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Boundedness and continuity of the time derivative in the parabolic Signorini problem

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We prove the boundedness of the time derivative in the parabolic Signorini problem, as well as establish its Hölder continuity at regular free boundary points.

1. Introduction and main results

Let v be a weak solution of the *parabolic Signorini problem*

$$\Delta v - \partial_t v = 0 \quad \text{in } Q_1^+ := B_1^+ \times (-1, 0], \tag{1.1}$$

$$v \geq \varphi, \quad -\partial_{x_n} v \geq 0, \quad (v - \varphi)\partial_{x_n} v = 0 \quad \text{on } Q_1' := B_1' \times (-1, 0], \tag{1.2}$$

$$v(\cdot, -1) = \varphi_0 \quad \text{in } B_1, \tag{1.3}$$

to be understood in the appropriate integral sense, where $\varphi : Q_1' \to \mathbb{R}$ is the *thin obstacle* and φ_0 is the initial data satisfying the compatibility condition $\varphi_0 \geq \varphi(\cdot, 0)$ on B_1'. This kind of unilateral problem appears in many applications, such as thermics (boundary heat control), biochemistry (semipermeable membranes and osmosis), and elastostatics (the original Signorini problem). It also serves as a prototypical example of parabolic variational inequalities. We refer to the book [7] for the derivation of such models as well as for some basic existence and uniqueness results, and to [5] for more recent results on the problem.

One of the main objects of study in the parabolic Signorini problem is the apriori unknown *free boundary*

$$\Gamma(v) := \partial_{Q_1'}(\{v > \varphi\} \cap Q_1'),$$

which separates the regions where $v = \varphi$ and $\partial_{x_n} v = 0$ (here $\partial_{Q_1'}$ denotes the boundary in the relative topology of Q_1').

It is known that if φ is sufficiently regular, namely $\varphi \in H^{2,1}(Q_1')$ (see the end of the introduction for the notations) then the Lipschitz regularity of φ_0 in $B_1^+ \cup B_1'$ implies the local boundedness of the spatial gradient ∇v in $Q_1^+ \cup Q_1'$ (see [2, Lemma 6]), which then implies the Hölder continuity $\nabla v \in H^{\gamma,\gamma/2}_{\mathrm{loc}}(Q_1^+ \cup Q_1')$ (see [1, Theorem 2.1]), for some $\gamma > 0$. Recently, it was shown in [5] that $v \in H^{3/2,3/4}_{\mathrm{loc}}(Q_1^+ \cup Q_1')$, which is the optimal regularity of v, at least in the space variables x. The paper [5] also gives a comprehensive treatment of the problem from the free boundary regularity point of view, based on Almgren-, Monneau-, and Weiss-type monotonicity formulas.

The aim of this paper is to obtain a better regularity in the time variable t for the solutions of the parabolic Signorini problem above and to complement the results of [5]. It is known already from [1, Lemma 7] that if the initial data $\varphi_0 \in W^2_\infty(B_1^+)$, then the time derivatives $\partial_t v$ will also be locally bounded in $Q_1^+ \cup Q_1'$. This assumption on the initial data φ_0, however, is rather restrictive and excludes a "standard" time-independent solution (for $\varphi \equiv 0$)

$$v(x,t) = \mathrm{Re}(x_{n-1} + i x_n)^{3/2}, \quad x_n \geq 0,$$

which is clearly not in W^2_∞.

Our first result shows that $\partial_t v$ is in fact bounded, without any extra assumptions on the initial data, even though we will require a bit more regularity on the thin obstacle φ.

Theorem 1.1. *Let $v \in H^{3/2,3/4}(Q_1^+ \cup Q_1')$ be a solution of the Signorini problem* (1.1)–(1.2) *with $\varphi \in H^{4,2}(Q_1')$. Then $\partial_t v$ is locally bounded in $Q_1^+ \cup Q_1'$ and moreover*

$$\|\partial_t v\|_{L_\infty(Q^+_{1/2})} \leq C_n \left(\|v\|_{L_2(Q_1^+)} + \|\varphi\|_{H^{4,2}(Q_1')} \right).$$

We prove this theorem in §2. In fact, instead of asking $\varphi \in H^{4,2}(Q_1')$ it is sufficient to assume that $\partial_t(\Delta_{x'}\varphi - \partial_t \varphi) \in L_\infty(Q_1')$.

