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Torsion classes in the cohomology of KHT Shimura varieties

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A particular case of Bergeron-Venkatesh's conjecture predicts that torsion classes in the cohomology of Shimura varieties are rather rare. According to this and for Kottwitz-Harris-Taylor type of Shimura varieties, we first associate to each such torsion class an infinity of irreducible automorphic representations in characteristic zero, which are pairwise non isomorphic and weakly congruent in the sense of [16]. Then, using completed cohomology, we construct torsion classes in regular weight and then deduce explicit examples of such automorphic congruences.

Introduction

Let $F = EF^+$ be a CM field and B/F a central division algebra of dimension d^2 equipped with an involution of second kind: we can then define a group of similitudes $G/\mathbb{Q}$ as explained in §1.2 whose unitary associated group has signature $(1, d-1)$ at one real place and $(0, d)$ at the others. We denote $X_{I,\eta} \to \operatorname{Spec} F$ the Shimura variety of Kottwitz-Harris-Taylor type associated to $G/\mathbb{Q}$ and an open compact subgroup I. For a fixed prime number l, consider the set $\operatorname{Spl}(I)$ of places v of F over a prime number $p \neq l$ such that

- $p = uu^c$ is split in the quadratic imaginary extension E/Q,
- $G(\mathbb{Q}_p)$ is split, i.e. of the following shape $\mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$,
- the local component at p of I, is maximal,
- $v|u$ and $B_v^\times \simeq GL_d(F_v)$.

Given an irreducible algebraic representation ξ of G which gives a $\overline{\mathbb{Z}}_l$-local system $V_{\xi,\overline{\mathbb{Z}}_l}$ over $X_{I,\bar\eta}$, if we believe in the general conjectures of [1], and as the defect equals 0, asymptotically as the level I increases, the torsion cohomology classes in $H^i(X_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Z}}_l})$ are rather rare. In this direction, the main theorem of [7] gives a way to cancel this torsion by imposing for a place $v \in \operatorname{Spl}(I)$ that the multiset of modulo l Satake parameters does not contain a sub-multiset of the form $\{\alpha, q_v\alpha\}$ where q_v is the cardinality of the residual field $\kappa(v)$ at v: put another way, for a torsion cohomology class to exist, the associated set of modulo l Satake parameters should, at every place $v \in \mathrm{Spl}(I)$, contain a subset of the shape $\{\alpha, q_v\alpha\}$. In this paper, in the opposite direction, we are interested in cohomology torsion classes and their arithmetic applications: the main result is corollary 2.9 which can be stated as follow.

Theorem. Let $\mathfrak{m}$ be a maximal ideal of some unramified Hecke algebra, associated to some non trivial torsion class in the cohomology of $X_{I,\bar\eta}$ with coefficients in $V_{\xi,\overline{\mathbb{Z}}_l}$: we denote by i the greatest integer such that the torsion of $H^{d-1-i}(X_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is non trivial. There exists then a set

$$\Big\{\Pi(v):\ v \in \mathrm{Spl}(I)\Big\}$$

of irreducible automorphic ξ-cohomological representations such that for all $w \in \mathrm{Spl}(I)$ distinct from v, the local component at w of $\Pi(v)$ is unramified, its modulo l Satake parameters being given by $\mathfrak{m}$. On the other hand, $\Pi(v)$ is ramified at v and more precisely, it's isomorphic to a representation of the following shape

$$\mathrm{St}_{i+2}(\chi_{v,0}) \times \chi_{v,1} \times \cdots \times \chi_{v,d-i-2}$$

where $\chi_{v,0}, \dots, \chi_{v,d-i-2}$ are unramified characters of F_v and, see notation 1.1.1, the symbol $\times$ means normalized parabolic induction.

Remarks. (a) As expected from the main result of [7], for a $\mathfrak{m}$-torsion cohomology class to exist, the Satake parameters modulo l at each $v \in \mathrm{Spl}(I)$ must contain a chain $\{\alpha, q_v\alpha, \dots, q_v^{a-1}\alpha\}$ with $a \geq 2$. Moreover if you want to have torsion at distance $i > 0$ from middle degree then these sets of Satake parameters, for each $v \in \mathrm{Spl}(I)$, have to contain such a chain with length at least $i + 2$.

(b) As showed in [9], torsion cohomology classes of compact Shimura varieties raise in characteristic 0 and one may asks about the precise level. The previous theorem tells you that, for a torsion cohomology class in H^{d-1-i}, beside the condition about the existence for every $v \in \mathrm{Spl}(I)$, of a chain $\{\alpha, q_v\alpha, \dots, \alpha^{i+1}\}$ of length $i + 2$ inside the multiset of modulo l Satake parameters, to raise in characteristic 0, it suffices to raise the level at one

$v \in \mathrm{Spl}(I)$ to the subgroup[1]

$$\ker\big(GL_d(\mathcal{O}_v) \longrightarrow P_{d-i-1,d-i,\dots,d}(\kappa(v))\big)$$

where $P_{d-i-1,1,\dots,1}$ is the standard parabolic subgroup with Levi $GL_{d-i-1} \times GL_1 \times \cdots \times GL_1$ of GL_d. In particular the more you go away from middle degree, to raise your torsion cohomology class in characteristic zero, the more you need, a priori, to raise the level.

(c) For $v \neq w \in \mathrm{Spl}(I)$, the representations $\Pi(v)$ and $\Pi(w)$ are not isomorphic but are weakly congruent in the sense of §3 [16], i.e. they share the same modulo l Satake parameters at each place you can define Satake parameters and so in particular at all places of $\mathrm{Spl}(I) - \{v, w\}$.

