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On the existence of ghost classes in the cohomology of the Shimura variety associated to $GU(2,2)$

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In this paper we study the existence of ghost classes in the cohomology of the Shimura variety attached to the group of unitary similitudes of signature $(2,2)$, denoted by $GU(2,2)$. We use considerations on the weights of the mixed Hodge structures attached to the cohomology spaces involved in their definition. The non-existence of ghost classes is known in the cases in which the highest weight of the irreducible representation is regular. We prove that for most irreducible representations with irregular highest weight there are no ghost classes and for the other cases we show that the possible weights in the mixed Hodge structure on the space of ghost classes belong always to the set consisting of the middle weight and the middle weight plus one.

1. Introduction

Let (G,X) be a Shimura pair, (ρ, V) an irreducible algebraic representation of G and $\mathbb{A}_f$ the ring of finite adeles of $\mathbb{Q}$, then for each compact open subgroup $K \subset G(\mathbb{A}_f)$, V induces a local system $\tilde{V}$ on the corresponding level variety $S_K = Sh_K(G,X)$ which is in fact a variation of Hodge structure of a given weight $wt(V)$. We denote by $\overline{S}_K$ the Borel-Serre compactification of S_K and by $\partial\overline{S}_K$ its boundary, then taking projective limits over the set of open compact subgroups $K \subset G(\mathbb{A}_f)$ we obtain the Shimura variety S, its Borel-Serre compactification $\overline{S}$ and its boundary $\partial\overline{S}$. Assume we have chosen a maximal torus T on G, a maximal $\mathbb{Q}$-split torus on G contained in T and a system of positive roots in the respective associated root systems such that they are compatible. By these choices in the $\mathbb{Q}$-root system we have an induced set $\mathcal{P}(G)$ of $\mathbb{Q}$-parabolic subgroups of G, the set of standard $\mathbb{Q}$-parabolic subgroups. Then $\partial\overline{S}$ can be written as a union of faces indexed

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by the standard $\mathbb{Q}$-parabolic subgroups of G, and for each $P \in \mathcal{P}(G)$ we will denote by ∂_P its corresponding face in $\partial\overline{S}$.

$\tilde{V}$ can be extended to the Borel-Serre compactification in a canonical way and we have an isomorphism in cohomology $H^q(S, \tilde{V}) \cong H^q(\overline{S}, \tilde{V})$ for each q. Considering its corresponding restriction to the boundary of the Borel-Serre compactification we have the following long exact sequence in cohomology

$$\cdots \to H_c^q(S, \tilde{V}) \to H^q(S, \tilde{V}) \to H^q(\partial\overline{S}, \tilde{V}) \to \cdots$$

where $H_c^*(S, \tilde{V})$ denotes the cohomology with compact support.

On the other hand, the covering of $\partial\overline{S}$ given by the faces attached to the standard $\mathbb{Q}$-parabolic subgroups induces a spectral sequence abutting to the cohomology of the boundary $H^*(\partial\overline{S}, \tilde{V})$.

If G has semisimple $\mathbb{Q}$-rank 2 then $\mathcal{P}(G)$ consists of three elements, one minimal standard $\mathbb{Q}$-parabolic subgroup denoted by P_0 and two maximal ones denoted by P_1 and P_2, this reduces the aforementioned spectral sequence to a long exact sequence in cohomology given by

$$\cdots \to H^q(\partial\overline{S}, \tilde{V}) \to H^q(\partial_{P_1}, \tilde{V}) \oplus H^q(\partial_{P_2}, \tilde{V}) \to H^q(\partial_{P_0}, \tilde{V}) \to \cdots$$

The space of q-ghost classes $Gh^q(V)$ is the subspace of $H^q(\partial\overline{S}, \tilde{V})$ given by the intersection of the image of the morphism

$$H^q(S, \tilde{V}) \to H^q(\partial\overline{S}, \tilde{V})$$

with the kernel of the morphisms

$$H^q(\partial\overline{S}, \tilde{V}) \to H^q(\partial_P, \tilde{V})$$

corresponding to each standard $\mathbb{Q}$-parabolic subgroup P of G, which is the same as the intersection of the image of the first map with the kernel of

$$H^q(\partial\overline{S}, \tilde{V}) \to H^q(\partial_{P_1}, \tilde{V}) \oplus H^q(\partial_{P_2}, \tilde{V})$$

Ghost classes have been first considered by A. Borel and have been treated by many mathematicians as G. Harder, J. Schwermer, J. Franke, C. Moeglin. The definition makes sense for any reductive group over $\mathbb{Q}$ and does not depend on the existence of a Hermitian structure. Ghost classes have been found, for example for GL_n by G. Harder in [5], using the combinatorics of Eisenstein series, and J. Franke developed a topological method to obtain such classes in [3], but few examples are known for Shimura varieties.

In this paper we study the existence, or not, of ghost classes for the Shimura variety associated to $GU(2,2)$.

By Saito's theory of mixed Hodge modules and by the results in [7], each term in the last two long exact sequences is endowed with a mixed Hodge structure and the given morphisms are morphisms of mixed Hodge structures, so the space of ghost classes has an induced mixed Hodge structure. By using information on the cohomology spaces appearing in the aforementioned long exact sequences we can obtain information on the possible types in the mixed Hodge structure on the spaces of ghost classes.

The information on the space $Gh^q(V)$ will be obtained by the study of the morphisms $H^q(S,\tilde{V}) \to H^q(\partial\overline{S},\tilde{V})$, $H^{q-1}(\partial_{P_1},\tilde{V}) \to H^{q-1}(\partial_{P_0},\tilde{V})$ and $H^{q-1}(\partial_{P_2},\tilde{V}) \to H^{q-1}(\partial_{P_0},\tilde{V})$ as well as by considerations on the mixed Hodge structures of the given cohomology spaces. The arguments for studying the first two morphisms come from weight considerations. The same method does not give much information on the morphism $H^{q-1}(\partial_{P_2},\tilde{V}) \to H^{q-1}(\partial_{P_0},\tilde{V})$, so its study will be based on the results in [4].

One knows that the weights in the mixed Hodge structure associated to $H^q(S,\tilde{V})$ are greater than or equal to $q + wt(V)$, where $wt(V)$ is the weight of the variation of (complex) Hodge structure defined by $\tilde{V}$, while the weights in $H_c^{q+1}(S,\tilde{V})$ are less than or equal to $(q+1)+wt(V)$. We call $q+wt(V)$ the middle weight. One interesting question is the study of the possible weights in the space of q-ghost classes, and in particular whether the only weights appearing are the middle weight and/or the middle weight plus one. In the case when $Gh^q(V)$ satisfies this property for every natural number q we will say that the representation V satisfies the 'weak middle weight property' and moreover we say that it satisfies the 'middle weight property' if in addition the only possible weight in such a mixed Hodge structure is the middle weight.

We present the main results of this paper. If the highest weight of the irreducible representation is regular then it is known, by combining Theorem 4.11 in [12] and Theorem 19 in [2], that there are no ghost classes. Then the new results of this paper come from the non regular highest weights. The irreducible algebraic representations of $GU(2,2)$ are parametrized by 5-tuples (a_1,a_2,a_3,a_4,c) where $a_1 \geq a_2 \geq a_3 \geq a_4$ and c is congruent to $\sum_{i=1}^4 a_i$ module 2. With this notation we prove in Theorem 8 that in the cases in which $a_1 \neq a_2$ or $a_3 \neq a_4$, these includes most of the irregular highest weights, there are no ghost classes and for the cases $a_1 = a_2$ and $a_3 = a_4$ we determine in Theorem 9 the weights and degree in cohomology of the possible ghost classes, obtaining that the corresponding local system satisfies the weak middle weight property.