Our second result is that $\partial_t v$ is continuous at so-called regular free boundary points (see §3 for the definition).

Theorem 1.2. *Let v be as in Theorem 1.1. Then $\partial_t v$ continuously equals to $\partial_t \varphi$ at regular free boundary points.*

In fact, in §3 we prove a more precise version of this theorem (Theorem 3.2), which shows the Hölder continuity of $\partial_t v$ at regular points.

At the end of the paper we state a direct corollary on the higher regularity of the free boundary in the t variable near regular points (see Corollary 3.3). When the thin obstacle $\varphi \equiv 0$, Theorem 1.2 can be used to make an iterative step in the application of a higher-order boundary Harnack principle for parabolic slit domains and establish the C^∞ regularity (both in x and t) of the free boundary near regular points (see [4]).

Remark. Shortly after this paper was completed and posted, results similar to our Theorems 1.2 and 3.2 have also appeared in [3]; particularly, see Theorem 4.8 there.

Notation

Throughout the paper we use the following conventions and notations.

- $\mathbb{R}^n$ stands for the n-dimensional Euclidean space. For

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

  we typically denote $x' = (x_1, \dots, x_{n-1})$ and $x'' = (x_1, \dots, x_{n-2})$. We also routinely identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.
- $B_r(x_0)$, $B'_r(x_0)$, $B''_r(x_0)$ stand for balls of radius $r > 0$ centered at x_0 in $\mathbb{R}^n$, $\mathbb{R}^{n-1}$, $\mathbb{R}^{n-2}$, respectively. We drop the center from the notation if $x_0 = 0$. We also denote $B^\pm_r(x_0) = B_r(x_0) \cap \{\pm x_n > 0\}$.
- $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0]$ is the parabolic cylinder, with similar definitions for Q'_r, Q''_r, $Q^\pm_r$.
- For parabolic functional spaces, we use notations similar to those in [9] and [5, §2.2]. In particular, $H^{\ell,\ell/2}(E)$ for $\ell = m + \gamma$, $m \in \mathbb{N} \cup \{0\}$, $\gamma \in (0, 1]$ is the space of functions such that the partial derivatives $\partial^\alpha_x \partial^j_t u$ are γ-Hölder in x and $\gamma/2$-Hölder in t for the derivatives of the parabolic order $|\alpha| + 2j \leq m$ and $(1 + \gamma)/2$-Hölder in t if $|\alpha| + 2j \leq m - 1$. $L_p(E)$ stands for the Lebesgue space, and $W^{2m,m}_p(E)$ is the Sobolev space of functions such that $\partial^\alpha_x \partial^j_t u \in L_p(E)$ for $|\alpha| + 2j \leq 2m$.

2. Boundedness of the time derivative

We first reduce the problem to the case of zero thin obstacle, at the expense of getting nonzero right hand side in the governing equation. Namely, let

$$u(x, t) := v(x, t) - \varphi(x', t).$$

Then we have

$$\Delta u - \partial_t u = f := \partial_t \varphi - \Delta_{x'} \varphi \quad \text{in } Q_1^+, \tag{2.1}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \partial_{x_n} u = 0 \quad \text{on } Q_1'. \tag{2.2}$$

It will also be convenient to extend the function u by the even symmetry in the x_n variable to the entire cylinder Q_1:

$$u(x', -x_n, t) = u(x', x_n, t).$$

Then the extended function will satisfy

$$\Delta u - \partial_t u = f + 2(\partial_{x_n}^+ u) \mathcal{H}^n \big|_{\Lambda(u)} \quad \text{in } Q_1,$$

in the sense of distributions, where f is also extended by the even symmetry in x_n to all of Q_1, $\partial_{x_n}^+ u(x', 0, t) = \partial_{x_n} u(x', 0+, t)$ for $(x', t) \in Q_1'$, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, and

$$\begin{aligned} \Lambda(u) &:= \{(x', t) \in Q_1' : u(x', 0, t) = 0\} \\ &= \{(x', t) \in Q_1' : v(x', 0, t) = \varphi(x', t)\} \end{aligned}$$

is the so-called *coincidence set.*

Proof of Theorem 1.1. For u solving (2.1)–(2.2) and a small $h > 0$ consider the incremental quotient in the time variable

$$U_h(x, t) = \frac{u(x, t) - u(x, t - h)}{h}, \quad (x, t) \in Q_{3/4}.$$

Let us also denote

$$F_h(x, t) = \frac{f(x, t) - f(x, t - h)}{h}, \quad (x, t) \in Q_{3/4}.$$

Note that $U_h \in H^{3/2, 3/4}(Q_{3/4}^{\pm} \cup Q_{3/4}')$ and $F_h \in H^{2,1}(Q_{3/4})$, from the assumption that the thin obstacle $\varphi \in H^{4,2}(Q_1')$.

