

Extremal weight projectors

HOEL QUEFFELEC AND PAUL WEDRICH

We introduce a quotient of the affine Temperley-Lieb category that encodes all weight-preserving linear maps between finite-dimensional \mathfrak{sl}_2 -representations. We study the diagrammatic idempotents that correspond to projections onto extremal weight spaces and find that they satisfy similar properties as Jones-Wenzl projectors, and that they categorify the Chebyshev polynomials of the first kind. This gives a categorification of the Kauffman bracket skein algebra of the annulus, which is well adapted to the task of categorifying the multiplication on the Kauffman bracket skein module of the torus.

1. Introduction

The Lie algebra \mathfrak{sl}_2 and its universal enveloping algebra $U(\mathfrak{sl}_2)$ are ubiquitous in representation theory, topology and physics, and their study typically starts with the description of the monoidal category $\text{Rep}(\mathfrak{sl}_2)$ of finite-dimensional \mathfrak{sl}_2 -representations. The *representation ring* of \mathfrak{sl}_2 , that is, the Grothendieck group $K_0(\text{Rep}(\mathfrak{sl}_2))$ with multiplication inherited from the tensor product of representations, is isomorphic to the polynomial ring $\mathbb{Z}[X]$ generated by the class of the vector representation $X = [V]$.

Two bases of $\mathbb{Z}[X]$, triangularly equivalent (over \mathbb{Q}) to the monomial basis, play a prominent role, especially in quantum topological applications:

- the Chebyshev polynomials of the first kind $L_m \in \mathbb{Z}[X]$, recursively defined as $L_0 = 2$, $L_1 = X$ and $L_m = XL_{m-1} - L_{m-2}$ for $m \geq 3$;
- the Chebyshev polynomials of the second kind $J_m \in \mathbb{Z}[X]$, defined to satisfy the same recurrence, but with initial conditions $J_0 = 1$, $J_1 = X$.

The basis elements J_m appear in $K_0(\text{Rep}(\mathfrak{sl}_2))$ as the classes of the simple representations $\text{Sym}^m(V)$. Since tensor powers of V can be decomposed into simple representations, this implies that the basis change in $\mathbb{Z}[X]$ from the monomial basis to $\{J_m\}$ has positive coefficients.

It is easy to see that the basis change from $\{J_m\}$ to $\{L_m\}$ also has positive coefficients, and in fact $J_m = L_m + J_{m-2}$ for $m \geq 2$. The element

L_m of the representation ring, thus, corresponds to the formal difference of the classes of $\text{Sym}^m(V)$ and $\text{Sym}^{m-2}(V)$, which we interpret as the class of the *extremal weight space* of $\text{Sym}^m(V)$, that is, the direct sum of the highest and lowest weight spaces. Another indication that the basis $\{L_m\}$ is more fundamental than the basis $\{J_m\}$ is given by comparing the multiplication rules:

$$(1.1) \quad \begin{aligned} J_m J_n &= J_{m+n} + J_{m+n-2} + \cdots + J_{|m-n|} \\ \text{and } L_m L_n &= L_{m+n} + L_{|m-n|}. \end{aligned}$$

The representation category $\text{Rep}(\mathfrak{sl}_2)$ admits a diagrammatic generators-and-relations description as the Karoubi envelope of the famous Temperley-Lieb category **TL**, which encodes intertwiners between tensor powers of the vector representation, see Section 2.1. In this description, the simple representations $\text{Sym}^m(V)$ are obtained through the Jones-Wenzl projectors $P_m \in \mathbf{TL}$, which correspond to the idempotent intertwiners:

$$V^{\otimes m} \twoheadrightarrow \text{Sym}^m(V) \hookrightarrow V^{\otimes m}.$$

In summary, the Karoubi envelope of the Temperley-Lieb category gives a diagrammatic categorification of the polynomial ring and the Jones-Wenzl projectors P_m categorify the polynomials J_m . The purpose of this paper is to give a description of analogs of the Jones-Wenzl projectors for the Chebyshev polynomials of the first kind L_m , which we call *extremal weight projectors*.

1.1. Main results

We define a diagrammatic extension $\mathbf{ATL}^{\text{ess}}$ of the Temperley-Lieb category, which utilizes diagrams in an annulus. More precisely, the monoidal category $\mathbf{ATL}^{\text{ess}}$ is defined as the quotient of the affine Temperley-Lieb category **ATL** by the ideal generated by the relation that an essential circle in the annulus is zero, see Definition 5 for details. In this category, we give a very simple recursive diagrammatic definition of morphisms T_m for $m \geq 0$, which is reminiscent of, but simpler than the recursive definition of the Jones-Wenzl projectors. Our main results are:

- $\mathbf{ATL}^{\text{ess}}$ describes the category of weight-preserving linear maps between \mathfrak{sl}_2 -representations, or in other words, the representation category $\text{Rep}(\mathfrak{h}) \supset \text{Rep}(\mathfrak{sl}_2)$ of the Cartan $\mathfrak{h} \subset \mathfrak{sl}_2$;
- the elements T_m are idempotents, corresponding to projections onto extremal weight spaces;

- in the Karoubi envelope of $\mathbf{ATL}^{\text{ess}}$, there are isomorphisms $T_m \otimes T_n \cong T_{m+n} \oplus T_{|m-n|}$, which categorify the multiplication rule in (1.1).

The simple definition of these idempotents makes them natural objects to study. Additionally, we expect that they will play a central role in categorifying skein algebras, as we will explain next.

1.2. Skein modules and skein algebras

Let M be an orientable 3-manifold and $R := \mathbb{Z}[q^{\pm 1/2}]$. Following Przytycki [18] and Turaev [24], the Kauffman bracket skein module $\text{Sk}_q(M)$ of M is constructed by forming the free R -module spanned by isotopy classes of framed links in M and taking the quotient by the Kauffman bracket skein relations [12] supported in 3-balls in M :

$$(1.2) \quad \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = q^{1/2} \left| \begin{array}{c} | \\ | \end{array} \right| + q^{-1/2} \begin{array}{c} \cup \\ \cup \end{array}, \quad \bigcirc = -(q + q^{-1})$$

For manifolds with boundary, one may take tangles with endpoints in a specified discrete set of boundary points as spanning set for a corresponding *relative skein module*. Thus, the categories \mathbf{TL} and $\mathbf{ATL}^{\text{ess}}$ can be seen as being built from skein modules of thickened squares or thickened annuli with prescribed boundary points, at $q = 1$. The multiplication is induced from gluing these 3-manifolds along the vertical part of their boundary.

We write $\text{Sk}_q(\Sigma)$ for the skein modules of thickened surfaces $\Sigma \times [0, 1]$. These become *skein algebras* with the multiplication operations obtained by stacking two copies of the thickened surface i.e. gluing along the horizontal part of their boundary. The skein algebra $\text{Sk}_q(\mathbb{R}^3)$ is free of rank one over R and the class of a framed link is given by its Kauffman bracket evaluation, which recovers the Jones polynomial. The skein algebra $\text{Sk}_q(A)$ of the annulus without boundary points is isomorphic to the polynomial ring $R[X]$ generated by the core curve of the annulus, and thus to the representation ring $K_0(\text{Rep}(\mathfrak{sl}_2)) \otimes R$. Elements of the skein of the annulus can be used for satellite operations on framed knots in skein modules. For example, decorating knots in $\text{Sk}_q(\mathbb{R}^3)$ by the element corresponding to $J_m \in \mathbb{Z}[X]$ produces the m -th colored Jones polynomial as evaluation.

1.3. Positive bases for skein algebras and categorification

The Kauffman bracket skein module of the torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ has a basis given by collections of parallel non-separating simple closed curves on T .