(d) In a recent preprint [8], this result is the main tool for generalizing Mazur's principle for G i.e. to do level lowering for automorphic representations of G. Roughly the idea goes like that: start from a maximal ideal $\mathfrak{m}$ which raises in characteristic 0 for some level I ramified at a fixed place w. The tricky part is to find some good hypothesis to force the existence of a non trivial torsion cohomology class with level $I' = I^w I'_w$ with $I_w \subsetneq I'_w$. Then use the previous theorem at a place $v \neq w$ to raise again in characteristic zero: then you succeed to lower the level at the place w but you might have to increase the level at v.

We are then lead to the construction of such torsion cohomology classes. In §3, we investigate this question with the help of the notion of completed cohomology

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_n \varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/\lambda^n}).$$

As they are independent of the choice of the weight ξ,

- by taking ξ the trivial representation and using the Hochschild-Serre spectral sequence which, starting from the completed cohomology, computes the cohomology at a finite level, we can show that for each divisor s of d, the free quotient of $\widetilde{H}^{d-s}_{I^l}$ is non trivial provided that the level I^l outside l is small enough: in fact here, we just prove an imprecise version of this fact, see proposition 3.3.
- Then taking ξ regular and as the free quotient of the finite cohomology outside the middle degree are trivial, we observe that, for each divisor

[1] It's well known that for $t_1 \geq t_2 \geq \cdots \geq t_a$, $\mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_a}(\chi_{v,a})$ has non trivial invariant vectors under $\ker(GL_{t_1+\cdots+t_a}(\mathcal{O}_v) \longrightarrow P_{s_1,\dots,s_b}(\kappa(v)))$ if $(s_1 \geq \cdots \geq s_b)$ is the conjugated partition associated to $(t_1 \geq \cdots \geq t_a)$ and this subgroup is, up to conjugacy, the minimal one among groups of this form.

s of d, we can find torsion classes in level I_l so that by considering their reduction modulo l^n for varying n, and through the process of completed cohomology i.e. taking first the limit on I_l and then on l^n, these classes organize themselves in some torsion free classes.

Automorphic congruences can be useful to construct non trivial elements in Selmer groups. For this one need to obtain such congruences between tempered and non tempered automorphic representations of same weight, which is not exactly the case here. In some forthcoming work, we should be able to do so using

- either torsion classes in the cohomology of Harris-Taylor's local system constructed in [6] and associated to non tempered automorphic representation. The idea is then to prove that this torsion classes give torsion classes in the cohomology of the whole Shimura variety and use the main theorem of this paper.
- or the cancellation of completed cohomology groups $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ when $i \geq d$, see [14] proposition IV.2.2.

1. Notations and background

1.1. Induced representations

Consider a local fiel K with its absolute value $|-|$: let q denote the cardinal of its residual field. For a representation π of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \operatorname{val} \circ \det}.$$

Notations 1.1.1. For π_1 and π_2 representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, we will denote by

$$\pi_1 \times \pi_2 := \operatorname{ind}_{P_{n_1,n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1 \left\{\frac{n_2}{2}\right\} \otimes \pi_2 \left\{-\frac{n_1}{2}\right\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \cdots \times GL_{r_k-r_{k-1}}.$$

Remind that an irreducible representation is called supercuspidal if it's not a subquotient of some proper parabolic induced representation.

Definition 1.1.2. (see [17] §9 and [4] §1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal representation of $GL_g(K)$. The induced representation

$$\pi\left\{\frac{1-s}{2}\right\} \times \pi\left\{\frac{3-s}{2}\right\} \times \cdots \times \pi\left\{\frac{s-1}{2}\right\}$$

holds a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi)$ (resp. $\mathrm{Speh}_s(\pi)$); it's a generalized Steinberg (resp. Speh) representation.

Remark. From a galoisian point of view through the local Langlands correspondence, the representation $\mathrm{Speh}_s(\pi)$ matches to the direct sum $\sigma(\frac{1-s}{2}) \oplus \cdots \oplus \sigma(\frac{s-1}{2})$ where σ matches to π. *More generally* for π any irreducible representation of $GL_g(K)$ associated to σ by the local Langlands correspondence, we will denote $\mathrm{Speh}_s(\pi)$ the representation of $GL_{sg}(K)$ matching, through the local Langlands correspondence, $\sigma(\frac{1-s}{2}) \oplus \cdots \oplus \sigma(\frac{s-1}{2})$.

Definition 1.1.3. A smooth $\overline{\mathbb{Q}}_l$-representation of finite length π of $GL_d(K)$ is said *entire* if there exist a finite extension $E/\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}}_l$, with ring of integers $\mathcal{O}_E$, and a $\mathcal{O}_E$-representation L of $GL_d(K)$, which is a free $\mathcal{O}_E$-module, such that $\overline{\mathbb{Q}}_l \otimes_{\mathcal{O}_E} L \simeq \pi$ and L is a $\mathcal{O}_E$ $GL_n(K)$-module of finite type. Let κ_E the residual field of $\mathcal{O}_E$, we say that

$$\overline{\mathbb{F}}_l \otimes_{\kappa_E} \kappa_E \otimes_{\mathcal{O}_E} L$$

is the modulo l reduction of L.

