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Spreading of the free boundary of relativistic Euler equations in a vacuum

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Thomas C. Sideris in [J. Differential Equations **257** (2014), no. 1, 1–14] showed that the diameter of a region occupied by an ideal fluid surrounded by vacuum will grow linearly in time provided the pressure is positive and there are no singularities. In this paper, we generalize this interesting result to isentropic relativistic Euler equations with pressure $p = \sigma^2 \rho$. We will show that the results obtained by Sideris still hold for relativistic fluids. Furthermore, a family of explicit spherically symmetric solutions is constructed to illustrate our result when $\sigma = 0$, which is different from Sideris's self-similar solution.

1. Introduction

Free boundary value problem for ideal fluids is important and interesting in physics, and has attracted much attention in the last decade, see [21] and [22] for 2D and 3D incompressible, irrotational full water wave equations with gravity; [6] for the well-posedness of compressible liquids and [5] for the well-posedness of compressible Euler equations in a physical vacuum. Recently, Sideris in [12] investigated the free boundary problem of compressible ideal gases and incompressible ideal fluids surrounded by vacuum. He showed that the diameter of the region will grow linearly in time provided the pressure is positive and the solution is smooth. The proof is based on some identities of integral averages introduced by himself in [13], which were utilized to show the formation of singularities in three-dimensional compressible fluids. This method has been further explored to study the singularity of solutions to various hyperbolic equations. One can refer to [7, 10, 17] for classical fluids, [14, 23, 24] for nonlinear wave equations, [2] and [9] for relativistic fluids. Inspired by Sideris's work [12] and the importance of the free boundary value problem in mathematics and physics, we focus our interest on the following relativistic Euler equations for a perfect fluid in $1 + n$-dimensional

Minkowski spacetime

$$\sum_{\mu=0}^{n} \partial_\mu T^{\mu\nu} = 0, \tag{1.1}$$

where

$$T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu + p(g^{-1})^{\mu\nu} \tag{1.2}$$

is the stress-energy tensor for a perfect fluid, and $(g^{-1})^{\mu\nu} = diag(-1, 1, \dots, 1)$ denotes the flat Minkowski metric for $\mu, \nu = 0, 1, \dots, n$, the coordinates $\mathbf{x} = (x^0, x^1, \dots, x^n)^T$ with $x^0 = ct$. ρ denotes the mass energy density, p the pressure, c the speed of light, and $\mathbf{u} = (u^0, \dots, u^n)^T = \frac{1}{c}\frac{d\mathbf{x}}{d\tau}$ (τ is the proper time) denotes the future-directed unit time-like $1+n$-vector in Minkowski spacetime and it satisfies

$$(u^0)^2 - \sum_{i=1}^{n} (u^i)^2 = 1, \quad u^0 > 0. \tag{1.3}$$

From (1.3), it is easy to see that only n variables of the quantities $u^0, u^1, \dots, u^n$ are independent. From now on, we adopt the normal space-time coordinates $(t, x^1, \dots, x^n)^T$. Set $x = (x^1, \dots, x^n)^T$, $u = (u^1, \dots, u^n)^T$ and let

$$v = \frac{cu}{\sqrt{1 + |u|^2}},$$

where $v = \frac{dx}{dt}$ denotes the speed of classical fluid and $|u|^2 = \sum_{i=1}^{n} (u^i)^2$. With above notations and (1.2), expanding (1.1) directly, we have

$$\begin{cases} \partial_t(\frac{\rho c^2+p}{c^2-v^2} - \frac{p}{c^2}) + \nabla_x \cdot (\frac{\rho c^2+p}{c^2-v^2} v) = 0, \\ \partial_t(\frac{\rho c^2+p}{c^2-v^2} v) + \nabla_x \cdot (\frac{\rho c^2+p}{c^2-v^2} v \otimes v) + \nabla_x p = 0, \end{cases} \tag{1.4}$$

where ∇_x denotes the spacial gradient operator, $a \otimes b = ab^T$ for two n-vectors a and b. One could easily see that once the pressure $p = p(\rho)$ is given, the above system is well determined and contains $n+1$ equations with $n+1$ unknowns $(\rho, v^1, \dots, v^n)$.

