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Automorphisms of Salem degree 22 on supersingular K3 surfaces of higher Artin invariant

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We give a short proof that every supersingular K3 surface (except possibly in characteristic 2 with Artin invariant $\sigma = 10$) has an automorphism of Salem degree 22. In particular an infinite subgroup of the automorphism group does not lift to characteristic zero. The proof relies on the case $\sigma = 1$ and the cone conjecture for K3 surfaces.

1. Introduction

A Salem number is a real algebraic integer $\lambda > 1$ which is conjugate to $1/\lambda$ and all whose other conjugates lie on the unit circle. Its minimal polynomial is called a Salem polynomial. Salem numbers arise naturally in algebraic geometry as follows: If X is a projective surface over an algebraically closed field k and $f\colon X \to X$ an automorphism, then the characteristic polynomial

$$\chi(f^*|H^2_{\acute{e}t}(X, \mathbb{Q}_\ell(1))) \quad (\ell \neq \operatorname{char} k)$$

factors as a product of cyclotomic polynomials and at most one Salem polynomial $s(x)$ [5]. We call the degree of the Salem factor $s(x)$ the *Salem degree* of f. Let H be an ample polarization of X. Since the order of f^* is finite on

$$\langle f^{*k}(H) \mid k \in \mathbb{Z}\rangle^\perp \subseteq H^2_{\acute{e}t}(X, \mathbb{Q}_\ell(1))$$

by [5], we get that $\ker s(f^*|H^2_{\acute{e}t}(X, \mathbb{Q}_\ell(1)))$ is contained in the (ℓ-adic) Néron-Severi group $\mathrm{NS}(X) \otimes \mathbb{Q}_\ell$ of X. In particular, we can bound the Salem degree of an automorphism by the Picard number $\rho(X)$. For a K3 surface X it is at most $\rho(X) \leq h^{1,1}(X) = 20$ in characteristic 0 by Lefschetz' Theorem on (1,1)-classes. However, in positive characteristic supersingular K3 surfaces have $\rho(X) = 22$. Indeed:

Theorem 1.1. *[1–4, 11, 12] The supersingular K3 surface X/k, $k = \overline{k}$, $\operatorname{char} k > 0$, of Artin invariant one has an automorphism of Salem degree 22.*

Note that the characteristic polynomial of f^* is stable under (good) specialization by standard comparison theorems. This observation leads to the interesting feature that an automorphism of Salem degree 22 is not geometrically liftable to characteristic zero (see [3] for details). This is in sharp contrast to the case of non-supersingular K3 surfaces in odd characteristic. There one can allways lift a finite index subgroup of the automorphism group to characteristic zero (cf [6, Thm. 3.2]).

Supersingular K3 surfaces are classified by their Artin invariant $1 \leq \sigma \leq 10$. For fixed Artin invariant σ they form a family of dimension $\sigma - 1$, while the supersingular K3 surface of Artin invariant $\sigma = 1$ is unique (cf. [8, 9]). The main purpose of this note is to extend Theorem 1.1 to *all* supersingular K3 surfaces.

Theorem 1.2 (Main Theorem). *Let Y/k be a supersingular K3 surface over an algebraically closed field such that the crystalline Torelli theorem holds for Y. Then Y has an automorphism of Salem degree 22.*

Remark 1.3. Set $p = \operatorname{char} k$ and $\sigma = \sigma(Y)$. The crystalline Torelli is proven for $p > 3$ in [8, Thm. I] and for $p = 2$ and $\sigma < 10$ and for $p = 3$ and $\sigma < 6$ (at the end of [9]). For $p = 3$ the main theorem is proved in [12]. Hence the only open case left is $p = 2$ and $\sigma = 10$. The main step in the proof is a reduction to Theorem 1.1.

In a recent preprint [14] Yu gives an independent proof of the main theorem for $p > 3$ using genus one fibrations. However, I believe the new proof to be of independent interest, as it is shorter and characteristic free. In particular the result for $p = 2, \sigma > 1$ is new.

2. Preliminaries

A lattice L is a finitely generated free abelian group equipped with a non-degenerate, integer valued bilinear form. It is called even if $x^2 \in 2\mathbb{Z}$ for all $x \in L$. The dual lattice is $L^\vee = \{x \in L \otimes \mathbb{Q} : x.L \subseteq \mathbb{Z}\}$ and the discriminant group $L^\vee/L$ of an even lattice L is equipped with the quadratic form

$$q : L^\vee/L \to \mathbb{Q}/2\mathbb{Z}, \quad \overline{x} \mapsto x^2 \mod 2\mathbb{Z}.$$

A supersingular K3 lattice N is an even lattice of signature $(1, 21)$ and discriminant group $N^\vee/N \cong \mathbb{F}_p^{2\sigma}$. If $p = 2$, we require furthermore that it is of type I, i.e. $x^2 \in \mathbb{Z}$ for $x \in N^\vee$. Such a lattice is determined up to isometry by p and σ (cf. [9, sect. 1]). Let X be a K3 surface defined over an algebraically closed field k of characteristic p. Recall that X is said to be *supersingular* if

$$\rho(X) = \operatorname{rk} \mathrm{NS}(X) = 22.$$

Then the Néron-Severi lattice $\mathrm{NS}(X)$ is a supersingular K3 lattice for $p = \operatorname{char} k$ and $1 \le \sigma \le 10$ (cf. [9, sect. 8]). We call σ the *Artin invariant* of X.