We then have the following key observation.

Lemma 2.1. *The positive and negative parts of U_h,*

$$U_h^{\pm} := \max\{\pm U_h, 0\},$$

satisfy

$$(\Delta - \partial_t)(U_h^{\pm}) \geq -F_h^{\mp} \quad \textit{in } Q_{3/4}.$$

Proof. It is clear that the inequality is satisfied in $Q^{\pm}_{3/4}$, so we will need to show the inequality near $(x_0, t_0) \in Q'_{3/4}$. Suppose first that $U_h(x_0, t_0) > 0$. Then, necessarily $u(x_0, t_0) > 0$ and therefore

$$(\Delta - \partial_t)u(x,t) = f(x,t) \quad \text{in } Q_\delta(x_0, t_0),$$

for some small $\delta > 0$. On the other hand,

$$(\Delta - \partial_t)u(x, t-h) \leq f(x, t-h) \quad \text{in } Q_\delta(x_0, t_0),$$

in the sense of distributions, and taking the difference, we obtain

$$(\Delta - \partial_t)U_h \geq F_h \quad \text{in } Q_\delta(x_0, t_0).$$

We thus have

$$(\Delta - \partial_t)U_h \geq -F_h^- \quad \text{in } \{U_h > 0\} \cap Q_{3/4}$$

and a standard argument now implies that

$$(\Delta - \partial_t)U_h^+ \geq -F_h^- \quad \text{in } Q_{3/4}.$$

Indeed, for nonnegative $\eta \in C_0^\infty(Q_{3/4})$ and $\varepsilon > 0$ let

$$\begin{aligned}
&\eta_\varepsilon = \eta\,\chi(U_h/\varepsilon),\\
&\text{where } \chi \in C^\infty(\mathbb{R}), \quad \chi\big|_{(-\infty,1]} = 0, \quad \chi\big|_{[2,\infty)} = 1, \quad \chi' \geq 0.
\end{aligned}$$

Since U_h is continuous, η_ε is supported in $\{U_h > 0\}$ and hence

$$\iint_{Q_{3/4}} (\nabla U_h \nabla \eta_\varepsilon + \partial_t U_h \eta_\varepsilon) \leq \iint_{Q_{3/4}} F_h^- \eta_\varepsilon \leq \iint_{Q_{3/4}} F_h^- \eta.$$

On the other hand,

$$\begin{aligned}
\iint_{Q_{3/4}} \nabla U_h \nabla \eta_\varepsilon &= \iint_{Q_{3/4}} (\nabla U_h \nabla \eta)\chi(U_h/\varepsilon) + \eta \frac{1}{\varepsilon}\chi'(U_h/\varepsilon)|\nabla U_h|^2\\
&\geq \iint_{Q_{3/4}} (\nabla U_h \nabla \eta)\chi(U_h/\varepsilon).
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0+$, using the Dominated Convergence Theorem, we then conclude

$$\iint_{Q_{3/4}} (\nabla U_h \nabla \eta + \partial_t U_h \eta)\chi_{\{U_h > 0\}} \leq \iint_{Q_{3/4}} F_h^- \eta,$$

which can be rewritten as

$$\iint_{Q_{3/4}} (\nabla U_h^+ \nabla \eta + \partial_t U_h^+ \eta) \le \iint_{Q_{3/4}} F_h^- \eta.$$

The proof for U_h^- is similar. □

We will also need the following known estimate.

Lemma 2.2. *Let u be a weak solution of* (2.1)–(2.2). *Then $u \in W_2^{2,1}(Q_\rho^+)$ for any $\rho < 1$ with*

$$\|D^2 u\|_{L_2(Q_\rho^+)} + \|\partial_t u\|_{L_2(Q_\rho^+)} \le C_{\rho,n} \left(\|u\|_{L_2(Q_1^+)} + \|f\|_{L_2(Q_1^+)} \right).$$

The proof can be found in [1, Lemma 6], and in the Gaussian-weighted case in [5].

