 follows from [15, Theorem 4.5] (by setting $D = -(K_Z + L)$), and it is also proved by Patakfalvi [31, Theorem 1.6] when Z is a curve.

Theorem 3.2. *With the same notation and assumptions as in Theorem 3.1, if Z is an elliptic curve, then L is semi-ample.*

Proof. By Theorem 3.1, we have $\deg L \geq 0$. We may assume that $\deg L = 0$, and it suffices to show that $L \sim_{\mathbb{Q}} 0$. Since $(K_X + \Delta)_{\bar\eta} \sim_{\mathbb{Q}} 0$, there is an ample Cartier divisor A on X such that $l(K_X + \Delta)_{\bar\eta} + A_{\bar\eta}$ is ample and free for every $l \in a\mathbb{Z}$. Recall that $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$ and $a\Delta$ is integral. By Fujita's vanishing theorem, there exist some $m_0 > 0$ such that for every nef Cartier divisor N on $X_{\bar\eta}$, $\mathcal{O}_{X_{\bar\eta}}((m_0 - 1)A_{\bar\eta} + N)$ is 0-regular with respect to $l(K_X + \Delta)_{\bar\eta} + A_{\bar\eta}$ for every $l \in a\mathbb{Z}$. Then the natural homomorphism

$$
\begin{aligned}
H^0(X_{\bar\eta}, l(K_X + \Delta)_{\bar\eta} + mA_{\bar\eta}) &\otimes H^0(X_{\bar\eta}, (m' - 1)A_{\bar\eta}) \\
\otimes H^0(X_{\bar\eta}, l'(K_X + \Delta)_{\bar\eta} + A_{\bar\eta}) & \\
\to H^0(X_{\bar\eta}, (l + l')(K_X + \Delta)_{\bar\eta} &+ (m + m')A_{\bar\eta})
\end{aligned}
$$

is surjective for every $l, l' \in a\mathbb{Z}$ and $m, m' \geq m_0$. Thus

$$
\begin{aligned}
H^0(X_{\bar\eta}, l(K_X + \Delta)_{\bar\eta} + mA_{\bar\eta}) &\otimes H^0(X_{\bar\eta}, l'(K_X + \Delta)_{\bar\eta} + m'A_{\bar\eta}) \\
\to H^0(X_{\bar\eta}, (l + l')(K_X + \Delta)_{\bar\eta} &+ (m + m')A_{\bar\eta})
\end{aligned}
$$

is also surjective, and hence the natural homomorphism

$$
\mathcal{G}(l, m) \otimes \mathcal{G}(l', m') \to \mathcal{G}(l + l', m + m')
$$

is generically surjective, where $\mathcal{G}(l, m) := f_*\mathcal{O}_X(l(K_{X/Z} + \Delta) + mA)$. From now on, we use the same notation as [14, Sections 2 or 3] or [15, Section 2].

Replacing m_0 if necessary, by [31, Corollary 2.23] we may assume that

$$H^0(X_{\overline{\eta}}, \phi^{(e)}_{(X_{\overline{\eta}},\Delta_{\overline{\eta}})} \otimes \mathcal{O}_{X_{\overline{\eta}}}(N + m_0 A_{\overline{\eta}})):$$
$$H^0(X_{\overline{\eta}}, (1-p^e)(K_X + \Delta)_{\overline{\eta}} + p^e(N + m_0 A_{\overline{\eta}})) \to H^0(X_{\overline{\eta}}, N + m_0 A_{\overline{\eta}})$$

is surjective for every $e > 0$ with $a|(p^e - 1)$ and for every nef Cartier divisor N on $X_{\overline{\eta}}$. Since $l(K_X + \Delta)_{\overline{\eta}}$ is nef,