For the readers' convenience we give a proof of the following well known

Lemma 2.1. *There is an embedding $N_{p,\sigma} \hookrightarrow N_{p',\sigma'}$ of supersingular K3 lattices if and only if $p = p'$ and $\sigma' \le \sigma$.*

Proof. The only if part follows from the fact that if $A \subset B$ are two lattices of the same rank, then

$$\det A = [B : A]^2 \det B.$$

In this situation

$$A \hookrightarrow B \hookrightarrow B^\vee \hookrightarrow A^\vee$$

and B/A is a totally isotropic subspace of $A^\vee/A$. Now, if A is 2-elementary of type I, then, since $B^\vee \subseteq A^\vee$, so is B. Let $p \ne 2$. Then the quadratic space $N_{p,10}^\vee/N_{p,10} \cong \mathbb{F}_p^{20}$ contains an isotropic line since it is of dimension greater two. As above this line corresponds to an overlattice N of $N_{p,10}$ which is hyperbolic and $|N^\vee/N| = p^{18}$. Since subquotients of vector spaces are vector spaces, we see that $N^\vee/N \cong \mathbb{F}_p^{18}$. Then $N \cong N_{p,9}$ is in fact a supersingular K3 lattice. Continuing in the same way, we get a chain of overlattices

$$N_{p,10} \subseteq N_{p,9} \subseteq \cdots \subseteq N_{p,1}.$$

Note that the process stops at $\sigma = 1$ since there is no isotropic line in the discriminant group. This is in accordance with the fact that there is no even unimodular lattice of signature $(1, 21)$. For $p = 2$ the discriminant form is isomorphic to a direct sum of forms of type $q(x, y) = x^2 + xy + y^2 \mod 2\mathbb{Z}$ and the existence of an isotropic vector follows as long as there are at least two summands, i.e., $\sigma > 1$. Since everything is contained in $N_{p,10}^\vee$, the constructed lattices stay of type I. □

Let L be an even lattice of signature $(1, n)$ and denote by $O^+(L)$ the subgroup of isometries preserving the two connected components of the positive cone. Set

$$V_L = \left\{x \in L \otimes \mathbb{R} \,|\, x^2 > 0 \text{ and } \forall r \in L \text{ with } r^2 = -2\colon\ (r, x) \neq 0\right\}.$$

According to [8, Proposition 1.10], the set V_L is open and each of its connected components meets $L \subset L \otimes \mathbb{R}$. These connected components of V_L are called *chambers* of V_L. Each point r of length -2 induces an orthogonal reflection

$$\delta_r : L \to L \quad x \mapsto x + \langle x.r\rangle r$$

along the hyperplane $r^\perp$. The Weyl group $W(L) \subseteq O(L)$ is the group generated by all orthogonal reflections along a (-2)-hyperplane. It acts transitively on the set of chambers.

If $L = \mathrm{NS}(X)$ for a K3 surface X, then one of the chambers is the ample cone. Its closure is the nef cone $\mathrm{Nef}(X)$. Classes of smooth rational curves are called nodal. By adjunction they are of square (-2) and they are exactly the rays of the effective cone. Note that if $r^2 = -2$, then by Riemann-Roch either r or $-r$ is effective but they are not necessarily nodal.

Theorem 2.2 (Cone conjecture). *[7, Thm. 6.1] Let X be a K3 surface over an algebraically closed field k. If X is supersingular suppose that crystalline Torelli holds for X. Let $\Gamma(X) \subseteq O^+(\mathrm{NS}(X))$ be the subgroup preserving the nef cone. Then $\Gamma(X) \cong O^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$ and*

1) *The natural map $\mathrm{Aut}(X) \to \Gamma(X)$ has finite kernel and cokernel.*
2) *The group $\mathrm{Aut}(X)$ is finitely generated.*
3) *The action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)$ has a rational polyhedral fundamental domain.*
4) *The set of orbits of $\mathrm{Aut}(X)$ in the nodal classes of X is finite.*

Over $\mathbb{C}$ the theorem follows from the strong Torelli theorem by work of Sterk [13, Thm. 01.]. Then, for K3 surfaces of finite height in arbitrary characteristic one can lift $X, \mathrm{NS}(X)$ and a finite index subgroup of $\mathrm{Aut}(X)$ to characteristic zero and apply the cone theorem there. For supersingular K3 surfaces one has to use the crystalline Torelli Theorem. In this case $\mathrm{Aut}(X) \to \Gamma$ is injective and its image contains the finite index subgroup $\ker(\Gamma \to O(\mathrm{NS}^\vee/\mathrm{NS}))$.

