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Local models for the moduli stacks of global $\mathfrak{G}$-shtukas

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In this article we develop the theory of local models for the moduli stacks of global $\mathfrak{G}$-shtukas, the function field analogs for Shimura varieties. Here $\mathfrak{G}$ is a smooth affine group scheme over a smooth projective curve. As the first approach, we relate the local geometry of these moduli stacks to the geometry of Schubert varieties inside global affine Grassmannian, only by means of global methods. Alternatively, our second approach uses the relation between the deformation theory of global $\mathfrak{G}$-shtukas and associated local $\mathbb{P}$-shtukas at certain characteristic places. Regarding the analogy between function fields and number fields, the first (resp. second) approach corresponds to Beilinson-Drinfeld-Gaitsgory (resp. Rapoport-Zink) type local model for (PEL-)Shimura varieties. This discussion will establish a conceptual relation between the above approaches. Furthermore, as applications of this theory, we discuss the flatness of these moduli stacks over their reflex rings, we introduce the Kottwitz-Rapoport stratification on them, and we study the intersection cohomology of the special fiber.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and let $\mathfrak{G}$ be a flat affine group scheme of finite type over C. A *global* $\mathfrak{G}$*-shtuka* $\underline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \underline{s}, \tau)$ consisting of a $\mathfrak{G}$-torsor $\mathcal{G}$ over $C_S := C \times_{\mathbb{F}_q} S$, an n-tuple of (characteristic) sections $\underline{s} := (s_i)_i \in C^n(S)$ and a Frobenius connection τ defined outside the graphs Γ_{s_i} of the sections s_i, that is, an isomorphism $\tau\colon \sigma^*\mathcal{G}|_{C_S \smallsetminus \cup_i \Gamma_{s_i}} \xrightarrow{\sim} \mathcal{G}|_{C_S \smallsetminus \cup_i \Gamma_{s_i}}$ where $\sigma^* = (\,\mathrm{id}_C \times \mathrm{Frob}_{q,S})^*$.

Philosophically, a global $\mathfrak{G}$-shtuka may be considered as a $\mathfrak{G}$-motive, in the following sense. It admits crystalline (resp. étale) realizations at characteristic places (resp. away from characteristic places) which are endowed with $\mathfrak{G}$-actions; see [AH14, Section 5.2], [AH19, Chapter 6] and also [AH15, Chapter 2]. Consequently, according to Deligne's motivic conception of Shimura varieties [Del70, Del71], the moduli stacks of global $\mathfrak{G}$-shtukas can be regarded as the function field analogue for Shimura varieties. Based on this philosophy, they play a central role in the Langlands program over function fields. Hereupon several moduli spaces (resp. stacks) parametrizing families of such objects have been constructed and studied by various authors. Among those one could mention the space of F-sheaves $FSh_{D,r}$ which was considered by Drinfeld [Dri87] and Lafforgue [Laf02] in their proof of the Langlands correspondence for $\mathfrak{G} = \mathrm{GL}_2$ and $\mathfrak{G} = \mathrm{GL}_r$ respectively, and which in turn was generalized by Varshavsky's [Var04] moduli stacks $FBun$. Likewise the moduli stacks $Cht_{\underline{\lambda}}$ of Ngô and Ngô Dac [NN08], $\mathcal{E}\ell\ell_{C,\mathscr{D},I}$ of Laumon, Rapoport and Stuhler [LRS93], and $\mathcal{A}b\text{-}\mathcal{S}h_H^{r,d}$ of Hartl [Har05] and also the moduli stacks $\nabla_n^{\underline{\omega}}\mathscr{H}_D^1(C,\mathfrak{G})$, constructed by Hartl and the first author, that generalizes the previous constructions; see [AH14] and [AH19].

The above moduli stack $\nabla_n^{\underline{\omega}}\mathscr{H}_D^1(C,\mathfrak{G})$ parametrizes $\mathfrak{G}$-shtukas bounded by $\underline{\omega}$ which are equipped with D-level structure. Here $\mathfrak{G}$ is a flat affine group scheme of finite type over C. The superscript $\underline{\omega}$ denotes an n-tuple of coweights of SL_r and D is a divisor on C. It can be shown that $\nabla_n^{\underline{\omega}}\mathscr{H}_D^1(C,\mathfrak{G})$ is Deligne-Mumford and separated over C^n; see [AH19, Theorem 3.15]. This construction depends on a choice of a faithful representation $\rho : \mathfrak{G} \to \mathrm{SL}_r$. Accordingly in [AH19] the authors also propose an intrinsic alternative definition for the moduli stack of global $\mathfrak{G}$-shtukas, in which they roughly replace $\underline{\omega}$ by an n-tuple $\widehat{Z}_{\underline{\nu}}$ of certain (equivalence class of) closed subschemes of twisted affine flag varieties and they further refine the D-level structure to H-level structure, for a compact open subgroup $H \subset \mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$. Here $\mathbb{A}_Q^{\underline{\nu}}$ is the ring of adeles of C outside the fixed n-tuple $\underline{\nu} := (\nu_i)_i$ of places ν_i on

C. For detailed account on H-level structures on a global $\mathfrak{G}$-shtuka, we refer the reader to [AH19, Chapter 6]. The resulting moduli stack is denoted by $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$. Note however that the boundedness conditions introduced in [AH19] has been established by imposing bounds to the associated tuple of local $\mathbb{P}_{\nu_i}$-shtukas, that are obtained by means of the global-local functor (4.2). For this reason, the moduli stack of "bounded" $\mathfrak{G}$-shtukas was defined only after passing to the formal completion at the fixed characteristic places $\underline{\nu} := (\nu_i)$; see Definition 4.1.1.

