An open adelic image theorem for motivic representations over function fields

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Let \( F \) be a field and \( k \) a function field of positive transcendence degree over \( F \). Let \( S \) be a smooth, separated, geometrically connected scheme of finite type over \( k \). If \( F \) is quasi-finite or algebraically closed we show that for motivic representations of the \( \acute{e} \)tale fundamental group \( \pi_1(S) \) of \( S \), \( \ell \)-Galois-generic points are Galois-generic. This is a geometric variant of a previous result of the author for representations of \( \pi_1(S) \) on the adelic Tate module of an abelian scheme \( A \to S \) when the base field \( k \) is finitely generated of characteristic 0. The procyclicity of the absolute Galois group of a quasi-finite field allows to reduce the assertion for \( F \) finite to the assertion for \( F \) algebraically closed. The assertion for \( F \) algebraically closed can then be deduced, using basically the same arguments as in the case of abelian schemes, from maximality results for the image of \( \pi_1(S) \) inside the group of \( \mathbb{Z}_\ell \)-points of its Zariski-closure.

1. Introduction

Let \( k \) be a field of characteristic \( p \geq 0 \), \( S \) a smooth, separated, geometrically connected scheme of finite type over \( k \) with generic point \( \eta \) and \( X \to S \) a smooth, proper morphism. For every \( s \in S \), fix a geometric point \( \overline{s} \) over \( s \) and an \( \acute{e} \)tale path from \( s \) to \( \eta \). For a prime \( \ell \neq p \), via the canonical isomorphism (smooth-proper base change) \( H^i(X_{\overline{s}}, \mathbb{Z}/\ell^n) \simeq H^i(X_{\overline{\eta}}, \mathbb{Z}/\ell^n) \), the Galois representation by transport of structure of \( \pi_1(s, \overline{s}) \) on \( H^i(X_{\overline{s}}, \mathbb{Z}/\ell^n) \) identifies with the restriction of the representation of \( \pi_1(S, \overline{\eta}) \) on \( H^i(X_{\overline{\eta}}, \mathbb{Z}/\ell^n) \) via the functorial morphism \( \sigma_s : \pi_1(s, \overline{s}) \to \pi_1(S, \overline{s}) \to \pi_1(S, \overline{\eta}) \). So, from now on, we omit base-points in our notation for \( \acute{e} \)tale fundamental groups and write

\[
H_{\ell^\infty} := H^*(X_{\overline{\eta}}, \mathbb{Z}_\ell) / \text{torsion}, \quad V_{\ell^\infty} := H_{\ell^\infty} \otimes \mathbb{Q}_\ell.
\]
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Let

\[ \rho_{\ell}\infty : \pi_1(S) \rightarrow \text{GL}(H_{\ell}\infty), \rho_{\infty} = \prod_{\ell \neq p} \rho_{\ell}\infty : \pi_1(S) \rightarrow \prod_{\ell \neq p} \text{GL}(H_{\ell}\infty) =: \text{GL}(H_{\infty}) \]

denote the resulting representations and set \( \Pi_{?} := \text{im}(\rho_{?}) \), \( ? = \infty, \ell\infty \). For \( s \in S \), also set \( \rho_{?,s} := \rho_{?} \circ \sigma_s \) and \( \Pi_{?,s} := \text{im}(\rho_{?,s}) \), \( ? = \infty, \ell\infty \).

Following the terminology of [CK16], we say that \( s \in S \) is \( \ell \)-Galois-generic (with respect to \( \rho_{\infty} \)) if \( \Pi_{\ell\infty,s} \) is open in \( \Pi_{\ell\infty} \) and that \( s \in S \) is Galois-generic (with respect to \( \rho_{\infty} \)) if \( \Pi_{\infty,s} \) is open in \( \Pi_{\infty} \).

Given a prime \( \ell \), we say that a field \( \mathbb{F} \) is \( \ell \)-non Lie semisimple if for every quotient \( \pi_1(\mathbb{F}) \rightarrow \Gamma_{\ell} \) with \( \Gamma_{\ell} \) a \( \ell \)-adic Lie group, none of the non-zero Lie subalgebra of \( \text{Lie}(\Gamma_{\ell}) \) is semisimple. Typical examples are algebraically closed fields and quasi-finite fields (in particular, finite fields), which are \( \ell \)-non Lie semisimple for every prime \( \ell \), or \( p \)-adic fields, which are \( \ell \)-non Lie semisimple for every prime \( \ell \neq p \).

