

Definable maximal discrete sets in forcing extensions

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Let \mathcal{R} be a Σ_1^1 binary relation, and recall that a set A is \mathcal{R} -discrete if no two elements of A are related by \mathcal{R} . We show that in the Sacks and Miller forcing extensions of L there is a Δ_2^1 maximal \mathcal{R} -discrete set. We use this to answer in the negative the main question posed in [7] by showing that in the Sacks and Miller extensions there is a Π_1^1 maximal orthogonal family (“mof”) of Borel probability measures on Cantor space. By contrast, we show that if there is a Mathias real over L then there are no Σ_2^1 mofs.

1. Introduction

(A) The present paper studies the definability of what in various contexts is called either *independent sets*, *orthogonal families*, or *antichains*. To capture these notions at once, we adopt the nomenclature of [21] and make the following definition:

Definition 1.1. Let \mathcal{R} be a binary relation on a set X . A set $A \subseteq X$ is called *\mathcal{R} -discrete* iff

$$(\forall x, y \in A) x \neq y \implies \neg(x \mathcal{R} y).$$

By a *maximal \mathcal{R} -discrete set* we mean an \mathcal{R} -discrete set which is maximal under inclusion among \mathcal{R} -discrete sets.

The above definition is familiar in the context of graphs, i.e., symmetric irreflexive relations, where discrete sets are often also called *independent sets*.

Another situation in which maximal discrete sets are of interest is when \mathcal{R} is a *compatibility relation*, i.e., when \mathcal{R} is symmetric and reflexive. Such relations often arise from a preorder: if \preceq is a preorder, then the associated

compatibility relation \mathcal{R}_{\preceq} is defined by

$$x \mathcal{R}_{\preceq} y \iff (\exists z)z \preceq x \wedge z \preceq y.$$

In this context, an \mathcal{R}_{\preceq} -discrete set is often called an *antichain* for \preceq .

A straight-forward transfinite induction shows that maximal discrete sets always exist for any binary relation \mathcal{R} . However, the *definability* of such maximal discrete sets may be contentious. In Gödel’s constructible universe L any Σ_1^1 (i.e., *effectively* analytic) binary relation admits a Δ_2^1 maximal discrete set, a fact that follows routinely from the existence of a Δ_2^1 wellordering of the reals in L of order type ω_1 with a good coding of initial segments. On the other hand, if we let Γ be the F_σ (in fact Σ_2^0) graph on 2^ω where $x \Gamma y$ iff x and y differ on exactly one bit, then a routine Baire category argument shows that Γ admits no Baire measurable maximal discrete set, and so by [13, Theorem 0.10] there is no Δ_2^1 maximal Γ -discrete set if there is a Cohen real over L . The situation is parallel with random reals.

The first goal of this paper is to show that the above failure does not persist in all forcing extensions of L with new reals.

Theorem 1.2. *Let \mathcal{R} be a Σ_1^1 binary relation on an effectively presented Polish space, and let x be a Sacks or Miller real over L . Then there is a Δ_2^1 maximal \mathcal{R} -discrete set in $L[x]$.*

A suitably relativized version of the previous theorem applies more generally to \mathcal{R} which are $\Sigma_1^1[a]$ for some real parameter a .

(B) Our main application of Theorem 1.2 is to the compatibility relation that comes from *absolute continuity* of Borel probability measures. Recall that if μ and ν are (non-trivial) measures on a measurable space X , then we write $\mu \ll \nu$ just in case every set which is null for ν is also null for μ . Two measures μ and ν that are not compatible in \ll (i.e., there is no measure \ll -below both of them) are called *orthogonal*, written $\mu \perp \nu$. By the Lebesgue decomposition theorem, for Borel probability measures this is equivalent to that there exists a Borel set $A \subseteq X$ such that $\nu(A) = 1$ and $\mu(A) = 0$. Also recall that $\mu \sim \nu$ means $\mu \ll \nu \wedge \nu \ll \mu$, i.e., μ and ν have the same null sets.

Orthogonal families of measures in the Polish space $P(X)$ of Borel probability measures on a Polish space X (see [17, Theorem 17.23, p.127]) show up in many different contexts, including representation theory, ergodic theory and operator algebras, see e.g. [25, 27]. Interest in the definability of

maximal orthogonal families of measures (abbreviated *mofs*) can be traced back to the following question posed by Mauldin: If X is a perfect Polish space, is there an *analytic* maximal orthogonal family in $P(X)$? The answer turns out to be no, as shown by Preiss and Rataj [23]. A new proof of this fact was provided by Kechris and Sofronidis [18] based on Hjorth’s turbulence theory. Later, it was shown by Fischer and the second author [7] that if $V = L$ then there is a Π_1^1 (lightface) mof in $P(2^\omega)$. On the other hand, [7] and [5] established that if there is a Cohen or random real over L there are no Π_1^1 mofs.

The seemingly restrictive nature of Π_1^1 mofs motivated the following question in [7]: *If there is a Π_1^1 mof in $P(2^\omega)$, must all reals be constructible?* Further compounding the intrigue, we will see in §2 below that no mof remains maximal after adding a real, eliminating any hope of constructing ‘indestructible’ mofs. Nevertheless, in this paper we will answer the above question in the negative by showing:

Theorem 1.3. *If x is a Sacks or Miller real over L , then in $L[x]$ there is a Π_1^1 mof in $P(2^\omega)$.*

A counterpoint to this is obtained in the last section, where the following is shown:

Theorem 1.4. *There are no Π_1^1 mofs in the Mathias extension of L .*

As another application of Theorem 1.2 we obtain a different proof of the following known result concerning maximal almost disjoint families (“mad” families) of subsets of ω . Such families are precisely the discrete sets for the compatibility relation that arises from the preorder \subseteq^* , inclusion modulo a finite set, in $[\omega]^\omega$. The study of the definability of mad families has a long history, see e.g. [4, 6, 8, 20, 29]. From Theorem 1.2 and [28] we get:

Fact 1.5. *If x is a Sacks or Miller real over L , then in $L[x]$ there is a Π_1^1 infinite mad family in $[\omega]^\omega$.*

We want to thank the anonymous referee for pointing out that a stronger result is already established by well-known methods: In any model of CH and for any reasonable forcing notion \mathbb{P} which does not add dominating reals one can construct an infinite \mathbb{P} -*indestructible* mad family (see, e.g., [3, 12]). In L , one can check that one can further demand that the indestructible family be Π_1^1 . Fact 1.5 follows, letting \mathbb{P} be Sacks or Miller forcing.

