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# Representation varieties detect essential surfaces

Stefan Friedl, Takahiro Kitayama, and Matthias Nagel

Extending Culler-Shalen theory, Hara and the second author presented a way to construct certain kinds of branched surfaces in a 3-manifold from an ideal point of a curve in the $\mathrm{SL}_n$-character variety. There exists an essential surface in some 3-manifold known to be not detected in the classical $\mathrm{SL}_2$-theory. We prove that every connected essential surface in a 3-manifold is given by an ideal point of a rational curve in the $\mathrm{SL}_n$-character variety for some $n$.

## 1. Introduction

In this paper we study an extension of Culler-Shalen theory for higher-dimensional representations. In their seminal work [CS] Culler and Shalen established a method to construct essential surfaces in a 3-manifold from an ideal point of a curve in the $\mathrm{SL}_2(\mathbb{C})$-character variety. The method is built on a beautiful combination of the theory of incompressible surfaces in a 3-manifold, the geometry of representation varieties, and Bass-Serre theory [Se1, Se2]. We refer the reader to the exposition [Sh] for literature and related topics on Culler-Shalen theory. Hara and the second author presented an analogous extension of the Culler-Shalen method to the case of higher-dimensional representations [HK]. They showed that certain kinds of branched surfaces (possibly without any branched points) are constructed from an ideal point of a curve in the $\mathrm{SL}_n(\mathbb{C})$-character variety for a general $n$. Such a branched surface corresponds to a nontrivial splitting of the 3-manifold group as a complex of groups [C, Ha].

The classical theory for 2-dimensional representations is not sufficient to detect all essential surfaces in Haken manifolds. Throughout the paper let $M$ be a compact connected orientable 3-manifold. We denote by

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$X_n(M)$ the $\mathrm{SL}_n(\mathbb{C})$-character variety of $\pi_1 M$. It was discovered by Boyer and Zhang [BZ], and Motegi [Mo] that there exist infinitely many Haken manifolds $M$, which are even hyperbolic, such that $X_2(M)$ has no irreducible component of positive dimension. See also [SZ] for further study on the topic. We say that *an essential surface $S$ in $M$ is given by an ideal point $\chi$ of a curve in $X_n(M)$* if $S$ is constructed from $\chi$ by the Culler-Shalen method or its extension developed in [HK] as described in Subsection 2.3. Hara and the second author formulated and raised the following question [HK, Question 6.1].

**Question 1.1.** *Does there exist an essential surface in some 3-manifold $M$ not given by any ideal point of curves in $X_2(M)$ but given by an ideal point of a curve in $X_n(M)$ for some $n$?*

The aim of this paper is to show that the extension of Culler-Shalen theory to the case of higher-dimensional representations [HK] detects all essential surfaces in Haken manifolds. The following is the main theorem of this paper, which, in particular, gives an affirmative answer to Question 1.1.

**Theorem 1.2.** *Every connected essential surface in $M$ is given by an ideal point of a rational curve in $X_n(M)$ for some $n$.*

The proof of Theorem 1.2 relies on the breakthroughs of Agol [A] and Wise [W], and the subsequent works of Przytycki and Wise [PW1, PW2] on the separability of subgroups in a 3-manifold group. For a given connected essential surface $S$ in $M$ there exists a non-separating lift $T$ of $S$ in some finite cover $N$ of $M$ by the separability of $\pi_1 S$ in $\pi_1 M$. The non-separating surface $T$ defines abelian representations $\pi_1 N \to \mathrm{SL}_2(\mathbb{C})$ parameterized in $\mathbb{C}^\times$, which induces an affine curve $D_T$ consisting of representations $\pi_1 M \to \mathrm{SL}_n(\mathbb{C})$ where $n$ is twice the degree of the cover $N$. The set of characters of representations in $D_T$ is a desired rational curve in $X_n(M)$ as in the statement of Theorem 1.2, which has a unique ideal point. Then analyzing the structure of the Bruhat-Tits building associated to the function field of $D_T$, we explicitly construct a PL-map from the universal cover of $M$ to the 1-skeleton of the building. Finally, we show that the inverse image of midpoints of edges by the PL-map is isotopic to parallel copies of $S$.