$$f_{Z^e*}(\phi^{(e)}_{(X/Z,\Delta)} \otimes \mathcal{O}_{X_{Z^e}}(l(K_{X/Z} + \Delta)_{Z^e} + m_0 A_{Z^e})):$$
$$\mathcal{G}((l-1)p^e + 1, m_0 p^e) \to f_{Z^e*}\mathcal{O}_{X_{Z^e}}(l(K_{X/Z} + \Delta)_{Z^e} + m_0 A_{Z^e})$$
$$\cong F_Z^{e*}\mathcal{G}(l, m_0)$$

is generically surjective. Let $b > 0$ be an integer such that $a|b$, that bL is integral and that $b(K_X + \Delta)$ is linearly equivalent to bf^*L. By Proposition 2.7, there exists a finite morphism $\pi : Z' \to Z$ from an elliptic curve Z' such that $\pi^*\mathcal{G}(r, m_0)$ is a direct sum of line bundles for each $0 \le r < b$ with $a|r$. By Lemma 2.3, we may replace L and $\mathcal{G}(r, m_0)$ respectively by its pullback by π. Set

$$\mathcal{F} := \bigoplus_{0 \le r < b, \ a|r} \mathcal{G}(r, m_0),$$
$$\mu := \min\{\deg \mathcal{M}|\mathcal{M} \in \mathrm{Pic}(Z) \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\}, \text{ and}$$
$$T := \{\mathcal{M} \in \mathrm{Pic}(Z)|\deg \mathcal{M} = \mu \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\}$$
$$= \{\mathcal{M}_1, \ldots, \mathcal{M}_\lambda\}.$$

Then for every $\mathcal{M}_i \in T$, there exists an $0 \le s < b$ with $a|s$ such that the composition

$$\mathcal{G}(s, m_0)^{\otimes p^e - 1} \otimes \mathcal{G}(r_{i,e}, m_0) \otimes \mathcal{O}_Z(-q_{i,e}bL)$$
$$\to \mathcal{G}((s-1)p^e + 1, p^e m_0) \to F_Z^{e*}\mathcal{G}(s, m_0) \twoheadrightarrow \mathcal{M}_i^{p^e}$$

is generically surjective for every $e > 0$ with $a|(p^e - 1)$. Here $q_{i,e}$ and $r_{i,e}$ are integers satisfying $1 + s - p^e = -q_{i,e}b + r_{i,e}$ and $0 \le r_{i,e} < b$. Then there exist a line bundle $\mathcal{M}$ which is a direct summand of $\mathcal{G}(s, m_0)^{p^e - 1} \otimes \mathcal{G}(r_{i,e}, m_0)$ and a non-zero morphism $\mathcal{M} \to \mathcal{M}_i^{p^e}(q_{i,e}bL)$. By considering the degree of the line bundles, we see that $\mathcal{M}_i^{p^e}(q_{i,e}bL) \cong \mathcal{M} \in T^{p^e}$, where

$$T^n := \{\bigotimes_{1 \le i \le \lambda} \mathcal{M}_i^{n_i} \in \mathrm{Pic}(Z)|n_i \ge 0, \ \sum_{1 \le i \le \lambda} n_i = n\}.$$

Fix an integer $e > 0$ such that $a | p^e - 1$. Set $n := \lambda(p^e - 1) + 1$. For every $\mathcal{N} \in T^n$, there exist $n_1, \ldots, n_\lambda \geq 0$ such that $\mathcal{N} \cong \bigotimes_{1 \leq i \leq \lambda} \mathcal{M}_i^{n_i}$ and $n_j' := n_j - p^e \geq 0$ for at least one j. Then

$$\mathcal{N}(q_{j,e}bL) \cong (\bigotimes_{i \neq j} \mathcal{M}_i^{n_i}) \otimes \mathcal{M}_j^{n_j'} \otimes \mathcal{M}_j^{p^e}(q_{j,e}bL).$$