Thus, as a R -module, it is isomorphic to a direct sum of countably many copies of $\mathrm{Sk}_q(\mathbf{A})$, one for each slope m/n on the torus. However, Frohman and Gelca [6] proved that the skein multiplication has a particular nice form in terms of the basis assembled from the Chebyshev polynomials L_d in the copies $\mathrm{Sk}_q(\mathbf{A})$. In fact, for a pair $(r, s) \in \mathbb{Z}^2$, they define $(r, s)_T$ to be the basis element $L_{\gcd(r, s)}$ of slope r/s in $\mathrm{Sk}_q(\mathbf{T})$ and show:

$$(1.3) \quad \begin{aligned} (m, n)_T * (r, s)_T &= q^{(ms-rn)/2} (m+r, n+s)_T \\ &\quad + q^{(rn-ms)/2} (m-r, n-s)_T \end{aligned}$$

This formula, taken at roots of unity, plays a central role in Bonahon and Wong's *miraculous cancellations* [3] and the related work of Lê [16], in connection to cluster algebra. Also in relation with cluster algebras, for arbitrary closed oriented surfaces Σ , Thurston showed that the elements $L_d \in \mathrm{Sk}(\mathbf{A})$, embedded along simple closed curves in Σ , give a basis for the skein algebra $\mathrm{Sk}_{q=1}(\Sigma)$ with positive structure constants [23]. He also conjectured that the same holds for generic q (see also [15]) and asked whether this positivity phenomenon might be the shadow of a monoidal categorification of $\mathrm{Sk}_q(\Sigma)$. We propose a definition.

Definition 1. A *monoidal categorification* of $\mathrm{Sk}_q(\Sigma)$ consists of a graded, monoidal, additive category $\mathcal{C}(\Sigma)$ with a categorical action of the mapping class group of Σ , and an isomorphism of unital R -algebras $\psi: \mathrm{Sk}_q(\Sigma) \cong K_0(\mathcal{C}(\Sigma))$ that intertwines the natural mapping class group action on $\mathrm{Sk}_q(\Sigma)$ with the decategorification of the categorical action.¹ Here, the grading shift auto-equivalences on $\mathcal{C}(\Sigma)$ decategorify to multiplication by powers of q .

Thurston's positivity conjecture suggests that one should look for monoidal categorifications of $\mathrm{Sk}_q(\Sigma)$ with simple objects categorifying the cablings of simple closed curves by L_d . This is the main motivation for our construction of the highest weight projectors. Indeed, in Section 2.7 we describe a subcategory of $\mathbf{ATL}^{\mathrm{ess}}$, whose Karoubi envelope gives a monoidal categorification of $\mathrm{Sk}_q(\mathbf{A})$ (after introducing an extra grading), and the idempotents T_d categorify the elements L_d .

A monoidal categorification of a skein algebra becomes even more interesting if one can evaluate links in it. To make this precise, let $\mathrm{Link}(\Sigma)$ denote the monoidal category whose objects are given by embedded links L in $\Sigma \times [0, 1]$ and with morphisms given by compact, oriented bordisms

¹Thanks to Ben Cooper for pointing out the relevance of compatibility with the mapping class group actions.

in $\Sigma \times [0, 1] \times [0, 1]$ between links, modulo isotopy. The composition is the usual gluing of bordisms (in the second copy of $[0, 1]$) and the tensor product is induced by stacking links (and bordisms) along the thickening direction (the first copy of $[0, 1]$) and rescaling: $[0, 2] \rightarrow [0, 1]$. Moreover, $\text{Link}(\Sigma)$ carries a categorical action of the mapping class group of Σ . We now define a categorification of a skein algebra to consist of a monoidal categorification as in Definition 1 together with a Khovanov-type link homology functor ([13]) that is compatible with the monoidal structure.

Definition 2. A *categorification of the skein algebra* $\text{Sk}_q(\Sigma)$ consists of a monoidal categorification as in Definition 1, together with a monoidal functor $\text{Kh}: \text{Link}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$ that is compatible with the categorical mapping class group actions, such that the class of each link $L \subset \Sigma \times [0, 1]$ in $\text{Sk}_q(\Sigma)$ is sent to $K_0(\text{Kh}(L))$ under ψ .

Recall that Bar-Natan's construction of Khovanov homology admits an extension to the case of links in $\Sigma \times [0, 1]$, see [2, Section 11.6]. The target category $\mathcal{C}(\Sigma)$ of this construction is the homotopy category of chain complexes with chain groups generated by 1-manifolds embedded in Σ and differentials generated by cobordisms in $\Sigma \times [0, 1]$, modulo certain relations. This gives a categorification of the skein module $\text{Sk}_{-q}(\Sigma)$ and a proposal for a (projective) Khovanov functor $\text{Kh}: \text{Link}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$ (see also [1] for a related construction). However, it is not known whether $\mathcal{C}(\Sigma)$ can be equipped with a monoidal structure making Kh monoidal.

In a future paper [21] we will examine this problem in the case of the torus $\Sigma = \mathbb{T}$. In this case, Bar-Natan's cobordism category contains what we call *slope subcategories* — subcategories of 1-manifolds and cobordisms which are \mathbb{S}^1 -equivariant along a fixed slope. Taking the quotient by the \mathbb{S}^1 -action, it is not hard to see that the degree zero part of this subcategory is closely related to the affine Temperley-Lieb category **ATL**. The additional relation in **ATL**^{ess} is realized by setting the endo-cobordism of \emptyset that is given by the boundary parallel torus equal to zero. (Setting the incompressible torus equal to a scalar is sensible, because otherwise the endomorphism algebra of the empty object would not be finite-dimensional in each degree. Zero is a distinguished choice, because the resulting relation is homogeneous for an additional grading on morphisms by (relative) second homology.) The homotopy category over a corresponding quotient cobordism category, thus, by the results in Section 1.1, contains idempotents which categorify the Frohman-Gelca basis elements $(m, n)_T$. Moreover, it follows from

this translation and our results in Section 2.6 that these idempotents satisfy a categorified version of the Frohman-Gelca formula (1.3) in the case when they lie in the same slope. In future work we treat the general case of tensoring such categorified Frohman-Gelca basis elements, although we have to work in a slightly different setting to avoid functoriality problems. We also speculate that a similar construction should be possible for other closed surfaces Σ .

2. Affine Temperley-Lieb and extremal weight projectors

2.1. Temperley-Lieb categories

We start by recalling classical results about the Temperley-Lieb category and its affine version at $q = 1$.

Definition 3. Let $\mathbf{TL}(m, n)$ denote the free \mathbb{C} -module spanned by planar matchings in a box with m boundary points on the bottom and n boundary points on the top (and no closed components). The Temperley-Lieb category \mathbf{TL} is the \mathbb{C} -linear strict monoidal category with objects $n \in \mathbb{N}$, morphism spaces $\mathbf{TL}(n, m)$, and composition $\mathbf{TL}(m, n) \times \mathbf{TL}(l, m) \rightarrow \mathbf{TL}(l, n)$ is defined on the bases of matchings by composing planar tangles and reducing each resulting closed components to -2 . The monoidal structure is defined on objects by $m \otimes n = m + n$ and on morphisms by placing diagrams side by side.

Example 4. We illustrate the tensor product of planar matchings

$$\boxed{\text{diagram 1}} \otimes \boxed{\text{diagram 2}} = \boxed{\text{diagram 3}},$$

and give an example for the composition:

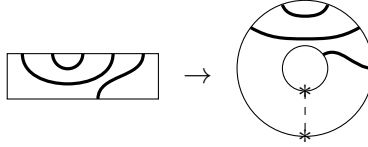
$$\boxed{\text{diagram 1}} \cdot \boxed{\text{diagram 2}} = -2 \boxed{\text{diagram 3}}$$

Stacking the two displayed matchings produces a closed component, which is reduced to -2 .

We write $\mathbf{TL}_n := \mathbf{TL}(n, n)$ for the Temperley-Lieb algebra on n strands. It has a presentation with generators U_i for $1 \leq i \leq n - 1$, which correspond to cap-cup matchings between the i -th and $i + 1$ -st strand. A complete set of relations is:

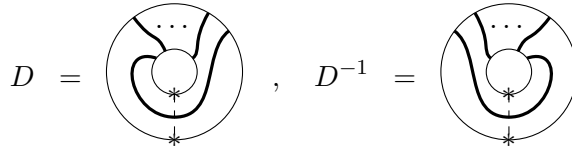
- $U_i U_{i+1} U_i = U_i$ for $1 \leq i \leq n-2$ and $U_i U_{i-1} U_i = U_i$ for $2 \leq i \leq n-1$,
- $[U_i, U_j] = 0$ for $|i-j| > 1$ and
- $U_i^2 = -2U_i$.

Gluing such boxes along their left and right boundaries, we get Temperley-Lieb diagrams in an annulus $A = \mathbb{S}^1 \times [0, 1]$ together with additional information: we remember the corners of the box as base points on the boundary circles and the glued edge as an arc connecting them.