Remark 1.1. For detailed derivation of (1.4) from (1.1), one can refer to Smoller and Temple [15] or our paper [8].

Remark 1.2. From (1.4), we see that when $c \to \infty$, the Newtonian limit of relativistic Euler equation is exactly the classical Euler equations. That is one motivation for us to investigate relativistic fluid. The other motivation is its importance in physics, especially in cosmology and general relativity.

In this paper, we assume

$$p = p(\rho) = \sigma^2 \rho, \tag{1.5}$$

which plays an important role in cosmology. For $\sigma^2 = \frac{1}{3}c^2$, it is used as a toy model for a "radiation-dominated" universe and can also be derived as a model for the equation of state in a dense Neutron star. For further discussions of this state equation, we refer to [20]. Due to its importance in physics, it attracts much attention of mathematicians and physicians. For instance, J. Smoller and B. Temple [15] proved the global weak solution of (1.4) with (1.5) by Glimm scheme for the one space dimension case. Later, their result was generated by B. D. Wissman [18] to the non-isentropic case. For $n = 3$, it was proved by Pan and Smoller [9] that the smooth solutions for (1.4) must blowup in finite time in the spirit of the work of Sideris [13] provided the smooth initial data has compact support and satisfies some largeness conditions. A great breakthrough has been made by Christodoulou in his monograph [1], in which he showed the formation of shocks in finite time for (1.1) under the assumption of irrotation by the techniques of differential geometry for $n = 3$. When $0 \leq \sigma^2 < \frac{c^2}{3}$, J. Speck in [4, 16] proved the global stability of Euler-Einstein system with a positive cosmological constant. The above results all focus on the non-vacuum Cauchy problem with compact initial data. For the free boundary problem, we need the following preparations before stating our result.

1.1. Free boundary problem

For convenience, we use similar symbols introduced by Sideris in [12]. At first, we need to emphasize that the local existence of the classical solution to the vacuum initial free boundary value problem for (1.4) is still unknown, which is an interesting and important problem in the research of relativistic fluids. In the following, all the results of this paper are based on the assumption that the classical solution of (1.4) exists in a bounded open region Ω_t with C^1 boundary $\partial\Omega_t$ for $0 \leq t \leq T$. Define the space-time region

$$\mathrm{S}_T = \{(t, x) : x \in \Omega_t, 0 < t < T\}$$

and its lateral free boundary

$$\mathtt{B}_T = \{(t, x) : x \in \partial\Omega_t,\ 0 < t < T\},$$

which is C^1 with unit outward normal vector $n(t, x) \in \mathbb{R}^{1+n}$ for $(t, x) \in \mathtt{B}_T$.

For simplicity, denote

$$\hat{\rho} = \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}, \quad \tilde{\rho} = \frac{\rho c^2 + p}{c^2 - v^2}. \tag{1.6}$$

Then, on $\mathtt{S}_T$, (1.4) can be rewritten as

$$\begin{cases} \partial_t \hat{\rho} + \nabla_x \cdot (\tilde{\rho} v) = 0, \\ \partial_t (\tilde{\rho} v) + \nabla_x \cdot (\tilde{\rho} v \otimes v) + \nabla_x p = 0. \end{cases} \tag{1.7}$$

Assume the fluid is surrounded by vacuum, then the appropriate boundary condition is

$$\rho = 0, \quad \text{on } \mathtt{B}_T. \tag{1.8}$$

It is easy to see that $\hat{\rho} = \tilde{\rho} = p = 0$ on $\mathtt{B}_T$.

We introduce some average quantities:

- Total mass:

$$M(t) = \int_{\Omega_t} \hat{\rho} dx. \tag{1.9}$$

- Center of mass:

$$\bar{x}(t) = M^{-1}(t) \int_{\Omega_t} \hat{\rho} x dx. \tag{1.10}$$

- Average velocity:

$$\bar{v}(t) = M^{-1}(t) \int_{\Omega_t} \tilde{\rho} v dx. \tag{1.11}$$

- Moment of inertia:

$$X(t) = \int_{\Omega_t} \frac{1}{2} \hat{\rho} |x - \bar{x}|^2 dx. \tag{1.12}$$

- Average radial momentum:

$$Y(t) = \int_{\Omega_t} [\tilde{\rho} \langle x - \bar{x}, v \rangle - \hat{\rho} \langle x - \bar{x}, \bar{v} \rangle] dx. \tag{1.13}$$

Remark 1.3. In contrast to classical fluids, $\hat{\rho}$ and $\tilde{\rho}$ depend not only on the density ρ but also on the velocity v.