Since $\mathcal{M}_j^{p^e}(q_{j,e}bL) \in T^{p^e}$, we have $\mathcal{N}(q_{j,e}bL) \in T^n$. Hence for every $m \geq q := \max\{q_{1,e}, \ldots, q_{\lambda,e}\}$,

$$\mathcal{N}(mbL) \in \{\mathcal{M}(kbL) \in \mathrm{Pic}(Z) | \mathcal{M} \in T^n, \ 0 \leq k < q\}.$$

Since T^n is a finite set, there are integers $m > m' > 0$ such that $\mathcal{N}(mbL) \cong \mathcal{N}(m'bL)$, and hence $(m - m')bL \sim 0$. $\qquad\qquad\square$

Proof of Theorem 1.2: the case $\kappa(X_{\overline{\eta}}) = 0$. As in the proof of Theorem 2.8, we may assume that X is minimal over Z and $K_{X_{\overline{\eta}}}$ is semi-ample, thus $K_{X_{\overline{\eta}}} \sim_{\mathbb{Q}} 0$. By Lemma 2.10, K_X is $\mathbb{Q}$-linearly equivalent to the pullback of $K_Z + L$ for some $\mathbb{Q}$-divisor L on Z. In particular $\kappa(X, K_X) = \kappa(Z, K_Z + L)$. It is enough to show that $\kappa(Z, K_Z + L) \geq \kappa(Z)$. By Lemma 3.1, we see that L is nef. Note that since $X_{\overline{\eta}}$ has at most rational double points as singularities, $X_{\overline{\eta}}$ is Gorenstein and $p > 5$, $X_{\overline{\eta}}$ is F-pure by [1, Section 3] and [16]. When Z is of general type, by Theorem 3.1, we have $K_Z + L$ is big, thus $\kappa(Z, K_Z + L) = \dim Z = \kappa(Z)$. And when Z is an elliptic curve, by Theorem 3.2, we have $\kappa(Z, K_Z + L) \geq \kappa(Z)$. This completes the proof. $\quad\square$

4. The case $\kappa(X_{\overline{\eta}}) = 1$

In this section, we consider the case when the Kodaira dimension of the geometric generic fiber is one.

Proof of Theorem 1.2: the case $\kappa(X_{\overline{\eta}}) = 1$. Let $f : X \to Z$ be a surjective morphism from a smooth projective 3-fold to a smooth projective curve of genus at least one, and let $\overline{\eta}$ be the geometric generic point of Z. Suppose that $\kappa(X_{\overline{\eta}}) = 1$. With the loss of smoothness, by Theorem 2.1 (2) we may assume that X is a minimal model. Then $X_{\overline{\eta}}$ has canonical singularities by the proof of Theorem 2.8.

If $g(Z) > 1$, then since $f_* \omega_{X/Z}^m$ contains a nef sub-bundle of rank $\geq cm$ for some $c > 0$ and any sufficiently divisible m (Theorem 2.8), by some

standard arguments (proof of [6, Proposition 5.1]), we can conclude that

$$\kappa(X) \geq 2 = \kappa(Z) + \kappa(X_{\overline{\eta}}).$$

So from now on, we assume $g(Z) = 1$. Then $\omega_X = \omega_{X/Z}$. We break the proof into several steps.

Step 1: By considering the relative Iitaka fibration and applying Lemma 2.10, we get the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

where Y is a smooth projective surface, and h is fibration with geometric fiber being a smooth elliptic curve by the proof of Theorem 2.8, such that $\sigma^* K_X \sim_{\mathbb{Q}} h^* D$ where D is a nef g-big divisor on Y.

If D is big, then we are done. From now on, we assume the numerical dimension $\nu(K_X) = \nu(D) = 1$. Then we claim that

Claim. If X has a fibration $f' : X \to W$ to a normal projective curve W such that $K_{F'}$ is numerically trivial, where F' denotes the generic fiber of f'. Assume moreover that there exist $L \in \mathrm{Pic}^0(Z)$ and an integer $m > 0$ such that $h^0(X, mK_X + f^*L) > 0$. Then K_X is semi-ample.