2. Preliminaries

2.1. Hodge theory and Shimura varieties

A real Hodge structure is a finite dimensional real vector space V together with a decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{p,q \in \mathbb{Z}} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p} \quad \forall p, q$.

By the characterization of the representations of an algebraic torus, to give a real Hodge structure on a real vector space V is the same as to give an algebraic homomorphism defined over $\mathbb{R}$ from the real algebraic group $\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ to $GL(V)$, where $Res_{\mathbb{C}/\mathbb{R}}$ denotes Weil restriction of scalars.

The main spaces that we are going to study are endowed with mixed Hodge structures, a more general structure defined as follows.

Definition 1. Let $H_{\mathbb{Z}}$ be a finitely generated abelian group, a mixed Hodge structure on $H_{\mathbb{Z}}$ consists of a finite decreasing filtration $F^{\bullet}H$ of $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, called the Hodge filtration, and a finite increasing filtration $W_{\bullet}H$ of the space $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, called the weight filtration, such that for each $k \in \mathbb{Z}$ the Hodge filtration induces a Hodge structure of weight k on the quotient space

$$Gr_k^W = W_k H / W_{k-1} H.$$

We will analyze certain local systems, most of them defining variations of Hodge structures (see [11] for this notion). In fact, all the local systems considered will define a complex variation of Hodge structure. These objects consist of families of complex Hodge structures satisfying certain conditions.

Definition 2. A complex Hodge structure is a finite dimensional $\mathbb{C}$-vector space E together with a decomposition $E = \bigoplus_{p,q \in \mathbb{Z}} E^{p,q}$ into a direct sum of $\mathbb{C}$-vector subspaces.

One can see that to give a complex Hodge structure is equivalent to give an algebraic representation of $\mathbb{S}_{\mathbb{C}}$.

We refer to [14] for the notion of a complex variation of Hodge structure.

We consider a Shimura pair (G, X), where G is a connected reductive algebraic group defined over $\mathbb{Q}$ and X is a complex space whose underlying set can be identified with a $G(\mathbb{R})$-conjugacy class of homomorphisms of real algebraic groups from $\mathbb{S}$ to $G_{\mathbb{R}}$. If $i : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ is the canonical homomorphism of real algebraic groups and $h \in X$ then one can see that the image of the morphism $h \circ i : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is central, hence it does not depend on the element $h \in X$ chosen. We call $h \circ i$ the weight homomorphism of the Shimura pair and we denote it by w_X.

The following proposition can be found in [10].

Proposition 3. *Let (G, X) be a Shimura datum and let $\rho : G_{\mathbb{R}} \to GL(V)$ be an algebraic representation of G defined over $\mathbb{R}$. Then X has a unique complex structure such that for every algebraic representation $\rho : G_{\mathbb{R}} \to GL(V)$ defined over $\mathbb{R}$, the assignment $h \in X \mapsto \rho \circ h$ defines a variation of Hodge structure.*

Moreover if we have an algebraic representation $\rho : G \to GL(V)$ on a complex vector space V, not necessarily defined over $\mathbb{R}$, then we obtain by the same procedure a complex variation of Hodge structure, i.e. to $h \in X$ corresponds the complex Hodge structure defined by $\rho \circ h$.

We define the level varieties as follows.

Definition 4. Let (G, X) be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup, the level variety associated to K is given by

$$Sh_K(G, X) = G(\mathbb{Q}) \backslash \left(X \times (G(\mathbb{A}_f)/K) \right).$$

For sufficiently small K the space $Sh_K(G, X)$ is a finite disjoint union of locally symmetric spaces given by arithmetic quotients of a hermitian symmetric domain.

As a conclusion, for sufficiently small K, the level variety $Sh_K(G, X)$ admits the structure of a complex quasi-projective variety.

If $K' \subset K$ then we have a canonical morphism $Sh_{K'}(G, X) \to Sh_K(G, X)$, moreover this morphism is regular.

Definition 5. Let (G, X) be a Shimura datum, we define the Shimura variety associated to the Shimura pair (G, X) by

$$Sh(G, X) = \varprojlim_{K} Sh_K(G, X).$$

If (ρ, V) is an irreducible algebraic representation of G then we can define local systems on the spaces $Sh_K(G, X)$ and also on the Shimura variety $Sh(G, X)$ as follows. For $K \subset G(\mathbb{A}_f)$ let $\tilde{V}$ be the local system on

$Sh_K(G,X)$ defined by

$$G(\mathbb{Q})\backslash X \times V \times G(\mathbb{A}_f)/K$$

while the local system on $Sh(G,X)$ is defined by

$$\lim_{\overleftarrow{K}} G(\mathbb{Q})\backslash X \times V \times G(\mathbb{A}_f)/K$$

which will also be denoted by $\tilde{V}$.

We denote by S_K the level variety associated to K and by S the Shimura variety $Sh(G,X)$.

Thus, given an irreducible algebraic representation of G we obtain local systems $\tilde{V}$ on S and on each level variety S_K; we will be interested in the cohomology spaces $H^*(S,\tilde{V})$ and $H^*(S_K,\tilde{V})$.

2.2. $\mathbb{Q}$-structure and Borel-Serre compactification

Let (G,X) be a Shimura datum and let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup. When the semisimple $\mathbb{Q}$-rank of G is not zero the level variety S_K will not be compact and we consider its Borel-Serre compactification denoted by $\overline{S}_K$ (see [1]). The inclusion $S_K \hookrightarrow \overline{S}_K$ is a homotopy equivalence, the local system $\tilde{V}$ can be naturally extended to a local system $\tilde{V}$ on $\overline{S}_K$ and we have an isomorphism

$$H^*(S_K,\tilde{V}) \cong H^*(\overline{S}_K,\tilde{V})$$

(see for example [4] or [13]). We denote the boundary of the Borel-Serre compactification by $\partial\overline{S}_K$.

At this point we can consider the long exact sequence in cohomology

$$\cdots \to H^q_c(S_K,\tilde{V}) \to H^q(S_K,\tilde{V}) \to H^q(\partial\overline{S}_K,\tilde{V}) \to \cdots \tag{1}$$

where $H^q_c(S_K,\tilde{V})$ denotes cohomology with compact support.

To describe the Borel-Serre compactification we need to understand the $\mathbb{Q}$-structure of G. We fix a maximal torus T and a maximal $\mathbb{Q}$-split torus $\tilde{T}$ of G such that $\tilde{T} \subset T$. We fix a set of positive roots on the root system $\Phi(G,T)$ and we denote by Δ the corresponding set of simple roots. We have a system of positive roots on the root system $\Phi(G,\tilde{T})$ that is compatible with the system of positive roots for $\Phi(G,T)$. We denote by $\Delta_{\mathbb{Q}}$ its corresponding set of simple roots.