1.2. Main Theorem

With the preliminaries above, we now state our main results as follows:

Theorem 1.4. Let $\rho \in C^0(\bar{\mathtt{S}}_T) \cap C^1(\mathtt{S}_T)$, $v \in C^1(\bar{\mathtt{S}}_T)$ be a solution of (1.7) and (1.8) with $\rho(0, \cdot) > 0$ in Ω_0. If $\sigma \neq 0$ and satisfies $D := \min\{c^2, n\sigma^2\} - \bar{v}^2(0) > 0$, then in fixed space-time coordinates (t, x), we have

$$[diam\, \Omega_t]^2 \geq [DM(0)t^2 + 2Y(0)t + 2X(0)]/M(0), \quad 0 \leq t < T.$$

If $\sigma = 0$, it holds that

$$[diam\, \Omega_t]^2 \geq [E(0)t^2 + 2Y(0)t + 2X(0)]/M(0), \quad 0 \leq t < T,$$

where $E(0) = \int_{\Omega_0} \hat{\rho}|v - \bar{v}|^2(0, y)dy > 0$.

Remark 1.5. In the above theorem, the assumption on the constant D is stronger than the subluminal condition $|v| \leq c$ in the sense of average since $\tilde{\rho} > \hat{\rho}$, and this assumption can be obtained by choosing the initial velocity appropriately. Moreover, we emphasize that $v \in C^1(\bar{S}_T)$ will always mean that v is a C^1 function on the closed set $\bar{S}_T$ and satisfies the subluminal condition $|v| < c$.

Remark 1.6. If the initial datas are smooth functions with compact support on Ω_0, then there can be no spreading and a singularity must develop in finite time, see [9] for details with general state equation $p(\rho)$.

Remark 1.7. When $\sigma = 0$, i.e., $p = 0$, it is known as the "pressureless dust" equation of state. In this case, $\hat{\rho} = \tilde{\rho}$, then (1.7) can be simplified and enjoys better structures, which can be seen in the last section.

In order to illustrate the sharpness of above spreading, we also obtain a family of spherically symmetric solutions for the case of $\sigma = 0$, which is different from the spherically symmetric, self-similar solutions of Sideris in [12].

Theorem 1.8. Assume that $\sigma = 0$ and let $\Omega_0 = \{|y| \leq 1\}$, $\rho(0, y) = \rho_0(|y|)$, $v(0, y) = v_0(|y|)\frac{y}{|y|}$ with $v_0(0)$, $v_0'(\cdot) \geq 0$, then the initial value problem (1.7), (1.8) admits a global, spherically symmetric solution and the fluid domain Ω_t is a ball which satisfies

$$diam\,\Omega_t = 2(1 + v_0(1)t).$$

Furthermore, we can get the explicit expressions

$$v\left(t, y + v_0(|y|)\frac{y}{|y|}t\right) = v_0(|y|)\frac{y}{|y|}$$

and

$$\rho\left(t, y + v_0(|y|)\frac{y}{|y|}t\right) = \frac{\rho_0(|y|)|y|^{n-1}}{\left(1 + v_0'(|y|)t\right)\left(|y| + v_0(|y|)t\right)^{n-1}}.$$

Remark 1.9. To make sure ρ does not tend to infinity in finite time, it is necessary to assume that $v_0(0)$, $v_0'(|y|) \geq 0$, which means that the characteristics arising from Ω_0 diverge with time. From the expression of v, we see that it satisfies the vectorial Burgers equation $\partial_t v + v \cdot \nabla_x v = 0$, since $p = 0$.

The strategy of the proof is similar to Sideris [12]. At first, we derive two conserved quantities $M(t)$ and $\bar{v}(t)$ along the fluid line according to equation (1.7). Then we prove the second derivative of $X(t)$ along the fluid line is positive by our assumption on the initial data. At last, via the relationship between $\hat{\rho}$ and $\tilde{\rho}$ and two conservation laws, we prove the lower bound of the diameter occupied by the relativistic fluids and explicitly construct a family of spherically symmetric solutions.