Definition 5. The affine Temperley-Lieb category, denoted **ATL**, is the \mathbb{C} -linear category with objects $n \in \mathbb{N}$ and with morphism spaces **ATL** (m, n) spanned by crossingless tangles in the annulus A with m and n endpoints on the inside and outside boundary respectively, without inessential (i.e. homologically trivial) circles. The multiplication is defined on basis elements by stacking an annular diagram inside another one such that endpoints and basepoints match up, and then reducing inessential circles to -2 . We write **ATL** $_n := \mathbf{ATL}(n, n)$ for the affine Temperley-Lieb algebra on n strands.

Note that the affine Temperley-Lieb category can be obtained from the usual one by simply adding a new type of invertible morphism:



We write $U_0 = DU_1D^{-1} = D^{-1}U_{n-1}D$ for the cap-cup between the first and last strand. A complete set of relations for **ATL** $_n$ in terms of the generators U_i and the invertible generator D is given by:

- $U_i U_{i+1} U_i = U_i = U_i U_{i-1} U_i$,
- $[U_i, U_j] = 0$ for $i \neq j \pm 1$,
- $U_i^2 = -2U_i$,
- $U_i D = DU_{i+1}$

where all indices are taken modulo n .

Remark 6. There exists a vast literature about the Temperley-Lieb category, the Temperley-Lieb algebras and their affine versions. Our version of \mathbf{ATL}_n agrees with the one defined by Graham and Lehrer [7] at $q = 1$. Jones and Reznikoff [11] also study this version, although they use shaded diagrams which encode a parity. Our \mathbf{ATL}_n should not be confused with the Temperley-Lieb quotients of the Iwahori-Hecke algebras of affine type A, which are studied e.g. in [4, 5, 8], and which appear as the proper subalgebras of our \mathbf{ATL}_n generated by the U_i , see also Section 2.7. Instead, our \mathbf{ATL}_n are related to the *extended* affine Hecke algebras of type A. Also note that \mathbf{ATL} encodes crossingless tangles in the annulus up to isotopies which are the identity on the boundary. Algebras of Temperley-Lieb diagrams where isotopies need not be the identity on the boundary have e.g. been studied in [10].

2.2. Link with \mathfrak{sl}_2 -representation theory

It is well-known that the Temperley-Lieb category \mathbf{TL} describes a category of \mathfrak{sl}_2 -representations and their intertwiners. This can be traced back to [22]. Indeed, there exists a fully faithful \mathbb{C} -linear monoidal functor

$$\varphi: \mathbf{TL} \rightarrow \mathrm{Rep}(\mathfrak{sl}_2),$$

which sends n to the n -th tensor power of the vector representation $V = \mathbb{C}\langle v_+, v_- \rangle$. In the following we describe a version of the functor φ using the representation theory of the associative algebra $U(\mathfrak{sl}_2)$.

Recall that $U(\mathfrak{sl}_2)$ is the \mathbb{C} -algebra generated by E, F, H subject to the following relations:

$$(2.1) \quad HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H.$$

It is a Hopf algebra with coproduct and antipode determined by $\Delta(x) = 1 \otimes x + x \otimes 1$ and $S(x) = -x$ for $x \in \mathfrak{sl}_2$.

Denote $V = \mathbb{C}\langle v_-, v_+ \rangle$ the vector representation, with

$$\begin{aligned} Ev_+ &= 0, & Fv_+ &= v_-, & Hv_+ &= v_+, \\ Ev_- &= v_+, & Fv_- &= 0, & Hv_- &= -v_-, \end{aligned}$$

The dual $V^* = \mathbb{C}\langle v_-^*, v_+^* \rangle$ is isomorphic to V via the map that send $v_-^* \mapsto v_+$ and $v_+^* \mapsto -v_-$.

The \mathbb{C} -linear monoidal functor φ is determined by its images on cap and cup morphisms in **TL**. It associates to those the negatives of the natural evaluation and co-evaluation maps for duals, composed with the isomorphism $V \cong V^*$:

$$\bigcup \xrightarrow{\varphi} \begin{cases} \mathbb{C} \rightarrow V \otimes V \\ 1 \mapsto v_+ \otimes v_- - v_- \otimes v_+ \end{cases}, \quad \bigcap \xrightarrow{\varphi} \begin{cases} V \otimes V \rightarrow \mathbb{C} \\ v_{\mp} \otimes v_{\pm} \mapsto \pm 1 \\ v_{\pm} \otimes v_{\pm} \mapsto 0 \end{cases}$$

It is an easy exercise to check that these intertwiners of $U(\mathfrak{sl}_2)$ -representations satisfy the relations of the Temperley-Lieb category and so φ is a well-defined monoidal functor. The exchange of two tensor factors

$$s: V \otimes V \rightarrow V \otimes V, \quad v_{\varepsilon_1} \otimes v_{\varepsilon_2} \mapsto v_{\varepsilon_2} \otimes v_{\varepsilon_1}$$

is the image under φ of the following morphism in **TL**:

$$(2.2) \quad \times := \left| \right| + \smile$$

This suggestive notation encodes useful relations in **TL** and is compatible with planar isotopies:

$$\frown = \smile, \quad \psi = \bar{\psi}$$

Lemma 7. *The following analogs of Reidemeister moves hold in **TL**:*

$$\bigcirc = - \left| \right|, \quad \times = \left| \right|, \quad \times \times = \times \times$$

This implies that **TL** is a symmetric monoidal category and that φ is a symmetric monoidal functor.

Remark 8. The tensor product $V^{\otimes m}$ decomposes into a direct sum of irreducible representations, amongst which $\text{Sym}^m(V)$ appears with multiplicity 1 and has highest weight. This decomposition can be translated in terms of idempotents in **TL**_{*n*} and the idempotent corresponding to the copy of $\text{Sym}^m(V)$ is the famous Jones-Wenzl projector P_m [9, 25] with diagrammatic representation recursively defined by $P_1 = \text{id}_1$ and:

$$(2.3) \quad \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline P_{m+1} \\ \hline \cdot \cdot \cdot \\ \hline \end{array} = \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline P_m \\ \hline \cdot \cdot \cdot \\ \hline \end{array} + \frac{m}{m+1} \begin{array}{|c|} \hline \cdot \cdot \cdot \\ \hline P_m \\ \hline \cdot \cdot \cdot \\ \hline \end{array}$$

Dual to the decomposition into simple representations, one can decompose $V^{\otimes m}$ as the direct sum of its weight spaces. Of course, the projections onto weight spaces are usually not $U(\mathfrak{sl}_2)$ -equivariant, and then cannot be realized as the images of idempotents in the Temperley-Lieb category **TL**.

We will thus extend the functor φ from **TL** to **ATL** (and denote the extension again by φ) by sending the morphisms $D \in \mathbf{ATL}_m$ to certain vector space endomorphisms of $V^{\otimes m}$ which respect the weight space decomposition, but which break the $U(\mathfrak{sl}_2)$ -action². Since we want the monoidal structure of **ATL** to extend the monoidal structure of **TL**, the φ -image of $D \in \mathbf{ATL}_1$ actually determines $\varphi(D)$ for all $D \in \mathbf{ATL}_m$. The requirement that $\varphi(D)$ is invertible and respects the weight space decomposition of V implies that $\varphi(D)(v_{\pm}) = c_{\pm}v_{\pm}$ for some $c_{\pm} \in \mathbb{C}^*$. The proof of Lemma 10 shows that we need $c_+ = c_-^{-1}$ in order to obtain a well-defined functor that respects all isotopies of diagrams. The reason for the following choice will become clear in Proposition 11.

Definition 9. Let $V \otimes W$ be the image under φ of the domain of D and $W \otimes V$ its co-domain. Then we define $\varphi(D)$ to be the linear map determined by $v_{\pm} \otimes w \mapsto i^{\pm 1}w \otimes v_{\pm}$ for $v_{\pm} \in V$ and any $w \in W$, where $i^{\pm 1}$ are primitive fourth roots of unity. Furthermore we set $\varphi(D^{-1}) = \varphi(D)^{-1}$.