Associated to these choices we have the set $\mathcal{P}(G)$ of standard $\mathbb{Q}$-parabolic subgroups of G. $\mathcal{P}(G)$ can be identified with the set of subsets of $\Delta_{\mathbb{Q}}$ where to each maximal standard $\mathbb{Q}$-parabolic subgroup corresponds a 1-element subset of $\Delta_{\mathbb{Q}}$. The boundary $\partial\overline{S}_K$ of the Borel-Serre compactification of S_K can be written as a union of faces indexed by the proper standard $\mathbb{Q}$-parabolic subgroups, and given $I \subset \Delta_{\mathbb{Q}}$ we denote by $\partial_{I,K}$ the corresponding face of $\partial\overline{S}_K$. This induces a spectral sequence in cohomology

$$E_1^{p,q} = \bigoplus_{|I|=p+1} H^q(\partial_{I,K}, \tilde{V}) \Rightarrow H^{p+q}(\partial\overline{S}_K, \tilde{V})$$

abutting to the cohomology of the boundary.

We assume from now on that G has semisimple $\mathbb{Q}$-rank two. So $\Delta_{\mathbb{Q}}$ has just two elements and we will have two standard maximal $\mathbb{Q}$-parabolic subgroups, to be denoted by P_1, P_2, together with a minimal one, denoted by P_0. Denote the corresponding faces in the boundary of the Borel-Serre compactification of S_K by $\partial_{1,K}, \partial_{2,K}$ and $\partial_{0,K}$ respectively. In this special case the aforementioned spectral sequence becomes a long exact sequence in cohomology, given by

$$\cdots \to H^q(\partial\overline{S}_K, \tilde{V}) \to H^q(\partial_{1,K}, \tilde{V}) \oplus H^q(\partial_{2,K}, \tilde{V}) \to H^q(\partial_{0,K}, \tilde{V}) \to \cdots \tag{2}$$

By Saito's theory of mixed Hodge modules and by [7], each term in the long exact sequences (1) and (2) admits a natural mixed Hodge structure. Moreover, (1) and (2) are long exact sequences of mixed Hodge structures. This will be one of the main tools in the study of the spaces of ghost classes.

If we work in the infinite level, we can write

$$H^*(S, \tilde{V}) = \varinjlim_{K} H^*(S_K, \tilde{V})$$

and, by taking the same direct limits, we can define the analogous objects at the infinite level $H^*(\partial\overline{S}, \tilde{V}), H_c^*(S, \tilde{V})$ and $H^*(\partial_i, \tilde{V})$ for $i \in \{0,1,2\}$ corresponding to the cohomology spaces in the long exact sequences (1) and (2). In this setting it is verified that the following sequences are exact:

$$\cdots \to H_c^q(S, \tilde{V}) \to H^q(S, \tilde{V}) \xrightarrow{r_q} H^q(\partial\overline{S}, \tilde{V}) \to \cdots$$

and

$$\cdots \to H^q(\partial\overline{S}, \tilde{V}) \xrightarrow{p_q} H^q(\partial_1, \tilde{V}) \oplus H^q(\partial_2, \tilde{V}) \to H^q(\partial_0, \tilde{V}) \to \cdots.$$

The two morphisms p_q and r_q in these sequences will be used in the next section to give the definition of ghost classes.

2.3. Ghost classes and mixed Hodge theory on cohomology

In the notation of the last subsection we define the space of q-ghost classes of the Shimura datum (G, X) with coefficients in the local system defined by the irreducible representation V by

$$Gh^q(\tilde{V}) = Im(r_q) \cap Ker(p_q).$$

In other words, $Gh^q(\tilde{V})$ is the space of cohomology classes in the cohomology of the boundary of the Borel-Serre compactification that lie in the image of $H^q(S, \tilde{V})$ by the canonical projection and also in the kernel of each projection to the cohomology of each face of the boundary of the Borel-Serre compactification $H^q(\partial_i, \tilde{V})$ corresponding to a standard $\mathbb{Q}$-parabolic subgroup.

Our approach is based on the facts that all the cohomology spaces considered are endowed with mixed Hodge structures (by Saito's theory of mixed Hodge modules and by [7]) and the long exact sequences in cohomology are exact sequences of mixed Hodge structures, which induces a mixed Hodge structure on the space of ghost classes.

The weight morphism ω_X of the Shimura datum defines the weight in the variation of Hodge structure defined by $\tilde{V}$ and all of this remains true for an algebraic representation not necessarily defined over $\mathbb{R}$ inducing a complex variation of Hodge structure. We denote by $wt(V)$ this weight and we say that $wt(V)$ is the weight of the representation V of G. By Saito's theory, we know that the weights in the mixed Hodge structure attached to $H^q(S, \tilde{V})$ are greater than or equal to $wt(V) + q$ and the weights in the mixed Hodge structure attached to $H^q_c(S, \tilde{V})$ are less than or equal to $wt(V) + q$. This gives the first implications on the mixed Hodge structure on the space of ghost classes since it implies that the weights in the mixed Hodge structure attached to $Gh^q(\tilde{V})$ are greater than or equal to $wt(V) + q$.

Here we introduce a question to be studied in what follows, that is whether the possible weights in the space of ghost classes $Gh^q(\tilde{V})$ belong to the set $\{q + wt(V),\ \ q + 1 + wt(V)\}$, in the case that this holds for the representation V we say that V satisfies the weak middle weight property. On the other hand, if V is such that the space of ghost classes $Gh^q(\tilde{V})$ has weight $q + wt(V)$ we say that V satisfies the middle weight property.

Another restriction can be obtained by studying the mixed Hodge structures on the spaces $H^q(\partial_0, \tilde{V})$ and the kernel of $H^q(\partial_0, \tilde{V}) \to H^{q+1}(\partial\overline{S}, \tilde{V})$, but this kernel is the image of the morphism

$$H^q(\partial_1, \tilde{V}) \oplus H^q(\partial_2, \tilde{V}) \to H^q(\partial_0, \tilde{V}).$$

To study this image we restrict the morphism to each term and study each map $H^q(\partial_i, \tilde{V}) \to H^q(\partial_0, \tilde{V})$ separately.

For this we use the decomposition given by any standard $\mathbb{Q}$-parabolic subgroup P of G

$$H^q(\partial_P, \tilde{V}) = \bigoplus_{i+j=q} Ind_{P(\mathbb{A}_f)\times\pi_0(P(\mathbb{R}))}^{G(\mathbb{A}_f)\times\pi_0(G(\mathbb{R}))} H^i(S^{M_P}, \widetilde{H^j(\mathfrak{u}_P, V)})$$

where $\mathfrak{u}_P$ denotes the Lie algebra of the unipotent radical of P, M_P is a Levi subgroup of P, S^{M_P} is an inverse limit of locally symmetric spaces associated to M_P and $\pi_0(P(\mathbb{R})), \pi_0(G(\mathbb{R}))$ denote the groups of connected components of $P(\mathbb{R})$ and $G(\mathbb{R})$ respectively (see for example [4] or [13]). From now on we will abbreviate $Ind_P^G = Ind_{P(\mathbb{A}_f)\times\pi_0(P(\mathbb{R}))}^{G(\mathbb{A}_f)\times\pi_0(G(\mathbb{R}))}$ for each standard parabolic $\mathbb{Q}$-subgroup P of G.

By Kostant's theorem ([9]) we have a decomposition of $H^j(\mathfrak{u}_P, V)$ as a direct sum of irreducible representations of M_P indexed by elements of the Weyl group.

Indeed, given the root system $\Phi(G,T)$, denote by Φ^+ the set of positive roots and by Φ^- the set of negative roots, then for each $w \in \mathcal{W}$ we define

$$\Phi_w = w(\Phi^-) \cap \Phi^+.$$

Given a root $\alpha \in \Phi$ we denote by $E_\alpha \subset \mathfrak{g}_\mathbb{C}$ the corresponding root space. If $\mathfrak{p}_\mathbb{C} = Lie(P)_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}$, then we have the decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{m} \oplus \mathfrak{u}$ corresponding to the Levi decomposition of P and we define $\Delta(\mathfrak{u}) = \{\alpha \in \Phi \mid E_\alpha \subset \mathfrak{u}\}$.