Proof of the claim. Take an effective divisor $D_L \sim mK_X + f^*L$. Since D_L is nef, effective and $D_L|_{F'} \sim_{num} 0$, we have

$$(mK_X + f^*L)|_{F'} \sim D_L|_{F'} \sim 0.$$

By Lemma 2.10 we can assume $D_L \sim_{\mathbb{Q}} f'^*A$ where A is a divisor on W, which is ample since $D_L \neq 0$. So we only need to show that $L \sim_{\mathbb{Q}} 0$.

Since X has at most terminal singularities, it is smooth in codimension one, so F' is a regular surface over the function filed $K(W)$ of W. Applying [34, Theorem 1.1], since $K_{F'}$ is numerically trivial, we have $K_{F'} \sim_{\mathbb{Q}} 0$. Therefore, we conclude that

$$f^*L|_{F'} \sim_{\mathbb{Q}} mK_{F'} + f^*L|_{F'} \sim_{\mathbb{Q}} (mK_X + f^*L)|_{F'} \sim_{\mathbb{Q}} D_L|_{F'} \sim_{\mathbb{Q}} 0.$$

On the other hand, F' is dominant over the curve $Z \otimes_k K(W)$, passing to the algebraic closure of $K(W)$ and applying Lemma 2.3, we show that L is torsion.   □

Step 2: By Theorem 2.8, there exists $c > 0$ such that for sufficiently divisible m_1, $f_*\omega_X^{m_1}$ contains a nef sub-bundle V of rank $r_{m_1} \geq cm_1$. If $\deg V > 0$, then we are done by some standard arguments ([6, Propostion 5.1]). So we assume that $\deg V = 0$, thus by Proposition 2.7 there exists a flat base change between two elliptic curves $\pi : Z_1 \to Z$ such that $\pi^*V = \oplus_{i=1}^n \mathcal{L}_i$ where $\mathcal{L}_i \in \mathrm{Pic}^0(Z_1)$. Let X_1 be the normalization of $X \times_Z Z_1$. Then we get the following commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\pi_1} & X \\
f_1 \downarrow & & \downarrow f \\
Z_1 & \xrightarrow{\pi} & Z
\end{array}
$$

where π_1 and f_1 denote the natural projections. We have that $\pi^* f_*\omega_X^{m_1} \subset f_{1*}\pi_1^*\omega_X^{m_1}$ by [20, Proposition 9.3], thus

$$
\pi^*V = \oplus_{i=1}^n \mathcal{L}_i \subset f_{1*}\pi_1^*\omega_X^{m_1}.
$$

So we conclude that

$$
h^0(X_1, \pi_1^* m_1 K_X - f_1^*\mathcal{L}_i) \geq 1,
$$

and if $\mathcal{L}_i = \mathcal{L}_j$ for some $j \neq i$ then the strict inequality holds. Since $\pi^* : \mathrm{Pic}^0(Z) \to \mathrm{Pic}^0(Z_1)$ is surjective, there exists L_i' such that $\mathcal{L}_i \sim \pi^*L_i'$, thus

$$
\pi_1^* m_1 K_X - f_1^*\mathcal{L}_i \sim \pi_1^*(m_1 K_X + f^*L_i').
$$

Applying Theorem 2.2, we can find a sufficiently divisible integer $l > 0$ such that

$$
h^0(X, l(m_1 K_X - f^*L_i')) \geq 1.
$$

Put $m = lm_1$ and $L_i = lL_i'$. Then $h^0(X, mK_X - f^*L_i) \geq 1$.