Before ending this section, we give the arrangement of this short paper. In Section 2, we study the properties of the average quantities and give the proof of Theorem 1.4. The spherically symmetric solution is constructed in Section 3 according to the conserved quantities $M(t)$ and $\bar{v}(t)$.

2. The proof of Theorem 1.4

In this section, we mainly prove Theorem 1.4 based on the properties of the integral average quantities and the relationship between $\hat{\rho}$ and $\tilde{\rho}$. As in [12], we need the following important identities:

Lemma 2.1. Let $\rho \in C^0(\bar{\mathtt{S}}_T) \cap C^1(\mathtt{S}_T)$, $v \in C^1(\bar{\mathtt{S}}_T)$ be the solution to (1.7) and (1.8), then

$$M(t) = M(0), \tag{2.1}$$

$$\bar{v}(t) = \bar{v}(0), \tag{2.2}$$

$$X^{'}(t) = Y(t), \tag{2.3}$$

$$Y^{'}(t) = \int_{\Omega_t} (\hat{\rho}(\bar{v})^2 + \tilde{\rho}|v|^2 - 2\tilde{\rho}\langle v, \bar{v}\rangle + np)dx. \tag{2.4}$$

Proof. Let $\hat{\rho}$, $\tilde{\rho}$, $p \in C^0(\bar{\mathtt{S}}_T) \cap C^1(\mathtt{S}_T)$, $v \in C^1(\bar{\mathtt{S}}_T)$ be a solution to (1.7) and (1.8). Define the flow line $x(t, y)$ as follows

$$\begin{cases} \frac{d}{dt}x(t, y) = v(t, x(t, y)), \\ x(0, y) = y. \end{cases} \tag{2.5}$$

Equation (2.5) defines a C^1 diffeomorphism from $\bar{\Omega}_0$ to $\bar{\Omega}_t$ with $0 \le t < T$ since $v \in C^1(\bar{\mathtt{S}}_T)$. Define the deformation $J(t, y) = detD_y x(t, y)$, it is easy to show that $J(t, y)$ satisfies

$$D_t J(t, y) = \nabla \cdot v(t, x(t, y))J(t, y), \quad J(0, y) = 1, \tag{2.6}$$

where $D_t = \partial_t + v \cdot \nabla$ denotes the usual material time derivative. For any C^1 function $f = f(t, x)$, by a direct calculation, we have

$$\begin{aligned} D_t \int_{\Omega_t} \hat{\rho} f dx &= D_t \int_{\Omega_0} \hat{\rho} f J(t, y)dy \\ &= \int_{\Omega_0} D_t f \hat{\rho} J(t, y)dy + \int_{\Omega_0} f D_t \hat{\rho} J(t, y)dy \\ &\quad + \int_{\Omega_0} f \hat{\rho} D_t J(t, y)dy \\ &= \int_{\Omega_t} D_t f \hat{\rho} dx + \int_{\Omega_t} [D_t \hat{\rho} + \nabla \cdot v\hat{\rho}] f dx \\ &= \int_{\Omega_t} D_t f \hat{\rho} dx + \int_{\Omega_t} [\partial_t \hat{\rho} + \nabla \cdot (v\hat{\rho})] f dx \\ &= \int_{\Omega_t} [\hat{\rho} D_t f + \nabla \cdot ((\hat{\rho} - \tilde{\rho})v) f] dx. \end{aligned} \tag{2.7}$$

Then, let $f = 1$, we have

$$D_t M(t) = \int_{\Omega_t} (\nabla_x \cdot [(\hat{\rho} - \tilde{\rho})v])dx = 0. \tag{2.8}$$

Thus, $M(t) = M(0)$.