Assume now that \( k \) is the function field of a smooth, separated, geometrically connected scheme of finite type and dimension \( \geq 1 \) over a field \( \mathbb{F} \). The main result of this note is

**Theorem 1.1.** Assume \( \mathbb{F} \) is \( \ell \)-non Lie semisimple. For a closed point \( s \in S \), the following are equivalent.

1) \( s \in S \) is \( \ell \)-Galois-generic;
2) \( s \in S \) is Galois-generic.

In particular, when \( \mathbb{F} \) is finite, this proves the abundance of closed Galois-generic points. More precisely, we have

**Corollary 1.2.** Assume \( \mathbb{F} \) is finite. Then

1) There exists an integer \( d \geq 1 \) such that there are infinitely many \((\ell\text{-})\)Galois-generic closed points \( s \in S \) with \( [k(s) : k] \leq d \).
2) Assume furthermore that \( S \) is a curve. Then all but finitely many \( s \in S(k) \) are \((\ell\text{-})\)Galois-generic.

**Proof.** Assertion (1) follows from [SS9] §10.6 while assertion (2) follows from [A17, Thm. 1.3 (3)], since motivic representations are GLP. \( \square \)
Theorem 1.1 is a geometric variant of a previous result of the author for representations of $\pi_1(S)$ on the adelic Tate module of an abelian scheme $A \to S$ when the base field $k$ is finitely generated of characteristic 0. The \( \ell\)-non Lie semisimple property allows to reduce Theorem 1.1 for $F$ \( \ell\)-non Lie semisimple to Theorem 1.1 for $F$ algebraically closed (Lemma 2.2.3). Theorem 1.1 for $F$ algebraically closed can then be deduced, following the guidelines of [C15], from maximality results for $\Pi_{\ell\infty}$ inside the group of $Z_{\ell}$-points of its Zariski-closure in $GL_{H_{\ell\infty}}$. For $p = 0$, the maximality result is the same as the one used in [C15]; it relies on a group-theoretical result of Nori ([N87]). For $p > 0$, the maximality result is due to Hui, Tamagawa and the author ([CHT17]).

It is reasonable to expect that Theorem 1.1 holds for $k$ a number field (hence, by Hilbert’s irreducibility theorem, for any finitely generated field of characteristic 0). This should follow from variants with $F_{\ell}$-coefficients of the Grothendieck-Serre-Tate conjectures.

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## 2. Proof

The implication (1.1.2) $\Rightarrow$ (1.1.1) is straightforward. We prove the converse implication. Fix a closed point $s \in S$. Without loss of generality, we may assume $s \in S(k)$.

### 2.1. Notation

Fix a smooth, separated, geometrically connected scheme $U$ over $F$ with generic point $\zeta$ such that there exists a model

\[
X \rightarrow S \rightarrow U \rightarrow F
\]
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of

\[
\begin{array}{c}
X \xrightarrow{s} S \xrightarrow{k} F
\end{array}
\]

in the sense that we have a cartesian diagram

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{X} \xrightarrow{s_{\mathcal{X}}} U \xrightarrow{\mathcal{F}} F \\
\downarrow \quad \downarrow \\
X \xrightarrow{s} k = k(\zeta) \xrightarrow{\mathcal{F}} F
\end{array}
\end{array}
\]

with \(\mathcal{X} \to S\) smooth, proper and \(S \to U\) smooth, separated, geometrically connected of finite type. In particular, the action of \(\pi_1(S), \pi_1(s)\) on \(H_\infty\) factor respectively through \(\pi_1(S) \to \pi_1(S)\) and \(\pi_1(s) \to \pi_1(U)\) so that