The paper is organized as follows. In Section 2 we review the extension of the Culler-Shalen method to the case of $\mathrm{SL}_n(\mathbb{C})$-representations in [HK]. Here we recall some of the standard facts on $\mathrm{SL}_n(\mathbb{C})$-character varieties and Bruhat-Tits buildings associated to the special linear group. In Subsection 2.3, we give the precise definition of the sentence '*an essential surface* *is given by an ideal point*'. Section 3 provides a brief exposition on the separability of surface subgroups by Przytycki and Wise [PW2]. Section 4 is devoted to the proof of Theorem 1.2.

## 2. $\mathrm{SL}_n$-Culler Shalen theory

We begin with an overview of the extension of Culler-Shalen theory to the case of higher-dimensional representations in [HK].

### 2.1. Character varieties

We briefly review the $\mathrm{SL}_n(\mathbb{C})$-character variety of a finitely generated group. See [LM, Si1, Si2] for more details.

Let $\pi$ be a finitely generated group. We define the following affine algebraic set

$$R_n(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}_n(\mathbb{C})).$$

The algebraic group $\mathrm{SL}_n(\mathbb{C})$ acts on the affine algebraic set $R_n(\pi)$ by conjugation. We denote by $X_n(\pi)$ the GIT quotient of the action [MFK]:

$$X_n(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}_n(\mathbb{C}))//\,\mathrm{SL}_n(\mathbb{C}).$$

The affine algebraic set $X_n(\pi)$ is called the $\mathrm{SL}_n(\mathbb{C})$-*character variety* of $\pi$. By definition the coordinate ring $\mathbb{C}[X_n(\pi)]$ is isomorphic to the subring $\mathbb{C}[R_n(\pi)]^{\mathrm{SL}_n(\mathbb{C})}$ of $\mathbb{C}[R_n(\pi)]$ consisting of $\mathrm{SL}_n(\mathbb{C})$-invariant functions. Procesi [P, Theorem 1.3] showed that $\mathbb{C}[R_n(\mathbb{C})]^{\mathrm{SL}_n(\mathbb{C})}$ is generated by *trace functions* $I_\gamma$ for $\gamma \in \pi$ defined by

$$I_\gamma(\rho) = \mathrm{tr}\,\rho(\gamma)$$

for $\rho \in R_n(\pi)$. Therefore $X_n(\mathbb{C})$ is identified with the set of *characters* $\chi_\rho$ for $\rho \in R_n(\pi)$ defined by

$$\chi_\rho(\gamma) = \mathrm{tr}\,\rho(\gamma)$$

for $\gamma \in \pi$. For a compact connected orientable 3-manifold $M$, we abbreviate $R_n(\pi_1 M)$ and $X_n(\pi_1 M)$ with $R_n(M)$ and $X_n(M)$ respectively to simplify notation.

Let $C$ be an affine variety, and denote by $\mathbb{C}(C)$ its field of rational functions. We call $C$ an *affine curve* if the transcendence degree of $\mathbb{C}(C)$ over $\mathbb{C}$ equals 1 [F, Section 6.5]. Consider an affine curve $C$ and its projectivisation $\overline{C}$. The projective curve $\overline{C}$ might not be smooth, but it has a unique smooth model, i.e., there is a smooth projective curve $\widetilde{C}$ together with a birational map $\widetilde{C} \dashrightarrow \overline{C}$ which is universal [F, Theorem 7.3]. Recall that a birational equivalence induces an isomorphism on the associated fields of rational functions [F, Proposition 6.12]. Thus their fields of rational functions all agree: $\mathbb{C}(C) = \mathbb{C}(\overline{C}) = \mathbb{C}(\widetilde{C})$. To a point $P$ of $\widetilde{C}$ the local ring $\mathcal{O}_{\widetilde{C},P}$ of $\widetilde{C}$ at $P$ is associated. As the point $P$ is a smooth point, the ring $\mathcal{O}_{\widetilde{C},P}$ is a discrete valuation ring, which induces a discrete valuation $v_P$ on $\mathbb{C}(C)$ [F, Section 7.1].

An *ideal point* $\chi$ of an affine curve $C$ is a point of its smooth projective model $\widetilde{C}$ corresponding to a point of $\overline{C} \setminus C$. We can equip the rational functions $\mathbb{C}(C)$ with the discrete valuation $v_\chi$ associated to an ideal point $\chi$ as described above.

## 2.2. Bruhat-Tits buildings

Following the exposition [G], we describe the Bruhat-Tits building [BT1, BT2, IM] associated to the special linear group over a discrete valuation field. See also [AB] for more details on buildings.