Remark. The *Brauer-Nesbitt principle* asserts that the semi-simplification of $\overline{\mathbb{F}}_l \otimes_{\mathcal{O}_E} L$ is a finite length $\overline{\mathbb{F}}_l$-representation of $GL_d(K)$ which is independent of the choice of L. Its image in the Grothendieck group will be denoted $r_l(\pi)$ and called the *modulo l reduction of* π.

Example. From [15] V.9.2 or [10] §2.2.3, we know that the modulo l reduction of $\mathrm{Speh}_s(\pi)$ is irreducible.

1.2. Geometry of KHT Shimura varieties

Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote

- F_v the completion of F at v,
- $\mathcal{O}_v$ the ring of integers of F_v,

- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F, of dimension d^2 such that at every place x of F, either B_x is split or a local division algebra.

Notation 1.2.4. Let denote Bad the set of places of F such that for any $w \notin \mathrm{Bad}$, we have $B_w^\times \simeq GL_d(F_w)$.

Further we assume B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and $G/\mathbb{Q}$ the group of similitudes, denoted G_τ in [12], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \tag{1.2.5}$$

where, identifying places of F^+ over x with places of F over y, $x = \prod_i z_i$ in F^+.

Convention. For $x = yy^c$ a place of $\mathbb{Q}$ split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (1.2.5) as well as

$$G(\mathbb{A}^z) := G(\mathbb{A}^x) \times \left(\mathbb{Q}_x^\times \times \prod_{z_i \neq z} (B_{z_i}^{op})^\times \right).$$

In [12], the author justify the existence of some G like before such that moreover

- if x is a place of $\mathbb{Q}$ non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [12] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

Notation 1.2.6. Denote $\mathcal{I}$ the set of open compact subgroups small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \longrightarrow \operatorname{Spec} F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.

From now on, we fix a prime number l unramified in E.

Definition 1.2.7. Define Spl the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write $\mathrm{Spl}(I)$ the subset of Spl of places which don't divide the level I.

Remark. For every $v \in \mathrm{Spl}$, the variety $X_{I,\eta}$ has a projective model $X_{I,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special fiber X_{I,s_v}. For I going through $\mathcal{I}$, the projective system $(X_{I,v})_{I\in\mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}^{-1}$ and $\mathrm{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin's isomorphism which sends geometric Frobenius to uniformizers.

Notations 1.2.8. (see [3] §1.3) For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $X_{I,\bar{s}_v}$ is denoted

$$X_{I,\bar{s}_v} =: X^{\geq 1}_{I,\bar{s}_v} \supset X^{\geq 2}_{I,\bar{s}_v} \supset \cdots \supset X^{\geq d}_{I,\bar{s}_v}$$

where $X^{=h}_{I,\bar{s}_v} := X^{\geq h}_{I,\bar{s}_v} - X^{\geq h+1}_{I,\bar{s}_v}$ is an affine scheme[2], smooth of pure dimension $d-h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h. For each $1 \leq h < d$, write

$$i_{h+1} : X^{\geq h+1}_{I,\bar{s}_v} \hookrightarrow X^{\geq h}_{I,\bar{s}_v}, \quad j^{\geq h} : X^{=h}_{I,\bar{s}_v} \hookrightarrow X^{\geq h}_{I,\bar{s}_v}.$$

1.3. Cohomology groups over $\overline{\mathbb{Q}}_l$

Let us begin with some known facts about irreducible algebraic representations of G, see for example [12] p.97. Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uples $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$.

For $K \subset \overline{\mathbb{Q}}_l$ a finite extension of $\mathbb{Q}_l$ such that the representation $\iota^{-1} \circ \xi$ of highest weight $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$, is defined over K, write $W_{\xi,K}$ the space of this representation and $W_{\xi,\mathcal{O}}$ a stable lattice under the action of the maximal

[2] see for example [13].

open compact subgroup $G(\mathbb{Z}_l)$, where $\mathcal{O}$ is the ring of integers of K with uniformizer λ.

Remark. If ξ is supposed to be l-small, in the sense that for all $\sigma \in \Phi$ and all $1 \leq i < j \leq n$ we have $0 \leq a_{\tau,j} - a_{\tau,i} < l$, then such a stable lattice is unique up to a homothety.

Notation 1.3.9. We will denote $V_{\xi,\mathcal{O}/\lambda^n}$ the local system on $X_{\mathcal{I}}$ as well as

$$V_{\xi,\mathcal{O}} = \lim_{\substack{\longleftarrow \\ n}} V_{\xi,\mathcal{O}/\lambda^n} \quad \text{and} \quad V_{\xi,K} = V_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} K.$$

For $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{Q}}_l$ version, we will write respectively $V_{\xi,\overline{\mathbb{Z}}_l}$ and $V_{\xi,\overline{\mathbb{Q}}_l}$.

Remark. The representation ξ is said *regular* if its parameter $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ verifies for all $\sigma \in \Phi$ that $a_{\sigma,1} < \cdots < a_{\sigma,d}$.

Definition 1.3.10. An irreducible automorphic representation Π is said ξ-cohomological if there exists an integer i such that

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

For Π an automorphic irreducible representation ξ-cohomological of $G(\mathbb{A})$, then, see for example lemma 3.2 of [6], for each $v \in \mathrm{Spl}$, the local component Π_v is isomorphic to some $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible non degenerate representation and $s \geq 1$ an integer which is independent of the place $v \in \mathrm{Spl}$.

Definition 1.3.11. The integer s mentioned above is called the degeneracy depth of Π.

From now on, we fix $v \in \mathrm{Spl}$.