Going back to the proof of Theorem 1.1, we can now use the interior L_∞-L_2 estimates for subsolutions (see [10, Theorem 6.17]) to write

$$\|U_h^\pm\|_{L_\infty(Q_{1/2})} \le C_n \left(\|U_h\|_{L_2(Q_{3/4})} + \|F_h^\mp\|_{L_\infty(Q_{3/4})} \right).$$

On the other hand, since

$$U_h(x,t) = \frac{1}{h} \int_{t-h}^{t} \partial_t u(x,s) ds,$$

we obtain that

$$\begin{aligned} \|U_h\|_{L_2(Q_{3/4})} &= 2\|U_h\|_{L_2(Q_{3/4}^+)} \le 2\|\partial_t u\|_{L_2(Q_{5/6}^+)} \\ &\le C_n \left(\|u\|_{L_2(Q_1^+)} + \|f\|_{L_2(Q_1^+)} \right), \end{aligned}$$

where in the last inequality we have applied Lemma 2.2. It is also clear that

$$\|F_h\|_{L_\infty(Q_{3/4})} \le \|\partial_t f\|_{L_\infty(Q_1^+)}.$$

Letting now $h \to 0$, we then obtain the estimate

$$\|\partial_t u\|_{L_\infty(Q_{1/2})} \le C_n \left(\|u\|_{L_2(Q_1^+)} + \|f\|_{L_2(Q_1^+)} + \|\partial_t f\|_{L_\infty(Q_1^+)} \right),$$

which readily implies the statement of Theorem 1.1. □

3. Hölder continuity of the time derivative at regular points

In formulation (2.1)–(2.2), the free boundary is given by

$$\Gamma(u) = \partial_{Q_1'}\{(x',t) \in Q_1' : u(x',0,t) > 0\}.$$

As shown in [5], a successful study of the properties of the free boundary near $(x_0, t_0) \in \Gamma(u)$ can be made by considering the rescalings

$$u_r(x,t) = u_r^{(x_0,t_0)}(x,t) := \frac{u(x_0 + rx, t_0 + r^2 t)}{(H_u^{(x_0,t_0)}(r))^{1/2}},$$

for $r > 0$, and then studying the limits of u_r as $r = r_j \to 0+$ (so-called *blowups*). Here

$$H_u^{(x_0,t_0)}(r) := \frac{1}{r^2}\int_{t_0-r^2}^{t_0}\int_{\mathbb{R}^n} u^2(x,t)\psi^2(x)G(x_0 - x, t_0 - t)\,dxdt,$$

where ψ is a cutoff function, which is supported in B_1 and equals 1 in a neighborhood of x_0, and

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, & t > 0,\\ 0, & t \le 0 \end{cases}$$

is the heat kernel. Then a free boundary point $(x_0, t_0) \in \Gamma(u)$ is called *regular* if u_r converges in the appropriate sense to

$$u_0(x,t) = c_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2},$$

as $r = r_j \to 0+$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$. Note that this does not depend on the choice of the cutoff function ψ above. See [5] for more details and for a finer classification of free boundary points based on a generalization of Almgren's and Poon's frequency formulas.

Thus, let $\mathcal{R}(u)$ be the set of regular free boundary points of u, also known as the *regular set* of the solution u. The following result has been proved in [5].

Proposition 3.1. *Let $u \in H^{3/2,3/4}(Q_1^+ \cup Q_1')$ be a solution of the parabolic Signorini problem (2.1)–(2.2) with $f \in H^{1,1/2}(Q_1^+ \cup Q_1')$. Then the regular set $\mathcal{R}(u)$ is a relatively open subset of $\Gamma(u)$. Moreover, if $(x_0, t_0) \in \mathcal{R}(u)$,* *then there exists* $\rho = \rho_u(x_0, t_0) > 0$ *and a parabolic Lipschitz function* $g : Q''_\rho(x''_0, t_0) \to \mathbb{R}$ *such that*

$$\begin{aligned}\Gamma(u) \cap Q'_\rho(x_0, t_0) &= \mathcal{R}(u) \cap Q'_\rho(x_0, t_0)\\ &= \{x_{n-1} = g(x'', t), x_n = 0\} \cap Q'_\rho(x_0, t_0),\\ \Lambda(u) \cap Q'_\rho(x_0, t_0) &= \{x_{n-1} \le g(x'', t), x_n = 0\} \cap Q'_\rho(x_0, t_0).\end{aligned}$$