Let $F$ be a commutative field equipped with a discrete valuation $v$ which is not necessarily complete. We denote by $\mathcal{O}_v$ the valuation ring associated to $v$. The *Bruhat-Tits building* associated to $\mathrm{SL}_n(F)$, which is an $(n-1)$-dimensional simplicial complex $B_v$, is defined as follows: A vertex of $B_v$ is the homothety class of a lattice in the $n$-dimensional vector space $F^n$, where a *lattice* in $F^n$ is a free $\mathcal{O}_v$-submodule of full rank, and two lattices $\Lambda$ and $\Lambda'$ are *homothetic* if $\Lambda = \alpha\Lambda'$ for some $\alpha \in F^\times$. A set of $(m+1)$ vertices $s_0, s_1, \ldots, s_m$ forms an $m$-simplex in $B_v$ if and only if there exist lattices $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ representing $s_0, s_1, \ldots, s_m$ respectively such that after relabeling indices we have the flag relation

$$\omega\Lambda_m \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_m,$$

where $\omega$ is an irreducible element of $\mathcal{O}_v$.

The simplicial complex $B_v$ is known to be an *Euclidean building*, and, in particular, a CAT(0)-space with respect to the standard metric. See for instance [AB, Definition 11.1] for the definition of an Euclidean building. Since $\mathrm{SL}_n(F)$ acts on the set of lattices in $F^n$ so that homothety classes and above flag relations are preserved, $\mathrm{SL}_n(F)$ acts also on $B_v$. This action is *type-preserving*, i.e., there exists an $\mathrm{SL}_n(F)$-invariant map $\tau\colon B_v^{(0)} \to \mathbb{Z}/n\mathbb{Z}$ such that $\tau|_{\Delta^{(0)}}$ is a bijection for each $(n-1)$-simplex $\Delta$ in $B_v$. Here for a simplicial complex $K$ we denote by $K^{(m)}$ the $m$-skeleton of $K$. In particular, for any subgroup $G$ of $\mathrm{SL}_n(F)$ the quotient $B_v/G$ is again an $(n-1)$-dimensional simplicial complex.

**Remark 2.1.** In the case of $n = 2$ the above construction is nothing but the one of the tree associated to $\mathrm{SL}_2(F)$ in [Se1, Se2].

### 2.3. An ideal point giving an essential surface

We summarize the construction in [HK] of a certain branched surface from an ideal point of a curve in the character variety. Here we restrict our attention to the case where such a branched surface has no branched points, and is an essential surface.

Let $C$ be a curve in $X_n(M)$ and $\chi$ an ideal point of $C$. We denote by $t\colon R_n(M) \to X_n(M)$ the quotient map. There exists a curve $D$ in $t^{-1}(C)$ such that $t|_D$ is not a constant map, and a regular map $\tilde{t}|_D\colon \widetilde{D} \to \widetilde{C}$ on the smooth projective models is induced by $t|_D$. We take a lift $\tilde{\chi} \in (\tilde{t}|_D)^{-1}(\{\chi\})$, and denote by $B_{\tilde{\chi}}$ the Bruhat-Tits building associated to $\mathrm{SL}_n(\mathbb{C}(D))$, where the function field $\mathbb{C}(D)$ is equipped with the discrete valuation at $\tilde{\chi}$. The *tautological representation* $\mathcal{P}\colon \pi_1 M \to \mathrm{SL}_n(\mathbb{C}(D))$ is defined by

$$\mathcal{P}(\gamma)(\rho) = \rho(\gamma)$$

for $\gamma \in \pi_1 M$ and $\rho \in D$. Pulling back the action of $\mathrm{SL}_n(\mathbb{C}(D))$ on $B_{\tilde{\chi}}$ by $\mathcal{P}$, we obtain the action of $\pi_1 M$ on $B_{\tilde{\chi}}$. Extending [CS, Theorem 2.2.1] to the case of a general $n$, Hara and the second author [HK, Corollary 4.5] proved that the action is *nontrivial*, i.e., for every vertex of $B_{\tilde{\chi}}$ its stabilizer subgroup of $\pi_1 M$ is proper.