Notation 1.3.12. For $1 \leq h \leq d$, let us denote $\mathcal{I}_v$ the set of open compact subgroups of the following shape

$$U_v(\underline{m}) := U_v(\underline{m}^v) \times K_v(m_1),$$

where $K_v(m_1) = \ker\big(GL_d(\mathcal{O}_v) \longrightarrow GL_d(\mathcal{O}_v/(\varpi_v^{m_1}))\big)$. The notation $[H^i(h,\xi)]$ (resp. $[H^i_!(h,\xi)]$) means the image of

$$\lim_{\substack{\longrightarrow \\ I\in\mathcal{I}_v}} H^i(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Q}}_l}) \qquad \text{resp. } \lim_{\substack{\longrightarrow \\ I\in\mathcal{I}_v}} H^i_c(X^{=h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Q}}_l})$$

in the Grothendieck group Groth(v) of admissible representations of $G(\mathbb{A}^{\infty,v}) \times GL_d(F_v) \times \mathbb{Z}$.

Remark. Recall that the action of $\sigma \in W_v$ on these $GL_d(F_v) \times \mathbb{Z}$-modules is given by those of $-\deg \sigma \in \mathbb{Z}$.

Notation 1.3.13. For $\Pi^{\infty,v}$ an irreducible representation of $G(\mathbb{A}^{\infty,v})$, let denote Groth$(h)\{\Pi^{\infty,v}\}$ the subgroup of Groth(v) generated by representations of the shape $\Pi^{\infty,v} \otimes \Psi_v \otimes \zeta$ where Ψ_v (resp. ζ) is any irreducible representation of $GL_d(F_v)$ (resp. of $\mathbb{Z}$). We will denote then

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\}$$

the projection of $[H^i(h,\xi)]$ on this direct factor.

We write

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\} = \Pi^{\infty,v} \otimes \left(\sum_{\Psi_v,\xi} m_{\Psi_v,\zeta}(\Pi^{\infty,v}) \Psi_v \otimes \zeta \right),$$

where Ψ_v (resp. ξ) goes through irreducible admissible representations of $GL_d(F_v)$, (resp. of $\mathbb{Z}$ which can be considered as an unramified representation of W_v).

Remark. Recall, cf. [4] where all these cohomology groups are explicitly computed, that if for some h and i, $[H^i(h,\xi)]\{\Pi^{\infty,v}\} \neq (0)$ then $\Pi^{\infty,v}$ is the component of a automorphic ξ-cohomological representation.

Proposition 1.3.14. *Let Π be an automorphic irreducible tempered representation ξ-cohomological.*

(i) *For all $h = 1, \dots, d$ and all $i \neq d-h$,*

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\} \qquad \text{and} \qquad [H^i_!(h,\xi)]\{\Pi^{\infty,v}\}$$

are trivial.

(ii) *If $[H^{d-h}(h,\xi)]\{\Pi^{\infty,v}\}$ (resp. $[H^{d-h}_!(h,\xi)]\{\Pi^{\infty,v}\}$) has non trivial invariants under the action of $GL_d(\mathcal{O}_v)$ then the local component Π_v of Π at v is isomorphic to a representation of the following shape*

$$\mathrm{St}_r(\chi_{v,0}) \times \chi_{v,1} \times \cdots \times \chi_{v,r}$$

where $\chi_{v,0}, \dots, \chi_{v,t}$ are unramified characters and $r = h$ (resp. $r \geq h$).

Proof. (i) This is exactly proposition 1.3.9 of [7].

(ii) The result for $H^i(h,\xi)$ is a particular case of proposition 3.6 of [6] (which proposition follows directly from proposition 3.6.1 of [4]) for the constant local system, i.e. when π_v is the trivial representation and $s=1$.

Concerning the cohomology with compact supports, we can use either proposition 3.12 of [6] or the description, given by corollary 5.4.1 of [3], of this extension by zero in terms of local systems on Newton strata with indices $h' \geq h$. □

Proposition 1.3.15. *(see [4] theorem 4.3.1) Let* Π *be an automorphic irreducible representation* ξ*-cohomological with depth of degeneracy* $s>1$*. Then for* ξ *the trivial character,* $[H_!^{d-2h+s}(h,\xi)]\{\Pi^{\infty,v}\}$ *is non trivial.*

Remark. If ξ is a regular parameter then the depth of degeneracy of any irreducible automorphic representation ξ-cohomological is necessary equal to 1. In particular theorem 4.3.1 of [4] is compatible with the classical result saying that for a regular ξ, the cohomology of the Shimura variety X_I with coefficients in $V_{\xi,\overline{\mathbb{Q}}_l}$, is concentrated in middle degree.

1.4. Hecke algebras

Consider the following set $\mathrm{Unr}(I)$ which is the union of

- places $q \neq l$ of $\mathbb{Q}$ inert in E not below a place of Bad and where I_q is maximal;
- places $w \in \mathrm{Spl}(I)$.

Notation 1.4.16. For $I \in \mathcal{I}$ a finite level, write

$$\mathbb{T}_I := \prod_{x \in \mathrm{Unr}(I)} \mathbb{T}_x$$

where for x a place of $\mathbb{Q}$ (resp. $x \in \mathrm{Spl}(I)$), $\mathbb{T}_x$ is the unramified Hecke algebra of $G(\mathbb{Q}_x)$ (resp. of $GL_d(F_x)$) over $\overline{\mathbb{Z}}_l$.