Lemma 2.3. *[10, p. 169] If λ is a Salem number of degree d then λ^n, $n \in \mathbb{N}$ is a Salem number of the same degree.*

Proof. Denote the Galois conjugates of $\lambda = \lambda_1$ by λ_i $i = 1, \ldots, n$. Then the Galois conjugates of λ_1^n are the λ_i^n. In particular λ_1^n is a Salem number. It remains to check that its conjugates are all distinct. Suppose that $\lambda_i^n = \lambda_k^n$. After applying a Galois conjugation we may assume that $i = 1$. In particular, $1 < \lambda_1^n = \lambda_k^n$. Now, $|\lambda_k| > 1$ is the unique conjugate of absolute value greater one, i.e. $k = 1$. □

Corollary 2.4. *The maximum occurring Salem degree of an automorphism of a K3 surface X over an algebraically closed field depends only on the isometry class of* $\mathrm{NS}(X)$*, given that the cone conjecture holds for X.*

Proof. Since any power of a Salem number of degree d remains a Salem number of this degree, we may pass to a finite index subgroup. Combining this with part (1) of the cone conjecture, we get that the maximum occurring Salem degree of an automorphism of X depends only on $\Gamma(X)$. Now, $\Gamma(X)$ depends up to conjugation by an element of the Weyl group only on the isometry class of $\mathrm{NS}(X)$. In particular, the maximal Salem degree of an automorphism of X depends only on $\mathrm{NS}(X)$. □

3. Proof of the main theorem

Lemma 3.1. *Let $N \subseteq L$ be two lattices of the same rank and $G \subseteq O(L)$ a subgroup. Then*

$$[G : O(N) \cap G] < \infty$$

where we view $O(N)$ and $O(L)$ as subgroups of $O(N \otimes \mathbb{R})$.

Proof. Since the ranks coincide, the index $n = [L : N]$ is finite and

$$nL \subseteq N \subseteq L.$$

Any isometry of L preserves nL hence we get a map

$$\varphi\colon G \to \mathrm{Aut}(L/nL).$$

Set $K = \ker \varphi$, which is a finite index subgroup of G. To see that $K \subseteq O(N)$ as well, recall that an isometry f of $O(nL)$ extends to $O(N)$ iff $f(N/nL) = N/nL$. Indeed, $f|L/nL = id|L/nL$ for $f \in K$, by definition. □

The following is a generalization of [14, Thm. 1.2] where the existence of at least one elliptic fibration on X with infinite automorphism group is assumed. We can drop this condition.

Theorem 3.2. *Let $X/k, Y/k'$ be two K3 surfaces over algebraically closed fields k, k' satisfying the cone conjecture. Suppose that $\rho(X) = \rho(Y)$ and that there is an isometric embedding*

$$\iota : \mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(X).$$

Then $\mathrm{sdeg}(X) \leq \mathrm{sdeg}(Y)$ *where*

$$\mathrm{sdeg}(X) = \max\{\textit{Salem degree of } f \mid f \in \mathrm{Aut}(X)\}.$$

Proof. Denote by $\mathrm{Nef}(X)$ and $\mathrm{Nef}(Y)$ the nef cones of X and Y. Any chamber of the positive cone of $\mathrm{NS}(X)$ is contained in the image of a unique chamber of the positive cone of $\mathrm{NS}(Y)$. Since the Weyl group acts transitively on the chambers, we can find an element $\delta \in W(\mathrm{NS}(X))$ of the Weyl group such that $\mathrm{Nef}(X) \subset \iota'_{\mathbb{R}}(\mathrm{Nef}(Y))$ where $\iota' = \delta \circ \iota$. To ease notation we identify $\mathrm{NS}(Y)$ with its image under ι'. By the preceding Lemma $[\Gamma(X) : \Gamma(X) \cap O(\mathrm{NS}(Y))]$ is finite, and since $\mathrm{Nef}(X) \subseteq \mathrm{Nef}(Y)$, we get that $\Gamma(X) \cap O(\mathrm{NS}(Y)) \subseteq \Gamma(Y)$. Now, by the cone Theorem 2.2 and the proof of Corollary 2.4

$$\begin{aligned}\mathrm{sdeg}(X) = \mathrm{sdeg}(\Gamma(X)) &= \mathrm{sdeg}(\Gamma(X) \cap O(NS(Y))) \\ &\leq \mathrm{sdeg}(\Gamma(Y)) = \mathrm{sdeg}(Y).\end{aligned}$$

□

Proof of Theorem. 1.2. If X/k and Y/k are supersingular K3 surfaces with $\sigma(X) \leq \sigma(Y)$, then $\mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(X)$ by Lemma 3.1. Combining the $\sigma = 1$ case (Thm. 1.1) and the previous theorem we get that $22 = \mathrm{sdeg}(X) \leq \mathrm{sdeg}(Y) \leq 22$. □

The converse inequality in Theorem 3.2 is false in general. See [14, rmk. 7.3] for examples.

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