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An open adelic image theorem for motivic representations over function fields

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Let $\mathbb{F}$ be a field and k a function field of positive transcendence degree over $\mathbb{F}$. Let S be a smooth, separated, geometrically connected scheme of finite type over k. If $\mathbb{F}$ is quasi-finite or algebraically closed we show that for motivic representations of the étale fundamental group $\pi_1(S)$ of S, ℓ-Galois-generic points are Galois-generic. This is a geometric variant of a previous result of the author for representations of $\pi_1(S)$ on the adelic Tate module of an abelian scheme $A \to S$ when the base field k is finitely generated of characteristic 0. The procyclicity of the absolute Galois group of a quasi-finite field allows to reduce the assertion for $\mathbb{F}$ finite to the assertion for $\mathbb{F}$ algebraically closed. The assertion for $\mathbb{F}$ algebraically closed can then be deduced, using basically the same arguments as in the case of abelian schemes, from maximality results for the image of $\pi_1(S)$ inside the group of $\mathbb{Z}_\ell$-points of its Zariski-closure.

1. Introduction

Let k be a field of characteristic $p \geq 0$, S a smooth, separated, geometrically connected scheme of finite type over k with generic point η and $X \to S$ a smooth, proper morphism. For every $s \in S$, fix a geometric point $\overline{s}$ over s and an étale path from $\overline{s}$ to $\overline{\eta}$. For a prime $\ell \neq p$, *via* the canonical isomorphism (smooth-proper base change) $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Z}/\ell^n) \simeq \mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}/\ell^n)$, the Galois representation by transport of structure of $\pi_1(s, \overline{s})$ on $\mathrm{H}^*(X_{\overline{s}}, \mathbb{Z}/\ell^n)$ identifies with the restriction of the representation of $\pi_1(S, \overline{\eta})$ on $\mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}/\ell^n)$ *via* the functorial morphism $\sigma_s : \pi_1(s, \overline{s}) \to \pi_1(S, \overline{s}) \tilde{\to} \pi_1(S, \overline{\eta})$. So, from now on, we omit base-points in our notation for étale fundamental groups and write

$$\mathrm{H}_{\ell^\infty} := \mathrm{H}^*(X_{\overline{\eta}}, \mathbb{Z}_\ell)/\text{torsion}, \ \mathrm{V}_{\ell^\infty} := \mathrm{H}_{\ell^\infty} \otimes \mathbb{Q}_\ell.$$

Let

$$\rho_{\ell^\infty} : \pi_1(S) \to \mathrm{GL}(\mathrm{H}_{\ell^\infty}),\ \rho_\infty = \prod_{\ell \neq p} \rho_{\ell^\infty} : \pi_1(S) \to \prod_{\ell \neq p} \mathrm{GL}(\mathrm{H}_{\ell^\infty}) =: \mathrm{GL}(H_\infty)$$

denote the resulting representations and set $\Pi_? := \mathrm{im}(\rho_?)$, $? = \infty, \ell^\infty$. For $s \in S$, also set $\rho_{?,s} := \rho_? \circ \sigma_s$ and $\Pi_{?,s} := \mathrm{im}(\rho_{?,s})$, $? = \infty, \ell^\infty$.

Following the terminology of [CK16], we say that $s \in S$ is ℓ-Galois-generic (with respect to ρ_∞) if $\Pi_{\ell^\infty,s}$ is open in Π_{ℓ^∞} and that $s \in S$ is Galois-generic (with respect to ρ_∞) if $\Pi_{\infty,s}$ is open in Π_∞.

Given a prime ℓ, we say that a field $\mathbb{F}$ is *ℓ-non Lie semisimple* if for every quotient $\pi_1(\mathbb{F}) \twoheadrightarrow \Gamma_\ell$ with Γ_ℓ a ℓ-adic Lie group, none of the non-zero Lie sub algebra of $\mathrm{Lie}(\Gamma_\ell)$ is semisimple. Typical examples are algebraically closed fields and quasi-finite fields (in particular, finite fields), which are ℓ-non Lie semisimple for every prime ℓ, or p-adic fields, which are ℓ-non Lie semisimple for every prime $\ell \neq p$.