Let \mathfrak{h} denote the Cartan subalgebra of diagonal matrices in \mathfrak{sl}_2 and consider $U(\mathfrak{h}) = \langle H \rangle \subset U(\mathfrak{sl}_2)$. We denote by $\text{Rep}(\mathfrak{h})$ the category of finite-dimensional $U(\mathfrak{h})$ -representations of integral weights, i.e. vector spaces with a \mathbb{Z} -grading. Note that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{sl}_2$ induces a restriction functor $\text{Rep}(\mathfrak{sl}_2) \rightarrow \text{Rep}(\mathfrak{h})$ and that $\varphi(D)$ and $\varphi(D^{-1})$ are morphisms in $\text{Rep}(\mathfrak{h})$.

Lemma 10. *The functor $\varphi: \mathbf{ATL} \rightarrow \text{Rep}(\mathfrak{h})$ is well-defined and fits into the commutative diagram:*

$$\begin{array}{ccc} \mathbf{TL} & \xrightarrow{\varphi} & \text{Rep}(\mathfrak{sl}_2) \\ \downarrow & & \downarrow \\ \mathbf{ATL} & \xrightarrow{\varphi} & \text{Rep}(\mathfrak{h}) \end{array}$$

Proof. All morphisms in **ATL** are compositions of caps or cups between adjacent strands, as well as the morphisms D and D^{-1} . Any relation satisfied

²Setting $\varphi(D)$ to be a certain $U(\mathfrak{sl}_2)$ -intertwiner can be used to construct evaluation representations of certain affine Lie algebras. We stress that this is not what we have in mind.

by compositions of these generating morphisms is either supported in $\mathbf{TL} \subset \mathbf{ATL}$ (and is thus respected by φ) or involves some generators D and D^{-1} . Since φ maps D and D^{-1} to inverse isomorphisms, it suffices to check that φ respects the relation that D intertwines adjacent caps and cups. The only interesting case corresponds to isotoping caps or cups across the dashed arc. For cups we need the relation:

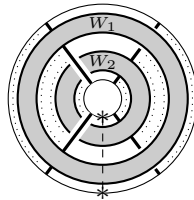
$$\varphi \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \varphi \left(\begin{array}{c} \text{Diagram 2} \end{array} \right)$$

Note that both sides annihilate vectors of the form $v_{\pm} \otimes w \otimes v_{\pm}$. On the other hand, the action on $v_{\pm} \otimes w \otimes v_{\mp}$ of the left-hand side and right-hand side, respectively, are given by:

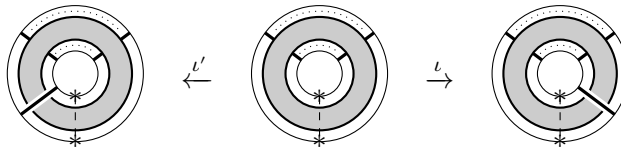
$$(2.4) \quad \begin{aligned} v_{\pm} \otimes w \otimes v_{\mp} &\mapsto i^{\pm 1} w \otimes v_{\mp} \otimes v_{\pm} \mapsto iw \quad \text{and} \\ v_{\pm} \otimes w \otimes v_{\mp} &\mapsto i^{\pm 1} v_{\mp} \otimes v_{\pm} \otimes w \mapsto iw \end{aligned}$$

The isotopy relation for cups is checked analogously. \square

Let $\otimes: \mathbf{ATL} \times \mathbf{ATL} \rightarrow \mathbf{ATL}$ denote the bi-functor given on objects by $(m, n) \mapsto m + n$ and on morphisms by superposing a pair of affine Temperley-Lieb diagrams (W_1, W_2) :



and resolving all crossings via (2.2). This is well-defined thanks to Lemma 7 and it induces a strict symmetric monoidal structure on \mathbf{ATL} such that both the inclusion $\mathbf{TL} \rightarrow \mathbf{ATL}$ and $\varphi: \mathbf{ATL} \rightarrow \text{Rep}(\mathfrak{h})$ become symmetric monoidal functors. Let ι and ι' denote the endo-functors of \mathbf{ATL} given by superposing a single strand on the right or left:

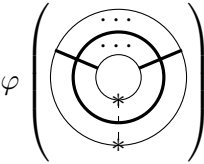


Note that these endo-functors can be written as $\iota'(-) = (-) \otimes 1$ and $\iota'(-) = D^{-1}\iota(-)D$.

2.3. A quotient

It is a key observation that the functor $\varphi: \mathbf{ATL} \rightarrow \text{Rep}(\mathfrak{h})$ is not faithful.

Proposition 11.

$$(2.5) \quad \varphi \left(\text{Diagram} \right) = 0.$$


Proof. This morphism is the composite of a cup, a sequence of crossings and a cap across the dashed arc. The first two components map the vector w to $v_+ \otimes w \otimes v_- - v_- \otimes w \otimes v_+$, which is killed by the cap across the dashed arc by (2.4). \square

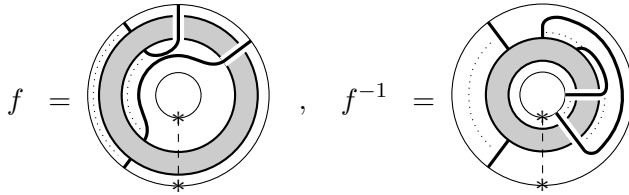
We define $\mathbf{ATL}^{\text{ess}}$ as the quotient of \mathbf{ATL} by the ideal generated by the elements shown in (2.5), with any number of through-strings. Note that the monoidal structure \otimes and the functor φ descend to the quotient $\mathbf{ATL}^{\text{ess}}$.

Theorem 12. *The functor $\varphi: \mathbf{ATL}^{\text{ess}} \mapsto \text{Rep}(\mathfrak{h})$ is fully faithful.*

Proof. We consider the isomorphisms

$$f: \mathbf{ATL}^{\text{ess}}(2n - m, m) \rightarrow \mathbf{ATL}^{\text{ess}}(0, 2n)$$

and their inverses, which are given by the following operations on diagrams.

$$f = \text{Diagram 1} , \quad f^{-1} = \text{Diagram 2}$$


Since these operations are given by tensoring with an identity morphism and then pre-composing with cups, or post-composing with caps, there are corresponding isomorphisms $\varphi(f^{\pm 1})$ between $\text{Hom}_{\text{Rep}(\mathfrak{h})}(V^{\otimes 2n-m}, V^{\otimes m})$ and

$\text{Hom}_{\text{Rep}(\mathfrak{h})}(\mathbb{C}, V^{\otimes 2n})$. To prove the theorem, it suffices to check that φ induces an isomorphism between the morphism spaces $\mathbf{ATL}^{\text{ess}}(0, 2n)$ and $\text{Hom}_{\text{Rep}(\mathfrak{h})}(\mathbb{C}, V^{\otimes 2n})$. Note that the former is spanned by annular crossingless matchings with no closed components, of which there are $\binom{2n}{n}$. To see this, we define a bijection between annular crossingless matchings and the set of labelings of their boundary points by an equal number of labels ‘in’ and ‘out’. This can be done by the following inverse rules:

- given a crossingless matching, find an isotopy representative and an orientation of the arcs so that they all turn clockwise around the annulus. Assign ‘in’ to the tail and ‘out’ to the head.
- given a labeling of boundary points, connect by an arc that only wraps in the clockwise direction around the annulus any pair of adjacent boundary points with labels ‘in’ and ‘out’ in that order. Then, iterate the process for all pairs of ‘in’- and ‘out’-labeled points in positions k and l whose neighbors in position $k+1$ and $l-1$ have already been connected by arcs.

On the $\text{Rep}(\mathfrak{h})$ side, $\text{Hom}_{\text{Rep}(\mathfrak{h})}(\mathbb{C}, V^{\otimes 2n}) \simeq (V^{\otimes 2n})_0$, the zero weight space, is of dimension $\binom{2n}{n}$, and spanned by vectors $v_\varepsilon := v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_{2n}}$ where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2n}) \in \{+, -\}^{2n}$ contains as many pluses as minuses. We shall argue that φ is surjective when restricted to $\mathbf{ATL}^{\text{ess}}(0, 2n)$, which also implies injectivity for dimension reasons.

