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# Poincaré inequality on complete Riemannian manifolds with Ricci curvature bounded below

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We prove that complete Riemannian manifolds with polynomial growth and Ricci curvature bounded from below, admit uniform Poincaré inequalities. A global, uniform Poincaré inequality for horospheres in the universal cover of a closed, $n$-dimensional Riemannian manifold with pinched negative sectional curvature follows as a corollary.

## 1. Introduction

**Statements of the main results.** In this paper, we will establish that complete Riemannian manifolds with Ricci curvature bounded below and having polynomial growth, admit a family of uniform Poincaré inequalities. To begin with, let $(M^n, g)$ be a complete $n$-dimensional Riemannian manifold. Henceforth, we will assume that $(M^n, g)$ satisfies the Ricci curvature lower bound

$$\text{Ricci}_{(M^n,g)} \geq -(n-1)\kappa, \text{ for some } \kappa \geq 0. \tag{1.1}$$

We will also assume that $(M^n, g)$ has $\alpha$-polynomial growth; this means that there exist constants $v > 0$, $\alpha > 0$ and $R_0 \geq 0$ such that for any $m \in M^n$ and $R > R_0$, the ball of radius $R$ centered at $m$ satisfies

$$\text{vol}\, B(m, R) \leq vR^\alpha, \tag{1.2}$$

where vol denotes the canonical measure on $(M^n, g)$. Recall that Bishop's Comparison Theorem (cf. [15, Section IV]) implies that when $\kappa = 0$, $(M^n, g)$ satisfies polynomial growth with $\alpha = n$.

We will finally assume that the local geometry of $(M^n, g)$ is controlled in the sense that there exists a constant $\omega > 0$ such that for any $m \in M^n$,

$$\operatorname{vol} B(m,1) \geq \omega. \tag{1.3}$$

The triple $(M^n, \mathrm{dist}, \mathrm{vol})$ with dist being the standard metric induced by the Riemannian metric is an example of a *metric measured space*. Throughout this paper $(X, \rho, \mu)$ will denote a metric space endowed with a Borel measure $\mu$. We will use the notation

$$u_A = \frac{1}{\mu(A)} \int_A u d\mu, \tag{1.4}$$

for every $A \subset X$ and measurable function $u : X \to [-\infty, \infty]$, and when $A$ is a ball $B(m, R)$, we will abuse the notation and write

$$u_R = \frac{1}{\mu(B(m,R))} \int_{B(m,R)} u d\mu. \tag{1.5}$$

We will say that a Riemannian manifold $(M^n, g)$ satisfies a $(\sigma, \beta, \sigma)$-*uniform* (with respect to balls) Poincaré inequality, for $\sigma \geq 1$, if there exists a constant $C$ such that for any $r_0 > 0$, there exists a constant $K$ such that for any $u \in C^1(M^n)$, any $R \geq r_0$ and any ball $B(m, R) \subset M^n$, we have

$$\int_{B(m,R)} |u(x) - u_R|^\sigma d\mathrm{vol}(x) \leq K R^\beta \int_{B(m,CR)} |\nabla u(x)|^\sigma d\mathrm{vol}(x). \tag{1.6}$$

Our main result is the following:

**Theorem 1.7 (Main Theorem).** *Let $(M^n, g)$ be a complete Riemannian manifold satisfying the Ricci curvature bound (1.1), the $\alpha$-polynomial growth assumption (1.2) and the assumption (1.3). Then, there exists a constant $C = C(n, \kappa)$ such that for any $\sigma \geq 1$ and $r_0 > 0$, there exists a constant $K = K(n, \sigma, r_0, R_0, \kappa, v, \omega)$ such that for any $u \in C^1(M^n)$, any $R \geq r_0$ and any ball $B(m, R) \subset M^n$, we have*

$$\int_{B(m,R)} |u(x) - u_R|^\sigma d\mathrm{vol}(x) \leq K R^{\alpha+\sigma-1} \int_{B(m,CR)} |\nabla u(x)|^\sigma d\mathrm{vol}(x), \tag{1.8}$$

*where $u_R = u_{B(m,R)}$.*

This theorem is meaningful for large balls. Indeed, since balls of small radii in $(M^n, g)$ are asymptotically Euclidean, they carry $(\sigma, \sigma, \sigma)$-uniform Poincaré inequalities for $\sigma \geq 1$. This is the reason for the restriction to balls of radius $R \geq r_0$, and it also explains why the constant $K$ depends, among other geometric quantities, on $r_0$. The local geometry assumption (1.3) is necessary as we can easily see by considering manifolds with arbitrary long and thin dumbells.

The exponent of $R$ in the above Theorem is optimal. For every $\alpha \in \mathbb{N}$, $\alpha \geq 1$, we construct examples of complete Riemannian manifolds $(M^n, g)$ with $\alpha$-polynomial growth $\operatorname{vol} B(m, R) \leq vR^\alpha$ and Ricci curvature bounded below, Ricci $\geq -(\mathrm{n}-1)\kappa$ such that there exist a function $u : M^n \to \mathbb{R}$ such that for any constants $C > 0$, $\sigma \geq 1$, and any $\beta < \alpha + \sigma - 1$,

$$\lim_{R\to\infty} \left( \int_{B(R)} |u - u_R|^\sigma \right) \left( R^\beta \int_{B(CR)} |\nabla(u)|^\sigma \right)^{-1} = \infty, \tag{1.9}$$

see Section 6.

**Remark 1.10.** Note that the constants of the Poincaré inequality are uniform among the set of all Riemannian manifolds with the same Ricci curvature lower bound and polynomial growth of order $\alpha$.

Next we apply our main theorem to special types of hypersurfaces. Let $\tilde{M}$ be the universal cover of a closed, complete, $n$-dimensional Riemannian manifold, $(M^n, g)$, whose sectional curvature satisfies

$$-b^2 \leq K_g \leq -a^2, \tag{1.11}$$

where $a, b$ are fixed positive constants. We will show that *horospheres* in $\widetilde{M}$ satisfy a uniform and *global* Poincaré inequality (1.6), where global means that the inequalities hold *independently* of the horosphere.

**Corollary 1.12.** *There exist positive constants $C$ and $K$, as in Theorem 1.7, depending only on $n$, $\sigma$, $a$ and $b$, such that every horosphere $\mathcal{H}$ in $\widetilde{M}$, endowed with the induced Riemannian metric, satisfies inequality* (1.8) *with*

$$\alpha = \frac{(n-1)b}{a}. \tag{1.13}$$

**Perspective.** Poincaré inequalities are central in the study of the geometrical analysis of manifolds. It is well known that carrying a Poincaré inequality has strong geometric consequences. For instance, a complete, doubling, non-compact, Riemannian manifold admitting a $(1,1,1)$-uniform Poincaré inequality satisfies an isoperimetric inequality. Moreover, a $(2,2,2)$-uniform Poincaré inequality is equivalent to Gaussian estimates for the heat kernel, [5], [8]. For comprehensive and detailed accounts of the subject, the reader is advised to consult for example [9] and [16].

To put our main theorem in perspective, let us recall a few classical results. The following theorem follows, for instance, from Buser's inequality, [1] (and an application of Minkowski inequality).

**Theorem 1.14.** *Assume that $M^n$ has Ricci curvature bounded below by $-(n-1)\kappa$. Then, for all $R>0$ and for all $\sigma \geq 1$, there exists a constant $C(n,R,\kappa)$ such that for all $u \in C^1(M)$ and for all $m \in M$, we have*

$$\int_{B(m,R)} |u(x) - u_R|^\sigma d\mathrm{vol}(x) \leq C(n,R,\kappa) \int_{B(m,3R)} |\nabla u(x)|^\sigma d\mathrm{vol}(x), \tag{1.15}$$

*where $u_R = u_{B(m,R)}$.*

A manifold for which (1.15) holds is said to carry a *local* Poincare inequality. For the proof see [2, Chapter VI.5], or [10, Lemma 2.9] for a different proof based on the Cheeger-Colding segment inequality, or [16, Theorem 5.6.6]. In fact, by [9, Section 10.1], and under the same assumptions, the following holds:

$$\int_{B(m,R)} |u - u_R| d\mathrm{vol} \leq C(n) \exp\left((n-1)\kappa R\right) R \int_{B(m,R)} |\nabla u| d\mathrm{vol}. \tag{1.16}$$

Hence, when $(M^n, g)$ has non-negative Ricci curvature, (i.e., a bound (1.1) with $\kappa = 0$), then for every $\sigma \geq 1$, $(M^n, g)$ satisfies a $(\sigma,\sigma,\sigma)$-uniform Poincaré inequality: For every $R > 0$,

$$\int_{B(m,R)} |u - u_R|^\sigma d\mathrm{vol} \leq K(n,\sigma) R^\sigma \int_{B(m,2R)} |\nabla u|^\sigma d\mathrm{vol} \tag{1.17}$$

(cf. [16, Theorem 5.6.6]).