Recall that a compact orientable properly-embedded surface $S$ in $M$ is called *essential* if for any component $S_0$ of $S$ the inclusion-induced homomorphism $\pi_1 S_0 \to \pi_1 N$ is injective, and $S_0$ is not boundary-parallel nor homeomorphic to the 2-sphere $S^2$. We say that *an essential surface $S$ is given by an ideal point* $\chi$ if for some lift $\tilde{\chi}$ of $\chi$ there exists a PL map $f\colon M \to B_{\tilde{\chi}}^{(1)}/\pi_1 M$ whose inverse image of the set of midpoints of the edges in $B_{\tilde{\chi}}$ is isotopic to some number of parallel copies of $S$.

When $n = 2$, since $\pi_1 M$ nontrivially acts on the tree $B_{\tilde{\chi}}$ without inversions, every ideal point $\chi$ gives some essential surface in $M$ [CS, Proposition 2.3.1]. In general, it follows from the proof of [HK, Theorem 4.7] that if $n = 3$ or if $\partial M$ is non-empty, then there exists a PL map $f\colon M \to B_{\tilde{\chi}}^{(2)}/\pi_1 M$ such that $f^{-1}(Y)$ is a certain branched surface called *essential tribranched* *surface* [HK, Definition 2.2], where $Y$ is the union of edges in the first barycentric subdivision of $B^{(2)}_{\tilde{\chi}}/\pi_1 M$ not contained in $B^{(1)}_{\tilde{\chi}}/\pi_1 M$. Note that an essential tribranched surface without any branched points is nothing but an essential surface in the usual sense.

**Remark 2.2.** The authors [FKN] showed that every closed 3-manifold $M$ with fundamental group of rank $\pi_1 M \geq 4$ contains an essential tribranched surface.

## 3. Surface subgroup separability

We recall the separability of surface subgroups in a 3-manifold group proved by Przytycki and Wise [PW2], which is a key ingredient of the proof of Theorem 1.2. A subgroup $H$ of a group $G$ is *separable* if $H$ equals the intersection of finite index subgroups of $G$ containing $H$.

**Theorem 3.1.** ([*PW2, Theorem 1.1*]) *Let $S$ be a connected essential surface $S$ in $M$. Then $\pi_1 S$ is separable in $\pi_1 M$.*

Theorem 3.1 was proved by Przytycki and Wise [PW1] when $M$ is a graph manifold, and by Wise [W] when $M$ is a hyperbolic manifold. In fact, every finitely generated subgroup of $\pi_1 M$ is separable when $M$ is a hyperbolic manifold, by Wise [W] in the case where $M$ has a non-empty boundary and by Agol [A] in the case where $M$ is closed. See also Liu [L] for a refinement of the separability.

The following is a topological interpretation of Theorem 3.1. While it is well-known for experts, nevertheless we give a proof for the sake of completeness. See also [Sc, Lemma 1.4].

**Corollary 3.2.** *For an essential surface $S$ in $M$ there exists some finite cover of $M$ where the inverse image of $S$ contains a non-separating component.*

*Proof.* We may assume that $S$ is connected and separating. Let $M_+$ and $M_-$ be the two components of the complement of $S$. It follows from [He, Theorem 10.5] that $\pi_1 S$ has index at least two in $\pi_1 M_-$ and in $\pi_1 M_+$. It follows from Theorem 3.1 that there exists an epimorphism $\varphi\colon \pi_1 M \to G$ to a finite group such that $\varphi(\pi_1 S) \neq \varphi(\pi_1 M_\pm)$. In particular, we have

$$[G\colon \varphi(\pi_1 S)] \geq 2[G\colon \varphi(\pi_1 M_\pm)].$$

Let $p\colon M_\varphi \to M$ be the covering corresponding to $\operatorname{Ker}\varphi$. The numbers of components of $p^{-1}(M_\pm)$ and $p^{-1}(S)$ are equal to $[G\colon \varphi(\pi_1 M_\pm)]$ and $[G\colon \varphi(\pi_1 S)]$ respectively, and the above inequality implies

$$[G\colon \varphi(\pi_1 S)] \geq [G\colon \varphi(\pi_1 M_+)] + [G\colon \varphi(\pi_1 M_-)].$$

Thus the number of components of $p^{-1}(S)$ is greater than or equal to that of its complement, which shows that some component of $p^{-1}(S)$ is non-separating. □

## 4. Proof of the main theorem

Now we prove the main theorem. For the readers' convenience we recall the statement.