Assume now that k is the function field of a smooth, separated, geometrically connected scheme of finite type and dimension ≥ 1 over a field $\mathbb{F}$. The main result of this note is

Theorem 1.1. *Assume $\mathbb{F}$ is ℓ-non Lie semisimple. For a closed point $s \in S$, the following are equivalent.*

1) $s \in S$ is ℓ-Galois-generic;

2) $s \in S$ is Galois-generic.

In particular, when $\mathbb{F}$ is finite, this proves the abundance of closed Galois-generic points. More precisely, we have

Corollary 1.2. *Assume $\mathbb{F}$ is finite. Then*

1) There exists an integer $d \geq 1$ such that there are infinitely many (ℓ-)Galois-generic closed points $s \in S$ with $[k(s) : k] \leq d$.

2) Assume furthermore that S is a curve. Then all but finitely many $s \in S(k)$ are (ℓ-)Galois-generic.

Proof. Assertion (1) follows from [S89, §10.6] while assertion (2) follows from [A17, Thm. 1.3 (3)], since motivic representations are GLP. □

Theorem 1.1 is a geometric variant of a previous result of the author for representations of $\pi_1(S)$ on the adelic Tate module of an abelian scheme $A \to S$ when the base field k is finitely generated of characteristic 0. The 'ℓ-non Lie semisimple' property allows to reduce Theorem 1.1 for $\mathbb{F}$ ℓ-non Lie semisimple to Theorem 1.1 for $\mathbb{F}$ algebraically closed (Lemma 2.2.3). Theorem 1.1 for $\mathbb{F}$ algebraically closed can then be deduced, following the guidelines of [C15], from maximality results for Π_{ℓ^∞} inside the group of $\mathbb{Z}_\ell$-points of its Zariski-closure in $\mathrm{GL}_{\mathrm{H}_{\ell^\infty}}$. For $p = 0$, the maximality result is the same as the one used in [C15]; it relies on a group-theoretical result of Nori ([N87]). For $p > 0$, the maximality result is due to Hui, Tamagawa and the author ([CHT17]).

It is reasonable to expect that Theorem 1.1 holds for k a number field (hence, by Hilbert's irreducibility theorem, for any finitely generated field of characteristic 0). This should follow from variants with $\mathbb{F}_\ell$-coefficients of the Grothendieck-Serre-Tate conjectures.

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2. Proof

The implication (1.1.2) $\Rightarrow$ (1.1.1) is straightforward. We prove the converse implication. Fix a closed point $s \in S$. Without loss of generality, we may assume $s \in S(k)$.

2.1. Notation

Fix a smooth, separated, geometrically connected scheme U over $\mathbb{F}$ with generic point ζ such that there exists a model

$$\mathcal{X} \longrightarrow \mathcal{S} \underset{\curvearrowleft s_U}{\longrightarrow} U \longrightarrow \mathbb{F}$$

of

$$X \longrightarrow S \underset{}{\overset{s}{\rightleftarrows}} k \longrightarrow \mathbb{F}$$

in the sense that we have a cartesian diagram

$$\begin{array}{ccccccc}
\mathcal{X} & \longrightarrow & \mathcal{S} & \overset{s_U}{\rightleftarrows} & U & \longrightarrow & \mathbb{F}\,, \\
\uparrow & \square & \uparrow & \square & \uparrow \zeta & & \Vert \\
X & \longrightarrow & S & \underset{s}{\rightleftarrows} & k = k(\zeta) & \longrightarrow & \mathbb{F}
\end{array}$$

with $\mathcal{X} \to \mathcal{S}$ smooth, proper and $\mathcal{S} \to U$ smooth, separated, geometrically connected of finite type. In particular, the action of $\pi_1(S)$, $\pi_1(s)$ on H_ℓ^∞ factor respectively through $\pi_1(S) \twoheadrightarrow \pi_1(\mathcal{S})$ and $\pi_1(s) \twoheadrightarrow \pi_1(U)$ so that