(C) The paper is organized as follows: In §2 we show there can be no indestructible $\Sigma_2^1[a]$ mof in $L[a]$. In §3, after we review some well-known facts about Sacks and Miller forcing, we prove a slightly more general version of Theorem 1.2. We also list some general properties of forcings which allow the proof to go through. In §4, we apply this to mofs and show that if there is a Σ_2^1 mof, there is a Π_1^1 mof. §5 presents an argument that a Mathias real over $L[a]$ rules out the existence of a $\Sigma_2^1[a]$ mof. Using the same ideas, we sketch a new proof that there is no analytic mof. We close in §6 with open questions.

2. There is no indestructible mof

In this brief section we prove that there is no hope of finding a mof that survives in an outer model which has new reals; in particular, there is no mof which is indestructible by any forcing adding a real.

A key property of \ll that we use is the so-called *ccc-below* property: If $\mu \in P(X)$, then any orthogonal family of measures \mathcal{F} such that $\nu \ll \mu$ for all $\nu \in \mathcal{F}$ must be countable. In fact, the argument only uses the ccc-below property and the simple definability of \ll , and that there is a perfect pairwise incompatible set; so it holds for a wide class of compatibility relations. We would like to thank the referee for suggesting the present form of the following theorem and its proof (these were also independently suggested by Benjamin D. Miller).

Theorem 2.1. *Let $V \subseteq W$ be transitive models of (enough of) ZFC. Assume that $\mathcal{A} \in V$ is an orthogonal family in $P(2^\omega)$ and W contains a new real, i.e., $(2^\omega)^W \setminus (2^\omega)^V \neq \emptyset$. Then \mathcal{A} is not maximal in W . In particular, if there is a mof \mathcal{A} such that $\mathcal{A} \in L$, then $\mathcal{P}(\omega) \subseteq L$.*

Proof. For the following, we view $P(2^\omega)$ as an effectively presented Polish space in precisely the manner described in [7]. From the product measure construction in [18, p. 1463] it follows easily that there is a Borel (in fact, Π_1^0) Cantor subset of $P(2^\omega)$ of pairwise orthogonal measures; let $Y \in V$ be such. Consider the predicate

$$R = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\ll \nu\}.$$

The ccc-below property of \ll implies that $R^{-1}(\nu) = \{\mu \in Y : \mu \not\ll \nu\}$ is countable for each $\nu \in P(2^\omega)$. Letting $\nu \in V$ and $\langle \mu_n : n \in \omega \rangle$ be the sequence in

V which enumerates $R^{-1}(\nu)$, observe that also

$$W \models R^{-1}(\nu) = \{\mu_n : n \in \omega\}$$

as the formula on the right is Π_2^1 (in fact Π_1^1 , since \perp is arithmetical) in a real coding $\langle \mu_n : n \in \omega \rangle$ and ν . In particular, $R^{-1}(\nu)^W \subseteq V$ for each $\nu \in V$. But since $(2^\omega)^W \setminus (2^\omega)^V \neq \emptyset$, W has a new branch ν' through Y and by the previous, $\nu' \perp \mu$ for every $\mu \in \mathcal{A}$. □

3. Definable maximal discrete sets in the Sacks or Miller extension

In this section, we prove Theorem 1.2 in the following, slightly stronger form:

Theorem 3.1. *Let $a \in \omega^\omega$. For any $\Sigma_1^1[a]$ binary relation \mathcal{R} on ω^ω there is a $\Delta_2^1[a]$ predicate which defines a maximal \mathcal{R} -discrete set in both $L[a]$ and $L[a][x]$, where x is any Sacks or Miller real over $L[a]$.*

The theorem applies to arbitrary effectively presented Polish spaces, since any two uncountable such spaces are Δ_1^1 isomorphic. The argument also applies more generally to *arboreal* forcing notions satisfying certain conditions, which we list in Theorem 3.11.

Following the anonymous referee’s suggestion, we mention the following version of Theorem 3.1, in which we allow an arbitrary model of CH (rather than a model of the form $L[a]$) as the ground model. The proof is exactly the same, minus the definability considerations. The reader can find similar results, e.g., in [2].

Theorem 3.2. *Assume CH holds in V , let x be a Sacks or Miller real over V , and let \mathcal{R} be a Σ_1^1 binary relation on ω^ω . Then there is a sequence $\langle T_\alpha : \alpha < \omega_1 \rangle \in V$ of trees on ω (i.e., of codes of closed subsets of ω^ω) such that $\bigcup_{\alpha < \omega_1} [T_\alpha]$ defines a maximal \mathcal{R} -discrete set in both V and $V[x]$.*

Before delving into the proof, we collect a few preliminaries about Sacks forcing \mathbb{S} and Miller forcing \mathbb{M} . Firstly, we need the following elementary fact about descriptive complexity calculations and the forcing relation. Let $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}\}$. In either case, we denote by \dot{x}_G the name for the generic real.

Fact 3.3. If $\varphi(x, y)$ is a Π_1^1 formula, the set

$$\{(p, a) \in \mathbb{P} \times \omega^\omega : p \Vdash_{\mathbb{P}} \varphi(\dot{x}_G, \check{a})\}$$

is Π_1^1 .

Proof. We treat the case \mathbb{S} in detail. Clearly, $p \Vdash_{\mathbb{S}} \varphi(\dot{x}_G, \check{a})$ if and only if the analytic set

$$(3.1) \quad A = \{x \in [p] : \neg\varphi(x, a)\}$$

is countable: If A is uncountable, then by the perfect set theorem there is a condition $q \leq p$ with $[q] \subseteq A$, and $q \Vdash_{\mathbb{S}} \neg\varphi(\dot{x}_G, \check{a})$ by Π_2^1 -absoluteness; if, on the other hand, A is countable, then $1_{\mathbb{S}} \Vdash \dot{x}_G \notin \check{A}$, and by Π_1^1 -absoluteness $p \Vdash (\forall x \in [\check{p}] \setminus \check{A}) \varphi(x, \check{a})$, proving the equivalence.

If A is countable, (the proof of) the effective perfect set theorem [22, Theorem 4F.1, p. 243] gives a sequence $(x_n)_{n \in \omega} \in L_{\omega_1^c}[a]$ such that

$$\{x \in 2^\omega : \neg\varphi(x, a)\} = \{x_n : n \in \omega\}.$$

Thus $p \Vdash_{\mathbb{S}} \varphi(\dot{x}_G, \check{a})$ if and only if

$$(\exists (x_n)_{n \in \omega} \in \Delta_1^1[a])(\forall x \in 2^\omega) [\neg\varphi(x, a) \implies (\exists n) x = x_n],$$

which is Π_1^1 by [19, Corollary 4.19, p. 53].