**Remark 1.18.** The constants which appear above do not depend on the point $m$.

Another important class of examples occur when $(M^n, g)$ is a unimodular connected Lie group equipped with a left invariant metric. Then, for every $\sigma \geq 1$, the following Poincaré inequality is known to hold (cf. [16, page 173]):

$$\int_{B(m,R)} |u - u_R|^\sigma d\mathrm{vol} \leq (2R)^\sigma \frac{\mathrm{vol}\, B(2R)}{\mathrm{vol}\, \mathrm{B(R)}} \int_{B(m,R)} |\nabla u|^\sigma d\mathrm{vol}. \tag{1.19}$$

Moreover, if the left invariant metric is doubling, (i.e., if there exists a constant $C$ such that for any $R > 0$, $\mathrm{vol}\, \mathrm{B(2R)} \leq \mathrm{C}\, \mathrm{vol}\, \mathrm{B(R)}$, then for every $\sigma \geq 1$, such a Lie group satisfies a $(\sigma, \sigma, \sigma)$-uniform Poincaré inequality [16, Theorem 5.6.1]:

$$\int_{B(m,R)} |u - u_R|^\sigma d\mathrm{vol} \leq C(2R)^\sigma \int_{B(m,R)} |\nabla u|^\sigma d\mathrm{vol}. \tag{1.20}$$

Lie groups equipped with doubling left invariant metrics have polynomial growth and examples of such groups are nilpotent ones. In [14], Kleiner proved analogous Poincaré inequalities to (1.19) and (1.20) for discrete finitely generated groups. Besides manifolds of non-negative Ricci curvature, unimodular Lie groups and discrete finitely generated groups with doubling property or manifolds which are roughly isometric to these (cf. Definition (2.1)), no other class of manifolds are known to satisfy $(\sigma, \sigma, \sigma)$-uniform Poincaré inequality.

**The scheme of the proof of our main theorem and the structure of this paper.** In [5], foundational work of Coulhon and Saloff-Coste shows that under two conditions on $(M^n, g)$, it admits a uniform Poincaré inequality (1.6) *if and only if* a graph approximation of $(M^n, g)$ admits a *discrete* version of this inequality (see Definition 2.25). The first of these conditions is a local Poincaré inequality (1.15) which, as we mentioned before, holds under the lower bound assumption of the Ricci curvature. The second condition in [5] is a local doubling property. It follows from the lower bound assumption on the Ricci curvature, that this property holds on $(M^n, g)$ as well as on any of its graph approximations. Thus, it is sufficient to prove a Poincaré inequality for any graph approximation of $(M^n, g)$.

Section 3 is devoted to a detailed exposition of the part of the work in [5] that we need in this paper. In Section 2, we describe a discretization scheme which is inspired by the seminal works of Kanai (cf. [12, 13]). Kanai's scheme provides a bounded valence graph approximation of $(M^n, g)$. His scheme relies on the Ricci curvature lower bound assumption and requires in addition positivity of the injectivity radius of $(M^n, g)$. However, Coulhon and Saloff-Coste [5] later made the important observation that a local doubling volume assumption is a sufficient one. In this section, we will also recall a theorem of Kanai and its improvement by Coulhon-Saloff-Coste relating the growth rate of the manifold and the growth of any of its graph approximation: the graph and the manifold are roughly quasi-isometric as metric measured spaces (see Definition 2.1), and it follows that the $\alpha$-polynomial growth property transfers from the manifold to any of its graph approximation.

In Section 4, we prove a discrete version of Poincaré inequality (1.6), which is a slight generalization of the one given in [6, 308–311], which holds for graphs having polynomial growth. In Section 5, we assemble the pieces together and provide the proofs of our main theorem and of Corollary 1.12. It is in this part that the assumption of the polynomial growth of the manifold is first used. Finally, in Section 6, we present examples elucidating the sharpness of our inequalities.

**Notation.** Henceforth, we will let $M$ denote $(M^n, g)$ and we let $d$ denote the dist function on $M$.

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## 2. Discretization of riemannian manifolds

In this section, we recall some basic definitions and lemmas that are needed before we state the main application that is needed in this paper, Corollary 2.22. This corollary relates the property of polynomial growth of a manifold to any of a tight discrete approximation of it. Throughout this section, we will closely follow the notation and logic as in [5] and in [12, 13].

**Definition 2.1 ([12, 13]).** Let $(X, \rho, \mu)$ and $(Y, d, \nu)$ be two metric measured spaces. A map $\phi : X \to Y$ is called a *rough isometry* if there exist constants $c_1 > 0$ and $c_2, c_3 > 1$ such that

$$Y = \cup_{x\in X} B(\phi(x), c_1), \tag{2.2}$$

$$c_2^{-1}(\rho(x,y) - c_1) \le d(\phi(x), \phi(y)) \le c_2(\rho(x,y) + c_1), \text{ and} \tag{2.3}$$

$$c_3^{-1}\mu(B(x,c_1)) \le \nu(B(\phi(x),c_1)) \le c_3\mu(B(x,c_1)). \tag{2.4}$$

If there exists a rough isometry between two metric measured spaces, they are said to be roughly isometric.

Let $M$ be a complete Riemannian manifold. A subset $\mathcal{G}$ of $M$ is said to be *$\epsilon$-separated*, for $\epsilon > 0$, if the Riemannian distance between any two distinct points of $\mathcal{G}$ is greater than or equal to $\epsilon$. An *$\epsilon$-discretization* of $M$ is an $\epsilon$-separated subset $\mathcal{G}$ of $M$ which is *maximal* with respect to inclusion of sets. The maximality implies that

$$M = \bigcup_{\xi \in \mathcal{G}} B(\xi, \epsilon), \tag{2.5}$$

and $\epsilon$ is then called the *covering radius* of the discretization. The graph structure on $\mathcal{G}$ is determined by defining the neighbors of $\xi \in \mathcal{G}$ to be the set

$$\mathbf{N}(\xi) = \{\mathcal{G} \cap B(\xi, 2\epsilon)\} \setminus \{\xi\}. \tag{2.6}$$

The multiplicity $\mathcal{M}(\mathcal{G}, \epsilon)$ of the covering $M = \bigcup_{\xi \in \mathcal{G}} B(\xi, \epsilon)$ is defined by

$$\mathcal{M}(\mathcal{G}, R) = \sup_{\xi \in \mathcal{G}} |\mathbf{N}(\xi)|. \tag{2.7}$$

In fact, this graph which we denote by $X$, carries a structure of a metric measured space $(X, \rho, \mu)$: The distance $\rho$ on $X$ is the canonical combinatorial distance, and the measure $\mu$ is defined by

$$\mu(x) = \operatorname{vol}(B(x, \epsilon)), \quad \text{for each } x \in X. \tag{2.8}$$

The following definition allows one to distinguish metric measured spaces with a special property of the measure.

**Definition 2.9.** A metric measured space $(X, \rho, \mu)$ satisfies the *local doubling condition*, $(DV)_{loc}$, if for all $r > 0$ there exists $C_r$ such that for all $x \in X$

$$\mu(B(x, 2r)) \leq C_r \mu(B(x, r)), \tag{2.10}$$

where the constant $C_r$ depends on $r$ but is independent of the point $x$.

The following lemma, which will be used in the proof of Theorem 3.1, asserts that the assumption of local doubling implies uniform control on the multiplicity of the covering $\{B(x, 3\epsilon)\}_{x \in X}$. It was first proved by Kanai [12, Lemma 2.3] under a Ricci lower bound curvature assumption. However, it turns out that the main ingredient in Kanai's proof is the $(DV)_{loc}$ property (which is implied by the Ricci lower bound curvature assumption).