In this article we first introduce a *global* notion of boundedness condition $\mathcal{Z}$ (which roughly is an equivalence class of closed subschemes of a global affine Grassmannian, satisfying some additional properties. This accredits us to define the moduli stack of bounded global $\mathfrak{G}$-shtukas $\nabla_n^{\mathcal{Z}}\mathscr{H}_D^1(C,\mathfrak{G})$ over n-fold fiber product $C_{\mathcal{Z}}^n := C_{Q_{\mathcal{Z}}}^n$ of the *reflex curve* associated to the *reflex field* $Q_{\mathcal{Z}}$; see Definition 3.1.3.

Now we come to the main theme of the present article, the theory of local models for the moduli stack $\nabla_n^{H,\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})$. Here it is necessary to assume that $\mathfrak{G}$ is a smooth affine group scheme over C. We study the local models for the moduli stacks of global $\mathfrak{G}$-shtukas according to the following two approaches.

As the first approach (global approach), based on the philosophy of Frobenius untwisting of $\mathfrak{G}$-shtukas, we prove that "global Schubert varieties" inside the global affine Grassmannian $GR_{\mathfrak{G},n}$, see Definition 2.0.7, may appear as a local model for the moduli stack of $\mathfrak{G}$-shtukas $\nabla_n^{\mathcal{Z}}\mathscr{H}_D^1(C,\mathfrak{G})$. We prove this in Theorem 3.2.1. This construction mirrors the Beilinson, Drinfeld and Gaitsgory type construction of the local model for Shimura varieties due to Pappas and Zhu [PZ].

Notice that, when $\mathfrak{G}$ is constant, i.e. $\mathfrak{G} := G_0 \times_{\mathbb{F}_q} C$, for a split reductive group G_0, our moduli stacks of global $\mathfrak{G}$-shtukas coincide the Varshavsky's moduli stacks of F-bundles. In this case the above observation was first formulated by Varshavsky [Var04, Theorem 2.20]. His proof relies on the well-known theorem of Drinfeld and Simpson [DS95], which assures that a G_0-bundle over C_S is Zariski-locally trivial after suitable étale base change $S' \to S$. The latter statement essentially follows from their argument about existence of B-structure on any G_0-bundle, where B is a Borel subgroup of G_0. Hence, one might not hope to implement this result in order to treat the present general case. Although, for the parahoric case, the Drinfeld-Simpson approach may be transposed, according to the normal forms of lattice chains, proved by De Jong and Rapoport-Zink; see [RZ96, Appendix to Chapter 3]. Nevertheless, to settle the general case, we modify Varshavsky's argument in chapter 3 and produce a proof which is independent of the theorem of Drinfeld and Simpson.

As the second approach (local approach), based on the analogue of Grothendiek-Messing theory for $\mathfrak{G}$-shtukas, see [AH14, Chapter 5], we use the relation between the deformation theory of global $\mathfrak{G}$-shtukas and local $\mathbb{P}$-shtukas, to relate the (local) geometry of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ to the (local) geometry of (a product of certain) Schubert varieties inside twisted affine flag varieties; see Proposition 4.4.2 and Theorem 4.4.6. To this goal, we basically implement the local theory of global $\mathfrak{G}$-shtukas, developed in [AH14], to produce the local model roof which relates the moduli stack of $\mathfrak{G}$-shutkas and the local model; see Theorem 4.4.6. This for instance allows to transpose the Satake perverse sheaves to the stack of $\mathfrak{G}$-shtukas. This construction of the local model is analogous to the Rapoport-Zink [RZ96] construction for PEL-type Shimura varieties.

Finally in section 4.5 we mention some applications of the above local model theories. In particular, using the local model roof, we introduce the analogue of *Kottwitz-Rapoport stratification* on the special fiber of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ and we observe that the corresponding intersection cohomology complex is of pure Tate type nature; see Proposition 4.5.2. Furthermore, in Proposition 4.5.3, we discuss the flatness of the moduli stack of $\mathfrak{G}$-shtukas over fiber product of reflex rings. This can be viewed as a function field analog of a result of Pappas and Zhu for Shimura varieties; see [PZ, Theorem 0.1 and Theorem 0.2].

Let us finally mention that our results in this article have number theoretic applications to the generalizations of the results of V. Lafforgue [Laf12] on Langlands parameterization, to the non-constant reductive case. One may already find such an application in [Laf12, Chapter 12], where he considers the situation which is slightly simplified by assuming that the paws (characteristic sections) are contained in the reductive locus of the group G. Also there are further applications to Varshavsky [Var04] results about the intersection cohomology of the moduli stacks of global G-shtukas. Additionally, according to the results obtained in [AH15], we expect further applications related to the possible description of the cohomology of affine Deligne-Lusztig varieties.

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1.1. Notation and conventions

Throughout this article we denote by

$\mathbb{F}_q$	a finite field with q elements of characteristic p,
C	a smooth projective geometrically irreducible curve over $\mathbb{F}_q$,
$Q := \mathbb{F}_q(C)$	the function field of C,
$\mathbb{F}$	a finite field containing $\mathbb{F}_q$,
$\widehat{A} := \mathbb{F}[\![z]\!]$	the ring of formal power series in z with coefficients in $\mathbb{F}$,
$\widehat{Q} := \mathrm{Frac}(\widehat{A})$	its fraction field,
ν	a closed point of C, also called a *place* of C,
$\mathbb{F}_\nu$	the residue field at the place ν on C,
$\widehat{A}_\nu$	the completion of the stalk $\mathcal{O}_{C,\nu}$ at ν,
$\widehat{Q}_\nu := \mathrm{Frac}(\widehat{A}_\nu)$	its fraction field,

$\mathbb{D}_R := \operatorname{Spec} R[\![z]\!]$	the spectrum of the ring of formal power series in z with coefficients in an $\mathbb{F}$-algebra R,
$\hat{\mathbb{D}}_R := \operatorname{Spf} R[\![z]\!]$	the formal spectrum of $R[\![z]\!]$ with respect to the z-adic topology.