We define the subset $\mathcal{W}^P \subset \mathcal{W}$ by

$$\mathcal{W}^P = \{w \in \mathcal{W} \mid \Phi_w \subset \Delta(\mathfrak{u})\} \tag{3}$$

Assume that the algebraic representation V of G is irreducible with highest weight λ, then we obtain a decomposition as representations of M_P

$$H^q(u_P, V) = \bigoplus_{w\in\mathcal{W}^P(q)} W_{w_*(\lambda)}.$$

In this decomposition, $\mathcal{W}^P(q) \subset \mathcal{W}^P$ is the subset of $\mathcal{W}^P$ consisting of the elements of length q, $w_*(\lambda) = w(\lambda + \delta) - \delta$ where $\delta = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ and $W_{w_*(\lambda)}$ is the irreducible representation of M_P with highest weight $w_*(\lambda)$.

Thus we obtain the decomposition

$$H^q(\partial_P, \tilde{V}_\lambda) = \bigoplus_{\substack{w \in \mathcal{W}^P \\ \ell(w) \leq q}} Ind_P^G H^{q-\ell(w)}(S^{M_P}, \widetilde{W}_{w_*(\lambda)})$$

where V_λ denotes the irreducible algebraic representation of G with highest weight λ.

There are uniquely determined subsets $\mathcal{W}_i^0$ of the Weyl group $\mathcal{W}$ (for $i = 1$ or 2) such that $\mathcal{W}^{P_0} = \mathcal{W}_i^0 \mathcal{W}^{P_i}$ and for each $w \in \mathcal{W}^{P_i}$ the restriction of the morphism $H^q(\partial_i, \tilde{V}) \to H^q(\partial_0, \tilde{V})$ to $Ind_{P_i}^G H^{q-\ell(w)}(S^{M_{P_i}}, \widetilde{W}_{w_*(\lambda)})$ has image in

$$\bigoplus_{\tilde{w} \in \mathcal{W}_i^0} Ind_{P_0}^G H^{q-\ell(w)-\ell(\tilde{w})}(S^{M_{P_0}}, \widetilde{W}_{\tilde{w}_*(w_*(\lambda))}).$$

Moreover, for $w \in \mathcal{W}^{P_0}$, $\tilde{w} \in \mathcal{W}_i^0$ and $w_i \in \mathcal{W}^{P_i}$ (for $i = 1$ or 2) such that $w = \tilde{w} w_i$ we have $\ell(w) = \ell(\tilde{w}) + \ell(w_i)$.

The study of the above morphism, since parabolic induction is exact, reduces to the study of the following morphism, for each $i \in \{1, 2\}$ and each $w \in \mathcal{W}^{P_i}$,

$$H^{q-\ell(w)}(S^{M_{P_i}}, \widetilde{W}_{w_*(\lambda)}) \to \bigoplus_{\tilde{w} \in \mathcal{W}_i^0} Ind_{P_0^i}^{M_{P_i}} H^{q-\ell(w)-\ell(\tilde{w})}(S^{M_{P_0}}, \widetilde{W}_{\tilde{w}_*(w_*(\lambda))})$$

where P_0^i denotes the $\mathbb{Q}$-parabolic subgroup $(P_0 \cap M_{P_i})$ of M_{P_i}. In order to apply similar strategies as in the case of the Shimura variety, we have to calculate the weights on the cohomology spaces $H^q(S^{M_{P_i}}, \widetilde{W}_{w_*(\lambda)})$. This will be done following Section 5 of [7] and [8].

For $i = 1$ or 2, P_i is a maximal standard $\mathbb{Q}$-parabolic subgroup. We have a decomposition of the Levi subgroup of P_i of the form $M_{P_i} = G_{P_i,h} G_{P_i,l}$ which is the product of the hermitian and the linear part, whose intersection is the subgroup A_{P_i} of M_{P_i} given by the product of the maximal $\mathbb{Q}$-split torus in the center of M_{P_i} and the center of G. $G_{P_i,h}$ is part of a Shimura datum and we denote by $h_{P_i} : \mathbb{S} \to G_{P_i,h}$ the corresponding morphism, which is determined up to $G_{P_i,h}(\mathbb{R})$-conjugation. The types of the mixed Hodge structure associated to $H^*(S^{M_{P_i}}, \widetilde{W}_{w_*(\lambda)})$ are determined by h_{P_i} and $W_{w_*(\lambda)}$, as one can see in (5.6.10) of [7].

For the minimal parabolic P_0 we take its hermitian part to be the hermitian part of P_2, in the more general case we have an association of a maximal standard $\mathbb{Q}$-parabolic subgroup to each standard $\mathbb{Q}$-parabolic subgroup of G (see for example [7]) and the hermitian part of a standard $\mathbb{Q}$-parabolic subgroup of G is that of the maximal standard $\mathbb{Q}$-parabolic associated. Finally, with these correspondences, the types of the mixed Hodge structure attached to the cohomology space $H^q(S^{M_{P_0}}, \widetilde{W}_{w_*(\lambda)})$ are determined by h_{P_2} and $W_{w_*(\lambda)}$, as one can see again in (5.6.10) of [7].

3. The case $GU(2,2)$

3.1. The Shimura variety involved

In this subsection we describe the Shimura variety to be studied in this paper. Let $\mathcal{K}$ be an imaginary quadratic field, let ι be the only nontrivial element of $Gal(\mathcal{K}/\mathbb{Q})$, $I_{(2,2)} = I_2 \times (-I_2) \in GL_4(\mathbb{Q})$ and let $GU(I_{(2,2)})$ be the connected reductive algebraic group defined over $\mathbb{Q}$ given by

$$GU(I_{(2,2)})(A) = \{g \in GL_n(A \otimes_{\mathbb{Q}} \mathcal{K}) \mid g^* I_{(2,2)} g = \nu(g) I_{(2,2)}, \nu(g) \in A^\times\}$$

for every $\mathbb{Q}$-algebra A, where $\overline{g}$ denotes $\iota(g)$ and g^* denotes the transpose of $\overline{g}$.

If $h : \mathbb{S} \to GU(I_{(2,2)})_{\mathbb{R}}$ is the algebraic homomorphism given by

$$h(z) = zI_2 \times (\overline{z}I_2) \quad \forall z \in \mathbb{S}(\mathbb{R})$$

and X is the $GU(I_{(2,2)})(\mathbb{R})$-conjugacy class of h, then the pair $(GU(I_{(2,2)}), X)$ is a Shimura datum.

For the calculations, it turns out to be more convenient to work with the algebraic group $GU(A_{(2,2)})$, that is isomorphic to $GU(I_{(2,2)})$, given by

$$GU(A_{(2,2)})(A) = \{g \in GL_n(A \otimes_{\mathbb{Q}} \mathcal{K}) \mid g^* A_{(2,2)} g = \nu(g) A_{(2,2)}, \nu(g) \in A^\times\}$$

for a given $\mathbb{Q}$-algebra A, where

$$A_{(2,2)} = \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix}, \text{ with } S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In what follows we will denote the group $GU(A_{(2,2)})$ by G. The complex points of G is the group $G(\mathbb{C}) = GL_4(\mathbb{C}) \times \mathbb{C}^\times$ where the map $G(\mathbb{R}) \to G(\mathbb{C})$ is given by $g \mapsto (g, \nu(g))$.