2.1.1. the groups $\Pi_?, \Pi_{?,s} \subset \mathrm{GL}(\mathrm{H}_?)$, $? = \infty, \ell^\infty$ identify with the images of the motivic representations attached to the smooth proper morphisms $\mathcal{X} \to \mathcal{S}$ and $\mathcal{X} \times_{\mathcal{S}, s_U} U \to U$ respectively. We write, again,

$$\rho_? : \pi_1(\mathcal{S}) \to \mathrm{GL}(\mathrm{H}_?),\ \rho_{?,s} : \pi_1(U) \to \mathrm{GL}(\mathrm{H}_{?,s}),\ ? = \infty, \ell^\infty$$

for the corresponding representations and set

$$\tilde{\Pi}_? := \rho_?(\pi_1(\mathcal{S}_{\overline{\mathbb{F}}})),\ \tilde{\Pi}_{?,s} := \rho_{?,s}(\pi_1(U_{\overline{\mathbb{F}}})),\ ? = \infty,\ \ell^\infty.$$

2.2.

We first reduce the assertion for $\mathbb{F}$ ℓ-non Lie semisimple to the assertion for $\mathbb{F}$ algebraically closed.

The introduction of the property 'ℓ-non Lie semisimple' comes from

2.2.1. Fact. *The following equivalent assertions hold:*

1) $\mathrm{Lie}(\tilde{\Pi}_{\ell^\infty})$ *and* $\mathrm{Lie}(\tilde{\Pi}_{\ell^\infty,s})$ *are semisimple Lie algebras;*
2) *The Zariski closure of* $\tilde{\Pi}_{\ell^\infty}$ *and* $\tilde{\Pi}_{\ell^\infty,s}$ *in* $\mathrm{GL}_{\mathrm{H}_{\ell^\infty}}$ *are semisimple algebraic groups.*

Proof. Recall 2.1.1. Then 2) follows from comparison between étale and singular cohomologies and [D71, Prop. (4.2.5), Thm. (4.2.6)] if $p = 0$ and from [D80, Cor. 3.4.13, Cor. 1.3.9] if $p > 0$. The equivalence of 1) and 2) follows from the general fact that if $\Pi \subset \mathrm{GL}_r(\mathbb{Q}_\ell)$ is a compact ℓ-adic Lie group whose Zariski closure $G \subset \mathrm{GL}_{r,\mathbb{Q}_\ell}$ is semi simple then Π is open in $G(\mathbb{Q}_\ell)$; this boils down to the fact that a semi simple Lie algebra over $\mathbb{Q}_\ell$ is algebraic - see *e.g.* [S66, §1, Cor.]. □

2.2.2. We begin with an elementary observation (a partial snake lemma in the category of profinite groups). Consider a commutative diagram of profinite groups with exact lines

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{\Pi} & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \tilde{\Pi}' & \longrightarrow & \Pi' & \longrightarrow & \Gamma' & \longrightarrow & 1 \end{array}$$

Assume the two left-hand vertical arrows are injective and the right-hand vertical arrow is surjective. Then the canonical map $\tilde{\Pi}/\tilde{\Pi}' \to \Pi/\Pi'$ is surjective and its fibers are isomorphic to $\tilde{\Pi} \cap \Pi'/\tilde{\Pi}'$. In particular,

1) $\tilde{\Pi}' \subset \tilde{\Pi}$ is open $\Rightarrow \Pi' \subset \Pi$ is open.
2) $\Pi' \subset \Pi$ is open and $\tilde{\Pi} \cap \Pi'/\tilde{\Pi}'$ is finite $\Rightarrow \tilde{\Pi}' \subset \tilde{\Pi}$ is open.

2.2.3. Lemma.

1) $\tilde{\Pi}_{\infty,s} \subset \tilde{\Pi}_\infty$ *is open* $\Rightarrow \Pi_{\infty,s} \subset \Pi_\infty$ *is open.*
2) *Fix a prime* $\ell \neq p$ *and assume* $\mathbb{F}$ *is* ℓ*-non Lie semisimple. Then* $\Pi_{\ell^\infty,s} \subset \Pi_{\ell^\infty}$ *is open* $\Rightarrow \tilde{\Pi}_{\ell^\infty,s} \subset \tilde{\Pi}_{\ell^\infty}$ *is open.*