When the ring R is obvious from the context we drop the subscript R from our notation.

For a formal scheme $\widehat{S}$ we denote by $\mathcal{N}ilp_{\widehat{S}}$ the category of schemes over $\widehat{S}$ on which an ideal of definition of $\widehat{S}$ is locally nilpotent. We equip $\mathcal{N}ilp_{\widehat{S}}$ with the étale topology. We also denote by

$n \in \mathbb{N}_{>0}$	a positive integer,
$\underline{\nu} := (\nu_i)_{i=1,\dots,n}$	an n-tuple of closed points of C,
$\mathbb{A}^{\underline{\nu}}$	the ring of integral adeles of C outside $\underline{\nu}$,
$\widehat{A}_{\underline{\nu}}$	the completion of the local ring $\mathcal{O}_{C^n,\underline{\nu}}$ of C^n at the closed point $\underline{\nu} = (\nu_i)$,
$\mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}} := \mathcal{N}ilp_{\operatorname{Spf} \widehat{A}_{\underline{\nu}}}$	the category of schemes over C^n on which the ideal defining the closed point $\underline{\nu} \in C^n$ is locally nilpotent,
$\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]} := \mathcal{N}ilp_{\hat{\mathbb{D}}}$	the category of $\mathbb{D}$-schemes S for which the image of z in $\mathcal{O}_S$ is locally nilpotent. We denote the image of z by ζ since we need to distinguish it from $z \in \mathcal{O}_{\mathbb{D}}$.
$\mathfrak{G}$	a smooth affine group scheme of finite type over C,
G	generic fiber of $\mathfrak{G}$,
$\mathbb{P}_\nu := \mathfrak{G} \times_C \operatorname{Spec} \widehat{A}_\nu,$	the base change of $\mathfrak{G}$ to $\operatorname{Spec} \widehat{A}_\nu$,
$P_\nu := \mathfrak{G} \times_C \operatorname{Spec} \widehat{Q}_\nu,$	the generic fiber of $\mathbb{P}_\nu$ over $\operatorname{Spec} Q_\nu$,
$\mathbb{P}$	a smooth affine group scheme of finite type over $\mathbb{D} = \operatorname{Spec} \mathbb{F}[\![z]\!]$,
P	the generic fiber of $\mathbb{P}$ over $\operatorname{Spec} \mathbb{F}(\!(z)\!)$.

Let S be an $\mathbb{F}_q$-scheme and consider an n-tuple $\underline{s} := (s_i)_i \in C^n(S)$. We denote by $\Gamma_{\underline{s}}$ the union $\bigcup_i \Gamma_{s_i}$ of the graphs $\Gamma_{s_i} \subseteq C_S$.

For an affine closed subscheme Z of C_S with sheaf $\mathcal{I}_Z$ we denote by $\mathbb{D}_S(Z)$ the scheme obtained by taking completion along Z and by $\mathbb{D}_{S,n}(Z)$ the closed subscheme of $\mathbb{D}_S(Z)$ which is defined by $\mathcal{I}_Z^n$. Moreover we set $\dot{\mathbb{D}}_S(Z) := \mathbb{D}_S(Z) \times_{C_S} (C_S \smallsetminus Z)$.

We denote by $\sigma_S \colon S \to S$ the $\mathbb{F}_q$-Frobenius endomorphism which acts as the identity on the points of S and as the q-power map on the structure sheaf. Likewise we let $\hat{\sigma}_S \colon S \to S$ be the $\mathbb{F}$-Frobenius endomorphism of an $\mathbb{F}$-scheme S. We set

$$\begin{aligned} &C_S := C \times_{\operatorname{Spec} \mathbb{F}_q} S, \\ \text{and} \quad &\sigma := \mathrm{id}_C \times \sigma_S. \end{aligned}$$

Let H be a sheaf of groups (for the étale topology) on a scheme X. In this article a *(right) H-torsor* (also called an *H-bundle*) on X is a sheaf $\mathcal{G}$ for the étale topology on X together with a (right) action of the sheaf H such that $\mathcal{G}$ is isomorphic to H on a étale covering of X. Here H is viewed as an H-torsor by right multiplication.

Definition 1.1.1. Assume that we have two morphisms $f, g \colon X \to Y$ of schemes or stacks. We denote by $\mathrm{equi}(f, g \colon X \rightrightarrows Y)$ the pull back of the diagonal under the morphism $(f, g) \colon X \to Y \times_{\mathbb{Z}} Y$, that is $\mathrm{equi}(f, g \colon X \rightrightarrows Y) := X \times_{(f,g), Y \times Y, \Delta} Y$ where $\Delta = \Delta_{Y/\mathbb{Z}} \colon Y \to Y \times_{\mathbb{Z}} Y$ is the diagonal morphism.

Definition 1.1.2. Assume that the generic fiber P of $\mathbb{P}$ over $\operatorname{Spec} \mathbb{F}(\!(z)\!)$ is connected reductive. Consider the base change P_L of P to $L = \mathbb{F}^{\mathrm{alg}}(\!(z)\!)$. Let S be a maximal split torus in P_L and let T be its centralizer. Since $\mathbb{F}^{\mathrm{alg}}$ is algebraically closed, P_L is quasi-split and so T is a maximal torus in P_L. Let $N = N(T)$ be the normalizer of T and let $\mathcal{T}^0$ be the identity component of the Néron model of T over $\mathcal{O}_L = \mathbb{F}^{\mathrm{alg}}[\![z]\!]$.