For \mathbb{M} , $p \Vdash_{\mathbb{M}} \varphi(\dot{x}_G, \check{a})$ precisely if $\{x \in 2^\omega : \neg\varphi(x, a)\}$ is contained in a K_σ set by [16] (or see [17, Corollary 21.23, p. 178]). The rest of the proof is analogous to the above. \square

Secondly, we will use the following well-known fact to give us a practical way of talking about names for reals. For the sake of completeness, we include a proof. For the rest of this section, let $\Omega = \omega$ when $\mathbb{P} = \mathbb{M}$, and $\Omega = 2$ when $\mathbb{P} = \mathbb{S}$.

Fact 3.4. Sacks and Miller forcing have *continuous reading of names for reals*: If $p \in \mathbb{P}$, \dot{x} is an \mathbb{P} -name and $p \Vdash \dot{x} \in \omega^\omega$, then there is $\eta : \Omega^\omega \rightarrow \omega^\omega$ continuous and $q \leq p$ such that $q \Vdash \eta(\dot{x}_G) = \dot{x}$.

Any continuous function $\eta : \Omega^\omega \rightarrow \omega^\omega$ arises from a monotone map between trees $\varphi : \Omega^{<\omega} \rightarrow \omega^{<\omega}$; in the notation of [17, Definition 2.5, p. 7], $\eta = \varphi^*$. So we can regard the countable object φ as a ‘code’ for η . To say that φ gives rise to a total function is Π_1^1 , whence absolute. We adopt the

convention that if η is coded by φ in this sense, i.e., $\eta = \varphi^*$, then after moving to a forcing extension we still use η to refer to the function coded by φ there, i.e., φ^* as interpreted in the current model. Without this convention the statement itself of Fact 3.4 makes little sense.

For the proof of Fact 3.4, we use the following terminology. Momentarily fix a tree $T \subseteq \Omega^{<\omega}$. Given $s \in \Omega^{<\omega}$, define $(T)_s = \{t \in T : s \subseteq t\}$. We call t an *immediate successor of s (in T)* if $s \subseteq t$, $t \in T$, and $\text{lh}(s) = \text{lh}(t) - 1$. Call $t \in T$ a *full splitting node (in T)* if the set of immediate successors of t in T has size Ω ; and for $n \in \omega \setminus \{0\}$ call t an *n -th full splitting node (in T)* if in addition $\{s \in T : s \subseteq t\}$ contains exactly n full splitting nodes. Finally, we write $(T)_{\leq n}^*$ for the set of immediate successors of n -th full splitting nodes.

Proof of Fact 3.4. For $n \in \omega$, the set D_n of $p \in \mathbb{P}$ which decide a value for $\dot{x}(\check{n})$ is dense and open. We construct q in a typical fusion argument: Let $p_0 = p$ and inductively find $p_{n+1} \leq p_n$ such that $(p_{n+1})_{\leq n+1}^* = (p_n)_{\leq n+1}^*$ and whenever $s \in (p_{n+1})_{\leq n+1}^*$, then $(p_{n+1})_s \in D_n$. Let q be the greatest lower bound in \mathbb{P} of the sequence p_0, p_1, \dots (it exists as for each k , $(p_n)_{\leq k+1}^*$ is eventually constant in n).

For each $n \in \omega$ there are sequences $\{s_0^n, s_1^n, \dots\} \subseteq \Omega^{<\omega}$ and $\{k_0^n, k_1^n, \dots\} \subseteq \omega$ such that $\{s_0^n, s_1^n, \dots\} = (q)_{n+1}^*$ and

$$q \Vdash \check{s}_i^n \subseteq \dot{x}_G \implies \dot{x}(\check{n}) = \check{k}_i^n.$$

For $r \in [q]$, defining $\eta(r)(n) = k_i^n$ whenever $s_i^n \subseteq r$, we have that $\eta: [q] \rightarrow \omega^\omega$ is continuous and $q \Vdash \eta(\dot{x}_G) = \dot{x}$. We can easily extend η to a continuous function defined on all of Ω^ω . □

Lastly, we need a Ramsey-theoretic statement, Corollary 3.6 below, which in the Sacks case follows from the following theorem due to Galvin (see [17, Theorem 19.7, p. 145]).

Theorem 3.5 (Galvin’s Ramsey theorem for Polish spaces). *Let Y be a perfect Polish space, and suppose*

$$[Y]^2 = P_0 \cup P_1$$

is a partition of $[Y]^2$ into Baire measurable pieces. Then there is $C \subseteq Y$ perfect and $i \in \{0, 1\}$ such that $[C]^2 \subseteq P_i$.

Given a binary relation \mathcal{R} , a set A is called \mathcal{R} -complete iff

$$(\forall x, y \in A) x \neq y \implies x \mathcal{R} y.$$

Corollary 3.6. *Let \mathcal{R} be an analytic relation on a standard Borel space X , let $\eta : \Omega^\omega \rightarrow X$ be a Borel function and let $p \in \mathbb{P}$. Then there is $q \in \mathbb{P}$ such that $q \leq p$ and either $\eta([q])$ is \mathcal{R} -complete, or $\eta([q])$ is \mathcal{R} -discrete.*

Proof. For $\mathbb{P} = \mathbb{S}$, assume \mathcal{R} is symmetric and let $Y = [p]$ and

$$P_0 = \{\{x, y\} \in [Y]^2 : \eta(x) \mathcal{R} \eta(y)\},$$

$P_1 = [Y]^2 \setminus P_0$. Let C be given by Galvin’s theorem and pick $q \in \mathbb{P}$ such that $q \leq p$ and $[q] = C$. If $[C]^2 \subseteq P_1$ then $\eta(C)$ is \mathcal{R} -discrete and if $[C]^2 \subseteq P_0$ then $\eta(C)$ is \mathcal{R} -complete.

For $\mathbb{P} = \mathbb{M}$, as was pointed out by Stevo Todorčević, we may use [26, Corollary 5.68, p. 121] instead of Theorem 3.5 to derive the result. \square

Remark 3.7. In the previous corollary, it is perfectly acceptable that η is constant, say. In that case $\eta(2^\omega)$ is both \mathcal{R} -complete and \mathcal{R} -discrete at the same time.

Definition 3.8. For \mathcal{R} and η as in Corollary 3.6, call $q \in \mathbb{P}$ a *Galvin witness* for η (and \mathcal{R}) if $\eta([q])$ is either \mathcal{R} -complete or $\eta([q])$ is \mathcal{R} -discrete.

Note that $\eta([q])$ being \mathcal{R} -discrete is $\Pi_1^1[a]$ uniformly in η and q , and $\eta([q])$ being \mathcal{R} -complete is $\Pi_2^1[a]$. In particular both are absolute for class models by Levy-Shoenfield, and thus so is the property of being a Galvin witness.