Proof. Since $s \in S(k)$, for $? = \infty, \ell^\infty$ the canonical morphism $\Pi_{?,s}/\tilde{\Pi}_{?,s} \to \Pi_?/\tilde{\Pi}_?$ is surjective and the short exact sequences of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{\Pi}_? & \longrightarrow & \Pi_? & \longrightarrow & \Pi_?/\tilde{\Pi}_? & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \tilde{\Pi}_{?,s} & \longrightarrow & \Pi_{?,s} & \longrightarrow & \Pi_{?,s}/\tilde{\Pi}_{?,s} & \longrightarrow & 1 \end{array}$$

is of the form considered in 2.2.2. So 1) follows from 2.2.2.1) while 2) would follow from 2.2.2.2) provided $\tilde{\Pi}_{\ell^\infty} \cap \Pi_{\ell^\infty,s}/\tilde{\Pi}_{\ell^\infty,s}$ is finite. This is where we use the assumption that $\mathbb{F}$ is ℓ-non Lie semisimple. Indeed, we have

$$\tilde{\Pi}_{\ell^\infty}\cap\Pi_{\ell^\infty,s}\twoheadrightarrow\tilde{\Pi}_{\ell^\infty}\cap\Pi_{\ell^\infty,s}/\tilde{\Pi}_{\ell^\infty,s}\hookrightarrow\Pi_{\ell^\infty,s}/\tilde{\Pi}_{\ell^\infty,s}\twoheadleftarrow\pi_1(\mathbb{F}).$$

By Fact 2.2.1, the Lie algebra of $\tilde{\Pi}_{\ell^\infty}\cap\Pi_{\ell^\infty,s}/\tilde{\Pi}_{\ell^\infty,s}$ is semisimple, being a quotient of $\mathrm{Lie}(\tilde{\Pi}_{\ell^\infty}\cap\Pi_{\ell^\infty,s})=\mathrm{Lie}(\tilde{\Pi}_{\ell^\infty})$. But this forces it to be 0, since $\mathbb{F}$ is ℓ-non Lie semisimple by assumption. □

Fix a prime $\ell\neq p$, assume $\mathbb{F}$ is ℓ-non Lie semisimple and $s\in S(k)$ is ℓ-Galois-generic. From (2.2.3.2), $\tilde{\Pi}_{\ell^\infty,s}\subset\tilde{\Pi}_{\ell^\infty}$ is open. If Theorem 1.1 holds for $\mathbb{F}$ algebraically closed, this would imply $\tilde{\Pi}_{\infty,s}\subset\tilde{\Pi}_\infty$ is open hence, from (2.2.3.1), $\Pi_{\infty,s}\subset\Pi_\infty$ is open. This observation reduces Theorem 1.1 for $\mathbb{F}$ ℓ-non Lie semisimple to Theorem 1.1 for $\mathbb{F}$ algebraically closed.

2.2.3 So, from now on, we assume $\mathbb{F}$ is *algebraically closed* hence

$$\tilde{\Pi}_?=\Pi_?,\ \tilde{\Pi}_{?,s}=\Pi_{?,s},\ ?=\infty,\ell^\infty,\ s\in S.$$

2.3.

Fix a prime $\ell_0\neq p$ and assume $s\in S(k)$ is ℓ_0-Galois-generic. We want to show $s\in S(k)$ is Galois-generic.

For every prime $\ell\neq p$ and profinite group Γ appearing as a subquotient of $\mathrm{GL}(\mathrm{H}_{\ell^\infty})$, let $\Gamma^+\subset\Gamma$ denote the (normal) subgroup of Γ generated by its ℓ-Sylow subgroups. Let $\mathfrak{G}_{\ell^\infty}$, $\mathfrak{G}_{\ell^\infty,s}$ denote respectively the Zariski-closure of Π_{ℓ^∞}, $\Pi_{\ell^\infty,s}$ in $\mathrm{GL}_{\mathrm{H}_{\ell^\infty}}$. Write G_{ℓ^∞} and $G_{\ell^\infty,s}$ for the generic fibers of $\mathfrak{G}_{\ell^\infty}$, $\mathfrak{G}_{\ell^\infty,s}$.