**Theorem 4.1 (Theorem 1.2).** *Every connected essential surface in $M$ is given by an ideal point of a rational curve in $X_n(M)$ for some $n$.*

Let $S$ be a connected essential surface in $M$. It follows from Corollary 3.2 that there exists a $d$-fold covering $p\colon N \to M$ for some $d$ such that $p^{-1}(S)$ contains a non-separating component $T$. Then the proof is divided into two parts: First, we construct a rational curve $C_T$ in $X_{2d}(M)$ with a unique ideal point $\chi_T$, which is determined by $T$. Second, for a lift $\tilde{\chi}_T$ of $\chi_T$ we construct a PL map $f\colon M \to B^{(1)}_{\tilde{\chi}_T}$ whose inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}$ is isotopic to two parallel copies of $S$.

### 4.1. Construction of a curve

We denote by $\psi\colon \pi_1 N \to \mathbb{Z}$ the epimorphism induced by the intersection pairing with $T$. For each $z \in \mathbb{C}^\times$ we define the representation $\tilde{\rho}_z\colon \pi_1 N \to \mathrm{SL}_2(\mathbb{C})$ to be the composition of $\psi$ and the homomorphism $\mathbb{Z} \to \mathrm{SL}_2(\mathbb{C})$ which sends an integer $k$ to the matrix

$$\begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix}.$$

We consider the induced representation $\rho_z\colon \pi_1 M \to \operatorname{Aut}(\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}^2)$ of $\tilde{\rho}_z$. Fixing representatives $\gamma_1, \ldots, \gamma_d \in \pi_1 M$ of the elements of

$\pi_1 M/p_*(\pi_1 N)$, we have the decomposition

$$\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}^2 = \bigoplus_{i=1}^{d} \gamma_i \otimes \mathbb{C}^2,$$

which is naturally identified with $\mathbb{C}^{2d}$. Thus we regard $\rho_z$ as a representation $\pi_1 M \to \mathrm{SL}_{2d}(\mathbb{C})$. We now set

$$D_T = \{\rho_z \in R_{2d}(M) : z \in \mathbb{C}^\times\},$$
$$C_T = \{\chi_{\rho_z} \in X_{2d}(M) : z \in \mathbb{C}^\times\}.$$

**Lemma 4.2.** 1) *The set $D_T$ is a curve in $R_{2d}(M)$ isomorphic to $\mathbb{C}^\times$.*

2) *The set $C_T$ is a rational curve in $X_{2d}(M)$ with a unique ideal point.*

*Proof.* These sets of representations and characters are constructed along the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{C}^\times & \longrightarrow & R_2(\mathbb{Z}) & \longrightarrow & R_2(N) & \longrightarrow & R_{2d}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & X_2(\mathbb{Z}) & \longrightarrow & X_2(N) & \longrightarrow & X_{2d}(M), \end{array}$$

where the first vertical map sends $z \in \mathbb{C}^\times$ to $z + z^{-1} \in \mathbb{C}$, and the first bottom horizontal map is an isomorphism which sends $w \in \mathbb{C}$ to the character of $\mathbb{Z}$ whose image of $1 \in \mathbb{Z}$ is $w$. The composition of the top horizontal maps is called $\Psi$ and the composition of the bottom horizontal maps is called $\Phi$. Then the sets $D_T$ and $C_T$ coincide with the images of $\Psi$ and $\Phi$

We may assume $\gamma_1 \in \pi_1 N$, and take $\mu \in \pi_1 N$ with $\psi(\mu) = 1$. Then we have

$$\rho_z(\mu) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \oplus \bigoplus_{i=2}^{d} \begin{pmatrix} z^{\psi(\gamma_i^{-1}\mu\gamma_i)} & 0 \\ 0 & z^{-\psi(\gamma_i^{-1}\mu\gamma_i)} \end{pmatrix},$$
$$\chi_{\rho_z}(\mu) = z + z^{-1} + \sum_{i=2}^{d} \left( z^{\psi(\gamma_i^{-1}\mu\gamma_i)} + z^{-\psi(\gamma_i^{-1}\mu\gamma_i)} \right).$$

Hence the restriction of the map $R_{2d}(M) \to \mathbb{C}^2$ sending a representation $\rho$ to the vector of the $(1,1)$- and $(2,2)$-entries of $\rho(\mu)$ gives the inverse regular map $D_T \to \mathbb{C}^\times$ of $\Psi$, where $\mathbb{C}^\times$ is identified with the curve $xy - 1$ in $\mathbb{C}^2$, and (1) is proved.