Now we are ready to prove the main theorem of this section:

Proof of Theorem 3.1. Let $\mathcal{R} \subseteq (\omega^\omega)^2$ be $\Sigma_1^1[a]$; the proof relativizes easily to the parameter a , so we suppress it below.

It suffices to produce a Σ_2^1 formula φ which defines a maximal \mathcal{R} -discrete set in any \mathbb{P} -generic extension of L , since if \mathcal{A} is Σ_2^1 and maximal \mathcal{R} -discrete set, then \mathcal{A} is in fact Δ_2^1 , since

$$x \notin \mathcal{A} \iff (\exists y \in \mathcal{A}) x \mathcal{R} y \wedge x \neq y.$$

Below we identify $\mathbb{P} \times C(\Omega^\omega, \omega^\omega)$ with a Π_1^1 subset of ω^ω , by identifying both $C(\Omega^\omega, \omega^\omega)$ and \mathbb{P} with subsets of ω^ω (see the remark after Fact 3.4) and identifying ω^ω and $(\omega^\omega)^2$ via some fixed effective bijection¹.

¹For the case $\Omega = 2$, we could alternatively use that the set of continuous functions $C(2^\omega, \omega^\omega)$ has an effective presentation as a Polish metric space, and so we can regard it as a Π_2^0 subset of 2^ω .

Working in L , fix an enumeration $\langle (p_\xi, \eta_\xi) : \xi < \omega_1 \rangle$ of $\mathbb{P} \times C(\Omega^\omega, \omega^\omega)$ such that

$$\xi < \delta \implies (p_\xi, \eta_\xi) <_L (p_\delta, \eta_\delta).$$

By recursion on ω_1 , we will define a sequence $\langle q_\xi : \xi < \omega_1 \rangle$, such that the following are satisfied:

- (i) q_ξ is a (not necessarily perfect) subtree of p_ξ .
- (ii) If p_ξ does not force

$$(3.2) \quad (\forall \delta < \xi)(\forall y \in [q_\delta]) \eta_\xi(\dot{x}_G) \mathcal{R} \eta_\delta(y)$$

then $q_\xi = \emptyset$.

- (iii) If p_ξ does force (3.2) and $q \in \mathbb{P}$ is \leq_L -least such that $q \leq p_\xi$ and

$$(3.3) \quad (\forall x \in [q])(\forall \delta < \xi)(\forall y \in [q_\delta]) \eta_\xi(x) \mathcal{R} \eta_\delta(y),$$

then

- (iii.a) if $\eta_\xi([q])$ is \mathcal{R} -discrete, $q_\xi = q$;
- (iii.b) if $\eta_\xi([q])$ is *not* \mathcal{R} -discrete, $q_\xi \subseteq q$ is the pruned subtree whose unique branch is the left-most branch of q .

- (iv) The set $\mathcal{A}^0 = \{(q_\xi, \eta_\xi) : \xi < \omega_1\}$ is Σ_2^1 .

Above it is implicit in (iii) that q exists; this follows since in case (iii) the analytic set

$$\{x \in [p_\xi] : (\exists \delta < \xi)(\exists y \in [q_\delta]) \eta_\xi(x) \mathcal{R} \eta_\delta(y)\}$$

must be countable since p_ξ forces (3.2) and by the proof of Fact 3.3.

Suppose for now that $\langle q_\xi : \xi < \omega_1 \rangle$ satisfies (i)–(iv) above. Then let $\varphi(y)$ be the Σ_2^1 formula

$$(\exists q, \eta)(\exists x \in [q])(q, \eta) \in \mathcal{A}^0 \wedge y = \eta(x).$$

Clause (iii) ensures that φ defines an \mathcal{R} -discrete set in any model. For maximality, suppose, seeking a contradiction, that

$$p \Vdash (\exists x \in \omega^\omega)(\neg \varphi(x) \wedge (\forall y)(\varphi(y) \implies x \mathcal{R} y)).$$

By Fact 3.4 we may assume (replacing p with a stronger condition if necessary) that there is a total continuous function $\eta : \Omega^\omega \rightarrow \omega^\omega$ such that

$$(3.4) \quad p \Vdash \neg \varphi(\eta(\dot{x}_G)) \wedge (\forall y \in \Omega^\omega)(\varphi(y) \implies \eta(\dot{x}_G) \mathcal{R} y).$$

By Corollary 3.6, we may assume that p is a Galvin witness for η , since otherwise, we can again replace p by a stronger condition.

Let $\xi < \omega_1$ be such that $(p, \eta) = (p_\xi, \eta_\xi)$. Then p_ξ forces (3.2) and so clause (iii) applies. Let $q \leq p_\xi$ be as in clause (iii).

If $\eta([q])$ is \mathcal{R} -discrete then $q_\xi = q$. Since $q \Vdash \dot{x}_G \in [q]$ it follows that $q \Vdash \varphi(\eta(\dot{x}_G))$, contradicting that $p \Vdash \neg\varphi(\eta(\dot{x}_G))$.

So it must be that $\eta([q])$ is not \mathcal{R} -discrete, and so clause (iii.b) applies. Since p is a Galvin witness for η , it follows that $\eta([p])$ is \mathcal{R} -complete. Let $z \in [q_\xi]$ be the unique branch through q_ξ . Then

$$(\forall x \in [p]) \eta(x) = \eta(z) \vee \eta(x) \mathcal{R} \eta(z),$$

and since this is $\Pi_2^1[z]$, it follows by Shoenfield's absoluteness theorem that $p \Vdash \eta(\dot{x}_G) = \eta(\dot{z}) \vee \eta(\dot{x}_G) \mathcal{R} \eta(\dot{z})$, contradicting (3.4).

It is routine that $\langle q_\xi : \xi < \omega_1 \rangle$ satisfying (i)–(iii) above can be found. In fact (ii) and (iii) determine the sequence $\langle q_\xi : \xi < \omega_1 \rangle$ uniquely. So proving the following claim will finish the proof.

Claim 3.9. *The set $\mathcal{A}^0 = \{(q_\xi, \eta_\xi) : \xi < \omega_1\}$ is Σ_2^1 .*

Proof of Claim. The set of sequences $\vec{x} = \langle (p_n^*, \eta_n^*) : n \in \alpha \rangle$, for $\alpha \leq \omega$, such that

$$(3.5) \quad (\exists \xi < \omega_1) \{(p_n^*, \eta_n^*) : n \in \alpha\} = \{(p_\delta, \eta_\delta) : \delta < \xi\}$$

is Σ_2^1 ; since \leq_L is a strongly Δ_2^1 well-ordering of ω^ω (see [10] or [30]) and $\mathbb{P} \times C(\Omega^\omega, \omega^\omega)$ was identified with a Π_1^1 subset of ω^ω , this follows easily by the proof of [22, Exercise 5A.1, p. 287]. Let $\Psi(\vec{x})$ be a Σ_2^1 formula equivalent to (3.5).