Similarly, for $\bar{v}(t)$, we have

$$
\begin{aligned}
D_t\bar{v}(t) &= D_t(M^{-1}(t)\int_{\Omega_t}\tilde{\rho}vdx) \\
&= D_t(M^{-1}(t))\int_{\Omega_t}\tilde{\rho}vdx \\
&\quad + M^{-1}(t)D_t\int_{\Omega_0}\tilde{\rho}v(t,x(t,y))J(t,y)dy \\
&= M^{-1}(t)\int_{\Omega_0}\left(D_t(\tilde{\rho}v)J + \tilde{\rho}v\nabla\cdot vJ\right)dy \\
&= M^{-1}(t)\int_{\Omega_t}[\partial_t(\tilde{\rho}v) + v\cdot\nabla(\tilde{\rho}v) + \tilde{\rho}v\nabla\cdot v]dx \\
&= M^{-1}(t)\int_{\Omega_t}[\partial_t(\tilde{\rho}v) + \nabla\cdot(\tilde{\rho}v\otimes v)]dx \\
&= -M^{-1}(0)\int_{\Omega_t}\nabla_x pdx = 0.
\end{aligned} \tag{2.9}
$$

Thus, $\bar{v}(t) = \bar{v}(0)$.

Before proving (2.3) and (2.4), we need to show that $D_t\bar{x}(t) = \bar{v}(t)$. Let $f = x(t,y)$ in (2.7), we have

$$
\begin{aligned}
D_t\bar{x}(t) &= D_t(M^{-1}(t))\int_{\Omega_0}\hat{\rho}x(t,y)J(t,y)dy \\
&\quad + M^{-1}(t)\int_{\Omega_0}D_t(\hat{\rho}x(t,y)J(t,y))dy \\
&= M^{-1}(t)\int_{\Omega_t}[\nabla_x\cdot((\hat{\rho}-\tilde{\rho})v)x + \hat{\rho}v]dx \\
&= M^{-1}(t)\int_{\Omega_t}[\nabla_x[(\hat{\rho}-\tilde{\rho})vx] - (\hat{\rho}-\tilde{\rho})v + \hat{\rho}v]dx \\
&= M^{-1}(t)\int_{\Omega_t}[-(\hat{\rho}-\tilde{\rho})v + \hat{\rho}v]dx \\
&= M^{-1}(t)\int_{\Omega_t}\tilde{\rho}vdx = \bar{v}(t).
\end{aligned} \tag{2.10}
$$

By (2.9) and (2.10), it is easy to see that

$$
\bar{x}(t) = \bar{x}(0) + \bar{v}(0)t. \tag{2.11}
$$

With (2.10) at hand, we are ready to prove (2.3) and (2.4). Let $f = \frac{1}{2}|x-\bar{x}|^2$ in (2.7), we have

(2.12)

$$
\begin{aligned}
&D_t X(t) = D_t \int_{\Omega_t} \frac{1}{2}\hat{\rho}|x-\bar{x}|^2 dx \\
&= \int_{\Omega_t} \left[\frac{1}{2}\nabla_x \cdot [(\hat{\rho}-\tilde{\rho})v]|x-\bar{x}|^2 + \hat{\rho}\langle x-\bar{x}, v-\bar{v}\rangle\right] dx \\
&= \int_{\Omega_t} \left[\frac{1}{2}\nabla_x \cdot [(\hat{\rho}-\tilde{\rho})v|x-\bar{x}|^2] - \langle(\hat{\rho}-\tilde{\rho})v, x-\bar{x}\rangle + \hat{\rho}\langle x-\bar{x}, v-\bar{v}\rangle\right] dx \\
&= \int_{\Omega_t} [-\langle(\hat{\rho}-\tilde{\rho})v, x-\bar{x}\rangle + \hat{\rho}\langle v-\bar{v}, x-\bar{x}\rangle] dx \\
&= \int_{\Omega_t} [\tilde{\rho}\langle x-\bar{x}, v\rangle - \hat{\rho}\langle x-\bar{x}, \bar{v}\rangle] dx = Y(t).
\end{aligned}
$$