The *Iwahori-Weyl group* associated with S is the quotient group $\widetilde{W} = N(L)/\mathcal{T}^0(\mathcal{O}_L)$. It is an extension of the finite Weyl group $W_0 = N(L)/T(L)$ by the coinvariants $X_*(T)_I$ under $I = \mathrm{Gal}(L^{\mathrm{sep}}/L)$:

$$0 \to X_*(T)_I \to \widetilde{W} \to W_0 \to 1.$$

By [HR03, Proposition 8] there is a bijection

$$L^+\mathbb{P}(\mathbb{F}^{\mathrm{alg}}) \backslash LP(\mathbb{F}^{\mathrm{alg}}) / L^+\mathbb{P}(\mathbb{F}^{\mathrm{alg}}) \xrightarrow{\sim} \widetilde{W}^{\mathbb{P}} \backslash \widetilde{W} / \widetilde{W}^{\mathbb{P}} \tag{1.1}$$

where $\widetilde{W}^{\mathbb{P}} := (N(L) \cap \mathbb{P}(\mathcal{O}_L))/\mathcal{T}^0(\mathcal{O}_L)$, and where $LP(R) = P(R(\!(z)\!))$ and $L^+\mathbb{P}(R) = \mathbb{P}(R[\![z]\!])$ are the loop group, resp. the group of positive loops of $\mathbb{P}$; see [PR08, § 1.a], or [BD, §4.5], [NP01] and [Fal03] when $\mathbb{P}$ is constant. Let $\omega \in \widetilde{W}^{\mathbb{P}} \backslash \widetilde{W} / \widetilde{W}^{\mathbb{P}}$ and let $\mathbb{F}_\omega$ be the fixed field in $\mathbb{F}^{\mathrm{alg}}$ of $\{\gamma \in \mathrm{Gal}(\mathbb{F}^{\mathrm{alg}}/\mathbb{F}) \colon \gamma(\omega) = \omega\}$. There is a representative $g_\omega \in LP(\mathbb{F}_\omega)$ of ω; see [AH14, Example 4.12]. The *Schubert variety* $\mathcal{S}(\omega)$ associated with ω is the ind-scheme theoretic closure of the $L^+\mathbb{P}$-orbit of g_ω in $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \mathbb{F}_\omega$. It is a reduced projective variety over $\mathbb{F}_\omega$. For further details see [PR08] and [Ric13a].

Finally by an IC-sheaf $IC(\mathcal{X})$ on a stack $\mathcal{X}$, we will mean the intermediate extension of the constant perverse sheaf $\overline{\mathbb{Q}}_\ell$ on an open dense substack $\mathcal{X}^\circ$ of $\mathcal{X}$ such that the corresponding reduced stack $\mathcal{X}^\circ_{red}$ is smooth. The IC-sheaf is normalized so that it is pure of weight zero.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, and let $\mathfrak{G}$ be a smooth affine group scheme of finite type over C.

Definition 2.0.1. We let $\mathscr{H}^1(C,\mathfrak{G})$ denote the category fibered in groupoids over the category of $\mathbb{F}_q$-schemes, such that the objects over S, $\mathscr{H}^1(C,\mathfrak{G})(S)$, are $\mathfrak{G}$-torsors over C_S (also called $\mathfrak{G}$-bundles) and morphisms are isomorphisms of $\mathfrak{G}$-torsors.

Remark 2.0.2. One can prove that the stack $\mathscr{H}^1(C,\mathfrak{G})$ is a smooth Artin-stack locally of finite type over $\mathbb{F}_q$. Furthermore, it admits a covering $\{\mathscr{H}^1_\alpha\}_\alpha$ by connected open substacks of finite type over $\mathbb{F}_q$. The proof for parahoric $\mathfrak{G}$ (with semisimple generic fiber) can be found in [Hei10, Proposition 1] and for general case we refer to [AH19, Theorem 2.5].

Remark 2.0.3. There is a faithful representation $\rho\colon \mathfrak{G} \hookrightarrow \mathrm{GL}(\mathcal{V})$ for a vector bundle $\mathcal{V}$ on C together with an isomorphism $\alpha\colon \wedge^{\mathrm{top}}\mathcal{V} \xrightarrow{\sim} \mathcal{O}_C$ such that ρ factors through $\mathrm{SL}(\mathcal{V}) := \ker\big(\det\colon \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(\wedge^{\mathrm{top}}\mathcal{V})\big)$ and the quotients $\mathrm{SL}(\mathcal{V})/\mathfrak{G}$ and $\mathrm{GL}(\mathcal{V})/\mathfrak{G}$ are quasi-affine schemes over C. Note that for the existence of such a representation it even suffices to assume that $\mathfrak{G}$ is a flat affine group scheme over C. For a detailed account, see [AH19][Proposition 2.2].

Definition 2.0.4. Let D be a proper closed subscheme of C. A *D-level structure* on a $\mathfrak{G}$-bundle $\mathcal{G}$ on C_S is a trivialization $\psi\colon \mathcal{G}\times_{C_S} D_S \xrightarrow{\sim} \mathfrak{G}\times_C D_S$ along $D_S := D\times_{\mathbb{F}_q} S$. Let $\mathscr{H}^1_D(C,\mathfrak{G})$ denote the stack classifying $\mathfrak{G}$-bundles with D-level structure, that is, $\mathscr{H}^1_D(C,\mathfrak{G})$ is the category fibred in groupoids over the category of $\mathbb{F}_q$-schemes, which assigns to an $\mathbb{F}_q$-scheme S the category whose objects are

$$\begin{aligned} &Ob\big(\mathscr{H}^1_D(C,\mathfrak{G})(S)\big)\\ :=& \left\{(\mathcal{G},\psi)\colon \mathcal{G}\in\mathscr{H}^1(C,\mathfrak{G})(S),\ \psi\colon \mathcal{G}\times_{C_S} D_S \xrightarrow{\sim} \mathfrak{G}\times_C D_S\right\},\end{aligned}$$

and whose morphisms are those isomorphisms of $\mathfrak{G}$-bundles that preserve the D-level structure.