If $h^0(X, mK_X - f^*L_i) > 1$, then $h^0(Y, mD - g^*L_i) > 1$ by the construction in Step 1. Since $mD - g^*L_i$ is nef and $\nu(mD - g^*L_i) = 1$, the movable part of $|mD - g^*L_i|$ has no base point, hence induces a fibration $g' : Y \to W'$ on Y to a curve W'. The Stein factorization of the composite morphism $g' \circ h : X' \to W'$ induces a fibration $f'' : X' \to W$ from X' to a normal curve W, which is defined by the base point free linear system $|\mu^*l(mK_X - f^*L_i)|$ for sufficiently divisible integer $l > 0$. Since $\sigma : X' \to X$ is a birational morphism such that $\sigma_*\mathcal{O}_{X'} = \mathcal{O}_X$, we conclude that $|\mu^*l(mK_X - f^*L_i)| = \mu^*|l(mK_X - f^*L_i)|$, thus $|l(mK_X - f^*L_i)|$ has no base point, hence defines

such a fibration $f' : X \to W$ as in Claim of Step 1. So K_X is semi-ample, and this completes the proof in this case.

From now on, we can assume

$$h^0(X, mK_X - f^*L_i) = 1 \quad \text{and} \quad h^0(X_1, \pi_1^*(mK_X - f^*L_i)) = 1.$$

For every i, we have unique effective divisors $B_i \sim mK_X - f^*L_i$. And by construction, we can assume $\pi_1^*B_i \neq \pi_1^*B_j$ if $i \neq j$, thus $L_i \neq L_j$. In the following, we only need to show that at least two of L_i are torsion.

Step 3: For every j, we have unique effective divisor $B_j \sim mK_X - f^*L_j$. Let B' be the reduced divisor supported on the union of components of $\sum_j B_j$. Take a smooth log resolution $\mu : \tilde{X} \to X$ of the pair (X, B'). Denote by $\tilde{f} : \tilde{X} \to Z$ the natural morphism. Let $\tilde{B}$ be the reduced divisor supported on the union components of $\sum_j \mu^*B_j$. Consider the dlt pair $(\tilde{X}, \tilde{B})$. Since X has terminal singularities, there exists an effective μ-exceptional divisor E on $\tilde{X}$ such that

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \mu^*K_X + E.$$

So $K_{\tilde{X}} + \tilde{B} \sim_{\mathbb{Q}} \mu^*K_X + E + \tilde{B}$ has a weak Zariski decomposition. By Theorem 2.1 (2), $(\tilde{X}, \tilde{B})$ has a minimal model $(\hat{X}, \hat{B})$ which is dlt, and there exists a natural morphism $\hat{f} : \hat{X} \to Z$. By the construction, we have the following:

(1) Note that $B_j|_{X_{\bar{\eta}}}$ is contained in finitely many fibers of the Iitaka fibration $I_{\bar{\eta}} : X_{\bar{\eta}} \to C_{\bar{\eta}}$, which implies that $\kappa(\tilde{X}_{\bar{\eta}}, (K_{\tilde{X}} + \tilde{B})|_{X_{\bar{\eta}}}) = 1$. Since the restriction $(K_{\hat{X}} + \hat{B})|_{\hat{X}_{\bar{\eta}}}$ is semi-ample by [33, Theorem 1.2], it induces an elliptic fibration on $\hat{X}_{\bar{\eta}}$ by construction. So applying Lemma 2.10 again, we get the following commutative diagram

$$
\begin{array}{ccc}
\hat{X}' & \xrightarrow{\hat{\sigma}} & \hat{X} \\
\hat{h} \downarrow & & \downarrow \hat{f} \\
\hat{Y} & \xrightarrow{\hat{g}} & Z
\end{array}
$$

where $\hat{Y}$ is a smooth projective surface and $\hat{h}$ is an elliptic fibration such that, $\hat{\sigma}^*(K_{\hat{X}} + \hat{B}) \sim_{\mathbb{Q}} \hat{h}^*\hat{D}$ where $\hat{D}$ is a nef and $\hat{g}$-big divisor on $\hat{Y}$.