**Lemma 2.11 ([12, Lemma 2.3]).** *Let $M^n$ be a complete Riemannian manifold which satisfies the local doubling condition $(DV)_{loc}$. Then, there exists $\mathcal{M} = \mathcal{M}(\epsilon)$, depending only on $\epsilon$, such that, for every $X \subset M^n$ an $\epsilon$-discretization of $M^n$, the multiplicity of the covering $\{B(x, 3\epsilon)\}_{x\in X}$ satisfies $\mathcal{M}(X, 3\epsilon) \leq \mathcal{M}$.*

*Proof.* Note that since $X$ is a $\epsilon$-separated set then $\{B(x, \frac{\epsilon}{2})\}_{x\in X}$ is a disjoint family of balls. Fix some ball $B(x, 3\epsilon)$, $x \in X$ and consider the subset $Y \subset X$ consisting of points $y$ such that $B(x, 3\epsilon) \cap B(y, 3\epsilon) \neq \emptyset$. Therefore, $Y \subset B(x, 6\epsilon)$ and $\{B(y, \frac{\epsilon}{2})\}_{y\in Y}$ is a disjoint family of balls contained in $B(x, 6\epsilon + \frac{\epsilon}{2})$. We have

$$\sum_{y\in Y} \mathrm{vol}\left(B\left(y, \frac{\epsilon}{2}\right)\right) \leq \mathrm{vol}\, B\left(x, 6\epsilon + \frac{\epsilon}{2}\right) \leq \mathrm{vol}\, B\left(x, 7\epsilon\right). \tag{2.12}$$

For each $y \in Y$, we have $B(x, 7\epsilon) \subset B(y, 13\epsilon)$ and by the local doubling assumption $(DV)_{loc}$ we obtain (using in the last step that $B\left(y, \frac{13\epsilon}{32}\right) \subset B\left(y, \frac{\epsilon}{2}\right)$)

$$\begin{aligned} \mathrm{vol}\, B\left(y, 13\epsilon\right) &\leq C_{\frac{13\epsilon}{2}} \mathrm{vol}\, B\left(y, \frac{13\epsilon}{2}\right) \\ &\leq \cdots \leq C_{\frac{13\epsilon}{2}} C_{\frac{13\epsilon}{4}} \ldots C_{\frac{13\epsilon}{32}} \mathrm{vol}\, B\left(y, \frac{\epsilon}{2}\right). \end{aligned} \tag{2.13}$$

This implies that

$$\mathrm{vol}\, B\left(y, \frac{\epsilon}{2}\right) \geq \left(C_{\frac{13\epsilon}{2}} C_{\frac{13\epsilon}{4}} \ldots C_{\frac{13\epsilon}{32}}\right)^{-1} \mathrm{vol}\, B\left(x, 7\epsilon\right). \tag{2.14}$$

Hence,

$$\sum_{y\in Y} \mathrm{vol}\, B\left(y, \frac{\epsilon}{2}\right) \geq |Y| \left(C_{\frac{13\epsilon}{2}} C_{\frac{13\epsilon}{4}} \ldots C_{\frac{13\epsilon}{32}}\right)^{-1} \mathrm{vol}\, B\left(x, 7\epsilon\right). \tag{2.15}$$

On the other hand, by (2.12) we have $\sum_{y\in Y} \mathrm{vol}\, B\left(y, \frac{\epsilon}{2}\right) \leq \mathrm{vol}\, B\left(x, 7\epsilon\right)$, therefore, we obtain

$$|Y| \leq C_{\frac{13\epsilon}{2}} C_{\frac{13\epsilon}{4}} \ldots C_{\frac{13\epsilon}{32}}. \tag{2.16}$$

This conclues the proof of the lemma with $\mathcal{M}(X,3\epsilon) = C_{\frac{13\epsilon}{2}} C_{\frac{13\epsilon}{4}} \ldots C_{\frac{13\epsilon}{32}}$. □

**Remark 2.17.** An obvious consequence of Lemma 2.11 is that the covering $\{B(x,\epsilon)\}_{x\in X}$ is locally finite. In the proof of Theorem 3.1, we will need to work with the cover induced by $\{B(x,3\epsilon)\}_{x\in X}$, which is obviously locally finite as well.

**Remark 2.18.** Let $M^n$ be a complete Riemannian manifold which satisfies the local doubling condition $(DV)_{loc}$ and $(X,\rho,\mu)$ an $\epsilon$-discretization of $M^n$. Then, $(X,\rho,\mu)$ satisfies the local doubling condition $(DV)_{loc}$.

As we recalled in the introduction, Kanai and Coulhon-Saloff-Coste described conditions under which an $\epsilon$-discretization of $M^n$ is roughly isometric to $(M^n, d, \mathrm{vol})$. Kanai assumed a lower bound on the Ricci curvature and positivity of the injectivity radius. Coulhon-Saloff-Coste refined Kanai's result by only requiring that the volume measure satisfies a local doubling condition.

The question of the invariance of polynomial growth under a rough isometry has been also worked out by these authors under the same assumptions as above. Coulhon and Saloff-Coste proved that if the volume measure is local doubling, then $\alpha$-polynomial growth is invariant under rough isometries. Let us summarize these results which will be used later in this section.

**Theorem 2.19 ([5, Section 2], [13, Section 6]).** *Let $M^n$ be a complete $n$-dimensional Riemannian manifold and suppose that $M^n$ satisfies the $(DV)_{loc}$ condition. Then, for any $\epsilon$-discretization $X$ of $M^n$, the metric measured space $(X,\rho,\mu)$ is roughly isometric to $M^n$. In particular, when $M^n$ satisfies the lower Ricci curvature bound (1.1), then the constants appearing in the definition of rough isometry (see Definition 2.1) depend only on $n$, $\kappa$ and $\varepsilon$.*

*Proof.* This Theorem was first proved by Kanai under two assumptions, a lower bound on the Ricci curvature and the positivity of the injectivity radius ([13, Lemma 3.6]). Coulhon and Saloff-Coste later observed that the $(DV)_{loc}$ condition is sufficient. More precisely, condition (2.2) holds by definition. The proof of condition (2.3) is derived exactly as the proof of Lemma 2.5 in [13] since the latter relies only on the $(DV)_{loc}$ condition (which is a consequence of the lower bound on the Ricci curvature). Condition (2.4), which was established in [13] Lemma 3.6 under the assumption that the injectivity radius is positive, is proved in [5], proposition 2.2, where only the condition of local doubling of the volume measure is assumed. □

The following theorem establishes an invariance property of polynomial growth under rough isometries.

**Theorem 2.20 ([5]** Proposition 2.2). *Let $(X_1, \rho_1, \mu_1)$ and $(X_2, \rho_2, \mu_2)$ be two metric measured spaces satisfying the $(DV)_{loc}$ condition. Suppose that $\Phi$ is a rough isometry between $(X_1, \rho_1, \mu_1)$ and $(X_2, \rho_2, \mu_2)$. Then, there exists a constant $C > 0$ depending on the constant of the rough isometry and the constant in the local doubling condition, such that*

$$C^{-1}\mu_1(B(x, C^{-1}R)) \leq \mu_2(B(\Phi(x), R)) \leq C\mu_1(B(x, CR)) \tag{2.21}$$

*for any $x \in X_1$ and $R \geq 1$. In particular, polynomial growth of order $\alpha$ is invariant under rough isometries.*

We can therefore deduce the following corollary.

**Corollary 2.22.** *Let $M^n$ be a complete $n$-dimensional Riemannian manifold and suppose that $M^n$ satisfies the $(DV)_{loc}$ condition and let $X$ be an $\epsilon$-discretization of $M^n$. Then $M^n$ has polynomial growth of order $\alpha$ if and only if $X$ has polynomial growth of order $\alpha$. In particular, when $M^n$ satisfies the lower Ricci curvature bound (1.1) and the $\alpha$-polynomial growth estimate (1.2), then the $\epsilon$-discretization $X$ has $\alpha$-polynomial growth, $\mu(B(x, R)) \leq v' R^\alpha$ for every $R \geq R_0'$, where the constants $v'$ and $R_0'$ depend only on $n$, $v$, $R_0$, $\kappa$ and $\epsilon$.*

*Proof.* Let $X$ be a $\epsilon$-discretization of $M^n$. By Theorem 2.19, $M^n$ and $X$ are roughly isometric. By Remark 2.18, $X$ satisfies the $(DV)_{loc}$ condition and therefore the assertion of the corollary follows upon applying Theorem 2.20. □

Henceforth, we will consider $\epsilon$-discretization subsets of $M^n$ such that the covering radius is $\epsilon$. We need two definitions before stating the main theorem of this section.

**Definition 2.23.** Let $X$ be an $\epsilon$-discretization of $M^n$. For a function $f : X \to \mathbb{R}$, the length of the discrete gradient of $f$ is defined by

$$\delta f(x) = \left( \sum_{y \sim x} |f(y) - f(x)|^2 \right)^{1/2} . \tag{2.24}$$

**Definition 2.25.** Given $\sigma \geq 1$ and $\beta \geq 1$, we say that a discrete metric measured space $(X, \rho, \mu)$ satisfies a uniform $(\sigma, \beta, \sigma)$-Poincaré inequality if there exist constants $r_1$, $C = C(\sigma, \beta)$ and $C' \geq 1$ such that for any function $f : X \to \mathbb{R}$, any $R \geq r_1$ and $x_0 \in X$ we have

$$\sum_{x \in B(x_0,R)} |f(x) - f_R|^\sigma \mu(x) \leq CR^\beta \sum_{x \in B(x_0,C'R)} (\delta f(x))^\sigma \mu(x), \tag{2.26}$$

where

$$f_R = f_{B(x_0,R)} = \frac{1}{\mu(B(x_0,R))} \sum_{x \in B(x_0,R)} \mu(x) f(x).$$

## 3. A criterion for a manifold to carry a uniform Poincaré inequality

In [5], Coulhon and Saloff-Coste studied a Poincaré inequality (1.6) with $\beta = 1$ and $\sigma = 1$, that is, a $(1, 1, 1)$-uniform Poincaré inequality. Nevertheless, the proof for an arbitrary $\beta \geq 1$ and $\sigma \geq 1$ works along the same lines of their arguments. Following [5] and in order to make this paper self-contained, we will now provide the part of the proof of Theorem 3.1 which is needed for our application: If $X$ satisfies a Poincaré inequality (2.26), then $M^n$ satisfies a Poincaré inequality (1.6). In addition, we will carefully keep track of the dependencies of the quantities appearing in the proof. The statement of the theorem is the following.