Let us recall the definition of the (unbounded ind-algebraic) Hecke stacks.

Definition 2.0.5. For each natural number n, let $Hecke_{n,D}(C,\mathfrak{G})$ be the stack fibered in groupoids over the category of $\mathbb{F}_q$-schemes, whose S valued points are tuples $\big((\mathcal{G},\psi),(\mathcal{G}',\psi'),\underline{s},\tau\big)$ where

- $(\mathcal{G},\psi)$ and $(\mathcal{G}',\psi')$ are in $\mathscr{H}^1_D(C,\mathfrak{G})(S)$,
- $\underline{s} := (s_i)_i \in (C \smallsetminus D)^n(S)$ are sections, and
- $\tau\colon \mathcal{G}'_{|C_S\smallsetminus\Gamma_{\underline{s}}} \xrightarrow{\sim} \mathcal{G}_{|C_S\smallsetminus\Gamma_{\underline{s}}}$ is an isomorphism preserving the D-level structures, that is, $\psi\circ\tau=\psi'$.

If $D=\emptyset$ we will drop it from the notation. Note that forgetting the isomorphism τ defines a morphism

$$Hecke_{n,D}(C,\mathfrak{G}) \to \mathscr{H}^1_D(C,\mathfrak{G}) \times \mathscr{H}^1_D(C,\mathfrak{G}) \times (C\smallsetminus D)^n. \tag{2.2}$$

Remark 2.0.6. A choice of faithful representation $\rho\colon \mathfrak{G}\to \mathrm{SL}(\mathcal{V})$ as in Remark 2.0.3 with quasi-affine (resp. affine) quotient $\mathrm{SL}(\mathcal{V})/\mathfrak{G}$, induces an ind-algebraic structure $\varinjlim Hecke_n^{\underline{\omega}}(C,\mathfrak{G})$ on the stack $Hecke_n(C,\mathfrak{G})$, which is relatively representable over $\mathscr{H}^1(C,\mathfrak{G})\times_{\mathbb{F}_q} C^n$ by an ind-quasi-projective (resp. ind-projective) morphism. Note that the limit is taken over n-tuples of coweights $\underline{\omega}=(\omega_i)$ of $\mathrm{SL}(\mathcal{V})$. For details see [AH19, Definition 3.8 and Proposition 3.9].

Definition 2.0.7. The *global affine Grassmannian* $GR_n(C,\mathfrak{G})$ is the stack fibered in groupoids over the category of $\mathbb{F}_q$-schemes, whose S-valued points are tuples $(\mathcal{G},\underline{s},\varepsilon)$, where $\mathcal{G}$ is a $\mathfrak{G}$-bundle over C_S, $\underline{s}:=(s_i)_i\in C^n(S)$ and $\varepsilon\colon \mathcal{G}|_{C_S\smallsetminus\Gamma_{\underline{s}}} \xrightarrow{\sim} \mathfrak{G}\times_C (C_S\smallsetminus\Gamma_{\underline{s}})$ is a trivialization. Since we fixed the curve C and the group $\mathfrak{G}$, we often drop them from notation and write $GR_n := GR_n(C,\mathfrak{G})$.

Remark 2.0.8. Notice that the global affine Grassmannian GR_n is isomorphic to the fiber product $Hecke_n(C,\mathfrak{G})\times_{\mathscr{H}^1(C,\mathfrak{G}),\mathfrak{G}} \operatorname{Spec}\mathbb{F}_q$ under the morphism sending $(\mathcal{G},\underline{s},\varepsilon)$ to $(\mathfrak{G}_S,\mathcal{G},\underline{s},\varepsilon^{-1})$. Hence, after we fix a faithful representation $\rho\colon\mathfrak{G}\hookrightarrow\mathrm{SL}(\mathcal{V})$ and coweights $\underline{\omega}$, as in Remark 2.0.3, the ind-algebraic structure on $Hecke_n(C,\mathfrak{G})$, induces an ind-quasi-projective ind-scheme structure on GR_n over C^n.

The following proposition explains the geometry of the stack $Hecke_n(C,\mathfrak{G})$ as a family over $C^n \times \mathscr{H}^1(C,\mathfrak{G})$.

Proposition 2.0.9. *Consider the stacks*

$$Hecke_n(C,\mathfrak{G}) \quad \text{and} \quad GR_n \times \mathscr{H}^1(C,\mathfrak{G})$$

as families over $C^n \times \mathscr{H}^1(C,\mathfrak{G})$, via the projections $(\mathcal{G},\mathcal{G}',\underline{s},\tau) \mapsto (\underline{s},\mathcal{G}')$ and $(\widetilde{\mathcal{G}},\underline{s},\widetilde{\tau}) \times \mathcal{G}' \mapsto (\underline{s},\mathcal{G}')$ respectively. They are locally isomorphic with respect to the étale topology on $C^n \times \mathscr{H}^1(C,\mathfrak{G})$.