Now assume $\Psi(\vec{x})$ holds and suppose $\vec{y} = \langle q_n^* : n \in \alpha \rangle$ enumerates the sequence $\{q_\delta : \delta < \xi\}$ in such a way that for all $\delta < \xi$ and $n \in \alpha$ we have $p_n^* = p_\delta \iff q_n^* = q_\delta$.

Observe that (ii) and (iii) are Σ_2^1 uniformly in $(\vec{x}, \vec{y}, p_\xi, \eta_\xi, q_\xi)$: (ii) is Π_1^1 in these parameters by 3.3. The property expressed in (3.3) is Π_1^1 in $(\vec{x}, \vec{y}, \eta_\xi, q)$. Thus, that q be minimal with that property is Σ_2^1 in these parameters, as \leq_L is strongly Δ_2^1 (by ‘closure under bounded quantification’). Clauses (iii.a) and (iii.b) are Boolean combinations of Σ_1^1 formulas in the parameters (η_ξ, q_ξ, q) by the remarks following Definition 3.8. Thus, (iii) is easily seen to be Σ_2^1 in $(\vec{x}, \vec{y}, p_\xi, \eta_\xi, q_\xi)$.

So we may express the conjunction of (ii) and (iii) by a Σ_2^1 formula

$$\Theta(q, p_\xi, \eta_\xi, \vec{x}, \vec{y}),$$

i.e., for \vec{x} and \vec{y} as above, $\Theta(q, p_\xi, \eta_\xi, \vec{x}, \vec{y})$ holds if and only if $q = q_\xi$.

Thus $\vec{y} = \langle q_n^* : n \in \alpha \rangle$, with $\alpha \leq \omega$, enumerates an initial segment of $\langle q_\xi : \xi < \omega_1 \rangle$ exactly if the following formula holds:

$$(3.6) \quad (\exists \vec{x}) \Psi(\vec{x}) \wedge \text{dom}(\vec{x}) = \text{dom}(\vec{y}) \wedge [(\forall n < \text{dom}(\vec{x})) \Theta(q_n^*, p_n^*, \eta_n^*, \vec{x} \upharpoonright n, \vec{y} \upharpoonright n)]$$

where we abuse notation by writing $\vec{x} \upharpoonright n$ for

$$\langle (p_m^*, \eta_m^*) : m < \text{dom}(\vec{x}), (p_m^*, \eta_m^*) \leq_L (p_n^*, \eta_n^*) \rangle$$

(similarly for $\vec{y} \upharpoonright n$). The formula (3.6) is easily seen to be equivalent to a Σ_2^1 formula; it follows that \mathcal{A}^0 is Σ_2^1 . Claim. \dashv

□

We also get the following effective corollary for Σ_1^1 relations:

Corollary 3.10. *Under the hypothesis of Theorem 3.1, if $a' \in L[x]$ and \mathcal{R} is a $\Sigma_1^1[a']$ binary relation on a effectively presented Polish space X , then in $L[x]$ there is a $\Delta_2^1[a']$ maximal \mathcal{R} -discrete set in X .*

Proof. We may assume $X = \omega^\omega$. If $a' \in L$, then Theorem 3.1 gives a $\Delta_2^1[a']$ formula φ defining a maximal \mathcal{R} -discrete set in $L[x]$. If $a' \notin L$ then $L[x] = L[a']$ (as x is of minimal degree, see [9]), and Theorem 3.1 provides a $\Delta_2^1[a']$ formula φ defining a maximal \mathcal{R} -discrete set in $L[a']$ —and, incidentally, its Sacks (resp. Miller) extension—starting from the strongly $\Delta_2^1[a']$ well-ordering of $L[a']$. □

It's simple to axiomatize a class of forcings for which the above proof goes through. A forcing \mathbb{P} is called *strongly arboreal* [14] if and only if its conditions are perfect trees on Ω where $\Omega \subseteq \omega$, ordered by inclusion, and for any $p \in \mathbb{P}$ and $s \in p$ also $(p)_s \in \mathbb{P}$. Any extension of V by a (\mathbb{P}, V) -generic filter G is generated by the single 'generic' real $\bigcap_{p \in G} [p]$; its name we denote by $\dot{x}_G^\mathbb{P}$. A real is called (\mathbb{P}, V) -generic over if and only if it arises in this way. For example, Sacks, Miller, Mathias and Laver are (equivalent to) strongly arboreal forcings (see e.g. [1] and [9]).

Theorem 3.11. *Let \mathbb{P} be a strongly arboreal forcing such that:*

- (A) \mathbb{P} has Borel reading of names (in the sense of [31, Proposition 2.3.1, p. 29]).
- (B) If $\varphi(x, y)$ is a Π_1^1 predicate then $\{(p, a) \in \mathbb{P} \times \omega^\omega : p \Vdash \varphi(\dot{x}_G^{\mathbb{P}}, \check{a})\}$ is Δ_2^1 .
- (C) The analogue of Galvin's theorem holds for \mathbb{P} : For \mathcal{R} as in Theorem 3.1, a Borel function $\eta: \Omega^\omega \rightarrow \omega^\omega$ and $p \in \mathbb{P}$, there is $q \in \mathbb{P}$, $q \leq p$ such that q is a Galvin witness for \mathcal{R} and η .

Then the analogue of Theorem 3.1 holds when x is a $(\mathbb{P}, L[a])$ -generic real.

We mention without proof that (B) can be replaced by: for all countable transitive M and $p \in \mathbb{P} \cap M$ there is $q \leq p$ s.t. any $r \in [q]$ is a (\mathbb{P}, M) -generic.

Proof of Theorem 3.11. One difference to the proof of Theorem 3.1 is how we obtain the enumeration $\langle p_\xi, \eta_\xi : \xi < \omega_1 \rangle$ at the beginning. The second coordinate now has to enumerate all codes for total Borel functions; the set of such codes is Π_1^1 (to see this, observe that if f is $\Delta_1^1[a]$ then $f(x)$ is $\Delta_1^1[x, a]$; now use [19, Corollary 4.19, p. 53]), so the proof goes through as before. It remains to notice that when even just requiring (B), clauses (ii) and (iii) remain Σ_2^1 . \square

4. A co-analytic mof in the Miller and Sacks extensions

Theorem 4.1. *If x is a Miller or Sacks real over $L[a]$, then*

$$L[a][x] \models \text{“there is a } \Pi_1^1[a] \text{ mof in } P(2^\omega)\text{”}.$$

This follows immediately from Theorem 1.2 together with the following lemma.