By post-composition with the \mathfrak{S}_{2n} -action on both sides of the map φ , it is actually enough to show that one vector v_ε is hit. For if one is hit, since all the other ones lie in the same \mathfrak{S}_{2n} -orbit, they will be hit as well. We consider the case of $v_{+-+--\dots+-}$, and we will prove the result first for $n=1$:

$$\varphi \left(\text{diagram 1} \right) (1) = v_{+-} - v_{-+} \quad , \quad \varphi \left(\text{diagram 2} \right) (1) = iv_{+-} + iv_{-+}$$

Denoting w_1 and w_2 the two generators of $\mathbf{ATL}^{\text{ess}}(0, 2)$ used above, we see that v_{+-} is obtained as the image of $\varphi((w_1 - iw_2)/2)$ applied to $1 \in \mathbb{C}$.

Now the n -fold diagrammatic tensor power of $(w_1 - iw_2)/2$ gives a morphism in $\mathbf{ATL}^{\text{ess}}(0, 2n)$ whose φ image evaluates to $v_{+-+--\dots+-}$ on $1 \in \mathbb{C}$. \square

Remark 13. In particular, Theorem 12 shows that crossingless matchings form a basis for the Hom-spaces $\mathbf{ATL}^{\text{ess}}(0, 2n)$ and $\mathbf{ATL}^{\text{ess}}(2n, 0)$ that can be used to deduce explicit bases for all Hom-spaces. This is in contrast

to other (relative) skein modules, where Diamond Lemma arguments are necessary in order to prove that certain crossingless diagrams are linearly independent.

Example 14. In $\mathbf{ATL}_1^{\text{ess}}$ we have $D = -D^{-1}$ and thus $D^2 = -1$.

Proof. This follows directly from resolving the crossing in the defining relation:

$$0 = \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \quad \square$$

2.4. Extremal weight projectors

Definition 15. The elements $T_m \in \mathbf{ATL}_m^{\text{ess}}$ are recursively defined via $T_{m+1} = \iota(T_m)s_m\iota(T_m)$ for $m \geq 2$ with initial conditions $T_1 = \text{id}_1$ and $T_2 = \text{id}_2 + U_1/2 + U_0/2$:

$$(2.6) \quad T_2 = \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} + \frac{1}{2} \text{Diagram 3} \\ = \frac{1}{2} \left(\text{Diagram 4} + \text{Diagram 5} \right)$$

The recursive relation can be depicted as follows:

$$(2.7) \quad \text{Diagram 1} := \text{Diagram 2} = \text{Diagram 3} + \text{Diagram 4}$$

Theorem 16. The element $\varphi(T_m)$ is the endomorphism of $V^{\otimes m}$ projecting onto the extremal weight space $\mathbb{C}\langle v_{+...+}, v_{-...-} \rangle$.

Proof. For $m = 1$ this is tautological and for $m = 2$ we use the expression $T_2 = \text{id}_2 + U_1/2 + U_0/2$ to compute:

$$\begin{aligned}\varphi(T_2): v_{\pm\pm} &\mapsto v_{\pm\pm} + 0 + 0 = v_{\pm\pm} \\ v_{\pm\mp} &\mapsto v_{\pm\mp} + (v_{\mp\pm} - v_{\pm\mp})/2 + (-v_{\mp\pm} - v_{\pm\mp})/2 = 0\end{aligned}$$

For the induction step, we see immediately from the recursion that $\varphi(T_{m+1})$ annihilates $v_{\varepsilon_1\varepsilon_2\cdots\varepsilon_{m+1}}$ unless $\varepsilon_1 = \cdots = \varepsilon_m$. In the remaining cases we have:

$$\begin{aligned}\varphi(\iota(T_m)s_m\iota(T_m))(v_{\pm\cdots\pm\mp}) &= \varphi(\iota(T_m)s_m)(v_{\pm\cdots\pm\mp}) = \varphi(\iota(T_m))(v_{\pm\cdots\mp\pm}) = 0 \\ \varphi(\iota(T_m)s_m\iota(T_m))(v_{\pm\cdots\pm\pm}) &= \varphi(\iota(T_m)s_m)(v_{\pm\cdots\pm\pm}) \\ &= \varphi(\iota(T_m))(v_{\pm\cdots\pm\pm}) = v_{\pm\cdots\pm\pm}\end{aligned}$$

So $\varphi(T_{m+1})$ is the extremal weight projector. \square

Theorem 16, combined with Theorem 12, shows that $\mathbf{ATL}^{\text{ess}}$ describes weight-preserving maps for tensor products of \mathfrak{sl}_2 -representations, and that the idempotents T_m realize extremal weight projectors.

Lemma 17. *The elements T_m satisfy the following properties:*

- 1) $T_m^2 = T_m$
- 2) $T_m s_i = s_i T_m = T_m$ for $m \geq 2$
- 3) $T_m U_i = U_i T_m = 0$ for $m \geq 2$
- 4) $T_m \iota^{m-n}(T_n) = \iota^{m-n}(T_n) T_m = T_m$ for $1 \leq n < m$
- 5) $D^{-1} T_m D = T_m$.

Proof. The proof simply consists in applying φ and checking that the result holds in $\text{Rep}(\mathfrak{h})$. For example, $T_m U_i = 0$ holds since U_i contains a cup morphism, whose image under φ maps $1 \mapsto v_{+-} - v_{-+}$, which is clearly annihilated by the extremal weight projector. \square

We leave it to the interested reader to prove these properties diagrammatically inside $\mathbf{ATL}^{\text{ess}}$.

2.5. Chebyshev recursion

We now explicitly show that the projectors T_m categorify the Chebyshev polynomials of the first kind in the same sense as the classical Jones-Wenzl P_m projectors categorify the Chebyshev polynomials of the second kind.

Note that a different approach to the categorification of the polynomial ring and to orthogonal polynomials appears in [14].

Definition 18. The split Grothendieck group of an additive category \mathcal{C} is the abelian group $K_0(\mathcal{C})$ defined as the quotient of the free abelian group spanned by the isomorphism classes $[X]$ of objects X of \mathcal{C} , modulo the ideal generated by relations of the form $[A \oplus B] = [A] + [B]$ for objects A, B of \mathcal{C} .

If \mathcal{C} is monoidal, then $K_0(\mathcal{C})$ inherits a unital ring structure with multiplication $[A] \cdot [B] := [A \otimes B]$.

We have already mentioned the classical fact that $K_0(\text{Rep}(\mathfrak{sl}_2))$ is isomorphic to the polynomial ring $\mathbb{Z}[X]$ generated by the class $X = [V]$ of the vector representation. The isomorphisms

$$\text{Sym}^m(V) \otimes V \cong \text{Sym}^{m+1}(V) \oplus \text{Sym}^{m-1}(V) \text{ for } m \geq 1$$

imply that classes of the simple representations are given by the Chebyshev polynomials of the second kind $[\text{Sym}^m V] = J_m \in \mathbb{Z}[X]$. This can also be seen in the Grothendieck group of **TL**, at the cost of passing to the Karoubi envelope.

Definition 19. The Karoubi envelope of a category \mathcal{C} is the category $\text{Kar}(\mathcal{C})$ with objects given by pairs (X, e) , where X is an object of \mathcal{C} and $e \in \text{Hom}_{\mathcal{C}}(X, X)$ an idempotent. Morphisms between (X, e) and (Y, f) are of the form $f \circ g \circ e$ with $g \in \text{Hom}_{\mathcal{C}}(X, Y)$. If \mathcal{C} is additive or monoidal, then $\text{Kar}(\mathcal{C})$ inherits these structures.

Recall that **TL** is equivalent to the full subcategory of $\text{Rep}(\mathfrak{sl}_2)$ with objects given by $V^{\otimes m}$. Since any objects of $\text{Rep}(\mathfrak{sl}_2)$ appears as a direct summand of some $V^{\otimes m}$, it can be picked out by an idempotent endomorphism of $V^{\otimes m}$. Thus $\text{Kar}(\text{TL})$ is equivalent to $\text{Rep}(\mathfrak{sl}_2)$ and $K_0(\text{Kar}(\text{TL})) \cong \mathbb{Z}[X]$. Analogously, $\text{Kar}(\mathbf{ATL}^{\text{ess}})$ is equivalent to $\text{Rep}(\mathfrak{h})$. We now explicitly compute that the classes of the Jones-Wenzl projectors satisfy the recursion relation of the Chebyshev polynomials of the second kind.