Proof. The proof proceeds in a similar way as [Var04, Lemma 4.1], only one has to replace S by $\mathscr{H}^1(C,\mathfrak{G})$ and take an étale cover $V \to C \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G})$ trivializing the universal $\mathfrak{G}$-bundle over $\mathscr{H}^1(C,\mathfrak{G})$ rather than a Zariski trivialization over S. Also one sets

$$\begin{aligned} & U = V \times_{\mathscr{H}^1(C,\mathfrak{G})} \cdots \times_{\mathscr{H}^1(C,\mathfrak{G})} V, \\ & U' = Hecke_n(C,\mathfrak{G}) \times_{C^n \times \mathscr{H}^1(C,\mathfrak{G})} U, \\ & U'' = GR_n \times_{C^n} U, \\ & V' = V \times_{C \times \mathscr{H}^1(C,\mathfrak{G}),\mathcal{G}'} C \times U' \\ \text{and} \quad & V'' = V \times_{C \times \mathscr{H}^1(C,\mathfrak{G})} C \times U''. \end{aligned}$$

□

Remark 2.0.10. Notice that when the group $\mathfrak{G}$ is parahoric then the ind-algebraic stack $Hecke_n(C,\mathfrak{G})$ is ind-projective over $C^n \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G})$. This follows from [Ric13b] Theorem 1.18.

Now we recall the construction of the (unbounded ind-algebraic) stack of global $\mathfrak{G}$-shtukas.

Definition 2.0.11. We define the *moduli stack* $\nabla_n\mathscr{H}_D^1(C,\mathfrak{G})$ *of global* $\mathfrak{G}$-*shtukas with* D*-level structure* to be the preimage in $Hecke_{n,D}(C,\mathfrak{G})$ of the graph of the Frobenius morphism on $\mathscr{H}^1(C,\mathfrak{G})$. In other words

$$\nabla_n\mathscr{H}_D^1(C,\mathfrak{G}) := \mathrm{equi}(\sigma_{\mathscr{H}_D^1(C,\mathfrak{G})} \circ pr_1, pr_2 \colon Hecke_{n,D}(C,\mathfrak{G}) \rightrightarrows \mathscr{H}_D^1(C,\mathfrak{G})),$$

where pr_i are the projections to the first, resp. second factor in (2.2). Each object $\underline{\mathcal{G}}$ of $\nabla_n\mathscr{H}_D^1(C,\mathfrak{G})(S)$ is called a *global* $\mathfrak{G}$*-shtuka with* D*-level structure over* S and the corresponding sections $\underline{s} := (s_i)_i$ are called the *characteristic sections* (or simply *characteristics*) of $\underline{\mathcal{G}}$.

More explicitly a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ with D-level structure over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \psi, \underline{s}, \tau)$ consisting of a $\mathfrak{G}$-bundle $\mathcal{G}$ over C_S, a trivialization $\psi\colon \mathcal{G} \times_{C_S} D_S \xrightarrow{\sim} \mathfrak{G} \times_C D_S$, an n-tuple of (characteristic) sections $\underline{s}$, and an isomorphism $\tau\colon \sigma^*\mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}} \xrightarrow{\sim} \mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}}$ with $\psi \circ \tau = \sigma^*(\psi)$. If $D = \emptyset$ we drop ψ from $\underline{\mathcal{G}}$ and write $\nabla_n \mathscr{H}^1(C, \mathfrak{G})$ for the *stack of global $\mathfrak{G}$-shtukas*. Sometimes we will fix the sections $\underline{s} := (s_i)_i \in C^n(S)$ and simply call $\underline{\mathcal{G}} = (\mathcal{G}, \tau)$ a global $\mathfrak{G}$-shtuka over S. The ind-algebraic structure $\varinjlim Hecke_n^{\underline{\omega}}(C, \mathfrak{G})$ on the stack $Hecke_n(C, \mathfrak{G})$ induces an ind-algebraic structure $\varinjlim_{\underline{\omega}} \nabla_n^{\underline{\omega}} \mathscr{H}_D^1(C, \mathfrak{G})$ on $\nabla_n \mathscr{H}_D^1(C, \mathfrak{G})$.

Theorem 2.0.12. *Let D be a proper closed subscheme of C. The stack $\nabla_n \mathscr{H}_D^1(C, \mathfrak{G}) = \varinjlim \nabla_n^{\underline{\omega}} \mathscr{H}_D^1(C, \mathfrak{G})$ is an ind-algebraic stack (see [AH19, Definition 3.14]) over $(C \smallsetminus D)^n$ which is ind-separated and locally of ind-finite type. The stacks $\nabla_n^{\underline{\omega}} \mathscr{H}_D^1(C, \mathfrak{G})$ are Deligne-Mumford. Moreover, the forgetful morphism*

$$\nabla_n \mathscr{H}_D^1(C, \mathfrak{G}) \to \nabla_n \mathscr{H}^1(C, \mathfrak{G}) \times_{C^n} (C \smallsetminus D)^n$$

is surjective and a torsor under the finite group $\mathfrak{G}(D)$.

Proof. See [AH19, Theorem 3.15]. □

3. Analogue Of Beilinson-Drinfeld-Gaitsgory local model

According to [PRS13, Definition 1.13], the local models for Shimura varieties (in the Beilinson-Drinfeld-Gaitsgory context) can be constructed as certain scheme-theoretic closures of orbits of positive loop groups inside a fiber product of global affine Grassmannian and a flag variety, which are associated with cocharacters $\lambda \in X_+(T)$, for a maximal split torus T of a constant split reductive group G.

This construction has been generalized to the case of Shimura varieties with parahoric level structure $K \subset G(\mathbb{Q}_p)$, for which G is non-split (split over tamely ramified extension) by Pappas and Zhu [PZ].