**Theorem 3.1 ([5], Proposition 6.10).** *Let $M^n$ be a complete Riemannian manifold which satisfies the local doubling condition, $(DV)_{loc}$, and the local Poincaré inequality* (1.15). *Let $X \subset M^n$ be a $\epsilon$-discretization of $M^n$. Then $M^n$ satisfies the uniform Poincaré inequality (1.6) if and only if the discretization $(X, \rho, \mu)$ of $M^n$ satisfies the uniform Poincaré inequality* (2.26).

*Proof.* Consider a complete Riemannian manifold $M^n$ and an $\epsilon$-discretization $X$ of $M^n$. We will prove the part of the Theorem which we will later need:

Given a function $\psi : M^n \to \mathbb{R}$ let the function $\tilde{\psi} : X \to \mathbb{R}$ be defined by

$$\tilde{\psi}(x) = \psi_{B(x,\epsilon)} = \frac{1}{\text{vol B(x, }\epsilon)} \int_{B(x,\epsilon)} \psi(z) d\text{vol}(z). \tag{3.2}$$

For $E \subset M^n$ and $F \subset X$, two functions $\psi : M^n \to \mathbb{R}$, and $f : X \to \mathbb{R}$ and $\sigma$ a positive integer, we define

$$\|\psi\|_{\sigma,E} = \left( \int_E |\psi(z)|^\sigma d\mathrm{vol}(z) \right)^{1/\sigma} \tag{3.3}$$

and

$$\|f\|_{\sigma,F} = \left( \sum_F |f(x)|^\sigma \mu(x) \right)^{1/\sigma} . \tag{3.4}$$

Let us recall the following lemma which relates the gradients of $\psi$ and $\tilde{\psi}$.

**Lemma 3.5 ([5], Lemma 6.4).** *For any $\sigma \geq 1$, there exist constants $\mathcal{T}$ and $\mathcal{T}'$ depending on $\sigma$ and the multiplicity of the covering associated to the $\epsilon$-discretization $X$ of $M^n$, such that for all smooth functions $\psi : M^n \to \mathbb{R}$, all $R \geq 1$, and all $x \in X$ the following holds*

$$\|\delta\tilde{\psi}\|_{\sigma,B(x,R)} \leq \mathcal{T} \|\nabla\psi\|_{\sigma,B(x,\mathcal{T}'R)}. \tag{3.6}$$

Inequality (3.6), the proof of which the authors referred to their Lemma 5.3 (which is not proved), is stated in Coulhon Saloff-Coste. Let us provide a proof of this Lemma.

*Proof.* The Lemma is a direct consequence of the following fact: for any $\epsilon > 0$, any $x, y \in M^n$ such that $d(x,y) \leq 2\epsilon$ and any $\psi : \mathcal{M}^n \to \mathbb{R}$,

$$|\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma V(x,\epsilon) \leq C \int_{B(x,6\epsilon)} |\nabla\psi|^\sigma, \tag{3.7}$$

where $C := C(n,\sigma,\epsilon,\kappa)$ and $V(x,\epsilon) = \mathrm{vol} B(x,\epsilon)$. Indeed, assuming (3.7) we have

$$\begin{aligned} \|\delta\tilde{\psi}\|^\sigma_{\sigma,B(x,R)} &= \sum_{z\in B(x,R)} |\delta\tilde{\psi}(z)|^\sigma V(z,\epsilon) \\ &\leq \sum_{z\in B(x,R)} \mathcal{M}(\epsilon)^{\sigma-1} \left( \sum_{z',\, d(z,z')\leq 2\epsilon} |\tilde{\psi}(z') - \tilde{\psi}(z)|^\sigma \right) V(z,\epsilon) \\ &\leq C\mathcal{M}(\epsilon)^{\sigma-1}\mathcal{M}(6\epsilon) \int_{B(x,C'R)} |\nabla\psi|^\sigma, \end{aligned} \tag{3.8}$$

where $\mathcal{M}(\epsilon)$ is the multiplicity of the covering of $M^n$ by balls of radius $\epsilon$ and $C' = 1 + 6\epsilon$; in (3.8), the first inequality is due to Jensen's inequality and the second follows from (3.7).

Let us conclude the proof of Lemma 3.5 by proving (3.7). Note that

$$\tilde{\psi}(x) - \tilde{\psi}(y) = \frac{1}{V(x,\epsilon)}\frac{1}{V(y,\epsilon)}\int_{B(x,\epsilon)}\int_{B(y,\epsilon)}(\psi(u) - \psi(z))dudz. \tag{3.9}$$

By Jensen's inequality, we get

$$|\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma \leq \frac{1}{V(x,\epsilon)}\frac{1}{V(y,\epsilon)}\int_{B(x,\epsilon)}\int_{B(y,\epsilon)}|\psi(u) - \psi(z)|^\sigma dudz, \tag{3.10}$$

and by Minkowski's inequality we get,

$$\begin{aligned} |\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma &\leq 2^{\sigma-1}\frac{1}{V(x,\epsilon)}\frac{1}{V(y,\epsilon)} \\ &\quad\times \int_{B(x,\epsilon)}\int_{B(y,\epsilon)}\left(|\psi(u) - \tilde{\psi}(x)|^\sigma + |\psi(z) - \tilde{\psi}(y)|^\sigma\right)dudz, \end{aligned} \tag{3.11}$$

which implies, by the local Poincaré inequality,

$$|\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma \leq 2^{\sigma-1}\frac{C}{V(x,\epsilon)}\int_{B(x,3\epsilon)}|\nabla\psi|^\sigma + 2^{\sigma-1}\frac{C}{V(y,\epsilon)}\int_{B(y,3\epsilon)}|\nabla\psi|^\sigma, \tag{3.12}$$

where $C =: C(n,\sigma,\epsilon,\kappa)$. We then deduce,

$$|\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma V(x,\epsilon) \leq 2^{\sigma-1}C\left(\int_{B(x,3\epsilon)}|\nabla\psi|^\sigma + \frac{V(x,\epsilon)}{V(y,\epsilon)}\int_{B(y,3\epsilon)}|\nabla\psi|^\sigma\right), \tag{3.13}$$

and by local doubling and the fact that $d(x,y) \leq 2\epsilon$,

$$|\tilde{\psi}(x) - \tilde{\psi}(y)|^\sigma V(x,\epsilon) \leq C'\int_{B(x,6\epsilon)}|\nabla\psi|^\sigma. \tag{3.14}$$

□

We now turn to the proof of the part of the theorem which will be needed for the applications of this paper.

Let $(X,\rho,\mu)$ be a fixed $\epsilon$-discretization of $M^n$ satisfying Poincaré inequality (2.26). After a normalization, which will affect the constants once and for all, we can assume that $\epsilon = 1$.

We need to prove that for a given $\sigma \geq 1$, there exists a constant $C > 0$ such that for any $r_0 > 0$, there exists a constant $K$ such that for any smooth function $\psi : M^n \to \mathbb{R}$ and any $R \geq r_0$, Inequality (1.6) holds, that is

$$\int_{B(m,R)} |\psi(x) - \psi_R|^\sigma d\mathrm{vol}(x) \leq K R^\beta \int_{B(m,CR)} |\nabla\psi(x)|^\sigma d\mathrm{vol}(x). \tag{3.15}$$

To this end, let us consider a smooth function $\psi$ on $M$, numbers $r_0 > 0$ and $\sigma \geq 1$, and a point $m \in M^n$. Let $R$ satisfy $R \geq r_0$. Let us define

$$R_1 = \max\{\epsilon, r_1\} = \max\{1, r_1\}, \tag{3.16}$$

where $r_1$ is determined by the discrete Poincaré inequality (2.26).

The radius $R$ can be either less than or equal to $R_1$, or larger than or equal to $R_1$. In the following, we will analyze these cases separately.