Lemma 20. *In $K_0(\text{Kar}(\text{TL}))$, we have $[P_m][P_1] = [P_m \otimes P_1] = [P_{m+1}] + [P_{m-1}]$. Here and in the following, we abuse notation and write P_m for the object of $\text{Kar}(\text{TL})$ given by the pair (m, P_m) .*

Proof. We rewrite the recurrence (2.3) by subtracting the term with two projectors from both sides. The result is a decomposition of the idempotent

$P_m \otimes P_1$ into a sum of orthogonal idempotents in \mathbf{TL} , which induces an isomorphism between objects in $\text{Kar}(\mathbf{TL})$. After applying K_0 , we get:

$$\left[\begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array} \right] = \left[\begin{array}{c} \vdots \\ \vdots \\ \boxed{P_{m+1}} \\ \vdots \\ \vdots \end{array} \right] + \left[-\frac{m}{m+1} \begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array} \right]$$

On the left-hand side we already see $[P_m \otimes P_1]$ and the first term on the right-hand side is $[P_{m+1}]$. It now remains to prove that the idempotent shown in the second bracket on the right-hand side is isomorphic to P_{m-1} in $\text{Kar}(\mathbf{TL})$. In order to avoid confusion, we return to the pair-notation in $\text{Kar}(\mathbf{TL})$. The desired isomorphism in $\text{Kar}(\mathbf{TL})$ is given by:

$$(2.8) \quad \left(m+1, -\frac{m}{m+1} \begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array} \right) \xleftrightarrow{\begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array}} \left(m-1, \begin{array}{c} \vdots \\ \vdots \\ \boxed{P_{m-1}} \\ \vdots \\ \vdots \end{array} \right)$$

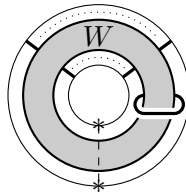
$-\frac{m}{m+1} \begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array}$

The verification that these maps give an isomorphism in $\text{Kar}(\mathbf{TL})$ uses the fact that $P_m \circ (P_{m-1} \otimes P_1) = P_m = (P_{m-1} \otimes P_1) \circ P_m$ in \mathbf{TL} as well as the partial trace formula:

$$\begin{array}{c} \vdots \\ \vdots \\ \boxed{P_m} \\ \vdots \\ \vdots \end{array} = -\frac{m+1}{m} \begin{array}{c} \vdots \\ \vdots \\ \boxed{P_{m-1}} \\ \vdots \\ \vdots \end{array}$$

which can be proved by induction on m using the defining recursion of P_m . \square

In order to prove that the extremal weight projectors categorify the Chebyshev polynomials L_m , we need an analogous partial trace formula for the T_m . We define the partial trace $\text{pTr}: \mathbf{ATL}_{n+1} \rightarrow \mathbf{ATL}_n$ on diagrams W by:



Lemma 21. *If we set $T_0 = 2 \text{id}_0$, then for $m \geq 1$ we have:*

$$(2.9) \quad \text{pTr}(T_m) = -T_{m-1}.$$

Proof. This is immediate for $m = 1$ and $m = 2$. For $m \geq 3$ it follows from the recursion (2.7) and the Reidemeister I relation from Lemma 7. \square

Lemma 22. *In $K_0(\text{Kar}(\mathbf{ATL}^{\text{ess}}))$, we have the following:*

$$\begin{aligned} [T_1][T_1] &= [T_1 \otimes T_1] = [T_2] + 2[\text{id}_0] \quad \text{and} \\ [T_m][T_1] &= [T_m \otimes T_1] = [T_{m+1}] + [T_{m-1}] \quad \forall m \geq 2. \end{aligned}$$

Proof. For $m \geq 2$, the proof is very analogous to the one of Lemma 20. The defining recursion (2.7) gives an orthogonal decomposition of the idempotent $T_m \otimes T_1$ into T_{m+1} and an idempotent containing a cap-cup. The latter is isomorphic to T_{m-1} in $\text{Kar}(\mathbf{ATL}^{\text{ess}})$ through an annular version of the isomorphism in (2.8), although with scalar -1 instead of $-\frac{m}{m+1}$. The verification that this gives an isomorphism relies on the fact that T_m absorbs lower order projectors, see (4) of Lemma 17 and the partial trace formula (2.9).

Finally, for $m = 1$ we recall that $\text{id}_2 = T_2 - U_1/2 - U_0/2$. This gives a decomposition of $\text{id}_2 = T_1 \otimes T_1$ into three orthogonal idempotents. It is also easy to check that the idempotents $-U_0/2$ and $-U_1/2$ are both isomorphic to id_0 in $\text{Kar}(\mathbf{ATL}^{\text{ess}})$. This finishes the proof. \square

Remark 23. The representation ring of \mathfrak{gl}_N is isomorphic to the tensor product of two copies of $\mathbb{Z}[[V], [\Lambda^2 V], \dots, [\Lambda^N V]]$ where V is the vector representation of \mathfrak{gl}_N . It is useful to identify each copy with $\mathbb{C}[x_1, x_2, \dots, x_N]^{\mathfrak{S}_N}$ by sending $[\Lambda^i V]$ to the elementary symmetric polynomials $e_i(x_1, x_2, \dots, x_N)$. The classes of the symmetric powers $[\text{Sym}^m V]$ then correspond to complete symmetric polynomials, the class of the simple representation indexed by the Young diagram λ corresponds to the Schur polynomial s_λ and the class of the extremal weight space of $V^{\otimes m}$ corresponds to the power sum symmetric polynomial $p_m = x_1^m + x_2^m + \dots + x_N^m$. Power sum symmetric polynomials, and thus extremal weight spaces, also play an important role in the HOMFLY-PT skein algebra of the torus and its relationship to the elliptic Hall algebra [17]. An extension of the present work to the \mathfrak{gl}_N case will appear in a follow-up paper [20].

2.6. Product formula

The subject of this section is the categorification of Equation (1.1), as stated in the following theorem.

Theorem 24. $T_m \otimes T_n \cong T_{m+n} \oplus T_{|m-n|}$ in $\text{Kar}(\mathbf{ATL}^{\text{ess}})$.

To prove this, we first split $T_m \otimes T_n$ into two orthogonal idempotents T_{m+n} and $e_{m,n}$ in Lemma 26. The Propositions 29 and 31 then identify $e_{m,n}$ with $T_{|m-n|}$ in the two distinct cases $m \neq n$ and $m = n$.

We start by noting that crossing-connected projectors can be combined.

Lemma 25. *Let $m, n \in \mathbb{N}$ with $m + n \geq 3$, then $(T_m \otimes T_n)s_m(T_m \otimes T_n) = T_{m+n}$.*

Proof. This follows again by application of φ . □

Lemma 26. *For $m, n \geq 1$ we have an orthogonal decomposition of idempotents $T_m \otimes T_n = T_{m+n} + e_{m,n}$ where $e_{1,1} = -U_1/2 - D^{-1}U_1D/2$ and $e_{m,n} = -(T_m \otimes T_n)U_m(T_m \otimes T_n)$ otherwise.*

Proof. For $m = n = 1$ this follows from the explicit description of T_2 . Otherwise we use Lemma 25:

$$T_{m+n} = (T_m \otimes T_n)s_m(T_m \otimes T_n) = (T_m \otimes T_n) + (T_m \otimes T_n)U_m(T_m \otimes T_n).$$

Clearly, $e_{m,n}$ and T_{m+n} are orthogonal by Lemma 17, (3), which implies the idempotency of $e_{m,n}$. □

As expected, $\varphi(e_{m,n})$ projects onto $\mathbb{C}\langle v_{+^m-n}, v_{-n+m} \rangle$. This could be used to rephrase the last proof.

Lemma 27. *For $1 \leq n, m$ and $n + m \geq 3$, the projector $e_{m,n}$ can alternatively be written as*

$$e_{m,n} = (-1)^r (T_m \otimes T_n)(T_{m-r} \otimes \text{Cu}_r \text{Ca}_r \otimes T_{n-r})(T_m \otimes T_n)$$

where $1 \leq r \leq \min(m, n)$, $r < \max(m, n)$ and $\text{Cu}_r \text{Ca}_r$ is a composition of r nested caps followed by r nested cups.

Proof. One simply computes the images under φ and checks they agree. □

Lemma 28. *For $1 \leq n \leq m$ we have*

$$(-1)^n (\text{id}_{m-n} \otimes \text{Ca}_n)(T_m \otimes T_n)(\text{id}_{m-n} \otimes \text{Cu}_n) = T_{m-n}$$

where Ca_n and Cu_n denote n nested caps and cups respectively and we set $T_0 = 2$.