The parabolic Lipschitz continuity of the function g above means that for some constant L (parabolic Lipschitz constant)

$$\begin{aligned}&|g(x'', t) - g(y'', s)| \le L(|x'' - y''|^2 + |t - s|)^{1/2},\\ &(x'', t), (y'', s) \in Q''_\rho(x''_0, t_0).\end{aligned}$$

We are now ready to prove the following more precise version of Theorem 1.2.

Theorem 3.2. *Let* $u \in H^{3/2,3/4}(Q_1^+ \cup Q'_1)$ *be a solution of the parabolic Signorini problem* (2.1)–(2.2) *with* $f \in H^{2,1}(Q_1^+ \cup Q_1)$, *extended by the even symmetry in* x_n *to* Q_1. *Then for any* $(x_0, t_0) \in \mathcal{R}(u) \cap Q'_{1/4}$ *we have*

$$|\partial_t u(x, t)| \le C(|x - x_0|^2 + |t - t_0|)^{\alpha/2}, \quad (x, t) \in Q_{1/2} \setminus \Lambda(u),$$

for some $\alpha = \alpha_u(x_0, t_0) > 0$ *and* $C = C_u(x_0, t_0)$.

Proof. Let $\rho = \rho_u(x_0, t_0) > 0$ be as in Proposition 3.1. Without loss of generality we may assume $\rho \le 1/4$. Consider then the incremental quotients U_h and F_h defined in the proof of Theorem 1.1. In addition to Lemma 2.1, we then also have that

$$U_h = 0 \quad \text{on } \Lambda_h,$$

where

$$\Lambda_h = \{(x', t) : x_{n-1} \le g(x'', t) - Lh^{1/2}\} \cap Q'_\rho(x_0, t_0).$$

Here g is the function in the representation of $\Lambda(u) \cap Q'_\rho(x_0, t_0)$ and L is the parabolic Lipschitz constant of g. Then Λ_h is a subgraph of a parabolic Lipschitz function in $Q'_\rho(x_0, t_0)$, with the same parabolic Lipschitz constant L as g (actually, just a shift of g). Besides, from the assumption $(x_0, t_0) \in$

$\Gamma(u)$, we have that

$$(x_h, t_0) := (x_0 - Lh^{1/2}e_{n-1}, t_0) \in \Lambda_h.$$

We then claim that

$$U_h^\pm(x,t) \le C(|x-x_h|^2 + |t-t_0|)^{\alpha/2}, \quad (x,t) \in Q_{1/2}, \tag{3.1}$$

with $\alpha > 0$ depending only on the parabolic Lipschitz norm of g, and C depending only on n, ρ, and the L_∞ norms of U_h and F_h. Since the latter are uniformly bounded by $\|u\|_{L_2(Q_1)}$ and $\|f\|_{H^{2,1}(Q_1)}$, we can pass to the limit as $h \to 0+$ to obtain

$$|\partial_t u(x,t)| \le C(|x-x_0|^2 + |t-t_0|)^{\alpha/2}, \quad (x,t) \in Q_{1/2} \setminus \Lambda(u).$$

Thus, to finish the proof, we need to establish (3.1). This, in principle, follows from [10, Theorem 6.32], but with the uniform density condition on the complement (condition (A)) replaced with a properly defined uniform thermal capacity condition satisfied by Λ_h (see e.g. [11, §3.2]). Nevertheless, we give a more direct proof below.