We deduce from the second equation above that the map $\Phi$ is not constant. Also by fixing an affine space $\mathbb{C}^N$ containing $X_{2d}(M)$, we regard $\Phi$ as a map $\mathbb{C} \to \mathbb{C}^N$. We denote by $\overline{\Phi}\colon \mathbb{P}^1 \to \mathbb{P}^N$ the projective extension of $\Phi$. Since $\overline{\Phi}$ is not a constant map, by the completeness of the projective line $\mathbb{P}^1$ [Mu, Section I.9, Theorem 1] the image $\overline{C}_T$ of $\overline{\Phi}$ is a projective curve, and by the Riemann-Hurwitz formula the curve $\overline{C}_T$ is rational. Therefore the set $C_T$, which coincides with the intersection of $\overline{C}_T$ and $\mathbb{C}^N$, is an affine rational curve. Since $\Phi$ induces a surjective regular map $\widetilde{\Phi}\colon \mathbb{P}^1 \to \widetilde{C}_T$ on the smooth projective models, the rational curve $C_T$ has a unique ideal point corresponding to the point at infinity of $\mathbb{P}^1$, which proves (2). □

It is a simple matter to check that both the two ideal points of $D_T$ corresponding to 0 and $\infty$ are lifts of the unique ideal point $\chi_T$ of $C_T$. Let $\tilde{\chi}_T$ be the one corresponding to 0. Then as in Section 2.3 we obtain the nontrivial action $\pi_1 M$ on the Bruhat-Tits building $B_{\tilde{\chi}_T}$ associated to $\mathrm{SL}_{2d}(\mathbb{C}(D_T))$. We identify $\mathbb{C}(D_T)$ with the standard function field $\mathbb{C}(t)$ and the valuation at $\tilde{\chi}_T$ with the lowest degree of the Laurent expansion of a rational function. Then the vector space $\mathbb{C}(t)^{2d}$ is decomposed into

$$\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2 \;=\; \bigoplus_{i=1}^{d} \gamma_i \otimes \mathbb{C}(t)^2,$$

where $\pi_1 N$ acts on $\mathbb{C}(t)^2$ by the representation $\mathcal{Q}\colon \pi_1 N \to \mathrm{SL}_2(\mathbb{C}(t))$ defined by

$$\mathcal{Q}(\gamma) = \begin{pmatrix} t^{\psi(\gamma)} & 0 \\ 0 & t^{-\psi(\gamma)} \end{pmatrix}$$

for $\gamma \in \pi_1 N$, and the tautological representation $\mathcal{P}\colon \pi_1 M \to \mathrm{SL}_{2d}(\mathbb{C}(t))$ is given by the left multiplication on $\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2$.

### 4.2. Construction of a PL-map

We take a triangulation of $M$ containing $S$ as a normal surface. We may assume that the intersection of each tetrahedron with $S$ is connected, if necessary, replacing the triangulation by an appropriate subdivision. The triangulation of $M$ induces ones of $N$ and the universal cover $\widetilde{M}$ of $M$, so that $T$ and its inverse image $\widetilde{T}$ by the covering $\widetilde{M} \to N$ are also normal surfaces. Then we take a cellular map $g\colon N \to \mathbb{R}/\mathbb{Z}$ such that $g^{-1}([\frac{1}{2}]) = T$, where we consider the cellular structure of $\mathbb{R}/\mathbb{Z}$ with one vertex corresponding to $\mathbb{Z}$. We define $\tilde{g}\colon \widetilde{M} \to \mathbb{R}$ to be the $\pi_1 N$-equivariant lift of $g$, so that $\tilde{g}^{-1}(\frac{1}{2} + \mathbb{Z}) = \widetilde{T}$.

We now define a map $\tilde{f}^{(0)}\colon \widetilde{M}^{(0)} \to B^{(0)}_{\tilde{\chi}_T}$ as follows. For $s \in \widetilde{M}^{(0)}$ we consider the lattice

$$\bigoplus_{i=1}^{d} \gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)}$$

in $\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2$, where $\Lambda_n$ is the lattice in $\mathbb{C}(t)^2$ generated by the vectors

$$\begin{pmatrix} t^n \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ t^{-n} \end{pmatrix}.$$

Note that $\tilde{g}(\gamma_i^{-1}s)$ is an integer for each $i$ by the construction of $\tilde{g}$. Then we set $\tilde{f}^{(0)}(s)$ to be the homothety class of the above lattice. In the following two lemmas we observe the key properties of $\tilde{f}^{(0)}$.