Example. For $w \in \mathrm{Spl}(I)$, we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l\big[T_{w,i} : \ i=1,\dots,d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w,\dots,\varpi_w}^{i},\overbrace{1,\dots,1}^{d-i})GL_d(\mathcal{O}_w)\subset GL_d(F_w).$$

More generally, the Satake isomorphism identifies $\mathbb{T}_x$ with $\overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ where

- T_x is a split torus,
- W_x is the spherical Weyl group
- and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Consider a fixed maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ and for every $x\in \mathrm{Unr}(I)$ let denote $S_{\mathfrak{m}}(x)$ the multi-set[3] of modulo l Satake parameters at x associated to $\mathfrak{m}$.

Example. For every $w\in \mathrm{Spl}(I)$, the multi-set of Satake parameters at w corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X):=\sum_{i=0}^{d}(-1)^i q_w^{\frac{i(i-1)}{2}}\overline{T_{w,i}}X^{d-i}\in\overline{\mathbb{F}}_l[X]$$

i.e.

$$S_{\mathfrak{m}}(w):=\big\{\lambda\in\mathbb{T}_I/\mathfrak{m}\simeq\overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda)=0\big\}.$$

To each Hecke polynomial $P_{\mathfrak{m},x}(X)$ at $x\in\mathrm{Unr}(I)$, one can associate its reciprocal $P^\vee_{\mathfrak{m},w}(X)$ polynomial whose roots are inverse of those of $P_{\mathfrak{m},x}(X)$. We then define $\mathfrak{m}^\vee$ to be the maximal ideal of $\mathbb{T}_I$ so that the roots of $P^\vee_{\mathfrak{m},w}(X)$ are those of $S_{\mathfrak{m}^\vee}(x)$ for every $x\in\mathrm{Unr}(I)$.

Example. For $x=w\in\mathrm{Spl}(I)$, with the previous notations, the image $\overline{T_{w,i}}$ of $T_{w,i}$ inside $\mathbb{T}_I/\mathfrak{m}$ can be written

$$\overline{T_{w,i}}=q_w^{\frac{i(1-i)}{2}}\sigma_i(\lambda_1,\dots,\lambda_d)$$

where we write $S_{\mathfrak{m}}(w)=\{\lambda_1,\dots,\lambda_d\}$ and where the σ_i are the elementary symmetric functions. Locally at w the maximal ideal $\mathfrak{m}^\vee$ is defined by

$$T_{w,i}\in\mathbb{T}_w\mapsto q_w^{\frac{i(1-i)}{2}}\sigma_i(\lambda_1^{-1},\dots,\lambda_d^{-1})\in\overline{\mathbb{F}}_l.$$

[3] A multi-set is a set with multiplicities.

2. Automorphic congruences

Consider from now on a fixed place $v \in \mathrm{Spl}(I)$.

Definition 2.1. A $\mathbb{T}_I$-module M is said to verify property **(P)**, if it has a finite filtration

$$(0) = \mathrm{Fil}^0(M) \subset \mathrm{Fil}^1(M) \cdots \subset \mathrm{Fil}^r(M) = M$$

such that for every $k = 1, \dots, r$, there exists

- an automorphic irreducible entire representation Π_k of $G(\mathbb{A})$, which appears in the cohomology of $(X_{I,\overline{\eta}_v})_{I \in \mathcal{I}}$ with coefficients in $V_{\xi,\overline{\mathbb{Q}}_l}$ and such that its local component $\Pi_{k,v}$ is ramified, i.e. $(\Pi_{k,v})^{GL_d(\mathcal{O}_v)} = (0)$;
- an unramified entire irreducible representation $\widetilde{\Pi}_{k,v}$ of $GL_d(F_v)$ with the same cuspidal support as $\Pi_{k,v}$ and
- a stable $\mathbb{T}_I$-lattice Γ of $(\Pi_k^{\infty,v})^{I^v} \otimes \widetilde{\Pi}_{k,v}^{GL_d(\mathcal{O}_v)}$ such that
  - either $\mathrm{gr}^k(M)$ is free, isomorphic to Γ,
  - or $\mathrm{gr}^k(M)$ is torsion and equals to some subquotient of Γ/Γ' where $\Gamma' \subset \Gamma$ is another stable $\mathbb{T}_I$-lattice.

We will say that $\mathrm{gr}^k(M)$ is of type i if moreover $\Pi_{k,v}$ looks like

$$\mathrm{St}_i(\chi) \times \chi_1 \times \cdots \times \chi_{d-i}$$

where $\chi, \chi_1, \dots, \chi_{d-i}$ are unramified characters. When all the $\mathrm{gr}^\bullet(M)$ are of type i, then we will say that M is of type i.

Remark. Property **(P)** is by definition stable through extensions and subquotients: replacing condition ξ-cohomological by $\xi^\vee$-cohomological, it is also stable by duality.