Fix $0 < R < \rho$ and let W solve the Dirichlet problem (see Fig. 1)

$$\begin{aligned} (\Delta - \partial_t)W = 0 &\quad \text{in } \Omega_h(R) := [B_R(x_h) \times (t_0 - R^2, t_0 + R^2)]^\dagger \cap [Q_1 \setminus \Lambda_h], \\ W = U_h^\pm &\quad \text{on } \partial_p \Omega_h(R). \end{aligned}$$

By using Lemma 2.1 and comparing W with $U_h^\pm + C(|x-x_h|^2 - (t-t_0) - 2R^2)$ in $\Omega_h(R)$ with $(2n-1)C \ge \|f\|_{H^{2,1}(Q_1)} \ge \|F_h^\mp\|_{L_\infty(Q_1)}$, we see that

$$U_h^\pm \le W + CR^2 \quad \text{on } \Omega_h(R).$$

On the other hand, using a comparison with a barrier function as in [11, Lemma 3.2], we have

$$W(x,t) \le C\left(\frac{|x-x_h|^2 + |t-t_0|}{R^2}\right)^{\beta/2} \sup_{\Omega_h(R)} U_h^\pm,$$

with C depending only on the parabolic Lipschitz constant L of g and the dimension n. Here we have used that $\sup_{\Omega_h(R)} W = \sup_{\Omega_h(R)} U_h^\pm$, by the

†Note that $B_R(x_h) \times (t_0 - R^2, t_0 + R^2)$ is the "full" parabolic cylinder at (x_h, t_0), while $Q_R(x_h, t_0)$ is the "backward-in-time" cylinder $B_R(x_h) \times (t_0 - R^2, t_0]$

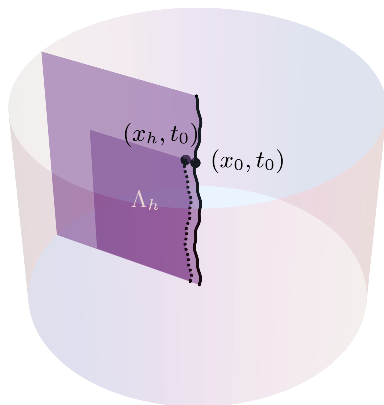

Figure 1: $\Omega_h(R)$.

maximum principle. Denoting

$$\omega(r) = \sup_{\Omega_h(r)} U_h^{\pm},$$

we then obtain

$$\omega(r) \leq C \left(\frac{r}{R}\right)^{\beta} \omega(R) + CR^2, \quad 0 < r \leq R.$$

Choosing $0 < \tau < 1$ small so that $\theta = C\tau^{\beta} < 1$, we then have

$$\omega(\tau R) \leq \theta\, \omega(R) + CR^2.$$

Then, a standard iterative argument (see [8, Lemma 8.23]) gives

$$\omega(R) \leq CR^{\alpha}, \quad R \leq \rho,$$

for $\alpha > 0$, which establishes (3.1) and completes the proof of the theorem. □

Corollary 3.3. *Let u, (x_0, t_0), ρ and g be as in Proposition 3.1. Then $\Gamma(u) \cap Q'_{\rho}(x_0, t_0)$ is an $(n-1)$-dimensional $C^{1,\alpha}$ surface both in the x and t variables.*

Proof. One argues precisely as in the proof of [5, Theorem 11.6] to show that

$$\frac{\partial_{x_j} u}{\partial_{x_{n-1}} u},\ j=1,\dots,n-2, \qquad \frac{\partial_t u}{\partial_{x_{n-1}} u} \in H^{\alpha,\alpha/2}(Q'_{\rho/2}(x_0,t_0)),$$

by the boundary Harnack principle in parabolic slit domains [11, §7]. The argument works for $\partial_t u$ since we now know that it continuously vanishes on $\Lambda(u)\cap Q_\rho(x_0,t_0)$ by Theorem 3.2. Consequently, the level sets $\{u=\varepsilon\}\cap Q'_{\rho/2}(x_0,t_0)$ are given as graphs

$$x_{n-1} = g_\varepsilon(x'', t)$$

with uniform estimates on the Hölder norms of $\partial_{x_j} g_\varepsilon$, $j=1,\dots,n-2$, and $\partial_t g_\varepsilon$. This then implies the Hölder continuity of $\partial_{x_j} g$ and $\partial_t g$ and completes the proof of the corollary. □

Remark 3.4. Very recently, in [4], it was proved that when the thin obstacle φ is identically zero, the free boundary is C^∞ both in the x and t variables near regular free boundary points. More precisely, the function g in the representation of $\Gamma(u)$ in Proposition 3.1 and Corollary 3.3 is C^∞. This is established by extending the higher-order boundary Harnack principle in [6] to parabolic slit domains, and using an argument similar to the proof of Corollary 3.3 above. An important ingredient in the proof is our Theorem 1.2, which allows to make the iteration steps in the t variable.

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