(2) We claim that $\nu(K_{\hat{X}} + \hat{B}) = \nu(\hat{D}) = 1$. Indeed, otherwise, $\hat{D}$ will be big. Note that the divisor $\mu^*\sum_j B_j - \tilde{B}$ is effective and $\mu^*\sum_j B_j \sim \mu^*nmK_X - \sum_j \tilde{f}^*L_j$. Then applying Theorem 2.2 we can get a contradiction

as follows

$$2 = \kappa(\hat{Y}, \hat{D}) = \kappa(\hat{X}', \hat{\sigma}^*(K_{\hat{X}} + \hat{B})) = \kappa(\hat{X}, K_{\hat{X}} + \hat{B}) = \kappa(\tilde{X}, K_{\tilde{X}} + \tilde{B})$$

$$\leq \kappa\left(\tilde{X}, K_{\tilde{X}} + \mu^* nmK_X - \sum_j \tilde{f}^* L_j\right)$$

$$= \kappa\left(\tilde{X}, \mu^* K_X + E + \mu^* nmK_X - \sum_j \mu^* f^* L_j\right)$$

$$= \kappa\left(X, (nm+1)K_X - \sum_j f^* L_j\right) = 1.$$

(3) For sufficiently divisible M and every $1 \leq i \leq n$, we get an effective Cartier divisor

$$\tilde{\Gamma}_i = M(\mu^* B_i + mE) + Mm\tilde{B} \sim M(mK_{\tilde{X}} - \tilde{f}^* L_i) + Mm\tilde{B}.$$

Denote by $\nu : \tilde{X} \dashrightarrow \hat{X}$ the natural birational map. Let $\hat{\Gamma}_i = \nu_* \tilde{\Gamma}_i$. Then

$$\hat{\Gamma}_i \sim M(mK_{\hat{X}} - \hat{f}^* L_i) + Mm\hat{B} \sim Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^* L_i.$$

Since E is contained in finitely many fibers of $\tilde{f}$, $\nu_* E$ is contracted by $\hat{f}$. So if a component of $\hat{\Gamma}_i$ is dominant over Z then it is contained in $\hat{B}$.

(4) Take an effective divisor $\hat{D}_i \sim Mm\hat{D} - M\hat{g}^* L_i$ for each i. Since $\hat{D}$ is nef and $\hat{D}^2 = 0$, so is $\hat{D}_i$. Considering connected components of the union of the $\hat{D}_i, 0 \leq i \leq n$, we see that there exist nef effective Cartier divisors $\hat{D}'_1, \ldots, \hat{D}'_k$ satisfying the conditions below:

- $\mathrm{Supp}(\hat{D}'_j)$ is connected for each j, and $\mathrm{Supp}(\hat{D}'_j) \cap \mathrm{Supp}(\hat{D}'_l) = \emptyset$ for each $j \neq l$;

- $(\hat{D}'_1)^2 = \cdots = (\hat{D}'_k)^2 = 0$;

- the greatest common divisor of the coefficients of every $\hat{D}'_j$ is equal to one;

- for each i, there exist $a_{i1}, \ldots, a_{ik} \in \mathbb{Z}^{\geq 0}$ such that $\hat{D}_i = a_{i1}\hat{D}'_1 + \cdots + a_{ik}\hat{D}'_k$.

Note that at least one of the $\hat{D}'_j$ is dominant over Z, and hence intersects every fiber of $\hat{f}$. From this we see that every $\hat{D}'_j$ is dominant over Z. Indeed, if a $\hat{D}'_j$ is contained in one fiber, then the support of $\hat{D}'_j$ is equal to the whole

fiber as shown by [3, VIII.4], which contradicts to the first condition above. Now we have $\hat{\sigma}^*\hat{\Gamma}_i = \hat{h}^*\hat{D}_i$ by the construction. Hence $\hat{h}^*\hat{D}'_1, \dots, \hat{h}^*\hat{D}'_k$ are disjoint connected components of $\hat{\sigma}^*(\sum \hat{\Gamma}_i)$. Let $\hat{G}_j := \hat{\sigma}_*\hat{h}^*\hat{D}'_j$. Then we have that $\mathrm{Supp}(\hat{G}_j) \cap \mathrm{Supp}(\hat{G}_l) = \emptyset$ for each $j \neq l$.