3.2. Maximal torus, root system and parabolic subgroups

Let $H \subset G(\mathbb{C})$ be the 5-dimensional complex torus consisting of the diagonal matrices, clearly a maximal torus for G which defines a root system of type A_3. We denote by $\mathcal{W}$ the corresponding Weyl group.

Let $\mathfrak{h} = Lie(H)$ be the complex Lie subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_4(\mathbb{C}) \oplus \mathbb{C}$ corresponding to H. We define for each $i \in \{1, 2, 3, 4\}$ the element $\epsilon_i \in \mathfrak{h}^*$ given by $\epsilon_i(h) = h_i$ for $h = (diag(h_1, h_2, h_3, h_4), x) \in \mathfrak{h}$.

Then the root system for these choices is given by

$$\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) = \{\epsilon_i - \epsilon_j \mid i \neq j\}$$

and taking the positive roots to be $\Phi^+ = \{\epsilon_i - \epsilon_j \mid i < j\}$ we have the system of simple roots given by $\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4\}$.

To give an irreducible algebraic representation of G is the same as giving a 5-tuple (a_1, a_2, a_3, a_4, c) with $a_1 \geq a_2 \geq a_3 \geq a_4$ and $c \equiv \sum_{i=1}^4 a_i \quad mod(2)$. We fix an irreducible algebraic representation $(\rho_\lambda, V_\lambda)$ with highest weight given by $\lambda = (a_1, a_2, a_3, a_4, c)$ and we denote by $\tilde{V_\lambda}$ the corresponding local system.

The given maximal torus on G gives a specific maximal $\mathbb{Q}$-split subtorus consisting of the subgroup of matrices of the form

$$\left\{ \begin{bmatrix} aa_1 & & & \\ & aa_2 & & \\ & & a_2^{-1} & \\ & & & a_1^{-1} \end{bmatrix} : a, a_1, a_2 \in \mathbb{G}_m \right\} \subset G$$

This $\mathbb{Q}$-split torus has rank 3 and it induces a $\mathbb{Q}$-root system of rank 2, of type C_2.

From the given maximal $\mathbb{Q}$-split torus and the corresponding system of positive $\mathbb{Q}$-roots induced by Φ^+ we have the associated standard $\mathbb{Q}$-parabolic subgroups, which in this case are given by

$$P_1 = \begin{bmatrix} Res_{K/\mathbb{Q}} GL_1 & * & * \\ 0 & GU(A_{(1,1)}) & * \\ 0 & 0 & Res_{K/\mathbb{Q}} GL_1 \end{bmatrix} \cap G,$$

$$P_2 = \begin{bmatrix} Res_{K/\mathbb{Q}} GL_2 & * \\ 0 & Res_{K/\mathbb{Q}} GL_2 \end{bmatrix} \cap G,$$

and

$$P_0 = \begin{bmatrix} Res_{K/\mathbb{Q}}GL_1 & * & * & * \\ 0 & Res_{K/\mathbb{Q}}GL_1 & * & * \\ 0 & 0 & Res_{K/\mathbb{Q}}GL_1 & * \\ 0 & 0 & 0 & Res_{K/\mathbb{Q}}GL_1 \end{bmatrix} \cap G$$

3.3. Weyl group and the subsets $\mathcal{W}^{P_0}$, $\mathcal{W}^{P_1}$ and $\mathcal{W}^{P_2}$

The Weyl group $\mathcal{W}$ can be identified with the permutation group S_4, where $\sigma \in S_4$ corresponds to the element w_σ satisfying

$$w_\sigma(\epsilon_i) = \epsilon_{\sigma^{-1}(i)} \quad \forall i \in \{1,2,3,4\}\,.$$

We give a description of the subsets $\mathcal{W}^{P_i} \subset \mathcal{W}$, $i = 0,1,2$, in the following proposition, which follows from the characterization (3) in Subsection 2.3.

Proposition 6. *We have that*

$$\mathcal{W}^{P_0} = \mathcal{W}, \quad \mathcal{W}^{P_1} = \{w_\sigma \in \mathcal{W} \mid \sigma(2) < \sigma(3)\} \;\; \textit{and}$$
$$\mathcal{W}^{P_2} = \{w_\sigma \in \mathcal{W} \mid \sigma(1) < \sigma(2) \textit{ and } \sigma(3) < \sigma(4)\}$$

We have to keep in mind that in this notation $w_\sigma w_\tau = w_{\tau\sigma} \quad \forall \sigma, \tau \in S_4$.

3.4. Weights on $W_{w_*(\lambda)}$

The aim of this section is to give a description of the weights of the mixed Hodge structure associated to the spaces of the form $H^q(S^{M_{P_i}}, \tilde{W}_{w_*(\lambda)})$, for $i \in \{0,1,2\}$.

We start with the case $i = 1$. We need to calculate the subgroup A_{P_1} of M_{P_1}, which is the maximal $\mathbb{Q}$-split torus in the center of M_{P_1} times the center of G. In this case this is given by

$$\left\{ a \begin{bmatrix} a_1 & & \\ & I_2 & \\ & & a_1^{-1} \end{bmatrix} : a \in Res_{K/\mathbb{Q}}\mathbb{G}_m, a_1 \in \mathbb{G}_m \right\}$$

For the calculation of the homomorphism $h_{P_1} : \mathbb{S} \to G_{P_1,h}$ defining the Shimura datum attached to the hermitian part $G_{P_1,h}$ of M_{P_1} we follow Section 5 of [6].

We now calculate the unique admissible Cayley morphism to be denoted by ω_1. $\omega_1 : \mathbb{G}_m \to A_{P_1}$ can be determined by the properties stated in the aforementioned paper, specially considering the filtration it induces in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by its composition with the adjoint representation. Also we use the fact that for every rational representation (σ, W) of G and for every $h \in X$, the Hodge filtration associated to the Hodge structure defined by the composition $\sigma \circ h$, together with the increasing filtration induced by the graduation of W given by the composition $\sigma \circ \omega_1$ defines a mixed Hodge structure on W.

By the first property one can see that ω_1 is given by

$$\omega_1(z) = \left\{ \begin{bmatrix} z^k & & \\ & z^{k+1} Id_2 & \\ & & z^{k+2} \end{bmatrix} : z \in \mathbb{G}_m \right\}$$

with $k \in \mathbb{Z}$. By the second property we take the inclusion $G \hookrightarrow Res_{\mathcal{K}/\mathbb{Q}} GL_4 \subset GL_8$ as an 8-dimensional rational representation of G, and one can see that $k = -2$ and $h_{P_1} : Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to (G_{P_1,h})_{\mathbb{R}}$ is given by

$$h_{P_1}(z) = \begin{bmatrix} z\overline{z} & & \\ & h_{(1,1)}(z) & \\ & & 1 \end{bmatrix} \in (G_{P_1,h})_{\mathbb{R}} \quad \forall z \in \mathbb{C}^{\times}.$$

where $h_{(1,1)} : \mathbb{S} \to GU(1,1)_{\mathbb{R}}$ is the map described in a way similar to h defining the Shimura variety for $GU(1,1)$.