In the first case, $r_0 \leq R \leq R_1$, the conclusion essentially follows from the local Poincaré inequality. Indeed, Inequality (1.15) yields that

$$\int_{B(m,R)} |\psi(x) - \psi_R|^\sigma d\mathrm{vol}(x) \leq C(n, \sigma, R) \int_{B(m,3R)} |\nabla\psi(x)|^\sigma d\mathrm{vol}(x), \tag{3.17}$$

and thus allows us to conclude that

$$\int_{B(m,R)} |\psi(x) - \psi_R|^\sigma d\mathrm{vol}(x) \leq K_1\, R^\beta \int_{B(m,3R)} |\nabla\psi(x)|^\sigma d\mathrm{vol}(x). \tag{3.18}$$

where

$$K_1 := \frac{1}{r_0^\beta} sup_{r_0 \leq R \leq R_1} C(n, \sigma, R) \tag{3.19}$$

is a constant which depends on $r_0$, $r_1$, as well as the local Poincaré function $C(n, \sigma, R)$ of $M^n$.

We now consider the second case where $R \geq R_1$. Let $\eta \in \mathbb{R}$ be a constant to be determined later. Let $\mathbb{1}_U$ denote that characteristic function of the set

$U$. Since $B(m,R) \subset \bigcup_{x\in X\cap B(m,R+\epsilon)} B(x,\varepsilon)$, we have

$$\begin{aligned}\int_{B(m,R)} &|\psi(z)-\eta|^\sigma d\mathrm{vol}(z)\\ &\le \int_{B(m,R)} \sum_{x\in X\cap B(m,R+\epsilon)} |\psi(z)-\eta|^\sigma \mathbb{1}_{B(x,\epsilon)}(z) d\mathrm{vol}(z)\\ &\le \sum_{x\in X\cap B(m,R+\epsilon)} \int_{B(x,\epsilon)} |\psi(z)-\eta|^\sigma d\mathrm{vol}(z).\end{aligned} \tag{3.20}$$

For any positive numbers $u, t$ and $\sigma$ an integer, Minkowski's inequality asserts that

$$|u-t|^\sigma \le 2^{\sigma-1}(|u|^\sigma + |t|^\sigma).$$

It then follows that

$$\begin{aligned}\int_{B(m,R)} &|\psi(z)-\eta|^\sigma d\mathrm{vol}(z)\\ &\le 2^{\sigma-1} \sum_{x\in X\cap B(m,R+\epsilon)} \int_{B(x,\epsilon)} |\psi(z)-\tilde{\psi}(x)|^\sigma d\mathrm{vol}(z)\\ &\quad + 2^{\sigma-1} \sum_{x\in X\cap B(m,R+\epsilon)} \mu(x)|\tilde{\psi}(x)-\eta|^\sigma.\end{aligned} \tag{3.21}$$

Let us denote by (I) and (II), the first term and the second term composing the right-hand side of the last inequality, respectively.

One can bound (I) from above by using the local Poincaré inequality (1.15) with $R=\epsilon$ and

$$C_1 = 2^{\sigma-1}C(n,\sigma,\epsilon), \tag{3.22}$$

to obtain

$$(I) \le C_1 \sum_{x\in X\cap B(m,R+\epsilon)} \int_{B(x,3\epsilon)} |\nabla\psi|^\sigma d\mathrm{vol}(z)\,. \tag{3.23}$$

By lemma 2.11, the multiplicity of the covering $\{B(x,3\epsilon)\}_{x\in X}$ is bounded by $\mathcal{M}(\epsilon)$. Since $\epsilon \le R$, we have that for each $x \in B(m,R+\epsilon)$, $B(x,3\epsilon) \subset$

$B(m, 5R)$. Therefore, we have

$$(I) \leq C_1 \mathcal{M}(\epsilon) \int_{B(m,5R)} |\nabla \psi|^\sigma d\mathrm{vol}(z). \tag{3.24}$$

We prove that the second term (II) is bounded in the following way. Since $(M, d, \mathrm{vol})$ and $(X, \rho, \mu)$ are roughly isometric, we can choose $x_0 \in X$ such that $d(m, x_0) \leq \varepsilon$. By choosing $r = R + 2\epsilon + 1$ and $C_3 = \frac{3\epsilon+1}{\epsilon} = 4$, we have (since $R \geq R_1 = \max\{\epsilon, r_1\} \geq \epsilon$ and $\epsilon = 1$)

$$R + 2\epsilon \leq r \leq C_3 R = 4R, \tag{3.25}$$

so that

$$X \bigcap B(m, R + \epsilon) \subset B(x_0, r), \tag{3.26}$$

and for any constant $C$ we have

$$B(x_0, Cr) \subset B(m, (CC_3 + 1)R) = B(m, (4C + 1)R). \tag{3.27}$$

In order to apply the discrete Poincaré inequality in the context of (II), let us choose

$$\eta = \tilde{\psi}_r = \frac{1}{\mu(B(x_0, r))} \sum_{x \in B(x_0, r)} \mu(x) \tilde{\psi}(x). \tag{3.28}$$

By assumption, $(X, \rho, \mu)$ satisfies a $(\sigma, \beta, \sigma)$-Poincaré inequality (see Definition 2.26) and since $1 < R_1 < R + \epsilon \leq r$, we obtain

$$(II) \leq 2^{\sigma-1} C r^\beta \sum_{x \in B(x_0, C' r)} \mu(x) |\delta \tilde{\psi}(x)|^\sigma. \tag{3.29}$$

Therefore, Inequality (3.6) of Lemma 3.5 and the fact that $r \leq C_3 R = 4R$ imply

$$(II) \leq 2^{\sigma-1} (C \mathcal{T}^\sigma 4^\beta) R^\beta \int_{B(x_0, C' \mathcal{T}' r)} |\nabla \psi(z)|^\sigma d\mathrm{vol}(z). \tag{3.30}$$

We now claim that

$$\int_{B(m,R)} |\psi(z) - \psi_R|^\sigma d\mathrm{vol}(z) \leq 2^\sigma \inf_{\tau \in \mathbb{R}} \int_{B(m,R)} |\psi(z) - \tau|^\sigma d\mathrm{vol}(z), \tag{3.31}$$

where $\psi_R = \psi_{B(m,R)}$.

Indeed, for any $\tau \in \mathbb{R}$, by applying Jensen's inequality we have

$$\begin{aligned}\|\tau - \psi_R\|_{\sigma,B(m,R)} \\ = \left(\int_{B(m,R)} \left(\frac{1}{\mathrm{vol\,B(m,R)}}\left|\int_{B(m,R)}(\psi(y)-\tau)d\mathrm{vol}(y)\right|\right)^{\sigma} d\mathrm{vol}(z)\right)^{\frac{1}{\sigma}} \\ \leq \|\psi - \tau\|_{\sigma,B(m,R)}.\end{aligned} \tag{3.32}$$

Furthermore, Minkowski's inequality implies that

$$\begin{aligned}\|\psi - \psi_R\|_{\sigma,B(m,R)} &\leq \|\psi - \tau\|_{\sigma,B(m,R)} + \|\tau - \psi_R\|_{\sigma,B(m,R)} \\ &\leq 2\|\psi - \tau\|_{\sigma,B(m,R)},\end{aligned} \tag{3.33}$$

hence, the claim follows.

Now let us define

$$C_2 = C_1\mathcal{M}(\epsilon), \tag{3.34}$$

$$C_4 = 2^{\sigma-1}C\mathcal{T}^{\sigma}C_3^{\beta} = 2^{\sigma-1}(C\mathcal{T}^{\sigma}4^{\beta}) \tag{3.35}$$

and

$$C_5 = \max\{C_2, C_4\}. \tag{3.36}$$

Inequalities (3.24), (3.30) and the claim imply that

$$\begin{aligned}\int_{B(m,R)} |\psi(z) - \psi_R|^{\sigma} d\mathrm{vol}(z) \\ \leq 2^{\sigma}\left(C_2\int_{B(m,5R)}|\nabla\psi(z)|^{\sigma}d\mathrm{vol}(z) + C_4R^{\beta}\int_{B(x_0,C'\mathcal{T}'r)}|\nabla\psi(z)|^{\sigma}d\mathrm{vol}(z)\right),\end{aligned} \tag{3.37}$$

hence, by applying (3.27) we obtain

$$\begin{aligned}\int_{B(m,R)} |\psi(z) - \psi_R|^{\sigma} d\mathrm{vol}(z) \\ \leq 2^{\sigma}C_5\left(\int_{B(m,5R)}|\nabla\psi(z)|^{\sigma}d\mathrm{vol}(z) + R^{\beta}\int_{B(m,(C'\mathcal{T}'C_3+1)R)}|\nabla\psi(z)|^{\sigma}d\mathrm{vol}(z)\right).\end{aligned} \tag{3.38}$$