Before the occurrence of the above constructions, the function fields analogous theory, for a split reductive group G over $\mathbb{F}_q$, was worked out by Varshavsky [Var04], following the original ideas of Beilinson-Drinfeld [BD] and Gaitsgory [Gai01]. In this chapter we generalize his construction of the local model to the case where $\mathfrak{G}$ is a smooth affine group scheme over C. Note in particular that this obviously includes the case where $\mathfrak{G}$ is parahoric. To this purpose, we must first introduce the notion of *global boundedness* *conditions*, and correspondingly, the notion of *reflex fields* for the moduli stacks of global $\mathfrak{G}$-shtukas.

3.1. Global boundedness conditions

In this section we establish an *intrinsic global boundedness condition* on the moduli stack $\nabla_n\mathscr{H}^1(C,\mathfrak{G})$ (resp. $Hecke_n(C,\mathfrak{G})$). The appellation *global boundedness condition* comes from the fact that it bounds the relative position of $\sigma^*\mathcal{G}$ and $\mathcal{G}$ (resp. $\mathcal{G}'$ and $\mathcal{G}$) under the isomorphism τ, globally on C. Note further that our proposed definition is intrinsic, in the sense that it is independent of the choice of representation $\rho : \mathfrak{G} \to SL(\mathcal{V})$; see [AH19, Remark 3.19]. The corresponding local version of this notion was previously introduced in [AH14, Definition 4.8].

Let us first recall the definition of global loop groups associated with $\mathfrak{G}$.

Definition 3.1.1. The group of (positive) loops $\mathfrak{L}_n\mathfrak{G}$ (resp. $\mathfrak{L}_n^+\mathfrak{G}$) of $\mathfrak{G}$ is an ind-scheme (resp. a scheme) representing the functor whose R-valued points consist of tuples $(\underline{s},\gamma)$ where $\underline{s} := (s_i)_i \in C^n(\operatorname{Spec} R)$ and $\gamma \in \mathfrak{G}(\dot{\mathbb{D}}(\Gamma_{\underline{s}}))$ (resp. $\gamma \in \mathfrak{G}(\mathbb{D}(\Gamma_{\underline{s}}))$). The projection $(\underline{s},\gamma) \mapsto \underline{s}$ defines morphism $\mathfrak{L}_n\mathfrak{G} \to C^n$ (resp. $\mathfrak{L}_n^+\mathfrak{G} \to C^n$).

Remark 3.1.2. Note that by the general form of the descent lemma of Beauville-Laszlo [BL95, Theorem 2.12.1], the map which sends $(\mathcal{G},\underline{s},\varepsilon) \in GR_n(S)$ to the triple $(\underline{s},\widehat{\mathcal{G}} := \mathcal{G}|_{\mathbb{D}(\Gamma_{\underline{s}})},\dot{\varepsilon} := \varepsilon|_{\dot{\mathbb{D}}(\Gamma_{\underline{s}})})$ is bijective. Thus the loop groups $\mathfrak{L}_n\mathfrak{G}$ and $\mathfrak{L}_n^+\mathfrak{G}$ act on GR_n by changing the trivialization on $\dot{\mathbb{D}}(\Gamma_{\underline{s}})$.

Definition 3.1.3. We fix an algebraic closure Q^{alg} of the function field $Q := \mathbb{F}_q(C)$ of the curve C. For a finite field extension $Q \subset K$ with $K \subset Q^{\mathrm{alg}}$ we consider the normalization $\widetilde{C}_K$ of C in K. It is a smooth projective curve over $\mathbb{F}_q$ together with a finite morphism $\widetilde{C}_K \to C$.

(a) For a finite extension K as above, we consider closed ind-subschemes Z of $GR_n \times_{C^n} \widetilde{C}_K^n$. We call two closed ind-subschemes $Z_1 \subseteq GR_n \times_{C^n} \widetilde{C}_{K_1}^n$ and $Z_2 \subseteq GR_n \times_{C^n} \widetilde{C}_{K_2}^n$ *equivalent* if there is a finite field extension $K_1.K_2 \subset K' \subset Q^{\mathrm{alg}}$ with corresponding curve $\widetilde{C}_{K'}$ finite over $\widetilde{C}_{K_1}$ and $\widetilde{C}_{K_2}$, such that $Z_1 \times_{\widetilde{C}_{K_1}^n} \widetilde{C}_{K'}^n = Z_2 \times_{\widetilde{C}_{K_2}^n} \widetilde{C}_{K'}^n$ in $GR_n \times_{C^n} C_{K'}^n$.

(b) Let $\mathcal{Z} = [Z_K]$ be an equivalence class of closed ind-subschemes $Z_K \subseteq GR_n \times_{C^n} \widetilde{C}_K^n$ and let $G_{\mathcal{Z}} := \{g \in \operatorname{Aut}(Q^{\mathrm{alg}}/Q) : g^*(\mathcal{Z}) = \mathcal{Z}\}$. We define the *field of definition* $Q_{\mathcal{Z}}$ of $\mathcal{Z}$ as the intersection of the fixed field of $G_{\mathcal{Z}}$ in Q^{alg} with all the finite extensions over which a representative of $\mathcal{Z}$ exists.

(c) We define a *bound* to be an equivalence class $\mathcal{Z} := [Z_K]$ of closed subscheme $Z_K \subset GR_n \times_{C^n} \widetilde{C}^n_K$, such that all the ind-subschemes Z_K are stable under the left $\mathfrak{L}_n^+\mathfrak{G}$-action on GR_n. The field of definition $Q_{\mathcal{Z}}$ (resp. the curve of definition $C_{\mathcal{Z}} := \widetilde{C}_{Q_{\mathcal{Z}}}$) of $\mathcal{Z}$ is called the *reflex field* (resp. *reflex curve*) of $\mathcal{Z}$.