Proof. We first verify $\text{Ca}_n(\text{id}_n \otimes T_n) = \text{Ca}_n(T_n \otimes \text{id}_n)$. This is trivial for $n = 1$ and easily verified through a short computation for $n = 2$. Indeed, $T_2 = \text{id}_2 + U_1/2 + U_0/2$ and the first two terms can be isotoped across Ca_2 at no cost. For the term $U_0/2$ we compare:

$$\begin{aligned} \text{Ca}_2(\text{id}_2 \otimes U_0) &= \text{Diagram 1} = - \text{Diagram 2} = - \text{Diagram 3} - \text{Diagram 4} \\ \text{Ca}_2(U_0 \otimes \text{id}_2) &= \text{Diagram 5} = - \text{Diagram 6} = - \text{Diagram 7} - \text{Diagram 8} \end{aligned}$$

The second terms on the right-hand sides of both equations are zero and the first terms are isotopic. The case of $n \geq 3$ then follows inductively from the recursive definition of the projectors T_n and the fact that crossings slide around caps. Having established that projectors slide around caps, we also have

$$\text{Ca}_n(T_n \otimes T_n) = \text{Ca}_n(T_n^2 \otimes \text{id}_n) = \text{Ca}_n(T_n \otimes \text{id}_n)$$

and the result follows from the partial trace formula (2.9). \square

Proposition 29. *For $m, n \geq 1$ and $m \neq n$ the idempotents $e_{m,n}$ and $T_{|m-n|}$ represent isomorphic objects in $\text{Kar}(\mathbf{ATL}^{\text{ess}})$.*

Proof. We may assume that $m > n$ as the case $m < n$ is completely analogous. We use Lemma 27 to write

$$e_{m,n} = (T_m \otimes T_n)(T_{m-n} \otimes \text{Cu}_n)(T_{m-n} \otimes \text{Ca}_n)(T_m \otimes T_n).$$

Then it is immediate from Lemma 28 that the maps $(T_m \otimes T_n)(T_{m-n} \otimes \text{Cu}_n)$ and $(T_{m-n} \otimes \text{Ca}_n)(T_m \otimes T_n)$ are inverse isomorphisms between the elements of the Karoubi element represented by the idempotents $e_{m,n}$ and T_{m-n} . \square

In order to describe $e_{m,m}$, we need the fact that overlapping projectors can be combined.

Lemma 30. *Let $1 \leq n \leq m$ and $0 \leq r < n$, then the following hold:*

$$\begin{aligned} (T_m \otimes \text{id}_r)(\text{id}_{m-n+r} \otimes T_n) &= T_{m+r} = (\text{id}_{m-n+r} \otimes T_n)(T_m \otimes \text{id}_r) \\ (\text{id}_r \otimes T_m)(T_n \otimes \text{id}_{m-n+r}) &= T_{m+r} = (T_n \otimes \text{id}_{m-n+r})(\text{id}_r \otimes T_m) \end{aligned}$$

Proof. Again, this follows by application of φ . \square

Proposition 31. *The idempotent $e_{m,m}$ is isomorphic to two copies of id_0 in the Karoubi envelope of $\mathbf{ATL}^{\text{ess}}$.*

Proof. For $m = 1$ we have $e_{1,1} = -U_1/2 - D^{-1}U_1D/2$. The two summands are orthogonal idempotents, each of which is isomorphic to id_0 in $\text{Kar}(\mathbf{ATL}^{\text{ess}})$. For $m > 1$ we rewrite:

$$\begin{aligned} e_{m,m} &= (-1)^{m-1}(T_m \otimes T_m)(\text{id}_1 \otimes (\text{Cu}_{m-1}\text{Ca}_{m-1}) \otimes \text{id}_1)(T_m \otimes T_m) \\ &= \underbrace{(-1)^m(T_m \otimes T_m)\text{Cu}_m}_{f_1} \circ \underbrace{\text{Ca}_m(T_m \otimes T_m)/2}_{g_1} \\ &\quad + \underbrace{(-1)^m(T_m \otimes T_m)D(\text{Cu}_1 \otimes \text{Cu}_{m-1})}_{f_2} \circ \underbrace{(\text{Ca}_1 \otimes \text{Ca}_{m-1})D^{-1}(T_m \otimes T_m)/2}_{g_2} \end{aligned}$$

The first equality comes from Lemma 27. The second equality can be verified by expanding T_2 in the equality

$$0 = (T_m \otimes T_m)D(T_2 \otimes \text{Cu}_{m-1}\text{Ca}_{m-1})D^{-1}(T_m \otimes T_m),$$

which follows from Lemma 30 and Lemma 17, (3). To prove the proposition, it remains to verify $g_i f_j = \delta_{i,j} \text{id}_0$. We give one example for orthogonality:

$$\begin{aligned} g_1 f_2 &= (-1)^m \text{Ca}_m(T_m \otimes T_m)D(\text{Cu}_1 \otimes \text{Cu}_{m-1})/2 \\ &= (-1)^m \text{Ca}_m(\text{id}_m \otimes T_m)D(\text{Cu}_1 \otimes \text{Cu}_{m-1})/2 = -\text{id}_0 \otimes (\text{Ca}_1 D \text{Cu}_1) = 0 \end{aligned}$$

Here we have used the proof of Lemma 28, partial trace relations and the essential torus relation. The proof of $g_2 f_1 = 0$ is analogous, while $g_1 f_1 = \text{id}_0$ follows from equation (2.9). It remains to check

$$g_2 f_2 = (-1)^m (\text{Ca}_1 \otimes \text{Ca}_{m-1})D^{-1}(T_m \otimes T_m)D(\text{Cu}_1 \otimes \text{Cu}_{m-1})/2 = \text{id}_0.$$

For $m = 2$ this follows by expanding the right copy of T_2 and seeing that all terms except the identity term die. The result is evaluated using the partial trace relation twice, which produces id_0 . For $m \geq 3$, we use the recursion on the right copy of T_m , absorb the resulting copies of T_{m-1} as in the proof of Lemma 28 and then apply the partial trace relation $m - 2$ times. The result is equal to $(\text{Ca}_1 \otimes \text{Ca}_1)D^{-1}(T_2 \otimes s)D(\text{Cu}_1 \otimes \text{Cu}_1)/2$, which evaluates to id_0 after expanding the crossing s . \square

2.7. Decategorification

We have observed that the extremal weight projectors in $\mathbf{ATL}^{\text{ess}}$ categorify the Chebyshev polynomials of the first kind in the sense that their K_0 -classes satisfy the appropriate recurrence relation and multiplication rule. However, the Grothendieck group of the Karoubi envelope of $\mathbf{ATL}^{\text{ess}}$, i.e. the representation ring of \mathfrak{h} is isomorphic to $\mathbb{Z}[Y^{\pm 1}]$, and is thus larger than the polynomial ring $\mathbb{Z}[X] \cong K_0(\text{Kar}(\mathbf{TL})) \cong K_0(\text{Rep}(\mathfrak{sl}_2))$, which appears as the subalgebra generated by $X = Y + Y^{-1}$. To see this, recall that the objects in $\text{Rep}(\mathfrak{h})$ are direct sums of integral \mathfrak{sl}_2 weight spaces. However, in the Grothendieck group, such direct sums can be written as formal differences of \mathfrak{sl}_2 -representations only if they are orbits of the action for the Weyl group \mathfrak{S}_2 . There are two ways to address this issue: in this section, we identify a sub-category of $\text{Kar}(\mathbf{ATL}^{\text{ess}})$ that is \mathfrak{S}_2 -equivariant, that contains the extremal weight projectors and has $\mathbb{Z}[X]$ as Grothendieck group. In the next section, on the other hand, we use the broken symmetry in $\mathbf{ATL}^{\text{ess}}$ to identify the projectors onto highest and lowest weight spaces.

Definition 32. We let $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ denote the subcategory of $\text{Rep}(\mathfrak{h})$ with objects that are invariant under \mathfrak{S}_2 and morphisms that are \mathfrak{S}_2 -equivariant.

Lemma 33. *The category $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ is semi-simple, $\text{im}(\varphi(T_m))$ for $m \geq 1$ are simple objects and the homomorphism*

$$\mathbb{Z}[X] \cong K_0(\text{Rep}(\mathfrak{sl}_2)) \rightarrow K_0(\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2})$$

induced by the inclusion is an isomorphism.