This implies that for any ball of radius $R \geq \epsilon$ we have

(3.39)
$$\int_{B(m,R)} |\psi(z) - \psi_R|^\sigma d\text{vol}(z) \leq K_2\, R^\beta \int_{B(m,(C'\mathcal{T}'C_3+5)R)} |\nabla\psi(z)|^\sigma d\text{vol}(z),$$

where

$$K_2 := 2^\sigma \left(\frac{C_5}{\epsilon^\beta} + C_5\right). \tag{3.40}$$

Inequalities (3.18) and (3.39) then give the required $(\sigma, \beta, \sigma)$-uniform Poincaré inequality

$$\int_{B(m,R)} |\psi(x) - \psi_R|^\sigma d\text{vol}(x) \leq K R^\beta \int_{B(m,C''R)} |\nabla\psi(x)|^\sigma d\text{vol}(x), \tag{3.41}$$

where

$$K = \max\{K_1, K_2\} \tag{3.42}$$

and

$$C'' = C'\mathcal{T}'C_3 + 5 = 4C'\mathcal{T}' + 5. \tag{3.43}$$

This concludes the proof of Theorem 3.1. □

Bishop-Gromov Comparison Inequality implies that for any complete Riemannian manifold $M^n$ with Ricci curvature bounded below $\text{Ricci}_{M^n} \geq -(n-1)k$, we have

$$\text{vol}\, B(m, 2R) \leq 2^n \exp((n-1)\sqrt{k}2R)\, \text{vol}\, B(m, R)), \tag{3.44}$$

i.e., $M^n$ is locally doubling. By Theorem 1.14, such a manifold also satisfies the local Poincaré inequality. We may therefore state

**Proposition 3.45.** *Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded below,* Ricci $\geq -(\text{n}-1)\kappa$*, then $M^n$ satisfies the local Poincaré inequality (1.15) and the local doubling property with constants depending on $n$ and $\kappa$.*

By applying the assertion of Theorem 3.1 we obtain

**Corollary 3.46.** *Let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded below. Then $M^n$ satisfies a $(\sigma, \beta, \sigma)$-uniform Poincaré inequality (1.6) if and only if an $\epsilon$-discretization $(X, \rho, \mu)$ of $M$ satisfies the discrete uniform analogue (2.26).*

## 4. Poincaré inequality for metric measured graphs

In this section, we prove that metric measured graphs which satisfy a certain growth condition, polynomial growth, support discrete versions of Poincaré inequalities as (2.26). In the applications, such graphs serve as discrete approximations to a complete Riemannian manifold. These graphs satisfy the conditions needed in order to apply the work in [5] to the proof of Theorem 1.7.

Let $(X, \rho, \mu)$ be a metric measured graph; $(X, \rho, \mu)$ will be said to have $\alpha$-polynomial growth if Inequality (1.2) holds with respect to the metric $\rho$ and the measure $\mu$. Let $V, E$ denote the set of vertices and (non-oriented) edges of $X$, respectively. We will write $x \sim y$ when $[x, y] \in E$, where $[x, y]$ denotes the directed edge from $x$ to $y$. Given a function $u : V \to \mathbb{R}$, we let $du : E \to \mathbb{R}$ denote the *gradient* of $u$ defined by $du([x, y]) = u(y) - u(x)$. Let us recall that we defined (see Definition 2.23)) the *length* of the gradient of $u$ at a vertex $x \in V$ to be

$$\delta u(x) = \left( \sum_{y \sim x} |u(y) - u(x)|^2 \right)^{1/2} . \tag{4.1}$$

Since $X$ is a discrete space, we can integrate any function $g$ on any subset $F \subset V$ with the restriction of $\mu$ to $F$. For the counting measure on $X$, we define the integration as $\int_F g = \sum_{x \in F} g(x)$.

We now establish a $(\sigma, \alpha + \sigma - 1, \sigma)$-Poincaré inequality of type (2.26). As mentioned in the introduction, the proof of the following Theorem is very similar to the one in [6, p. 310]

**Theorem 4.2.** *Let $(X, \rho, \mu)$ be a metric measured graph with $\alpha$-polynomial growth, namely, for some $R_0 > 0$ and any $R \geq R_0$, we have $\mu(B(x, R)) \leq v' R^\alpha$. We also assume that $\mu(x) \geq \omega$. Then for any $\sigma \geq 1$, for any function $u : X \to \mathbb{R}$, $R \geq R_0$ and any ball $B(p, R) \subset X$, we have*

$$\int_{B(p,R)} |u(x) - u_R|^\sigma d\mu(x) \leq 6^{\sigma - 1} v' \omega^{-1} R^{\alpha + \sigma - 1} \int_{B(p,3R)} |\delta u(x)|^\sigma d\mu(x), \tag{4.3}$$

*where* $u_R = u_{B(p,R)}$.

*Proof.* Let $\gamma_{x,y}$ is a minimizing geodesic joining $x$ to $y$. By the definition of the length of the gradient of $u$, we have

$$|u(x) - u(y)| \leq \int_{\gamma_{x,y}} |\delta u|. \tag{4.4}$$

We also have

$$\begin{aligned}\int_{B(p,R)} &|u(x) - u_R|^\sigma d\mu(x) \\ &= \frac{1}{\mu(B(p,R))^\sigma} \int_{B(p,R)} \left| \int_{B(p,R)} (u(x) - u(y)) d\mu(y) \right|^\sigma d\mu(x).\end{aligned} \tag{4.5}$$

Hence, by normalizing the measures involved to have total mass equal to one and then applying Jensen's inequality twice, we obtain

$$\begin{aligned}\int_{B(p,R)} &|u(x) - u_R|^\sigma d\mu(x) \\ &\leq \frac{1}{\mu(B(p,R))} \int_{B(p,R)} \left( \int_{B(p,R)} \left( \int_{\gamma_{x,y}} |\delta u| \right)^\sigma d\mu(y) \right) d\mu(x).\end{aligned} \tag{4.6}$$

By applying Jensen's Inequality again to the innermost integral in equation (4.6) we get

$$\begin{aligned}\int_{B(p,R)} &|u(x) - u_R|^\sigma d\mu(x) \\ &\leq \frac{1}{\mu(B(p,R))} \int_{B(p,R)} \int_{B(p,R)} \ell_{x,y}^{\sigma-1} \left( \int_{\gamma_{x,y}} |\delta u|^\sigma \right) d\mu(y) d\mu(x),\end{aligned} \tag{4.7}$$

where $\ell_{x,y}$ is the length of the geodesic segment $\gamma_{x,y}$. Since $\gamma_{x,y} \subset B(p, 3R)$ for any $x, y \in B(p,R)$, it follows that $\ell_{x,y} \leq 6R$. Hence,

$$\int_{\gamma_{x,y}} |\delta u|^\sigma \leq \int_{B(p,3R)} |\delta u|^\sigma \leq \omega^{-1} \int_{B(p,3R)} |\delta u(x)|^\sigma d\mu(x), \tag{4.8}$$

the last inequality coming from our assumption $\mu(x) \geq \omega$.

It follows that

$$\int_{B(p,R)} |u(x) - u_R|^\sigma d\mu(x) \leq (6R)^{\sigma-1}\,\omega^{-1}\mu(B(p,R)) \int_{B(p,3R)} |\delta u(x)|^\sigma d\mu(x). \tag{4.9}$$

The polynomial growth assumption implies that

$$\int_{B(p,R)} |u(x) - u_R|^\sigma d\mu(x) \leq 6^{\sigma-1}v'\omega^{-1}R^{\alpha+\sigma-1} \int_{B(p,3R)} |\delta u(x)|^\sigma d\mu(x). \tag{4.10}$$

This ends the proof of the theorem. □

**Remark 4.11.** Inequality 4.8 can be stated because $X$ is discrete. Indeed, on a manifold the geodesic $\gamma_{x,y}$ and the ball $B(p, 4R)$ would have different dimensions. This inequality may seem crude, but we will show that it is in fact *optimal*. Indeed, in Section 6, we exhibit an example of a graph $X$ and a function $u : X \to \mathbb{R}$, such that for every $R$ and for a given $x_0 \in X$, the support of $\delta u$ in the ball $B(x_0, R)$ is a diameter $L$ and for most of couples $(x, y)$ in $B(x_0, R)$, the geodesic $\gamma_{xy}$ goes through $L$ so that $\int_{\gamma_{xy}} |\nabla u|^\sigma \approx \int_{B(x_0,R)} |\nabla u|^\sigma$. Consequently, Inequality 4.8 is essentially an equality.

## 5. Proofs of the main results

We start this section by proving Theorem 1.7. After recalling a few basic definitions from the general setting of Riemannian manifolds, we turn to the proof of Corollary 1.12.