2.1. the groups \(\Pi_?, \Pi_{?,s} \subset \text{GL}(H_?)\), \(? = \infty, \ell\infty\) identify with the images of the motivic representations attached to the smooth proper morphisms \(\mathcal{X} \to S\) and \(\mathcal{X} \times_{S, s_U} U \to U\) respectively. We write, again,

\[
\rho_? : \pi_1(S) \to \text{GL}(H_?), \rho_{?,s} : \pi_1(U) \to \text{GL}(H_{?,s}), \(? = \infty, \ell\infty\)
\]

for the corresponding representations and set

\[
\tilde{\Pi}_? := \rho_!(\pi_1(S_?), \tilde{\Pi}_{?,s} := \rho_{?,s}(\pi_1(U_?)), \(? = \infty, \ell\infty\).
\]

2.2.

We first reduce the assertion for \(F\) \(\ell\)-non Lie semisimple to the assertion for \(F\) algebraically closed.

The introduction of the property ‘\(\ell\)-non Lie semisimple’ comes from

2.2.1. Fact. The following equivalent assertions hold:

1) \(\text{Lie}(\tilde{\Pi}_{\ell\infty})\) and \(\text{Lie}(\tilde{\Pi}_{\ell\infty,s})\) are semisimple Lie algebras;

2) The Zariski closure of \(\tilde{\Pi}_{\ell\infty}\) and \(\tilde{\Pi}_{\ell\infty,s}\) in \(\text{GL}(H_{\ell\infty})\) are semisimple algebraic groups.

Proof. Recall 2.1. Then 2) follows from comparison between étale and singular cohomologies and [D71 Prop. (4.2.5), Thm. (4.2.6)] if \(p = 0\) and
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from [D80] Cor. 3.4.13, Cor. 1.3.9] if $p > 0$. The equivalence of 1) and 2) follows from the general fact that if $\Pi \subset \text{GL}_r(\mathbb{Q}_\ell)$ is a compact $\ell$-adic Lie group whose Zariski closure $G \subset \text{GL}_r(\mathbb{Q}_\ell)$ is semi simple then $\Pi$ is open in $G(\mathbb{Q}_\ell)$; this boils down to the fact that a semi simple Lie algebra over $\mathbb{Q}_\ell$ is algebraic - see e.g. [S66] §1, Cor.]. □

2.2.2. We begin with an elementary observation (a partial snake lemma in the category of profinite groups). Consider a commutative diagram of profinite groups with exact lines

$$
\begin{array}{c}
1 \rightarrow \tilde{\Pi} \rightarrow \Pi \rightarrow \Gamma \rightarrow 1 \\
1 \rightarrow \tilde{\Pi}' \rightarrow \Pi' \rightarrow \Gamma' \rightarrow 1
\end{array}
$$

Assume the two left-hand vertical arrows are injective and the right-hand vertical arrow is surjective. Then the canonical map $\tilde{\Pi}' / \Pi' \rightarrow \Pi / \Pi'$ is surjective and its fibers are isomorphic to $\tilde{\Pi} \cap \Pi' / \Pi'$. In particular,

1) $\tilde{\Pi}' \subset \tilde{\Pi}$ is open $\Rightarrow$ $\Pi' \subset \Pi$ is open.

2) $\Pi' \subset \Pi$ is open and $\tilde{\Pi} \cap \Pi' / \Pi'$ is finite $\Rightarrow$ $\tilde{\Pi}' \subset \tilde{\Pi}$ is open.

2.2.3. Lemma.

1) $\tilde{\Pi}_{\infty,s} \subset \tilde{\Pi}_{\infty}$ is open $\Rightarrow$ $\Pi_{\infty,s} \subset \Pi_{\infty}$ is open.

2) Fix a prime $\ell \neq p$ and assume $F$ is $\ell$-non Lie semisimple. Then $\Pi_{\ell,\infty,s} \subset \Pi_{\ell,\infty}$ is open $\Rightarrow$ $\tilde{\Pi}_{\ell,\infty,s} \subset \tilde{\Pi}_{\ell,\infty}$ is open.