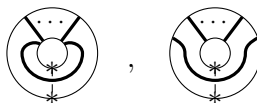
Proof. It is clear that the indecomposable objects in $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ are of the form

$$\mathbb{C}\langle v_{\varepsilon_1, \dots, \varepsilon_{n+m}}, v_{-\varepsilon_1, \dots, -\varepsilon_{n+m}} \rangle.$$

Through the permutation action, such an object is isomorphic to an object of the form $\mathbb{C}\langle v_{+, -, m, -n}, v_{-, +, m, n} \rangle$ with $m \geq n$ and then further to $\mathbb{C}\langle v_{+, m-n}, v_{-, m-n} \rangle = \varphi(T_{m-n})$ if $m > n$. There are no morphisms between distinct such objects and their endomorphism algebras are 1-dimensional over \mathbb{C} . This shows that $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ is semi-simple. The isomorphism follows since extremal weight spaces can be expressed as formal differences of the classes of simple representations in $K_0(\text{Rep}(\mathfrak{sl}_2))$. \square

We aim to describe the subcategory $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ of $\text{Rep}(\mathfrak{h})$ by a subcategory of $\mathbf{ATL}^{\text{ess}}$.

Definition 34. Let $\mathbf{ATL}_s^{\text{ess}}$ denote the symmetric monoidal \mathbb{C} -linear subcategory of $\mathbf{ATL}^{\text{ess}}$ with the same objects, but with morphisms spaces spanned by compositions of the cap-cups U_i together with the wrap-around caps and cups:



Note that the restriction of φ to the subcategory $\mathbf{ATL}_s^{\text{ess}}$ has image contained in $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$.

Remark 35. The endomorphism algebra $\mathbf{ATL}_s^{\text{ess}}(n, n)$ is isomorphic to the essential-circle quotient of the Temperley-Lieb quotient of the Hecke algebra of type \hat{A}_{n-1} .

Lemma 36. T_m , $e_{m,n}$ and the isomorphisms between $e_{m,n}$ and $T_{|m-n|}$ are contained in $\mathbf{ATL}_s^{\text{ess}}$.

Proof. This is clear for T_m and $e_{m,n}$ from their definitions. The isomorphisms used in the proof of Proposition 29 are not in $\mathbf{ATL}_s^{\text{ess}}$. However, they can be twisted by a suitable power of D , which turns caps and cups into wrap-around caps and cups, which are in $\mathbf{ATL}_s^{\text{ess}}$. \square

Proposition 37. The functor $\varphi: \mathbf{ATL}_s^{\text{ess}} \rightarrow \text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$ is fully faithful and induces an equivalence between $\text{Kar}(\mathbf{ATL}_s^{\text{ess}})$ and $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$.

Proof. Faithfulness is inherited from Theorem 12. Fullness follows since the image of φ contains the projections onto the simple objects in $\text{Rep}(\mathfrak{h})^{\mathfrak{S}_2}$. \square

Corollary 38. $\text{Kar}(\mathbf{ATL}_s^{\text{ess}})$ categorifies the polynomial ring $\mathbb{Z}[X]$ and its objects T_m categorify the Chebyshev polynomials of the first kind.

2.8. Highest and lowest weight projectors

Because of the desired topological applications, we have up to now focused on the idempotents that project onto the extremal weight space in a \mathfrak{sl}_2 -representation. Actually, by Theorem 12, we can split T_m in $\mathbf{ATL}^{\text{ess}}$ into orthogonal idempotents that project onto the highest weight space and the lowest weight space separately. This is reminiscent of the splitting of the Chebyshev polynomials used in [19].

Proposition 39. *The following defines a family of idempotents in $\mathbf{ATL}^{\text{ess}}$ that correspond under φ to projectors onto the highest and lowest weight spaces respectively:*

$$T_{++} = \frac{1}{4} \left(2 \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + 2i \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} + i \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} + i \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right)$$

$$T_{--} = \frac{1}{4} \left(2 \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} - 2i \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} + \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} + \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} - i \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} - i \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right)$$

The projectors T_{+^m} and T_{-^m} for $m \geq 3$ are obtained via recursions of type (2.7) with base cases T_{++} and T_{--} .

Proof. The proof is again based on the application of the functor φ . □

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References

- [1] M. M. Asaeda, J. H. Przytycki, and A. S. Sikora, *Categorification of the Kauffman bracket skein module of I-bundles over surfaces*, Algebraic & Geometric Topology **4** (2004), no. 2, 1177–1210.
- [2] D. Bar-Natan, *Khovanov’s homology for tangles and cobordisms*, Geom. Topol. **9** (2005) 1443–1499.
- [3] F. Bonahon and H. Wong, *Representations of the Kauffman skein algebra I: invariants and miraculous cancellations*, Inventiones mathematicae **204** (2016), no. 1, 195–243.

- [4] K. Erdmann and R. M. Green, *On representations of affine Temperley-Lieb algebras. II*, Pacific J. Math. **191** (1999), no. 2, 243–273.
- [5] C. K. Fan and R. M. Green, *On the affine Temperley-Lieb algebras*, J. London Math. Soc. (2) **60** (1999), no. 2, 366–380.
- [6] C. Frohman and R. Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4877–4888.
- [7] J. J. Graham and G. I. Lehrer, *The representation theory of affine Temperley-Lieb algebras*, Enseign. Math. (2) **44** (1998), no. 3-4, 173–218.
- [8] R. M. Green, *On representations of affine Temperley-Lieb algebras*, in Algebras and modules, II (Geiranger, 1996), Vol. 24 of *CMS Conf. Proc.*, 245–261, Amer. Math. Soc., Providence, RI (1998).
- [9] V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111.
- [10] V. F. R. Jones, *A quotient of the affine Hecke algebra in the Brauer algebra*, Enseign. Math. (2) **40** (1994), no. 3-4, 313–344.
- [11] V. F. R. Jones and S. A. Reznikoff, *Hilbert space representations of the annular Temperley-Lieb algebra*, Pacific J. Math. **228** (2006), no. 2, 219–249.
- [12] L. Kauffman, *State Models and the Jones Polynomial*, New developments in the theory of knots **11** (1990) 162.
- [13] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.
- [14] M. Khovanov and R. Sazdanovic, *Categorification of the polynomial ring*, Fundamenta Mathematicae **230** (2015), no. 3, 251–280.
- [15] T. T. Lê, *On positivity of Kauffman bracket skein algebras of surfaces*, to appear in IMRN (2016). [arXiv:1603.08265](#).
- [16] T. T. Lê, *Triangular decomposition of skein algebras*, to appear in Quantum Topology (2016). [arXiv:1609.04987](#).
- [17] H. Morton and P. Samuelson, *The HOMFLYPT skein algebra of the torus and the elliptic Hall algebra*, Duke Math. J. **166** (2017), no. 5, 801–854.

- [18] J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Polish Acad. Sci. Math. **39** (1991), no. 1-2, 91–100.
- [19] H. Queffelec and H. M. Russell, *Chebyshev polynomials and the Frohman–Gelca formula*, Journal of Knot Theory and Its Ramifications **24** (2015), no. 04, 1550023.
- [20] H. Queffelec and P. Wedrich, *Extremal weight projectors II*. [arXiv:1803.09883](#).
- [21] H. Queffelec and P. Wedrich, *Khovanov homology and categorification of skein modules*. [arXiv:1806.03416](#).
- [22] G. Rumer, E. Teller, and H. Weyl, *Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten*, Nachrichten von der Ges. der Wiss. Zu Göttingen. Math.-Phys. Klasse (1932) 498–504. In German.
- [23] D. P. Thurston, *Positive basis for surface skein algebras*, Proc. Natl. Acad. Sci. USA **111** (2014), no. 27, 9725–9732.
- [24] V. G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 6, 635–704.
- [25] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada **9** (1987), no. 1, 5–9.

IMAG, UNIVERSITÉ DE MONTPELLIER
 CNRS, MONTPELLIER, FRANCE
E-mail address: hoel.queffelec@umontpellier.fr

MATHEMATICAL SCIENCES INSTITUTE
 THE AUSTRALIAN NATIONAL UNIVERSITY
 CANBERRA ACT 2601, AUSTRALIA
E-mail address: p.wedrich@gmail.com

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