### 5.1. A proof of Theorem 1.7.

Henceforth, we let $M^n$ be a complete Riemannian manifold with Ricci curvature bounded below, Ricci $\geq -(\mathrm{n}-1)\kappa$, and polynomial growth of order $\alpha$ $\mathrm{vol}B(m, R) \leq vr^\alpha$ for every $R \geq R_0$ (see (1.1) and (1.2), respectively) for some positive constants $v, \alpha$. By Proposition 3.45, $M^n$ satisfies the local doubling condition, $(DV)_{loc}$, and a local Poincaré inequality. We now consider an $\epsilon$-discretization $X$ of $M^n$ with $\epsilon = 1$.

Corollary 2.22 implies that $X$ has polynomial growth of order $\alpha$, i.e., there exists $R_0'$ such that for any $R \geq R_0'$, $\mu(B(x,R)) \leq v' R^\alpha$, where $v'$ depends on $n$, $v$ and $\kappa$, and $R_0'$ depends on $n$, $R_0$ and $\kappa$. By (1.3) and (2.8), we also have $\mu(x) = \mathrm{vol}(B(x,1)) \geq \omega$ for every $x \in X$. Let $p \in X$ be arbitrary. By Theorem 4.2, for every $R \geq r_1 = R_0'$, $X$ satisfies the Poincaré inequality (4.3):

$$(5.1)\quad \int_{B(p,R)} |u(x) - u_R|^\sigma d\mu(x) \leq 6^{\sigma-1} v' \omega^{-1} R^{\alpha+\sigma-1} \int_{B(p,3R)} |\delta u(x)|^\sigma d\mu(x).$$

Hence, we are in position to apply Theorem 3.1 and obtain

$$(5.2)\quad \int_{B(m,R)} |\psi(x) - \psi_R|^\sigma d\mathrm{vol}(x) \leq K R^\beta \int_{B(m,C''R)} |\nabla \psi(x)|^\sigma d\mathrm{vol}(x).$$

Let us explicitly summarize what $K$ in the above inequality depends on. Recall that the constant $K$ satisfies

$$K = \max\{\mathrm{K}_1, \mathrm{K}_2\}, \quad \mathrm{K}_1 = (1/\mathrm{r}_0^\beta)\mathrm{sup}_{\mathrm{r}_0 \leq \mathrm{R} \leq \mathrm{R}_1} \mathrm{C}(\mathrm{n}, \sigma, \mathrm{R})$$
$$\text{and} \quad K_2 = 2^\sigma \left( \frac{C_5}{\epsilon^\beta} + C_5 \right)$$

where $C_5 = \max\{C_2, C_4\}$, $C_2 = C_1 \mathcal{M}(\epsilon)$, $C_4 = 2^{\sigma-1} C \mathcal{T}^\sigma C_3^\beta = 2^{\sigma-1} 4^\beta C \mathcal{T}^\sigma$ (cf. (3.42), (3.19), (3.40), (3.36), (3.34) and (3.35)) with $C = 6^{\sigma-1} v' \omega^{-1}$ (cf. (4.3)). The constant $C''$ in (5.2) has been defined by $C'' = C' \mathcal{T}' C_3 + 5 = 4C'\mathcal{T}' + 5$ (cf. (3.43)) with $C' = 3$.

We deduce that the constant $K$ depends on $n, \sigma, r_0, R_0, \kappa$, and $v$ and the constant $C''$ depends on $n, \kappa$. This ends the proof of Theorem 1.7. □

## 5.2. Uniform and global Poincaré inequality for horospheres

We now turn to the proof of Corollary 1.12. Let $M^n$ be a $n$-dimensional closed Riemannian manifold with its negative sectional curvature uniformly satisfying

$$(5.3)\quad -a^2 \leq K \leq -b^2 < 0.$$

Let $\widetilde{M}^n$ be the universal cover of $M^n$, $T^1\widetilde{M}^n$ its unit tangent bundle, and $\pi : T^1\widetilde{M}^n \to \widetilde{M}^n$ the canonical projection. We denote by $\partial\widetilde{M}^n$ the ideal boundary of $\widetilde{M}^n$. For $v \in T^1\widetilde{M}^n$, let $\gamma_v(t)$ be the geodesic in $\widetilde{M}^n$ such that $\gamma_v(0) = \pi(v)$ and $\dot{\gamma}(0) = v$. Given a point $\xi = \gamma_v(-\infty) \in \partial\widetilde{M}^n$, and a base point $x_0 \in \widetilde{M}^n$, for all $\xi \in \partial\widetilde{M}^n$ and for all $x \in \widetilde{M}^n$, the Busemann function $B_\xi(\cdot)$ is then defined by $B_\xi(x) = \lim_{t\to-\infty}(d(x,\gamma_v(t)) - d(x_0,\gamma_v(t)))$. it is known that since $M^n$ is a closed negatively curved manifold, for each $\xi \in \partial\widetilde{M}^n$, the Busemann function $B_\xi(\cdot)$ is smooth. Furthermore, for any $t \in \mathbb{R}$, the level set $H_\xi(t) = \left\{x \in \widetilde{M}^n;\, B_\xi(x) = t\right\}$ is a smooth submanifold of $\widetilde{M}^n$ which is diffeomorphic to $\mathbb{R}^n$ and is called a *horosphere* centred at $\xi$ (the reader is referred to [7] for the necessary background). For each $v \in T^1\widetilde{M}^n$, let $W^{su}(v)$ denote the *strong unstable leave* of the geodesic flow on $T^1\widetilde{M}^n$. Recall that $\pi(W^{su}(v))$ can be identified with the horosphere $H_\xi(0)$ centered at $\xi = \gamma_v(-\infty)$ and passing through $\pi(v)$, that is $\pi(W^{su}(v)) = H_\xi(0)$.

For $t \in \mathbb{R}$, let $\exp_t : W^{su}(v) \to \widetilde{M}^n$ be the restriction of the exponential map to $H_\xi(0)$, i.e., for any unit vector $u \in W^{su}(v)$, $\mathrm{exp_t(u) = \gamma_u(t)}$.

**A proof of Corollary 1.12.** Let us consider a horosphere, $H := H_\xi$, centered at $\xi \in \partial\widetilde{M}^n$. We let $\rho$ denote the distance on $H$ determined by the induced Riemannian metric on $H$. Let us prove that $H$, endowed with $\rho$ and the corresponding induced vol measure (which by abuse of notation we will denote by vol), has the following polynomial growth: For every $R \geq 1$,

$$\operatorname{vol} B(p,R) \leq DR^\alpha, \quad \text{with } \alpha = \frac{(n-1)a}{b}, \tag{5.4}$$

where $B(p,R)$ is a ball in $H$ centered at $p$ and having radius $R$, and $D$ is a constant.

Our starting point is a distance comparison proposition due to E. Heintze and H. ImHof; the proof is a consequence of Rauch's comparison theorem, which can be applied due to the assumption on the sectional curvature of $\widetilde{M}$.

**Proposition 5.5 ([11], Proposition 4.1).** *Let $u \in T^1\widetilde{M}^n$ be a unit tangent vector on $\widetilde{M}^n$ and let $v$, $w \in W^{su}(u)$ be two unit vectors in the strong unstable leaf of $u$. Then for all $t \geq 0$, the distance between $\gamma_v(t)$ and $\gamma_w(t)$ satisfies*

$$e^{bt}\rho(\gamma_v(0),\gamma_w(0)) \leq \rho(\gamma_v(t),\gamma_w(t)) \leq e^{at}\rho(\gamma_v(0),\gamma_w(0)). \tag{5.6}$$

The two following properties are immediate consequences of this proposition:

$$B(\gamma_u(t), e^{bt}) \subset \pi\left(\exp_t(\pi^{-1}B(\gamma_u(0),1))\right), \tag{5.7}$$

and

$$\text{vol}\, B(\gamma_u(t), e^{bt}) \leq \text{vol}\, \pi\left(\exp_t(\pi^{-1}B(\gamma_u(0),1))\right) \leq e^{(n-1)at}\text{vol}\, B(\gamma_u(0),1), \tag{5.8}$$

where $B(\gamma_u(0),1)$ and $B(\gamma_u(t)), e^{bt})$ are balls on the horosphere $H_\xi(0)$ and $H_\xi(t)$, respectively, with $\xi = \gamma_u(-\infty)$.