**Lemma 4.3.** *The map $\tilde{f}^{(0)}$ is $\pi_1 M$-equivariant.*

*Proof.* For $\gamma \in \pi_1 M$ there exist a permutation $\sigma$ of degree $d$ and $\delta_i \in \pi_1 N$ such that

$$\gamma\gamma_i = \gamma_{\sigma(i)}\delta_i$$

for each $i$. Then

$$\begin{aligned}
\bigoplus_{i=1}^{d} \gamma\gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)} &= \bigoplus_{i=1}^{d} \gamma_{\sigma(i)}\delta_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)} = \bigoplus_{i=1}^{d} \gamma_{\sigma(i)} \otimes \mathcal{Q}(\delta_i) \cdot \Lambda_{\tilde{g}(\gamma_i^{-1}s)} \\
&= \bigoplus_{i=1}^{d} \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)+\psi(\delta_i)} = \bigoplus_{i=1}^{d} \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\delta_i\gamma_i^{-1}s)} \\
&= \bigoplus_{i=1}^{d} \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\gamma_{\sigma(i)}^{-1}\gamma s)} = \bigoplus_{i=1}^{d} \gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}\gamma s)}
\end{aligned}$$

for $\gamma \in \pi_1 M$ and $s \in \widetilde{M}^{(0)}$, which implies that $\tilde{f}^{(0)}$ is a $\pi_1 M$-equivariant map. □

**Lemma 4.4.** *For each tetrahedron $\Delta$ in $\widetilde{M}$ the set $\tilde{f}^{(0)}(\Delta^{(0)})$ consists of one vertex if $\gamma_i^{-1} \cdot \Delta$ does not intersect with $\widetilde{T}$ for any $i$, and two vertices of distance 2 with respect to the graph metric on $B^{(1)}_{\tilde{\chi}}$ otherwise.*

*Proof.* If $\gamma_i^{-1} \cdot \Delta$ does not intersect with $\widetilde{T}$ for any $i$, then it follows from the choice of $g$ that there exists some $n_i \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \{n_i\}$$

for each $i$, and hence we obtain

$$\tilde{f}^{(0)}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \left\{ \left[ \bigoplus_{i=1}^{d} \gamma_i \otimes \Lambda_{n_i} \right] \right\}.$$

In the following we consider the case where $\gamma_i^{-1} \cdot \Delta$ intersects with $\widetilde{T}$ for $i = i_1, \ldots, i_m$. Then the intersections of $\gamma_{i_k}^{-1} \cdot \Delta$ with $\widetilde{T}$ are all connected and of same type for $k = 1, \ldots, m$, since otherwise $p(T) = S$ implies that the intersection of some tetrahedron in $M$ with $S$ is not a normal disc, which contradicts the choice of the triangulation of $M$. Thus $\Delta^{(0)}$ is divided into two subsets $\Delta_+^{(0)}$ and $\Delta_-^{(0)}$ satisfying the following:

1) there exists some $n_{i_k} \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_{i_k}^{-1} \cdot \Delta_+^{(0)}) = \{n_{i_k} + 1\} \text{ and } \tilde{g}(\gamma_{i_k}^{-1} \cdot \Delta_-^{(0)}) = \{n_{i_k}\}$$

for $k = 1, \ldots, m$;

2) there exists some $n_i \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \{n_i\}$$

for $i \neq i_1, \ldots, i_m$.

Hence we obtain

$$\tilde{f}^{(0)}(\Delta_+^{(0)}) = \{[\Lambda_+]\} \text{ and } \tilde{f}^{(0)}(\Delta_-^{(0)}) = \{[\Lambda_-]\},$$

where

$$\Lambda_+ = \left( \bigoplus_{k=1}^{m} \gamma_{i_k} \otimes \Lambda_{n_{i_k}+1} \right) \oplus \left( \bigoplus_{i \neq i_1, \ldots, i_m} \gamma_i \otimes \Lambda_{n_i} \right),$$

$$\Lambda_- = \left( \bigoplus_{k=1}^{m} \gamma_{i_k} \otimes \Lambda_{n_{i_k}} \right) \oplus \left( \bigoplus_{i \neq i_1, \ldots, i_m} \gamma_i \otimes \Lambda_{n_i} \right).$$