At last, we prove (2.4), differentiating (2.12) again, we have

$$
\begin{aligned}
D_t Y(t) &= D_t \left[\int_{\Omega_0} \langle \tilde{\rho} v J, x-\bar{x}\rangle dy - \int_{\Omega_0} \langle \hat{\rho} J \bar{v}, x-\bar{x}\rangle dy\right] \\
&= \int_{\Omega_0} \langle D_t(\tilde{\rho} v J), x-\bar{x}\rangle dy + \int_{\Omega_0} \langle \tilde{\rho} v, v-\bar{v}\rangle J dy \\
&\quad - \int_{\Omega_0} \langle D_t(\hat{\rho} J)\bar{v}, x-\bar{x}\rangle dy - \int_{\Omega_0} \langle \hat{\rho}\bar{v}, v-\bar{v}\rangle J dy \\
&= \int_{\Omega_t} \langle -\nabla_x p, x-\bar{x}\rangle dx + \int_{\Omega_t} \langle \tilde{\rho} v, v-\bar{v}\rangle dx \\
&\quad - \int_{\Omega_t} \langle \nabla_x \cdot [(\hat{\rho}-\tilde{\rho})v]\bar{v}, x-\bar{x}\rangle dx - \int_{\Omega_t} \langle \hat{\rho}\bar{v}, v-\bar{v}\rangle dx \\
&= \int_{\Omega_t} [-\nabla_x \cdot [p(x-\bar{x})] + np] dx + \int_{\Omega_t} \langle \tilde{\rho} v, v-\bar{v}\rangle dx \\
&\quad - \int_{\Omega_t} [\nabla_x \cdot ((\hat{\rho}-\tilde{\rho})v\langle x-\bar{x}, \bar{v}\rangle) - (\hat{\rho}-\tilde{\rho})\langle v, \bar{v}\rangle] dx \\
&\quad - \int_{\Omega_t} \langle \hat{\rho}\bar{v}, v-\bar{v}\rangle dx \\
&= \int_{\Omega_t} [np + \tilde{\rho}\langle v, v-\bar{v}\rangle + (\hat{\rho}-\tilde{\rho})\langle v, \bar{v}\rangle - \hat{\rho}\langle \bar{v}, v-\bar{v}\rangle] dx \\
&= \int_{\Omega_t} (\hat{\rho}(\bar{v})^2 + \tilde{\rho}|v|^2 - 2\tilde{\rho}\langle v, \bar{v}\rangle + np) dx.
\end{aligned} \tag{2.13}
$$

Then, the proof of the lemma is completed. □

Repeating the above processes again, we easily get the following corollary for the case of $\sigma = 0$.

Corollary 2.2. When $\sigma = 0$, let $\rho \in C^0(\bar{\mathtt{S}}_T) \cap C^1(\mathtt{S}_T)$, $v \in C^1(\bar{\mathtt{S}}_T)$ be the solution to (1.7) and (1.8), then

$$\begin{aligned} M(t) &= M(0), \\ \bar{v}(t) &= \bar{v}(0), \\ X^{''}(t) &= \int_{\Omega_t} (\tilde{\rho}|v-\bar{v}|^2)dx = X^{''}(0) = E(0), \end{aligned}$$

where $E(0)$ is defined in Theorem 1.4.

Proof. When $\sigma = 0$, then $p = 0$, $\hat{\rho} = \tilde{\rho}$. (1.7) can be equivalently rewritten as

$$\begin{cases} \partial_t \tilde{\rho} + \nabla_x \cdot (\tilde{\rho} v) = 0, \\ \partial_t (\tilde{\rho} v) + \nabla_x \cdot (\tilde{\rho} v \otimes v) = 0. \end{cases}$$

Then we have

$$D_t(\tilde{\rho}(t, x(t,y))J(t,y)) = \partial_t \tilde{\rho} + \nabla_x \cdot (\tilde{\rho} v) = 0,$$

i.e., along the fluid line, $M(t) = M(0)$. Furthermore, it holds that

$$\tilde{\rho}(t, x(t,y))J(t,y) = \tilde{\rho}(0,y). \tag{2.14}$$

Similarly, we have $\bar{v}(t) = \bar{v}(0)$ and

$$\tilde{\rho}(t, x(t,y))v(t, x(t,y))J(t,y) = \tilde{\rho}v(0,y). \tag{2.15}$$

At last, we prove the conservation of $X^{''}(t) = \int_{\Omega_t} \tilde{\rho}|v-\bar{v}|^2 dx$. Differentiating $X^{''}(t)$ by D_t gives

$$\begin{aligned} D_t X^{''}(t) &= \int_{\Omega_0} D_t(\tilde{\rho}J)|v-\bar{v}|^2 dy + 2\int_{\Omega_0} \tilde{\rho}J\langle D_t(v-\bar{v}), v-\bar{v}\rangle \\ &= 2\int_{\Omega_t} \tilde{\rho}\langle \partial_t(v-\bar{v}) + v\cdot\nabla(v-\bar{v}), v-\bar{v}\rangle dx = 0, \end{aligned}$$