Proof. Since $s \in S(k)$, for $? = \infty, \ell^{\infty}$ the canonical morphism $\Pi_{?,s} / \Pi_{?,s} \rightarrow \Pi_{?,s} / \Pi_{?,s}$ is surjective and the short exact sequences of profinite groups

$$
\begin{array}{c}
1 \rightarrow \tilde{\Pi}_{?,s} \rightarrow \Pi_{?,s} \rightarrow \Pi_{?,s} / \Pi_{?,s} \rightarrow 1 \\
1 \rightarrow \tilde{\Pi}_{?,s} \rightarrow \Pi_{?,s} \rightarrow \Pi_{?,s} / \Pi_{?,s} \rightarrow 1
\end{array}
$$

is of the form considered in 2.2.2. So 1) follows from 2.2.2.1) while 2) would follow from 2.2.2.2) provided $\tilde{\Pi}_{\ell,\infty} \cap \Pi_{\ell,\infty,s} / \tilde{\Pi}_{\ell,\infty,s}$ is finite. This is where we
use the assumption that $F$ is $\ell$-non Lie semisimple. Indeed, we have
\[ \tilde{\Pi}^{\infty} \cap \Pi^{\infty,s} / \tilde{\Pi}^{\infty,s} \rightarrow \Pi^{\infty,s} / \tilde{\Pi}^{\infty,s} \leftarrow \pi_1(F). \]
By Fact 2.2.1, the Lie algebra of $\tilde{\Pi}^{\infty} \cap \Pi^{\infty,s} / \tilde{\Pi}^{\infty,s}$ is semisimple, being a quotient of $\text{Lie}(\tilde{\Pi}^{\infty} \cap \Pi^{\infty,s}) = \text{Lie}(\Pi^{\infty})$. But this forces it to be 0, since $F$ is $\ell$-non Lie semisimple by assumption. □

Fix a prime $\ell \neq p$, assume $F$ is $\ell$-non Lie semisimple and $s \in S(k)$ is $\ell$-Galois-generic. From (2.2.3.2), $\tilde{\Pi}^{\infty,s} \subset \tilde{\Pi}^{\infty}$ is open. If Theorem 1.1 holds for $F$ algebraically closed, this would imply $\Pi^{\infty,s} \subset \Pi^{\infty}$ is open hence, from (2.2.3.1), $\Pi^{\infty,s} \subset \Pi^{\infty}$ is open. This observation reduces Theorem 1.1 for $F$ $\ell$-non Lie semisimple to Theorem 1.1 for $F$ algebraically closed.

2.2.3 So, from now on, we assume $F$ is algebraically closed hence
\[ \tilde{\Pi}^{?} = \Pi^{?}, \tilde{\Pi}^{?,s} = \Pi^{?,s}, ? = \infty, \ell^{\infty}, s \in S. \]

2.3.

Fix a prime $\ell_0 \neq p$ and assume $s \in S(k)$ is $\ell_0$-Galois-generic. We want to show $s \in S(k)$ is Galois-generic.

For every prime $\ell \neq p$ and profinite group $\Gamma$ appearing as a subquotient of $\text{GL}(H^{\infty})$, let $\Gamma^+ \subset \Gamma$ denote the (normal) subgroup of $\Gamma$ generated by its $\ell$-Sylow subgroups. Let $\Theta^{\infty}, \Theta^{\infty,s}$ denote respectively the Zariski-closure of $\Pi^{\infty}, \Pi^{\infty,s}$ in $\text{GL}_{n^{\infty}}$. Write $G^{\infty}$ and $G^{\infty,s}$ for the generic fibers of $\Theta^{\infty}, \Theta^{\infty,s}$.

2.3.1. Fact. The dimensions of $G^{\infty}, G^{\infty,s}$ are independent of $\ell(\neq p)$.

Proof. This follows from comparison between étale and singular cohomologies if $p = 0$ and from [LaP95, Thm. 2.4] if $p > 0$. More precisely, [LaP95, Thm. 2.4] implies that, if $Y \rightarrow C$ is a smooth proper morphism with $C$ a smooth, separated, geometrically connected curve over the algebraic closure $\overline{F}$ of $F_p$ then the dimension of the Zariski closure of the image of
\[ \pi_1(C) \rightarrow \text{GL}(H^*(Y_\ell, Q_\ell)) \]
is independent of $\ell$. To apply this to the setting of (2.1.1), we need the generalization of [LaP95, Thm. 2.4] for $C$ of arbitrary dimension. This can be
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deduced from the case of curves by Jouanolou’s version of Bertini’s theorem
[Jou83 Thm. 6.10, 2), 3)] and the smooth proper base change theorem. We
refer to the Claim in the proof of [CT17 Prop. 3.2] for details.