Lemma 4.2. *If there is a $\Sigma_2^1[a]$ mof in $P(2^\omega)$, then there is a $\Pi_1^1[a]$ mof.*

Proof. We suppress the parameter a below.

The proof is based on a slight simplification of the coding method from [7]. Let $P_c(2^\omega)$ denote the set of *atomless* Borel probability measures on 2^ω , i.e., $\mu \in P(2^\omega)$ such that $\mu(\{x\}) = 0$ for any x . This set is Π_2^0 as a subset of $P(2^\omega)$, see [7, Lemma 2.1]. Given $\mu \in P_c(2^\omega)$, let y be the left-most branch of the tree

$$\{t \in 2^{<\omega} : \mu(N_t) > 0\},$$

where $N_s = \{x \in 2^\omega : s \subseteq x\}$ is the basic open neighborhood determined by $s \in \omega^n$.

For $\mu \in P_c(2^\omega)$ given, let $n(0), n(1), \dots$ enumerate the infinite set of n such that $\mu(N_{y \upharpoonright n \smallfrown 0}) > 0$ and $\mu(N_{y \upharpoonright n \smallfrown 1}) > 0$ (here we use that μ is atomless) and define $G(\mu) \in 2^\omega$ by

$$G(\mu)(i) = \begin{cases} 0 & \text{if } \mu(N_{y \upharpoonright n(i) \smallfrown 0}) \geq \mu(N_{y \upharpoonright n(i) \smallfrown 1}) \\ 1 & \text{otherwise.} \end{cases}$$

We say $G(\mu)$ is “coded” by μ . As in [7], we can find a Δ_1^1 coding function $F: P_c(2^\omega) \times 2^\omega \rightarrow P_c(2^\omega)$ such that for all $\mu \in P_c(2^\omega)$ and $y \in 2^\omega$, $F(\mu, y)$ is absolutely equivalent to μ and codes y , that is, $G(F(\mu, y)) = y$.

Now let \mathcal{A} be a Σ_2^1 mof in $P(2^\omega)$. By subtracting from each measure in \mathcal{A} a measure concentrating on a countable set and re-normalizing the remaining non-zero measures, we may assume $\mathcal{A} \subseteq P_c(2^\omega)$ (here we use the ccc-below property) and that \mathcal{A} is maximal among orthogonal families of atomless measures. Let R be Π_1^1 such that $\mu \in \mathcal{A} \iff (\exists y) R(\mu, y)$. By Π_1^1 uniformization, we can assume R is a functional relation, i.e.

$$(\forall x \in \text{dom}(R)) (\exists! y) R(x, y).$$

Fix a Σ_1^0 bijection $(x, y) \mapsto x \oplus y$ from $(2^\omega)^2$ to 2^ω , and let $x \mapsto (x)_i$, for $i \in \{0, 1\}$ be the pair of maps such that for all $z \in 2^\omega$, $z = (z)_0 \oplus (z)_1$ (i.e., the components of the inverse of our bijection). Let $g: 2^\omega \rightarrow P_c(2^\omega)$ be a Δ_1^1 bijection.

Define $\mathcal{A}' \subseteq P_c(2^\omega)$ by letting $\mu' \in \mathcal{A}'$ just in case $\mu' \in P_c(2^\omega)$ and

$$(4.1) \quad (\forall z, \mu, y) [z = G(\mu') \wedge \mu = g((z)_0) \wedge y = (z)_1] \implies [\mu' = F(\mu, z) \wedge R(\mu, y)].$$

Clearly \mathcal{A}' is Π_1^1 and moreover \mathcal{A}' is an orthogonal family of atomless measures which is maximal among the atomless measures since

$$\mathcal{A}' = \{F(\mu, g^{-1}(\mu) \oplus y) : \mu \in \mathcal{A}, y \in 2^\omega, R(\mu, y)\}.$$

By enlarging \mathcal{A}' to contain all Dirac measures (i.e., probability measures concentrating on a single point), we obtain a Π_1^1 mof. □

5. No co-analytic mofs in the Mathias extension

The purpose of this section is to complement Theorem 4.1 by showing that its conclusion fails when r is a Mathias real over L .

Theorem 5.1. *If all $\Sigma_2^1[a]$ (equivalently, all $\Delta_2^1[a]$) sets of reals are Ramsey there is no $\Sigma_2^1[a]$ mof. In particular, if r is a Mathias real over $L[a]$, then*

$$L[a][r] \models \text{“there is no } \Sigma_2^1[a] \text{ mof”}.$$

In the process of proving Theorem 5.1 we will also obtain a new proof that there are no analytic mofs, see the end of this section.

The proof of Theorem 5.1 requires multiple steps. We start by defining a way of assigning to each $x \in [\omega]^\omega$ a product measure on 2^ω . Let $f_x : \omega \rightarrow \omega$ denote the unique increasing function such that $x = f_x[\omega]$. Then define a sequence $\alpha^x \in [\frac{1}{4}, \frac{3}{4}]^\omega$ by

$$\alpha_n^x = \begin{cases} \frac{1}{4} + \frac{1}{2\sqrt{f_x^{-1}(n)+1}} & n \in x, \\ \frac{1}{4} & n \notin x, \end{cases}$$

and define $\mu^x \in P(2^\omega)$ by

$$\mu^x = \prod_{n \in \omega} (\alpha_n^x \delta_0 + (1 - \alpha_n^x) \delta_1),$$

where $\delta_i \in P(\{0, 1\})$ is the Dirac measure concentrated at $i \in \{0, 1\}$. For $x, y \in [\omega]^\omega$, let

$$\rho(x, y) = \sum_{n \in \omega} (\alpha_n^x - \alpha_n^y)^2.$$

Note that if $z \subseteq y \subseteq x$ then $\rho(y, x) \leq \rho(z, x)$.

The intention behind the definition of α^x is to be able to use Kakutani’s theorem on equivalence and orthogonality of product measures. Specifically, [15, Corollary 1, p. 222] gives

$$(5.1) \quad \mu^x \sim \mu^y \iff \rho(x, y) < \infty$$

and

$$\mu^x \perp \mu^y \iff \rho(x, y) = \infty.$$

Let then

$$\mathcal{F} = \{g \in [\omega]^\omega : \rho(g, \omega) < \infty\},$$

and define a binary operation \cdot on $[\omega]^\omega$ by

$$x \cdot y = f_y \circ f_x[\omega].$$

We can think of the operation $x \cdot y$ as follows: f_y identifies ω and y , and $x \cdot y$ is the copy of x inside of y under this identification.

Proposition 5.2.