By the same procedure we obtain the morphism $h_{P_2} : Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to (G_{P_2,h})_{\mathbb{R}}$, given by

$$h_{P_2}(z) = \begin{bmatrix} z\overline{z} Id_2 & \\ & Id_2 \end{bmatrix} \in (G_{P_2,h})_{\mathbb{R}} \quad \forall z \in \mathbb{C}^{\times}.$$

To finalize this section we calculate the weight of the representations V_λ of G and $W_{w_*(\lambda)}$ of M_P for $w \in \mathcal{W}^P$ for each P standard $\mathbb{Q}$-parabolic subgroup of G (see 2.3 for this notion).

Let $i \in \{0, 1, 2\}$. For each $w \in \mathcal{W}^{P_i}$, we have $w_*(\lambda) = w(\lambda + \delta) - \delta$.

If we abbreviate by writing (a_1, a_2, a_3, a_4) the weight $\lambda = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\epsilon_4$, then we have that $(w_\sigma)_*(\lambda)$ is given by

$$(a_{\sigma(1)} + 1 - \sigma(1), a_{\sigma(2)} + 2 - \sigma(2), a_{\sigma(3)} + 3 - \sigma(3), a_{\sigma(4)} + 4 - \sigma(4)).$$

From the description of h we have that the weight homomorphism of the Shimura variety defined by (G, h) is given by

$$\mathbb{R}^{\times} \to G(\mathbb{R}), t \mapsto tId_4$$

so, by the description of λ, the weight of the representation V_λ is given by $wt(V_\lambda) = -c$.

On the other hand, from the definition of h_{P_1} we have that the weight homomorphism of the Shimura datum defined by $G_{P_1,h}$ and h_{P_1} is given by

$$t \mapsto \begin{bmatrix} t^2 & & \\ & tId_2 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} t & & \\ & tId_2 & \\ & & t \end{bmatrix} \begin{bmatrix} t & & \\ & Id_2 & \\ & & t^{-1} \end{bmatrix} \in G_{P_1,h}(\mathbb{R})$$

and the weight of the representation of $G_{P_1,h}$ determined by $W_{(w_\sigma)_*(\lambda)}$, for $w_\sigma \in \mathcal{W}^{P_1}$, is

$$wt(w_\sigma, P_1) = -c + ((a_{\sigma(4)} + 4 - \sigma(4)) - (a_{\sigma(1)} + 1 - \sigma(1)))$$

By the same procedure we can verify that for a given $w_\sigma \in \mathcal{W}^{P_2}$ the weight $wt(w_\sigma, P_2)$ of the representation of $G_{P_2,h}$ defined by $W_{(w_\sigma)_*(\lambda)}$ is given by the formula

$$\begin{aligned} -c + (a_{\sigma(4)} + 4 - \sigma(4)) + (a_{\sigma(3)} + 3 - \sigma(3)) \\ -(a_{\sigma(1)} + 1 - \sigma(1)) - (a_{\sigma(2)} + 2 - \sigma(2)) \end{aligned}$$

By [7] the weight $wt(w_\sigma, P_0)$ of the representation on the hermitian part of P_0 defined by the representation $W_{(w_\sigma)_*(\lambda)}$ of M_{P_0}, for $w_\sigma \in \mathcal{W}^{P_0}$, is given by

$$\begin{aligned} -c + (a_{\sigma(4)} + 4 - \sigma(4)) + (a_{\sigma(3)} + 3 - \sigma(3)) \\ -(a_{\sigma(1)} + 1 - \sigma(1)) - (a_{\sigma(2)} + 2 - \sigma(2)) \end{aligned}$$

3.5. The decompositions $\mathcal{W}^{P_0} = \mathcal{W}_1^0 \mathcal{W}^{P_1}$ and $\mathcal{W}^{P_0} = \mathcal{W}_2^0 \mathcal{W}^{P_2}$

We have the decompositions $\mathcal{W}^{P_0} = \mathcal{W}_1^0 \mathcal{W}^{P_1}$ and $\mathcal{W}^{P_0} = \mathcal{W}_2^0 \mathcal{W}^{P_2}$, where $\mathcal{W}_i^0$ (for $i = 1$ or 2) is the set of Weyl representatives for the unique standard $\mathbb{Q}$-parabolic subgroup of M_{P_i} defined by the root system induced from that of G. Using the fact that $w_\sigma w_{\tilde{\sigma}} = w_{\tilde{\sigma}\sigma}$ one can see, by the same arguments as in Proposition 6 in section 3.3, that $\mathcal{W}_1^0 = \{w_\sigma \in \mathcal{W} \mid \sigma \in \{1, s_{2,3}\}\}$ and that $\mathcal{W}_2^0 = \{w_\sigma \in \mathcal{W} \mid \sigma \in \{1, s_{1,2}, s_{3,4}, s_{1,2} \circ s_{3,4}\}\}$,
where $s_{i,j}$ denotes the transposition (ij).

3.6. The image of the morphism $r^*_{1,0} : H^*(\partial_1, \tilde{V}_\lambda) \to H^*(\partial_0, \tilde{V}_\lambda)$

As explained in the preliminaries, this morphism respects the direct sum decomposition indexed by the set of Weyl representatives $\mathcal{W}^{P_1}$ and $\mathcal{W}^{P_0}$. In other words, we can study the image of $r^*_{1,0}$ by restricting this morphism to each factor corresponding to a given $w \in \mathcal{W}^{P_1}$, obtaining a morphism from the space

$$Ind^G_{P_1} H^{q-\ell(w)}(S^{M_{P_1}}, \tilde{W}_{w_*(\lambda)})$$

to the space

$$\bigoplus_{\substack{w' \in \mathcal{W}^0_1 \\ \ell(w') \leq q-\ell(w)}} Ind^G_{P_0} H^{q-\ell(w)-\ell(w')}(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)}).$$

One observation here is that $H^k(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)}) = 0$ for all $k > 0$, so in the right hand side we only have to consider the elements $w' \in \mathcal{W}^0_1$ satisfying $q - \ell(w) - \ell(w') = 0$.

In this case we have that M_{P_1} coincides with its hermitian part $G_{P_1,h}$. Fix $w \in \mathcal{W}^{P_1}$, then we have the long exact sequence in cohomology

$$\cdots \to H^{q-\ell(w)}(S^{M_{P_1}}, \tilde{W}_{w_*(\lambda)}) \to \bigoplus_{\substack{w' \in \mathcal{W}^0_1 \\ \ell(w') = q-\ell(w)}} Ind^{M_{P_1}}_{P^1_0} H^0(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)})$$
$$\to H^{q-\ell(w)+1}_c(S^{M_{P_1}}, \tilde{W}_{w_*(\lambda)}) \to \cdots$$

where P^1_0 denotes the $\mathbb{Q}$-parabolic subgroup $P_0 \cap M_{P_1}$ of M_{P_1}, $Ind^{M_{P_1}}_{P^1_0}$ denotes $Ind^{M_{P_1}(\mathbb{A}_f)\times\pi_0(M_{P_1}(\mathbb{R}))}_{P^1_0(\mathbb{A}_f)\times\pi_0(P^1_0(\mathbb{R}))}$ and the term involving the direct sum corresponds to the cohomology of the boundary of the Borel-Serre compactification of the Shimura variety attached to the Levi subgroup M_{P_1} of the $\mathbb{Q}$-parabolic subgroup P_1.