Also, to prove Theorem 1.1, we may freely replace
\( U \) and \( S \) by connected étale covers. In particular,

2.3.2. Fact. We may assume the following holds.

1) \( \Pi_{\ell^{\infty}} = \Pi_{\ell^{\infty}, s} \cap \Pi_{\ell^{\infty}} \) for \( \ell \gg 0 \);
2) \( \Pi_{\ell^{\infty}} = \prod_{\ell \neq p} \Pi_{\ell^{\infty}, s} \cap \Pi_{\ell^{\infty}} \);
3) \( G_{\ell^{\infty}}, G_{\ell^{\infty}, s} \) are connected for every prime \( \ell \neq p \);
4) \( \Pi_{\ell^{\infty}} = \mathfrak{S}_{\ell^{\infty}}(\mathbb{F}_{\ell}) + \mathfrak{S}_{\ell^{\infty}}(\mathbb{Z}_{\ell})^{+} \) for \( \ell \gg 0 \).

Proof. Recall 2.1.1 and 2.2.3. Then 1) follows from [CT17 Thm. 1.1] while
2) is [CT17 Cor. 4.6]. 3) follows from comparison between étale and singular
cohomologies if \( p = 0 \) and from [LaP95 Prop. 2.2] if \( p > 0 \). For 4), assume
first \( p = 0 \) (see [C15, §2.3] for details). Let \( \Pi_{\ell} \subset \mathfrak{S}_{\ell^{\infty}}(\mathbb{F}_{\ell}) \) denote the image
of \( \Pi_{\ell^{\infty}} \) via the reduction-modulo-\( \ell \) morphism \( \mathfrak{S}_{\ell^{\infty}}(\mathbb{Z}_{\ell}) \to \mathfrak{S}_{\ell^{\infty}}(\mathbb{F}_{\ell}) \). Then,
from [N87 Thm. 5.1], \( \Pi_{\ell} = \prod_{\ell \neq p} \mathfrak{S}_{\ell^{\infty}}(\mathbb{F}_{\ell})^{+} \) for \( \ell \gg 0 \). This forces \( \Pi_{\ell^{\infty}} = \mathfrak{S}_{\ell^{\infty}}(\mathbb{Z}_{\ell})^{+} \) since, by [C15 Fact 2.3, Lemma 2.4], \( \mathfrak{S}_{\ell^{\infty}}(\mathbb{Z}_{\ell})^{+} \to \mathfrak{S}_{\ell^{\infty}}(\mathbb{F}_{\ell})^{+} \) is
Frattini for \( \ell \gg 0 \). Eventually, 4) for \( p > 0 \) is [CHT17 Thm. 7.3.2].

2.4.

We can now conclude the proof. From (2.3.2.2), it is enough to show that

1) \( \Pi_{\ell^{\infty}, s} \subset \Pi_{\ell^{\infty}} \) is open for every prime \( \ell \neq p \);
2) \( \Pi_{\ell^{\infty}, s} = \Pi_{\ell^{\infty}} \) for \( \ell \gg 0 \).

Since \( s \in S(k) \) is \( \ell_{0} \)-Galois-generic, (2.3.2.3) for \( \ell_{0} \) ensures \( G_{\ell^{\infty}, s} = G_{\ell^{\infty}} \). As \( G_{\ell^{\infty}, s} \) is always a subgroup of \( G_{\ell^{\infty}} \), Fact 2.3.1 and (2.3.2.4) also ensure
\( G_{\ell^{\infty}, s} = G_{\ell^{\infty}} \) hence \( \mathfrak{S}_{\ell^{\infty}, s} = \mathfrak{S}_{\ell^{\infty}} \) for every prime \( \ell \neq p \). Then 1) follows from
(2.2.1.1) while 2) follows from (2.3.2.4).

References

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