(5) Take two divisors $\hat{D}_1, \hat{D}_2$. Since $\hat{D}_1 \neq \hat{D}_2$, we may assume that $a_{11} > a_{21} \geq 0$. We may further assume that $a_{22} > a_{12} \geq 0$ because $\hat{D}_1 \sim_{num} \hat{D}_2$. We can write that

$$\hat{\Gamma}_1 = a_{11}\hat{G}_1 + a_{12}\hat{G}_2 + \hat{G}'_3 \text{ and } \hat{\Gamma}_2 = a_{21}\hat{G}_1 + a_{22}\hat{G}_2 + \hat{G}''_3$$

where neither of $\hat{G}_1$ and $\hat{G}_2$ intersects $\hat{G}'_3 \cup \hat{G}''_3$.

Step 4: Take two reduced, irreducible and dominant over Z components $\hat{C}_1, \hat{C}_2$ of $\hat{G}_1, \hat{G}_2$ respectively. Then $\hat{C}_1, \hat{C}_2$ are contained in $\hat{B}$ by the construction of $\hat{\Gamma}_i$ in Step 3 (3). Since $(\hat{X}, \hat{B})$ is dlt and $\hat{B}$ is a reduced integral divisor, so $\hat{C}_1, \hat{C}_2$ are log canonical centers of $(\hat{X}, \hat{B})$. By Step 3 (4), since $\hat{D}$ is nef and $\hat{D} \cdot \hat{D}_i = 0$, so $\hat{D}|_{\hat{D}_i} \sim_{num} 0$. For $j = 1, 2$, since $\hat{h}(\hat{\sigma}^{-1}\hat{C}_j)$ is a component of some $\hat{D}_i$, by $\hat{\sigma}^*(K_{\hat{X}} + \hat{B}) \sim_{\mathbb{Q}} \hat{h}^*\hat{D}$, we conclude that

$$\hat{\sigma}^*(K_{\hat{X}} + \hat{B})|_{\hat{\sigma}^{-1}\hat{C}_j} \sim_{num} 0.$$

Denote by $\hat{C}'_i$ the normalization of $\hat{C}_i$. Then $(K_{\hat{X}} + \hat{B})|_{\hat{C}'_i} \sim_{num} 0$, thus $(K_{\hat{X}} + \hat{B})|_{\hat{C}'_i} \sim_{\mathbb{Q}} 0$ by Lemma 2.4. Therefore,

(1)
$$\begin{aligned} -a_{21}M\hat{f}^*L_1|_{\hat{C}'_1} &\sim_{\mathbb{Q}} a_{21}(Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^*L_1)|_{\hat{C}'_1} \\ &\sim_{\mathbb{Q}} a_{21}\hat{\Gamma}_1|_{\hat{C}'_1} \sim_{\mathbb{Q}} a_{11}a_{21}\hat{G}_1|_{\hat{C}'_1} \\ &\sim_{\mathbb{Q}} a_{11}\hat{\Gamma}_2|_{\hat{C}'_1} \sim_{\mathbb{Q}} -a_{11}M\hat{f}^*L_2|_{\hat{C}'_1} \end{aligned}$$

which, by Lemma 2.3, implies that

$$a_{21}ML_1 \sim_{\mathbb{Q}} a_{11}ML_2.$$

In the same way, restricting on $\hat{C}'_2$ gives

$$a_{22}ML_1 \sim_{\mathbb{Q}} a_{12}ML_2.$$

Finally by conditions $a_{11} > a_{21}$ and $a_{12} < a_{22}$, we conclude that $L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} 0$, and this completes the proof. □

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