2.3.1. Fact. *The dimensions of G_{ℓ^∞}, $G_{\ell^\infty,s}$ are independent of $\ell(\neq p)$.*

Proof. This follows from comparison between étale and singular cohomologies if $p=0$ and from [LaP95, Thm. 2.4] if $p>0$. More precisely, [LaP95, Thm. 2.4] implies that, if $Y\to C$ is a smooth proper morphism with C a smooth, separated, geometrically connected curve over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ then the dimension of the Zariski closure of the image of

$$\pi_1(C)\to\mathrm{GL}(\mathrm{H}^*(Y_{\overline{c}},\mathbb{Q}_\ell))$$

is idependant of ℓ. To apply this to the setting of (2.1.1), we need the generalization of [LaP95, Thm. 2.4] for C of arbitrary dimension. This can be deduced from the case of curves by Jouanolou's version of Bertini's theorem [Jou83, Thm. 6.10, 2), 3)] and the smooth proper base change theorem. We refer to the Claim in the proof of [CT17, Prop. 3.2] for details. □

Also, to prove Theorem 1.1, we may freely replace U and $\mathcal{S}$ by connected étale covers. In particular,

2.3.2. Fact. *We may assume the following holds.*

1) $\Pi_{\ell^\infty} = \Pi^+_{\ell^\infty}$, $\Pi_{\ell^\infty,s} = \Pi^+_{\ell^\infty,s}$ *for* $\ell \gg 0$*;*
2) $\Pi_\infty = \prod_{\ell\neq p} \Pi_{\ell^\infty}$, $\Pi_{\infty,s} = \prod_{\ell\neq p} \Pi_{\ell^\infty,s}$*;*
3) G_{ℓ^∞}, $G_{\ell^\infty,s}$ *are connected for every prime* $\ell \neq p$*;*
4) $\Pi_{\ell^\infty} = \mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell)^+$, $\Pi_{\ell^\infty,s} = \mathfrak{G}_{\ell^\infty,s}(\mathbb{Z}_\ell)^+$ *for* $\ell \gg 0$*;*

Proof. Recall 2.1.1 and 2.2.3. Then 1) follows from [CT17, Thm. 1.1] while 2) is [CT17, Cor. 4.6]. 3) follows from comparison between étale and singular cohomologies if $p = 0$ and from [LaP95, Prop. 2.2] if $p > 0$. For 4), assume first $p = 0$ (see [C15, §2.3] for details). Let $\Pi_\ell \subset \mathfrak{G}_{\ell^\infty}(\mathbb{F}_\ell)$ denote the image of Π_{ℓ^∞} *via* the reduction-modulo-ℓ morphism $\mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell) \to \mathfrak{G}_{\ell^\infty}(\mathbb{F}_\ell)$. Then, from [N87, Thm. 5.1], $\Pi_\ell = \Pi_\ell^+ = \mathfrak{G}_{\ell^\infty}(\mathbb{F}_\ell)^+$ for $\ell \gg 0$. This forces $\Pi_{\ell^\infty} = \mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell)^+$ since, by [C15, Fact 2.3, Lemma 2.4], $\mathfrak{G}_{\ell^\infty}(\mathbb{Z}_\ell)^+ \to \mathfrak{G}_{\ell^\infty}(\mathbb{F}_\ell)^+$ is Frattini for $\ell \gg 0$. Eventually, 4) for $p > 0$ is [CHT17, Thm. 7.3.2]. □

2.4.

We can now conclude the proof. From (2.3.2.2), it is enough to show that

1) $\Pi_{\ell^\infty,s} \subset \Pi_{\ell^\infty}$ *is open for every prime* $\ell \neq p$*;*
2) $\Pi_{\ell^\infty,s} = \Pi_{\ell^\infty}$ *for* $\ell \gg 0$.

Since $s \in S(k)$ is ℓ_0-Galois-generic, (2.3.2.3) for ℓ_0 ensures $G_{\ell_0^\infty,s} = G_{\ell_0^\infty}$. As $G_{\ell^\infty,s}$ is always a subgroup of G_{ℓ^∞}, Fact 2.3.1 and (2.3.2.3) also ensure $G_{\ell^\infty,s} = G_{\ell^\infty}$ hence $\mathfrak{G}_{\ell^\infty,s} = \mathfrak{G}_{\ell^\infty}$ for every prime $\ell \neq p$. Then 1) follows from (2.2.1.1) while 2) follows from (2.3.2.4).

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