where we have used

$$\partial_t v + v \cdot \nabla v = 0$$

and

$$D_t \bar{v} = 0.$$

Thus, the proof is completed. □

In order to prove Theorem 1.4, we also need the following two lemmas

Lemma 2.3. The following equality holds

$$\hat{\rho} = \frac{1}{c^2}\tilde{\rho}v^2 + \rho. \tag{2.16}$$

Proof. The proof of this lemma comes from Pan and Smoller [9]. For completeness of this paper, we give the proof here.

It is easy to see that

$$\begin{aligned}
\hat{\rho} &= \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \\
&= \frac{\rho c^4 + pc^2 - pc^2 + pv^2}{c^2(c^2 - v^2)} \\
&= \frac{\rho c^2 v^2 + \rho c^2(c^2 - v^2) + pv^2}{c^2(c^2 - v^2)} \\
&= \frac{1}{c^2}\frac{\rho c^2 + p}{c^2 - v^2}v^2 + \rho \\
&= \frac{1}{c^2}\tilde{\rho}v^2 + \rho.
\end{aligned}$$

□

Next lemma comes from a geometric fact of Sideris [12]

Lemma 2.4. We have

$$\sup_{x\in\Omega_t} |x - \bar{x}| \leq diam\Omega_t. \tag{2.17}$$

Proof. Similar to [12], it suffices to prove

$$\int_{\Omega_t} \hat{\rho}(x - \bar{x})dx = 0.$$

This follows by definition that

$$\int_{\Omega_t} \hat{\rho}x dx = M(t)\bar{x}(t),$$

and

$$\int_{\Omega_t} \hat{\rho}\bar{x}dx = \bar{x}(t)\int_{\Omega_t} \hat{\rho}dx = M(t)\bar{x}(t).$$

Then by the same discussions as in [12], this lemma holds. □

Based on Lemmas 2.1-2.4, we give the proof of Theorem 1.4 below.

Proof of Theorem 1.4. From Lemmas 2.1 and 2.3, we see that

$$\begin{aligned} X^{''}(t) &= \int_{\Omega_t} (\hat{\rho}(\bar{v})^2 + \tilde{\rho}|v|^2 - 2\tilde{\rho}\langle v, \bar{v}\rangle + np)dx \\ &= M(0)\bar{v}^2(0) + \int_{\Omega_t} \tilde{\rho}v^2 dx - 2M(0)\bar{v}^2(0) + \int_{\Omega_t} npdx \\ &= \int_{\Omega_t} c^2\frac{\tilde{\rho}v^2}{c^2} + n\sigma^2\rho dx - M(0)\bar{v}^2(0) \\ &\geq \min\{c^2, n\sigma^2\}\int_{\Omega_t}\left(\frac{\tilde{\rho}v^2}{c^2} + \rho\right)dx - M(0)\bar{v}^2(0) \\ &= (\min\{c^2, n\sigma^2\} - \bar{v}^2(0))M(0) := DM(0). \end{aligned} \tag{2.18}$$

Integrating the above second order ODE twice, we have

$$X(t) \geq \frac{1}{2}DM(0)t^2 + Y(0)t + X(0). \tag{2.19}$$

On the other hand, by Lemma 2.4 we have

$$\begin{aligned} X(t) = \frac{1}{2}\int_{\Omega_t} \hat{\rho}|x - \bar{x}|^2 dx &\leq \frac{1}{2}\sup_{x\in\Omega_t}|x - \bar{x}|^2\int_{\Omega_t}\hat{\rho}dx \\ &\leq \frac{1}{2}(diam\Omega_t)^2 M(0). \end{aligned} \tag{2.20}$$

Combining (2.19) with (2.20), we easily see

$$(diam\Omega_t)^2 \geq \frac{DM(0)t^2 + 2Y(0)t + 2X(0)}{M(0)}.$$

Then Theorem 1.4 holds when $\sigma > 0$.