Lemma 2.2. *Consider $h \geq 1$ and M an irreducible subquotient of*

$$H_c^{d-h}(X_{I,\bar{s}_v}^{=h}, V_{\xi,\overline{\mathbb{Q}}_l}) \quad \textit{resp. of } H^{d-h}(X_{I,\bar{s}_v}^{\geq h}, V_{\xi,\overline{\mathbb{Q}}_l}),$$

then

- *either M verify property **(P)** and then is of type h or $h+1$ (resp. of type h),*
- *or M is not a subquotient of $H^{d-h-1}(X_{I,\bar{s}_v}^{\geq h+1}, V_{\xi,\overline{\mathbb{Q}}_l})$.*

Proof. The result follows from explicit computations of these $\overline{\mathbb{Q}}_l$- cohomology groups with infinite level given in [4]: the reader can see a presentation of them at §3.3 (resp. §3.2) of [6]. Precisely for Π^∞ an irreducible representation of $G(\mathbb{A}^\infty)$, the isotypic component

$$\varinjlim_{I\in\mathcal{I}} H_c^{d-h}(X_{\mathcal{I},\bar{s}_v}^{=h}, V_\xi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)\{\Pi^{\infty,v}\},$$
$$\text{resp. } \varinjlim_{I\in\mathcal{I}} H_c^{d-h}(X_{\mathcal{I},\bar{s}_v}^{=h}, V_\xi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)\{\Pi^{\infty,v}\}$$

is zero if Π^∞ is not the component outside ∞ of an automorphic ξ-cohomological representation Π. Otherwise, we distinguish three cases according to the local component Π_v of Π at v:

(i) $\Pi_v \simeq \mathrm{St}_r(\chi_v) \times \pi'_v$ with $h \leq r \leq d$,

(ii) $\Pi_v \simeq \mathrm{Speh}_r(\chi_v) \times \pi'_v$ with $h \leq r \leq d$,

(iii) Π_v is not of the two previous shapes,

where χ_v is an unramified character of $F_v^\times$ and π'_v is an irreducible admissible unramified representation of $GL_{d-h}(F_v)$. Then this isotypic component, as a $GL_d(F_v) \times \mathbb{Z}$-representation, is of the following shape:

- in case (i) we obtain $\big(\mathrm{Speh}_h(\chi\{\frac{h-r}{2}\}) \times \mathrm{St}_{r-h}(\chi\{\frac{h}{2}\})\big) \times \pi'_v \otimes \Xi^{\frac{r-h}{2}}$ (resp. zero if $r \neq h$ and otherwise $\mathrm{St}_h(\chi_v)$);
- zero in the case (ii) if $r \neq h$ and otherwise $\mathrm{Speh}_h(\chi_v) \times \pi'_v$.
- Finally in case (iii), the obtained $GL_d(F_v)$-representation won't have non trivial invariants under $GL_d(\mathcal{O}_v)$.

Thus taking invariants under I and because v doesn't divide I,

- case (i): we obtain a $\mathbb{T}_I$-module verifying property **(P)** which is of type h ou $h+1$ according $r = h$ or $h+1$.
- case (ii): the obtained $\mathbb{T}_I$-module is then not a subquotient of the cohomology group $H^{d-h-1}(X_{I,\bar{s}_v}^{\geq h+1}, V_{\xi,\mathbb{Q}_l})$,
- and case (iii): as it doesn't have non trivial invariants under $GL_d(\mathcal{O}_v)$, we obtain nothing else than zero. □

From now on we assume that there exists i such that the torsion submodule of $H^{d-1+i}(X_{I,\bar{\eta}_v}, V_\xi)$ is non trivial and we fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ such that the torsion of $H^{d-1+i}(X_{I,\bar{\eta}_v}, V_\xi)_\mathfrak{m}$ is non trivial. Let $1 \leq$

$h \leq d$ be maximal such that there exists i for which the torsion subspace $H^{d-h+i}(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor}$ of $H^{d-h+i}(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m}}$ is non reduced to zero. Notice that

- since the dimension of $X^{\geq d}_{I,\bar{s}_v}$ equals zero then we have $h < d$;
- by the smooth base change theorem $H^\bullet(X_{I,\bar{\eta}_v}, V_\xi) \simeq H^\bullet(X^{\geq 1}_{I,\bar{s}_v}, V_\xi)$ so that $h \geq 1$.

Lemma 2.3. *With the previous notations and assuming the existence of non trivial torsion cohomology classes in $H^\bullet(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m}}$, then 0 is the smallest indice i such that $H^{d-h+i}(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor} \neq (0)$. Moreover every irreducible non trivial submodule of $H^{d-h}(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor}$ verifies property **(P)** being of type $h+1$.*

Proof. Consider the following short exact sequence of perverse sheaves[4]

$$0 \to i_{h+1,*} V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h+1}_{I,\bar{s}_v}}[d-h-1] \longrightarrow j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h] \longrightarrow V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h] \to 0. \tag{2.4}$$

Indeed as the strata $X^{\geq h}_{I,\bar{s}_v}$ are smooth and $j^{\geq h}$ is affine, the three terms of this exact sequence are perverse and even free in the sense of the natural torsion theory from the linear $\overline{\mathbb{Z}}_l$-linear structure, see [5] §1.1-1.3.

Moreover from Artin's theorem, see for example theorem 4.1.1 of [2], using the affiness of $X^{=h}_{I,\bar{s}_v}$, we deduce that

$$H^i(X^{\geq h}_{I,\bar{s}_v}, j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h])$$

is zero for every $i < 0$ and without torsion for $i = 0$, so that for $i > 0$, we have

$$0 \to H^{-i-1}(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \to 0, \tag{2.5}$$

and for $i = 0$,

$$0 \to H^{-1}(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \longrightarrow H^0(X^{\geq h}_{I,\bar{s}_v}, j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \to \cdots \tag{2.6}$$

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[4] As we are dealing with perverse sheaves, note the shifts of the grading of cohomology groups.