We have the following list of facts that will be useful for describing the image of $r^*_{1,0} : H^q(\partial_{1,}, \tilde{V}_\lambda) \to H^q(\partial_0, \tilde{V}_\lambda)$,

- The weights in the mixed Hodge structure on $H^{q-\ell(w)}(S^{M_{P_1}}, \tilde{W}_{w_*(\lambda)})$ are greater than or equal to $q - \ell(w) + wt(w, P_1)$.
- The weights in the mixed Hodge structure on $H^{q-\ell(w)+1}_c(S^{M_{P_1}}, \tilde{W}_{w_*(\lambda)})$ are less than or equal to $q - \ell(w) + 1 + wt(w, P_1)$
- The weight on $Ind^{M_{P_1}}_{P^1_0} H^0(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)})$ is $wt(w'w, P_0)$.

In particular, if the weight in $Ind_{P_0^1}^{M_{P_1}} H^0(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)})$ is greater than $q-\ell(w)+1+wt(w,P_1)$ then such space is in the image of $r_{1,0}^*$.

To make the calculations, let σ, σ' be the elements in S_4 such that $w = w_\sigma$ and $w' = w_{\sigma'}$. Then we know that

$$wt(w_\sigma, P_1) = -c + (a_{\sigma(4)} + 4 - \sigma(4)) - (a_{\sigma(1)} + 1 - \sigma(1))$$

while

$$\begin{aligned} wt(w_{\sigma\sigma'}, P_0) = -c &+ (a_{\sigma\sigma'(4)} + 4 - \sigma\sigma'(4)) + (a_{\sigma\sigma'(3)} + 3 - \sigma\sigma'(3)) \\ &- (a_{\sigma\sigma'(1)} + 1 - \sigma\sigma'(1)) - (a_{\sigma\sigma'(2)} + 2 - \sigma\sigma'(2)) \end{aligned}$$

but as $w_{\sigma'} \in \mathcal{W}_1^0$, we have that $\sigma\sigma'(n) = \sigma(n)$ and $\sigma\sigma'(1) = \sigma(1)$, thus

$$wt(w'w, P_0) = wt(w, P_1) + (a_{\sigma\sigma'(3)} + 3 - \sigma\sigma'(3)) - (a_{\sigma\sigma'(2)} + 2 - \sigma\sigma'(2)).$$

As a consequence we have that the weight on $H^0(S^{M_{P_0}}, \tilde{W}_{(w'w)_*(\lambda)})$ is greater than $q - \ell(w) + 1 + wt(w, P_1)$ if and only if $(a_{\sigma\sigma'(3)} + 3 - \sigma\sigma'(3)) - (a_{\sigma\sigma'(2)} + 2 - \sigma\sigma'(2)) > \ell(w') + 1$, from this we can say that if $w_\sigma \in \mathcal{W}^{P_1}$, $w_{\sigma'} \in \mathcal{W}_1^0$ and

$$(a_{\sigma\sigma'(3)} + 3 - \sigma\sigma'(3)) - (a_{\sigma\sigma'(2)} + 2 - \sigma\sigma'(2)) > \ell(w_{\sigma'}) + 1$$

then $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma\sigma'})_*(\lambda)})$ is in the image of the morphism $r_{1,0}^*$. Taking all this into account we have the following proposition.

Proposition 7. *Let $w_\sigma \in \mathcal{W}$ such that $\sigma(2) > \sigma(3)$. If*

$$(a_{\sigma(3)} - a_{\sigma(2)}) + (\sigma(2) - \sigma(3)) > 1$$

then $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ lies in the image of $r_{1,0}^*$.

Proof. If $\sigma(2) > \sigma(3)$ and w_σ satisfies the hypotheses in the proposition, then we write $w_\sigma = w_{\tilde{\sigma}} w_{\sigma_1}$ with respect to the decomposition $\mathcal{W} = \mathcal{W}_1^0 \mathcal{W}^{P_1}$. Thus one can see that

$$(a_{\sigma(3)} + 3 - \sigma(3)) - (a_{\sigma(2)} + 2 - \sigma(2)) > 1 + l(w_{\tilde{\sigma}}).$$

and by the previous observations we obtain the desired result. □

A notation we will use from now on is the following, a permutation $\sigma \in S_4$ will be denoted by the 4-tuple $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$.

3.7. Case $a_1 > a_2$ or $a_3 > a_4$

By a case by case treatment we arrive at the following result.

Theorem 8. *If the highest weight $\lambda = (a_1, a_2, a_3, a_4, c)$ of the irreducible representation V_λ of G satisfies $a_1 > a_2$ or $a_3 > a_4$, then there are no ghost classes in the cohomology space of the local system defined by V_λ.*

Although the nonexistence of ghost classes is known for regular highest weights, most of the irregular highest weights satisfy the hypothesis of this theorem, the original part of this result is then the nonexistence of ghost classes for these irregular highest weights.

Proof. We prove the result assuming $a_1 > a_2$. We can use almost the same arguments to prove the statement in the case $a_3 > a_4$.

A direct use of the previous proposition and the assumption $a_1 > a_2$, gives as a result that for σ in the set

$$\big\{(2,3,1,4),(2,4,1,3),(3,2,1,4),(3,4,1,2),\\ (4,2,1,3),(4,3,1,2),(1,4,2,3),(3,4,2,1)\big\}$$

the space $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class because all these spaces are in the image of $r^*_{1,0} : H^*(\partial_1, \tilde{V}_\lambda) \to H^*(\partial_0, \tilde{V}_\lambda)$.

If we consider $\sigma = (1,2,3,4)$ then the weight of $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ is $-c + (a_4 + a_3) - (a_1 + a_2)$, while $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ is a direct summand of $H^0(\partial_0, \tilde{V}_\lambda)$ and so in order to contribute to a ghost class in the image of the morphism $H^0(\partial_0, \tilde{V}_\lambda) \to H^1(\partial\overline{S}, \tilde{V}_\lambda)$, its weight must be greater than or equal to the middle weight which in this case is $-c + 1$. We conclude that $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class. By a similar argument one can see that $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class for σ in the set

$$\big\{(1,2,4,3),(2,1,3,4),(2,1,4,3),(1,3,2,4),(1,3,4,2),\\ (3,1,2,4),(3,1,4,2),(1,4,3,2),(4,1,2,3),(4,1,3,2)\big\}.$$

We will use the results in [4] in order to analyze the image of the morphism $r^*_{2,0} : H^*(\partial_2, \tilde{V}_\lambda) \to H^*(\partial_0, \tilde{V}_\lambda)$.

For the case $\sigma = (3,2,4,1)$, we consider the expression $w_\sigma = w_{\tilde{\sigma}} w_{\sigma_2}$ given by the decomposition $\mathcal{W}^{P_0} = \mathcal{W}_2^0 \mathcal{W}^{P_2}$, then in the notation of Subsection 3.5 we have $\tilde{\sigma} = s_{1,2} \circ s_{3,4}$ and $\sigma_2 = (2,3,1,4)$.

We have to analyse the restriction of the morphism $r^*_{2,0}$ to the subspace $Ind^G_{P_2} H^2(S^{M_{P_2}}, \widetilde{W}_{(w_{\sigma_2})_*(\lambda)})$ of $H^4(\partial_2, \tilde{V}_\lambda)$ and we just need to consider the morphism

$$H^2(S^{M_{P_2}}, \widetilde{W}_{(w_{\sigma_2})_*(\lambda)}) \to Ind^{M_{P_2}}_{P_0^2} H^0(S^{M_{P_0}}, \widetilde{W}_{(w_\sigma)_*(\lambda)})$$

where P_0^2 denotes the $\mathbb{Q}$-parabolic subgroup $P_0 \cap M_{P_2}$ of M_{P_2}.