Therefore, if we let $R = e^{bt}$, $t \geq 0$ it follows that

$$\text{vol}\, B(\gamma_u(t), R) \leq R^{\alpha}\text{vol}\, B(\gamma_u(0),1). \tag{5.9}$$

Consider now the ball $B(x,R) \subset H_\xi(0)$ centered at $x = \pi v$ the base point of the unit tangent vector $v$. Let $u = \dot\gamma_v(-t)$ be the unit tangent vector such that $\gamma_u(t) = x$ From (5.9) we have

$$\text{vol}\, B(x, R) \leq R^{\frac{(n-1)a}{b}}\text{vol}\, B(\gamma_u(0),1). \tag{5.10}$$

The set $\{(\tilde H, p) \,|\, p \in \tilde H, \tilde H \in \widetilde{M^n}\}$ of pointed horospheres of $\widetilde{M^n}$ is homeomorphic to $T^1\widetilde{M^n}$ and, therefore, since $M$ is closed, is co-compact. It follows that there exists a positive constant $D$ such that

$$D = \sup\{\text{vol}\, B(p,1)\} < \infty, \tag{5.11}$$

where the supremum is taken over all balls of radius 1 on all horospheres. We then conclude that for every $R \geq 1$,

$$\text{vol}\, B(x,R) \leq DR^{\alpha}. \tag{5.12}$$

Hence, horospheres in $\widetilde{M^n}$ have uniform polynomial growth. Furthermore, since $M^n$ is closed, horospheres have uniform bounded sectional curvature (for the induced Riemannian metric) and in particular will have a uniform lower bound on their Ricci curvature. Therefore, any horosphere $H$ satisfies Poincaré inequality (1.6) with $\alpha$ as defined above. □

## 6. Examples

It is natural to ask if the inequalities we derived in Theorem 1.7 can be improved. In this section, we show that the assertions of this theorem is optimal in the sense that, when $\alpha \geq 1$ and $\sigma \geq 1$, one can construct a Riemannian manifold $M^n$ of Ricci curvature bounded below and polynomial growth of order $\alpha$ which does not carry a $(\sigma, \beta, \sigma)$ Poincaré inequality with $\beta < \alpha + \sigma - 1$. In fact, Theorem 4.2 is also optimal: we will construct a graph of polynomial growth of order $\alpha \geq 1$, which does not carry a $(\sigma, \beta, \sigma)$ Poincaré inequality with $\beta < \alpha + \sigma - 1$, for any $\sigma \geq 1$. Following this, we will construct a manifold $M^n$ that is roughly isometric to the graph. In these examples, we will assume for simplicity that $\alpha = 2$, the general case can be done in the same way.

To this end, let us first construct a planar embedded graph $G$ with a quadratic growth. Let $\mathbb{R}^2$ be endowed with the Euclidean metric. The graph $G$ is the following "antenna like" embedded in this $\mathbb{R}^2$ as

$$G = \{x = 0\} \cup_{n \in \mathbb{Z}} \{y = n\}. \tag{6.1}$$

The vertex set $V$ of $G$ is defined by $V = \{(m, n) \mid m,\, n \in \mathbb{Z}\}$. An edge of $G$ is either a vertical segment joining $(0, n)$ and $(0, n+1)$, $n \in \mathbb{Z}$, or a horizontal segment joining $(n, m)$ and $(n+1, m)$, $n,\, m \in \mathbb{Z}$. In particular, there is no vertical edge joining $(n, m)$ and $(n, m+1)$ for $n \neq 0$.

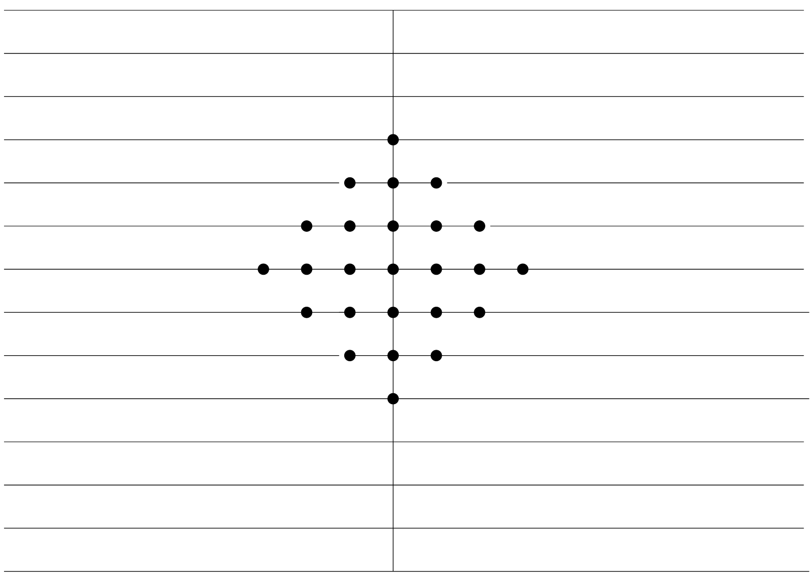

Figure 6.2: A ball of radius 4 in $G$.

The distance $d$ on $G$ is the intrinsic distance induced by the embedding of $G$ in $\mathbb{R}^2$,

$$d((m,n),(m',n')) = |m| + |m'| + |n-n'|, \tag{6.3}$$

and the measure on $G$ is the counting measure.

Given two functions $f$, $g : [0,+\infty[\to [0,+\infty[$, we will write $f \asymp g$ if there exists a constant $c > 0$ such that $f(R) \leq cg(R)$ and $g(R) \leq cf(R)$ for $R$ large enough. It is easy to check that the volume of an open ball of radius $R$ in $G$ centered on the $\{x = 0\}$ axis satisfies

$$V(R) = 2\big(1 + 3 + \ldots + (2R+1)\big) - (2R+1) \asymp R^2, \tag{6.4}$$

and that balls far away from this axis have linear growth.

We now construct a manifold model for $G$. Consider $G \subset \mathbb{R}^2 \subset \mathbb{R}^3$. For $\epsilon > 0$ small enough, the set $S_\epsilon$ of points in $\mathbb{R}^3$ at distance $\epsilon$ from $G$ is a smooth surface. The surface $S_\epsilon$ inherits a Riemannian metric induced by the metric of $\mathbb{R}^3$ so that $S_\epsilon$ is made of flat cylinders attached together at the vertices $(0,n)$, $n \in \mathbb{Z}$.

Note that the graph and the surface are embedded in $\mathbb{R}^3$ and that the projection of the surface on the graph is Lipschitz. Consider now the graph in $\mathbb{R}^2$ (as a horizontal plane in $\mathbb{R}^3$). Then, the vertical projection up from the graph to the surface is also Lipschitz. The first map is surjective and the second map has an image whose $2\epsilon$-neighborhood covers the surface $S_\epsilon$, so the graph and the surface are roughly isometric.

Let us now define a function $u : V \to \mathbb{R}$ such that for any positive constant $C > 0$ and any $\beta < \alpha + \sigma - 1$, $\sigma \geq 1$,

$$\lim_{R\to\infty} \left(\int_{B(R)} |u - u_R|^\sigma\right)\left(R^\beta \int_{B(CR)} |\nabla(u)|^\sigma\right)^{-1} = \infty. \tag{6.6}$$

The function $u$ is given by

$$u(m,n) = n, \quad \text{for all } m,n \in \mathbb{Z}, \tag{6.7}$$

for any horizontal edge, $u$ is defined to be its value on one of the endpoints. Finally, on vertical edges, $u$ is defined by extending its value at the endpoints linearly.

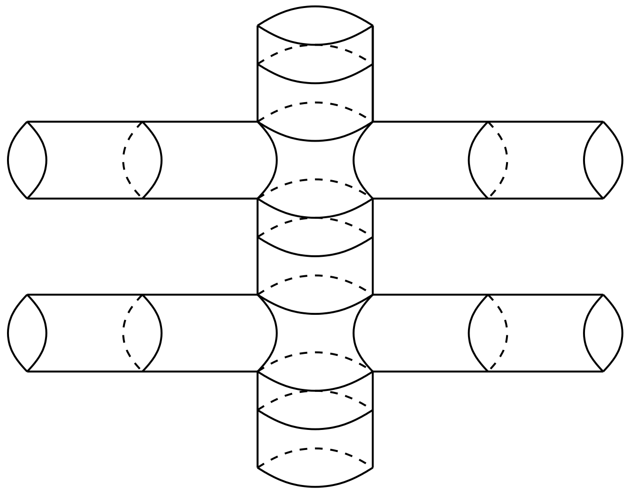

Figure 6.5: Part of the surface $S_\epsilon$.

**Lemma 6.8.** *For any positive number $C$,*

$$\int_{B(CR)} |\nabla(u)|^\sigma \asymp R, \tag{6.9}$$

*and*

$$\int_{B(R)} |u - u_R|^\sigma \asymp R^{\sigma+2}, \tag{6.10}$$

*where the balls $B(R)$ and $B(CR)$ are centered at $(0,0)$. The relation (6.6) follows immediately.*

*Proof.* The first estimate follows by a simple counting argument. The second estimate follows by comparing the sum with (since $u_R = 0$) $\int_0^N (2x+1)(N-x)^\sigma dx$. □

With $S_\epsilon$ as defined as above, one argues as before that a relation analogous to (6.6) holds. Thus, the assertion of Theorem 1.7 is optimal.

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