Thus if the torsion of $H^i(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])$ is non trivial then $i \geq 0$ and thanks to Grothendieck-Verdier duality, the smallest such indice is necessary $i = 0$. Furthermore the torsion of $H^0(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])$ raises both into $H^0_c(X^{=h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])$ and $H^0(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])$, which are both free. Thus by the previous lemma, the torsion of $H^0(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h])_{\mathfrak{m}}$ verifies property **(P)** being of type $h+1$. □

Lemma 2.7. *With previous notations, for all $1 \leq h' \leq h$, the greatest i such that the torsion of $H^{d-h'-i}(X^{\geq h'}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m},tor}$ is non zero, equals $h-h'$. Moreover this torsion verifies property **(P)** being of type $h+1$.*

Proof. We argue by induction on h' from h to 1. The case $h' = h$ follows directly from the previous lemma so that we suppose the result true up to $h'+1$ and consider the cas of h'. Resume the spectral sequences (2.4) with h'. Then the result follows from (2.5) and the induction hypothesis. □

Using the smooth base change theorem, the case $h' = 1$ of the previous lemma, then gives the following proposition.

Proposition 2.8. *Let i be maximal, if it exists, such that the torsion submodule of $H^{d-1-i}(X_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is non zero. Then it verifies property **(P)** being of type $i+2$.*

Corollary 2.9. *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ and i maximal, if it exists, such that the torsion of $H^{d-1-i}(X_{I,\bar{\eta}}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is non zero. Then there exists a set $\{\Pi(v) : v \in \mathrm{Spl}(I)\}$ of irreducible automorphic ξ-cohomological representations such that*

- *for any $w \in \mathrm{Spl}(I)$ different from v, the local component at w of $\Pi(v)$ is unramified with modulo l Satake parameters given by $S_{\mathfrak{m}}$;*
- *the local component $\Pi(v)_v$ of Π at v is isomorphic to a representation of the following shape*

$$\mathrm{St}_{i+2}(\chi_{v,0}) \times \chi_{v,1} \times \cdots \times \chi_{v,d-i-2},$$

where $\chi_{v,0}, \ldots, \chi_{v,d-i-2}$ are unramified characters of F_w.

Remark. In the first point of the previous corollary, we can of course say that for all finite places not dividing $I \cup \mathrm{Bad} \cup \{l\}$, and different from v in the sense of the formula (1.2.5), the modulo l Satake parameters of $\Pi(v)$ and Π are the same: but the place v where the level increase must belong to $\mathrm{Spl}(I)$. For Π_1 and Π_2 irreducible representations of $G(\mathbb{A})$ and S the set of finite places of ramification of either Π_1 or Π_2, if the modulo l Satake parameters outside $S \cup \mathrm{Bad}$ of Π_1 and Π_2 are the same then we say that they are weakly congruent.

Proof. Consider an irreducible $\mathbb{T}_I$-submodule M of $H^{d-1-i}(X_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m},tor}$. For any place $v \in \mathrm{Spl}(I)$, thanks to the smooth base change theorem, we have

$$H^{d-1-i}(X_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}} \simeq H^{d-1--i}(X^{\geq 1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}.$$

From the previous proposition, this module M verifies property **(P)** being of type $i+2$ so that it exists an automorphic irreducible ξ-cohomological representation $\Pi(v)$ verifying the required properties. □

3. Completed cohomology and torsion classes

Given a level $I^l \in \mathcal{I}$ maximal at l, recall that the completed cohomology groups are

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}) := \lim_{\substack{\longrightarrow \\ I_l}} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/\lambda^n})$$

and

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) := \lim_{\substack{\longleftarrow \\ n}} \widetilde{H^i}_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}),$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_l$ on which the representation ξ is defined.

Notation 3.1. When $\xi = 1$ is the trivial representation, we will denote

$$\widetilde{H}^i_{I^l} := \widetilde{H}^i_{I^l}(V_{1,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l.$$

Remark. For n fixed, there exists an open compact subgroup $I_l(n)$ such that, using the notations below 1.3.9, every $I_l \subset I_l(n)$ acts trivialy on $W_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}/\lambda^n$. We then deduce that the completed cohomology groups don't depend of the choice of ξ in the sense where, see theorem 2.2.17 of [11]:

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l \simeq \widetilde{H}^i_{I^l} \otimes W_\xi \tag{3.2}$$

where $G(\mathbb{Q}_l)$ acts diagonally on the right side.

Scholze, see [14] proposition IV.2.2, has showed that the $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ are trivial for all $i > d-1$. In our situation we can prove that for all divisor s of d, there are non zero for $i = d - s$: the argument is quite simple but it uses some particular results about entire notions of intermediate extension of Harris-Taylor's local systems. As we don't really need such precision, we only prove the following property.

Proposition 3.3. *For each divisor s of $d = sg$ and for a level I^l outside l small enough, there exists $i \leq d-s$ such that $\widetilde{H}^i_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ has, as a $GL_d(F_v)$-representation, an irreducible quotient with degeneracy depth equals to s.*

Proof. Recall the Hochschild-Serre spectral sequence allowing to compute the cohomology at finite level from completed one

$$E_2^{i,j} = H^i(I_l, \widetilde{H}^j_{I_l} \otimes V_{\xi,\overline{\mathbb{Z}}_l}) \Rightarrow H^{i+j}(X_{I^l I_l}, V_{\xi,\overline{\mathbb{Z}}_l}). \tag{3.4}$$