Since

$$t\Lambda'_n \subsetneq \Lambda_{n+1} \subsetneq \Lambda'_n,$$
$$t\Lambda'_n \subsetneq \Lambda_n \subsetneq \Lambda'_n$$

for $n \in \mathbb{Z}$, where $\Lambda'_n$ is the lattice in $\mathbb{C}(t)^2$ generated by the vectors

$$\begin{pmatrix} t^n \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ t^{-n-1} \end{pmatrix},$$

we have

$$t\Lambda' \subsetneq \Lambda_+ \subsetneq \Lambda',$$
$$t\Lambda' \subsetneq \Lambda_- \subsetneq \Lambda',$$

where

$$\Lambda' = \left( \bigoplus_{k=1}^{m} \gamma_{i_k} \otimes \Lambda'_{n_{i_k}} \right) \oplus \left( \bigoplus_{i \neq i_1, \ldots, i_m} \gamma_i \otimes \Lambda_{n_i} \right).$$

By the definition of the building $B_{\tilde{\chi}}$ these relations imply that there exist edges in $B_{\tilde{\chi}}$ connecting $[\Lambda_+]$ and $[\Lambda_-]$ with $[\Lambda']$, and hence the distance between $[\Lambda_+]$ and $[\Lambda_-]$ is at most 2 in $B^{(1)}_{\tilde{\chi}}$. We further observe that the matrix

$$\left( \bigoplus_{i=1}^{n} \gamma_{i_k} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \oplus \left( \bigoplus_{i \neq i_1, \ldots, i_m} \gamma_i \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

in $\mathrm{SL}_{2d}(\mathbb{C}(t))$ sends $[\Lambda_-]$ to $[\Lambda_+]$. Since the action of $\mathrm{SL}_{2d}(\mathbb{C}(t))$ on $B_{\tilde{\chi}}$ is type-preserving, the distance between them is exactly equal to 2, and the lemma follows. □

We are now in position to construct a desired PL map $f\colon M \to B^{(1)}_{\tilde{\chi}_T}/\pi_1 M$. We consider the 1st barycentric subdivision of the triangulation of $M$ and the induced subdivision of that of $\widetilde{M}$. It follows from Lemmas 4.3 and 4.4 that $\tilde{f}^{(0)}$ extends to a $\pi_1 M$-equivariant simplicial map $\tilde{f}\colon \widetilde{M} \to B^{(1)}_{\tilde{\chi}_T}$ with respect to the subdivision. Let $\Delta$ be a tetrahedron in the original triangulation of $\widetilde{M}$. If $\gamma_i^{-1} \cdot \Delta$ intersects with $\widetilde{T}$ for some $i$, then the simplicial subsurface in the subdivision of $\Delta$ separating the sets $\Delta_+^{(0)}$ and $\Delta_-^{(0)}$ is mapped by $\tilde{f}$ to the vertex $[\Lambda']$ of $B_{\tilde{\chi}_T}$ as in the proof of Lemma 4.4. Here the subsurface in $\Delta$ is isotopic to the intersection of $\Delta$ and $\gamma_i \cdot \widetilde{T}$. Thus the inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}$ by $\tilde{f}$ is isotopic to two parallel copies of the subsurface $\bigcup_{i=1}^{d} \gamma_i \cdot \widetilde{T}$, which coincides with the preimage of $S$ in $\widetilde{M}$.

We define a PL map $f\colon M \to B^{(1)}_{\tilde{\chi}_T}/\pi_1 M$ with respect to the subdivision to be the quotient of $\tilde{f}$ by $\pi_1 M$. Then the inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}/\pi_1 M$ by $f$ is isotopic to two parallel copies of $S$. Therefore $S$ is given by $\chi_T$, which completes the proof.

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Department of Mathematics, University of Regensburg
Raum M118, 93047 Regensburg, Germany
*E-mail address*: `sfriedl@gmail.com`

Graduate School of Mathematical Sciences
The University of Tokyo, Tokyo 113-8654, Japan
*E-mail address*: `kitayama@ms.u-tokyo.ac.jp`

Department of Mathematics & Statistics, McMaster University
1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada
*E-mail address*: `nagel@cirget.ca`

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