(i) For all $x, y, z \in [\omega]^\omega$

$$\sqrt{\rho(x, y)} \leq \sqrt{\rho(x, z)} + \sqrt{\rho(z, y)}$$

holds, even if $\rho(x, y)$ is infinite. In particular, $\sqrt{\rho}$ is finite on \mathcal{F} and defines a complete metric on \mathcal{F} , inducing a Polish topology on \mathcal{F} . In this topology, \mathcal{F} is a perfect Polish space.

(ii) The operation \cdot is associative and makes $[\omega]^\omega$ a monoid with the unit being ω .

(iii) For all $g \in \mathcal{F}$ and $x \in [\omega]^\omega$ we have $\rho(g \cdot x, x) = \rho(g, \omega)$ and $\mu^{g \cdot x} \sim \mu^x$. It follows that \mathcal{F} is closed under the operation \cdot , and so is a monoid with unit ω .

(iv) If $|x \Delta y| < \infty$ then $\rho(x, y) < \infty$.

(v) For any $g \in \mathcal{F}$ and $k \in \omega$, it holds that $g \cdot (\omega \setminus \{k\}) \in \mathcal{F}$, and

$$\lim_{k \rightarrow \infty} \rho(g \cdot (\omega \setminus \{k\}), g) = 0.$$

(vi) For any $x \in [\omega]^\omega$, the equivalence relation \sim^x , defined in $[x]^\omega$ by $z \sim^x y \iff \mu^z \sim \mu^y$, has meagre, dense classes in the Polish topology on $[x]^\omega$.

Proof. (i) Note that $\sqrt{\rho(x, y)} = \|\alpha^x - \alpha^y\|_2$, where $\|\cdot\|_2$ is the 2-norm. The norm inequality

$$\|\alpha^x - \alpha^y\|_2 \leq \|\alpha^x - \alpha^z\|_2 + \|\alpha^z - \alpha^y\|_2$$

holds in the strong sense that if the left hand side is infinite, then so is the right hand side (use that $\ell_2(\omega)$ is closed under addition). This establishes the inequality in (i). The map $g \mapsto \alpha^g - \alpha^\omega$ is then an isometric embedding of \mathcal{F} into the Hilbert space $\ell_2(\omega)$. It is straight-forward to check that the image under $g \mapsto \alpha^g - \alpha^\omega$ is closed in $\ell_2(\omega)$. Finally, (v) implies that \mathcal{F} has no isolated points.

(ii) follows immediately from the definitions. (iii) is follows easily from (i), eq. (5.1), and the definition of ρ . For (iv) and (v) we need:

Claim. $\omega \setminus k \in \mathcal{F}$ for any $k \in \omega$.

Proof of Claim.

$$\sum_{n \geq k} (\alpha_n^\omega - \alpha_n^{\omega \setminus k})^2 = \sum_{n \geq k} \left(\frac{1}{2\sqrt{n+1}} - \frac{1}{2\sqrt{n+1-k}} \right)^2,$$

and the right hand side above converges since, using a small amount of calculus, we have $(n - k)^{-\frac{1}{2}} - n^{-\frac{1}{2}} \leq n^{-\frac{3}{2}}$ for $n > k$ sufficiently large. Claim.1

(iv) Suppose $|x \Delta y| < \infty$ and let $z = x \cup y$. Let k be such that $z \setminus k \subseteq x \cap y$ and $k_0 = |z \cap k|$. Then

$$\rho(z, x), \rho(z, y) \leq \rho(z, x \cap y) \leq \rho(z, z \setminus k) = \rho(z, (\omega \setminus k_0) \cdot z) < \infty,$$

with the last inequality following from the previous claim and (iii).

(v) Since $\omega \setminus \{k\} \supseteq \omega \setminus (k + 1)$ the claim gives $\omega \setminus \{k\} \in \mathcal{F}$ for all $k \in \omega$. Now

$$\rho(g \cdot (\omega \setminus \{k\}), g) = \sum_{n \geq k} (\alpha_n^{g \cdot (\omega \setminus \{k\})} - \alpha_n^g)^2 = \sum_{n \geq k} (\alpha_n^{g \cdot (\omega \setminus \{0\})} - \alpha_n^g)^2,$$

and the last sum tends to 0 as $k \rightarrow \infty$ since $\rho(g \cdot (\omega \setminus \{0\}), g) < \infty$.

(vi) By eq. (5.1) the relation \sim^x is F_σ . (iv) immediately gives that all \sim^x classes are dense in $[x]^\omega$. On the other hand, it is easy to check that if $y \in [x]^\omega$ and the complement of $f_x^{-1}(y)$ is not in the summable ideal, then $\rho(y, x) = \infty$. So \sim^x has at least two (necessarily dense) classes. It follows that the complement of any \sim^x class is a dense G_δ set in $[x]^\omega$, whence each \sim^x class is meagre in $[x]^\omega$. \square

Remark 5.3. (v) in the previous proposition intends to say that multiplication on the right in the monoid \mathcal{F} has at least *some* amount of continuity at ω (the identity). By contrast, (iii) shows that left multiplication in \mathcal{F} is continuous at ω . We do not know if right multiplication is actually continuous at the identity, but it seems unlikely.

Let X be a Polish space. Recall that the equivalence relation F_2^X on X^ω is defined by

$$\vec{x} F_2^X \vec{y} \iff \{\vec{x}_n : n \in \omega\} = \{\vec{y}_n : n \in \omega\}.$$

Lemma 5.4. *Let $\vartheta : [\omega]^\omega \rightarrow X^\omega$ be a continuous function (w.r.t. the Polish topologies), and suppose ϑ is (\mathcal{F}, F_2^X) -equivariant, that is,*

$$(\forall g \in \mathcal{F})(\forall x \in [\omega]^\omega) \vartheta(g \cdot x) F_2^X \vartheta(x).$$

Then there is a non-empty open set U_0 such that $\vartheta_0 \upharpoonright U_0$ is constant, where, in general, ϑ_l is defined by $\vartheta_l(x) = \vartheta(x)_l$ for $l \in \omega$.

Proof (à la Hjorth). For the purpose of this proof, we identify $[\omega]^\omega$ with a G_δ subset of 2^ω in the natural way. Define for each $l \in \omega$ a closed set

$$A_l = \{(g, x) \in \mathcal{F} \times [\omega]^\omega : \vartheta_l(g \cdot x) = \vartheta_0(x)\},$$

and let l_0 be least such that A_{l_0} is non-meagre; this exists because $\mathcal{F} \times [\omega]^\omega = \bigcup_{l \in \omega} A_l$. Let $V \times U \subseteq A_{l_0}$ be open and non-empty, and fix $g_0 \in V$. Using (v) of Proposition 5.2, find k_0 such that

$$(\forall k > k_0) g_0 \cdot (\omega \setminus \{k\}) \in V.$$

Let $s_0 \in 2^{<\omega}$ be such that $N_{s_0} \subseteq U$. By either making s_0 longer or k_0 larger, we may assume that s_0 is the characteristic function of a set with k_0 elements.