Let $v \in \mathrm{Spl}(I^l)$ be a fixed place over some prime number $p \neq l$. Consider then a divisor s of $d = sg$ and an automorphic representation Π which is cohomological relatively to a algebraic representation ξ of G and such that its local component at the place v is isomorphic to $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible cuspidal representation of $GL_g(F_v)$. As before we choose a finite level I^l outside l so that Π has non trivial invariant vectors under I^l. According to [4], the Π^∞-isotypic factor of the $\overline{\mathbb{Q}}_l$-cohomology group of indice $d-s$ is non trivial for $I = I^l I_l$ with I_l small enough. The result then follows from the spectral sequence (3.4). $\square$

Let $\widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ be the p-adic completion of $H^i_{I^l}(V_{\xi,\mathcal{O}}) := \varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}})$ that is

$$\widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}}) = \varprojlim_n \left(\varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1]) \,/\, \lambda^n H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1]) \right).$$

It kills the p-divisible part of $H^i_{I^l}(V_{\xi,\mathcal{O}})$. Consider also the p-adic Tate module of $H^i_{I^l}(V_{\xi,\mathcal{O}})$

$$T_p H^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_n H^i_{I^l}(V_{\xi,\mathcal{O}})[\lambda^n].$$

whom knows only about torsion. Recall then the short exact sequence

$$0 \to \widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \longrightarrow \widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \longrightarrow T_p H^{i+1}_{I^l}(V_{\xi,\mathcal{O}}) \to 0. \tag{3.5}$$

When ξ is a regular algebraic representation, the cohomology of X_I with coefficients in $V_{\xi,\overline{\mathbb{Q}}_l}$, is concentrated in middle degree and so $\widehat{H}^i_{I^l}(V_{\xi,\overline{\mathbb{Q}}_l})$ is trivial for all $i \neq d-1$. Let $s \geq 2$ be a divisor of d and $i \leq d-s < d-1$

such that, thanks to the previous proposition, for some finite level I^l outside l small enough

$$\widetilde{H}^i_{I^l}(V_{\xi,\overline{\mathbb{Z}}_l}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq T_p H^{i+1}_{I^l}(V_{\xi,\overline{\mathbb{Z}}_l})$$

is non trivial. It means then that for all $n \geq 1$, there exists an open compact subgroup I_l small enough, for which $H^{i+1}(X_{I^l I_l}, V_{\xi,\overline{\mathbb{Z}}_l})$ has a class of exactly λ^n-torsion so that through the process of completed cohomology, i.e. when you first take the limit on I_l and then on λ^n, the reductions modulo λ^n for varying n, of these torsion classes give torsion free classes generating an automorphic representation Π with depth of degeneracy equals to s and trivial weight. From proposition 3.3, for $\mathfrak{m}$ a maximal ideal of $\mathbb{T}_I$ associated to this Π, there exists a set $\{\Pi(v);\ v \in \mathrm{Spl}(I)\}$ such that the properties of corollary 2.9 hold:

- in particular these $\Pi(v)$ are tempered representations, non isomorphic and weakly congruent in twos; there are all of the same regular weight;
- each of these tempered irreducible representations $\Pi(v)$ of regular weight ξ is also weakly congruent with Π, an irreducible representation of trivial weight with degeneracy depth > 1.

More generally using the isomorphism (3.2) for any ξ not necessarily trivial or regular, take as before $i \leq d-s$ minimal such that the free quotient of $\widetilde{H}^i_{I^l}$ has an irreducible quotient Π with depth of degeneracy s and let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_{I^l}$ associated to such a Π. Thus for any irreducible algebraic representation ξ we have again $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}})_{\mathfrak{m}} \neq (0)$ so that for every I_l small enough and for all n, we have $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/(\lambda^n)})_{\mathfrak{m}} \neq (0)$ so, using (3.5),

(i) either $\widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ is non trivial so that there exists a ξ-cohomological automorphic representation Π' which is weakly congruent with Π: note that, by minimality of i, such a Π' is necessary of degeneracy depth s;

(ii) or, $T_p H^{i+1}_{I^l}(V_{\xi,\mathcal{O}}) \neq (0)$ so thanks to proposition 2.8 there exists an ξ-cohomological automorphic tempered representation Π' whose modulo l Satake parameters at places of $\mathrm{Unr}(I) \backslash \{v\}$ are given by $\mathfrak{m}$.

So for any weights $\xi_1 \neq \xi_2$, we can construct weakly automorphic congruences between representations Π'_1 and Π'_2 respectively of weight ξ_1 and ξ_2, governed by a maximal ideal $\mathfrak{m}$ attached to some irreducible automorphic representation, cohomological for the trivial character, with degeneracy depth $s \geq 2$. Concerning the degeneracy depth of Π'_1 and Π'_2 it is equal to s or 1 according to they fall in case (i) or (ii).

Example. Consider the most trivial case where ξ_1 is the trivial character and ξ_2 is a regular one and take $\mathfrak{m}$ the maximal ideal associated to the trivial representation Π of $G(\mathbb{A})$ with degeneracy depth $s = d$ which is trivially, as $\widetilde{H}^0_{I^l} \simeq \mathbb{Z}_l$, ξ_1-cohomological. As ξ_2 is regular we are then in case (ii), that is for any deep enough I_l, the torsion of $H^1(X_{I^l I_l}, V_{\xi,\mathcal{O}})$ is non trivial. Thanks to proposition 2.8, this torsion raises in characteristic zero to a tempered automorphic representation ξ-cohomological Π such that:

- its local component at v is a Steinberg representation $\mathrm{St}_d(\chi_v)$ with χ_v congruent to the trivial character modulo l,
- its local Satake parameters at every place of $\mathrm{Unr}(I)\backslash\{v\}$ are those of the trivial character of $G(\mathbb{A})$.

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