When $\sigma = 0$, by Corollary 2.2, we have

$$X(t) = \frac{1}{2}E(0)t^2 + Y(0)(t) + X(0). \tag{2.21}$$

Combining (2.20) with (2.21) gives,

$$(diam\Omega_t)^2 \geq \frac{E(0)t^2 + 2Y(0)t + 2X(0)}{M(0)}.$$

Thus, the proof of Theorem 1.4 is completed. □

Remark 2.5. In the proof of Theorem 1.4, we need $\sigma^2 > \frac{\bar{v}^2(0)}{n}$. From (2.16), we can also show that

$$X^{''}(t) \leq (\max\{c^2, n\sigma^2\} - \bar{v}^2(0))M(0).$$

Intergrating twice, we have

$$X(t) \leq \frac{1}{2}(\max\{c^2, n\sigma^2\} - \bar{v}^2(0))M(0)t^2 + Y(0)t + X(0).$$

Remark 2.6. The isothermal equation of state $p = \sigma^2\rho$ plays an important role in the proof of the inequality (2.18). This is a technical reason for us to choose this equation of state and we hope the main result still holds for other ideal fluids such as polytropic gas and so on.

3. The proof of Theorem 1.8

Based on the conserved quantity $M(t)$ and $\bar{v}(t)$ in last section, especially (2.14) and (2.15), we can construct a class of spherically symmetric solution, which shows that the diameter of the region surrounded by vacuum for isentropic relativistic Euler equations grows linearly in time. In this section, we focus on Theorem 1.8 to illustrate the result stated in Theorem 1.4 with $\sigma = 0$.

Proof. We assume that system (1.7), (1.8) has a spherically symmetric solution, which satisfies the two conservation laws (2.14) and (2.15) along the fluid line. Then we have

$$\tilde{\rho}(t, x(t, y))J(t, y) = \tilde{\rho}(0, y) = \left[\frac{c^2\rho(t, x(t, y))}{c^2 - v^2(t, x(t, y))}\right] J(t, y), \tag{3.1}$$

and

$$\tilde{\rho}v(t, x(t, y))J(t, y) = \tilde{\rho}v(0, y) = \frac{c^2\rho v(t, x(t, y))}{c^2 - v^2(t, x(t, y))} J(t, y). \tag{3.2}$$

From (3.1) and (3.2), it is easy to show that

$$v(t, x(t, y)) = v(0, y) = v_0(|y|)\frac{y}{|y|}. \tag{3.3}$$

Via (3.3), we consider the flow line. By the spherically symmetric assumption, we have

$$\begin{cases} \frac{d}{dt}|x| = v_0(|y|), \\ t = 0 : |x| = |y|. \end{cases} \tag{3.4}$$

Thus,

$$|x(t, y)| = |y| + v_0(|y|)t. \tag{3.5}$$

Then we have by (2.6) and (3.3) and simple integration

$$J(t, y) = detD_y x(t, y) = \left(1 + v_0'(|y|)t\right)\left(1 + \frac{v_0(|y|)t}{|y|}\right)^{n-1}. \tag{3.6}$$

We get easily from (3.1)

$$\rho\left(t, y + v_0(|y|)t\frac{y}{|y|}\right) = \frac{\rho_0(|y|)|y|^{n-1}}{\left(1 + v_0'(|y|)t\right)\left(|y| + v_0(|y|)t\right)^{n-1}}. \tag{3.7}$$

In this case, the diameter of the region becomes

$$diam\Omega_t = 2(1 + v_0(1)t).$$

Thus, the proof of Theorem 1.8 is completed. □

Remark 3.1. From Theorem 1.8, we can see that the conclusion of the linear growth with time of the diameter of the region, occupied by the perfect fluid surrounded by a vacuum in the Minkowski spacetime, is sharp.

Remark 3.2. Due to the complexity of the relativistic Euler equations, when $\sigma \neq 0$, we could not find the explicit spherically symmetric solution taking the form of Sideris's or Theorem 1.8. When $\sigma = 0$, the velocity v satisfies the vectorial Burgers equation and the modified density $\tilde{\rho}$ is simply transported. Then by the standard method of characteristics, we can also get the solution formulas to the system (1.4) without the assumption of the spherical symmetry. The main idea can be found in [3, 11] for general case with non-negative spectrum of the gradient of the initial velocity fields or [19] for 3D radial case.

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