Claim. If $y, z \in N_{s_0}$ differ on only one bit then $\vartheta_0(y) = \vartheta_0(z)$.

Proof of Claim. Suppose $n \in y$ and $n \notin z$. Let $k = f_y^{-1}(n)$, and note that $z = (\omega \setminus \{k\}) \cdot y$ and $k > k_0$. Since $g_0 \cdot (\omega \setminus \{k\}) \in V$ we have

$$\vartheta_0(z) = \vartheta_0(\omega \setminus \{k\} \cdot y) = \vartheta_{l_0}(g_0 \cdot (\omega \setminus \{k\}) \cdot y) = \vartheta_0(y).$$

Claim. \dashv

Now $U_0 = N_{s_0}$ works, since the claim implies that the continuous function ϑ_0 is constant on a dense set in N_{s_0} . □

Proof of Theorem 5.1. Assume that every Σ_2^1 set is completely Ramsey; in particular by [13, 0.9], this holds in $L[r]$, where r is a Mathias real over L . By Lemma 4.2, it is enough to show that there is no Π_1^1 mof. (For notational convenience, we suppress the parameter a .)

Suppose for a contradiction that $\mathcal{A} \subseteq P(2^\omega)$ is a Π_1^1 mof. Let $Q \subseteq [\omega]^\omega \times P(2^\omega)^\omega$ be

$$Q = \{(x, (\nu_n)) : (\forall n)(\nu_n \in \mathcal{A} \wedge \nu_n \not\perp \mu^x) \wedge (\forall \mu)(\mu \not\perp \mu^x \longrightarrow (\exists n)\nu_n \not\perp \mu)\}.$$

(Thus $(x, (\nu_n)) \in Q$ iff (ν_n) enumerates the countably many measures in \mathcal{A} that are not orthogonal to μ^x .) Since \mathcal{A} is maximal the sections Q_x are

never empty, and by the Π_1^1 uniformization theorem, we can find a function $\vartheta : [\omega]^\omega \rightarrow P(2^\omega)^\omega$ which has a Π_1^1 graph, and such that $(x, \vartheta(x)) \in Q$ for all $x \in [\omega]^\omega$. Note that if $\mu^x \sim \mu^y$ then

$$\{\vartheta_n(x) : n \in \omega\} = \{\vartheta_n(y) : n \in \omega\},$$

so that by (iii) of Proposition 5.2, ϑ is $(\mathcal{F}, F_2^{P(2^\omega)})$ -equivariant.

It is easy to check that then $\vartheta^{-1}(U)$ is Σ_2^1 for every basic open set $U \subseteq P(2^\omega)^\omega$. Since every Σ_2^1 set is completely Ramsey, a standard argument [17, Exercise 19.19, p. 134] shows that there is $x \in [\omega]^\omega$ such that $\vartheta \upharpoonright [x]^\omega$ is continuous (w.r.t. the Polish topology on $[x]^\omega$.)

From Lemma 5.4 it follows that there is a non-empty open set $U \subseteq [x]^\omega$ and $\nu \in P(2^\omega)$ such that $\vartheta_0(x) = \nu$ for all $x \in U$. By (vi) of Proposition 5.2 there is an uncountable (indeed a perfect) set $P \subseteq U$ such that if $x, y \in P$ and $x \neq y$, then $\mu^x \perp \mu^y$. Now for every $x \in P$ we have $\mu^x \not\ll \nu$, contradicting the ccc-below property of \ll . \square

The above line of argument also gives a new proof of the theorem of Preiss and Rataj, which we sketch below. Unlike the new proof that was given by Kechris and Sofronidis in [18], the proof below does not rely directly on Hjorth’s turbulence theory. All the same, Lemma 5.4 above owes a debt to [11, Lemma 3.14, p. 42] that can scarcely be ignored.

Theorem 5.5 ([23]). *There are no analytic mofs in $P(2^\omega)$.*

Sketch of proof. Suppose \mathcal{A} were an analytic mof. By maximality, \mathcal{A} would be Borel, and maximality along with the ccc-below property gives that the Borel set

$$Q' = \{(x, \nu) \in [\omega]^\omega \times P(2^\omega) : \mu^x \not\ll \nu \wedge \nu \in \mathcal{A}\}$$

would have all vertical sections Q'_x non-empty and countable. Then we could find countably many Borel functions $\vartheta_l : [\omega]^\omega \rightarrow P(2^\omega)$ such that

$$Q' = \bigcup_{l \in \omega} \text{graph}(\vartheta_l).$$

The equivariance

$$\mu^x \sim \mu^y \implies \{\vartheta_n(x) : n \in \omega\} = \{\vartheta_n(y) : n \in \omega\}$$

is clear. Again, [17, Exercise 19.19, p.134] would allow us to find $x \in [\omega]^\omega$ such that $\vartheta_l \upharpoonright [x]^\omega$ is continuous (w.r.t. the Polish topology) for all l . A contradiction is then obtained in exactly the same way it was in the proof of Theorem 5.1. □

6. Open problems

Given the results of this paper, we pose the following questions:

- 1) Does the analogue of Theorem 3.1 hold for the Laver extension? (Note that the analogue of Galvin’s theorem is false for Laver forcing.)
- 2) Is there a model with a Π_1^1 mof such that in addition, for any $r \in 2^\omega$, there is a Miller real over $L[r]$? (For Sacks forcing, this question has been answered in [24].)
- 3) Is the existence of a Π_1^1 mof consistent with $2^\omega = \omega_3$? (For $2^\omega = \omega_2$, this question has also been answered in [24].)
- 4) Does Theorem 3.1 fail for \mathcal{R} which are Π_1^1 ? What about Theorem 3.2?
- 5) Are there natural forcing notions other than Sacks and Miller to which the hypothesis of Theorem 3.11 applies?

Acknowledgements

We would like to thank Stevo Todorčević for making us aware of the analogue of Galvin’s theorem for Miller forcing, which we use in the proof of Corollary 3.6. Further, we would like to thank the anonymous referee for their diligent corrections and many insightful suggestions.

The authors gratefully acknowledge the generous support from Sapere Aude grant no. 10-082689/FNU from Denmark’s Natural Sciences Research Council, and the first author gratefully acknowledges generous support from the DNRF Niels Bohr Professorship of Lars Hesselholt.

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RECEIVED OCTOBER 26, 2015