By [4], as we are in the unbalanced case, $Ind^{M_{P_2}}_{P_0^2} H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ is in the image of $r^*_{2,0}$ unless $W_{(w_{\sigma_2})_*(\lambda)}$ is one dimensional and, on the other hand, one can see that this representation is not one dimensional. As a result $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class. The same procedure proves this result for

$$\sigma \in \{(4,2,3,1), (4,3,2,1)\}\,.$$

In the case $\sigma = (2,3,4,1)$, if we write $w_\sigma = w_{\tilde{\sigma}} w_{\sigma_2}$ with respect to the decomposition $\mathcal{W}^{P_0} = \mathcal{W}^0_2 \mathcal{W}^{P_2}$, one can see that, in the notation of Subsection 3.5, we have $\tilde{\sigma} = s_2$ and $\sigma_2 = (2,3,1,4)$.

We consider the restriction of the morphism $H^3(\partial_2, \tilde{V}_\lambda) \to H^3(\partial_0, \tilde{V}_\lambda)$ to the corresponding subspace $Ind^G_{P_2} H^1(S^{M_{P_2}}, \widetilde{W}_{(w_{\sigma_2})_*(\lambda)})$. Its image is inside the space

$$Ind^G_{P_0}(H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)}))$$

where $\sigma' = (3,2,1,4)$. As we know, we just need to consider the morphism

$$H^1(S^{M_{P_2}}, \widetilde{W}_{{w_{\sigma_2}}_*(\lambda)}) \to Ind^{M_{P_2}}_{P_0^2}(H^0(S^{M_{P_0}}, \tilde{W}_{{w_\sigma}_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \tilde{W}_{{w_{\sigma'}}_*(\lambda)})).$$

By Theorem 2 in [4], for the balanced case one can see that the image of the aforementioned morphism plus $Ind^{M_{P_2}}_{P_0^2} H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)})$ is all of

$$Ind^{M_{P_2}}_{P_0^2}(H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)}) \oplus H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)}))$$

because σ is in the fundamental chamber (with respect to [4]). Finally, as $Ind^G_{P_0} H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)})$ is in the image of the cohomology space of ∂_1, we have that $Ind^G_{P_0}(H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)}))$ is in the image of the morphism $H^*(\partial_1, \tilde{V}_\lambda) \oplus H^*(\partial_2, \tilde{V}_\lambda) \to H^*(\partial_0, \tilde{V}_\lambda)$ and therefore it does not contribute to a ghost class. By the same arguments one can see that in the case $\sigma = (2,4,3,1)$, the corresponding space $Ind^G_{P_0}(H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)}))$ does not contribute to a ghost class.

So we have proved the stated result assuming $a_1 > a_2$. By the same procedure one can prove this statement assuming $a_3 > a_4$. $\square$

3.8. Case $a_1 = a_2$ and $a_3 = a_4$

Theorem 9. *If the highest weight $\lambda = (a_1, a_2, a_3, a_4, c)$ of the irreducible representation $\tilde{V}_\lambda$ satisfies $a_1 = a_2$ and $a_3 = a_4$, the only possible ghost classes in the cohomology space of the local system defined by V_λ are in $H^2(\partial\overline{S}, \tilde{V}_\lambda)$ (this case is only possible if $a_1 = a_2 = a_3 = a_4$) with weight $-c+2$, in the space $H^4(\partial\overline{S}, \tilde{V}_\lambda)$ with weight $-c+4$ and $H^5(\partial\overline{S}, \tilde{V}_\lambda)$ with weight $-c+6$. In particular the weak middle weight property is always satisfied for $GU(2,2)$.*

Proof. By Proposition 7 one can see that for σ in the set

$$\{(1,4,2,3),(2,3,1,4),(2,4,1,3),(3,4,1,2),(3,4,2,1),(4,3,1,2)\}$$

the space $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class. By the same proposition we have that for $\sigma = (1,3,2,4)$, if $a_2 > a_3$ then the corresponding space does not contribute to a ghost class.

On the other hand, if $\sigma = (1,2,3,4)$ we have $\ell(w_\sigma) = 0$, and then the space $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ is a direct summand of $H^0(\partial_0, \tilde{V}_\lambda)$ and we conclude that this space does not contribute to a ghost class under the morphism $H^0(\partial_0, \tilde{V}_\lambda) \to H^1(\partial\overline{S}, \tilde{V}_\lambda)$ because its weight is lower than the middle weight. By the same argument we can show that $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to a ghost class for

$$\sigma \in \{(1,2,4,3),(2,1,3,4),(2,1,4,3),(1,3,4,2),(3,1,2,4),(3,1,4,2)\}.$$

By Theorem 2 in [4] for the unbalanced case and using the same procedure as in the last subsection, we have that $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ doesn't contribute to a ghost class for σ in the set

$$\{(3,2,4,1),(4,1,3,2),(4,2,3,1)\}$$

because these spaces are included in the image of $H^*(\partial_2, \tilde{V}_\lambda) \to H^*(\partial_0, \tilde{V}_\lambda)$. The same conclusion is obtained if we apply this procedure to $(4,3,2,1)$ unless $a_1 = a_2 = a_3 = a_4$.

On the other hand, if $a_1 = a_2 = a_3 = a_4$, let σ be the permutation defined by $(4,3,2,1)$, then w_σ has length 6 and the space

$$Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$$

is a direct summand of $H^6(\partial_0, \tilde{V}_\lambda)$. By applying Poincaré duality we can show that the morphism $H^7(S, \tilde{V}_\lambda) \to H^7(\partial\overline{S}, \tilde{V}_\lambda)$ is zero and then the space $H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ does not contribute to ghost classes.

There are seven other elements in the Weyl group, namely:

- w_σ and $w_{\sigma'}$ for $\sigma = (1,4,3,2)$ and $\sigma' = (4,1,2,3)$. They have length 3 and so the spaces $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ and $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)})$ are summands of the space $H^3(\partial_0, \tilde{V}_\lambda)$ and they both have weight $-c+4$ which is the middle weight of $H^4(S, \tilde{V}_\lambda)$.
- w_σ and $w_{\sigma'}$ for $\sigma = (2,3,4,1)$ and $\sigma' = (3,2,1,4)$. They have length 3 and so the spaces $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ and $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)})$ are summands of the space $H^3(\partial_0, \tilde{V}_\lambda)$ and they both have weight $-c+4$ which is the middle weight of $H^4(S, \tilde{V}_\lambda)$.
- w_σ and $w_{\sigma'}$ for $\sigma = (2,4,3,1)$ and $\sigma' = (4,2,1,3)$. They have length 4 and so the spaces $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ and $Ind_{P_0}^G H^0(S^{M_{P_0}}, \tilde{W}_{(w_{\sigma'})_*(\lambda)})$ are summands of the space $H^4(\partial_0, \tilde{V}_\lambda)$ and they both have weight $-c+6$ which is the middle weight of $H^5(S, \tilde{V}_\lambda)$ plus 1.
- w_σ for $\sigma = (1,3,2,4)$ (when $a_2 = a_3$). Its length is 1 and so the space $Ind_{P_0}^G H^0(S^{M_0}, \tilde{W}_{(w_\sigma)_*(\lambda)})$ is a summand of the space $H^1(\partial_0, \tilde{V}_\lambda)$ and its weight is $-c+2$ which is the middle weight of $H^2(S, \tilde{V}_\lambda)$. $\square$

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