(d) Let $\mathcal{Z}$ be a bound in the above sense. Let S be an $\mathbb{F}_q$-scheme equipped with an $\mathbb{F}_q$-morphism $\underline{s}' = (s'_1, \ldots, s'_n) \colon S \to C^n_{\mathcal{Z}}$ and let $s_i \colon S \to C$ be obtained by composing $s'_i \colon S \to C_{\mathcal{Z}}$ with the morphism $C_{\mathcal{Z}} \to C$. Let $\mathcal{G}$ and $\mathcal{G}'$ be two $\mathfrak{G}$-bundles over C_S. Consider an isomorphism $\varphi : \mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}} \to \mathcal{G}'|_{C_S \smallsetminus \Gamma_{\underline{s}}}$ defined outside the graph of the sections s_i. Take an *fppf*-cover $T \to S$ in such a way that the induced *fppf* cover $\mathbb{D}_T(\Gamma_{\underline{s}_T}) \to \mathbb{D}_S(\Gamma_{\underline{s}})$ trivializes $\widehat{\mathcal{G}}'$. Here $\underline{s}_T := (s_{T,i})$ denotes the n-tuple of morphisms $s_{T,i} : T \to C$, obtained by composing the covering morphism $T \to S$ with the morphism s_i. For existence of such trivializations see Lemma 3.1.6 below. Fixing a trivialization $\widehat{\alpha} : \widehat{\mathcal{G}}' \times_{\mathbb{D}(\Gamma_{\underline{s}})} \mathbb{D}_T(\Gamma_{\underline{s}_T}) \tilde{\to} \mathfrak{G} \times_C \mathbb{D}_T(\Gamma_{\underline{s}_T})$ we obtain a morphism $T \to GR_n$ which is induced by the tuple

$$\left(\underline{s}_T, \widehat{\mathcal{G}} := \mathcal{G}|_{\mathbb{D}_T(\Gamma_{\underline{s}_T})}, \dot{\widehat{\alpha}} \circ \varphi|_{\dot{\mathbb{D}}_T(\Gamma_{\underline{s}_T})} : \dot{\widehat{\mathcal{G}}} := \widehat{\mathcal{G}}|_{\dot{\mathbb{D}}_T(\Gamma_{\underline{s}_T})} \tilde{\to} \mathfrak{G} \times_C \dot{\mathbb{D}}_T(\Gamma_{\underline{s}_T})\right);$$

see Remark 3.1.2. We say that $\varphi : \mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}} \to \mathcal{G}'|_{C_S \smallsetminus \Gamma_{\underline{s}}}$ satisfies the *global boundedness condition* (**GBC**) by $\mathcal{Z}$ if for all representative Z_K of $\mathcal{Z}$ over K the induced morphism

$$T \times_{C^n_{\mathcal{Z}}} \widetilde{C}^n_K \to GR_n \times_{C^n} \widetilde{C}^n_K$$

factors through Z_K. Note that since Z_K is invariant under the left $\mathfrak{L}_n^+\mathfrak{G}$-action, this definition is independent of the choice of the trivialization $\widehat{\alpha}$.

(e) We say that a tuple $(\mathcal{G}, \mathcal{G}', (s_i), \tau)$ in $(Hecke_n(C, \mathfrak{G}) \times_{C^n} C^n_{\mathcal{Z}})\,(S)$ is *bounded by* $\mathcal{Z}$ if τ^{-1} satisfies (**GBC**) by $\mathcal{Z}$ in the above sense. This consequently establishes the boundedness condition on $\mathfrak{G}$-shtukas in

$$\left(\nabla_n \mathscr{H}^1(C, \mathfrak{G}) \times_{C^n} C^n_{\mathcal{Z}}\right)(S).$$

We denote the corresponding moduli stacks, obtained by imposing the bound $\mathcal{Z}$, respectively by $Hecke_n^{\mathcal{Z}}(C, \mathfrak{G})$ and $\nabla_n^{\mathcal{Z}} \mathscr{H}^1(C, \mathfrak{G})$. These stacks naturally lie over the n-fold fiber product $C_{\mathcal{Z}}^n$ of the reflex curve $C_{\mathcal{Z}}$ over $\mathbb{F}_q$.

Remark 3.1.4. Note that in contrary to [AH19] and [AH14], in the above Definition 3.1.3(e) (and also in Definition 4.3.2(c) bellow) we impose the boundedness conditions to τ^{-1} instead of τ. The main reason for switching to this convention is to keep our terminology consistent with previous literature such as [Var04].

Remark 3.1.5. In a similar way as explained in [AH14, Remark 4.7] one can compute the reflex field of $\mathcal{Z}$ in the following concrete sense. We choose a finite extension $K \subset Q^{\mathrm{alg}}$ of Q over which a representative Z_K of $\mathcal{Z}$ exists, and for which the inseparability degree $\iota(K)$ of K over Q is minimal. Then the reflex field $Q_{\mathcal{Z}}$ equals $K^{G_{\mathcal{Z}}}$ and $C_{\mathcal{Z}} = C_{K^{G_{\mathcal{Z}}}}$. Moreover, let $\widetilde{K}$ be the normal closure of K. Then $\iota(\widetilde{K}) = \iota(K)$ and therefore $\widetilde{K}$ is Galois over $Q_{\mathcal{Z}}$ with Galois group

$$\mathrm{Gal}(\widetilde{K}/Q_{\mathcal{Z}}) \;=\; \{\gamma \in \mathrm{Aut}_Q(\widetilde{K}) \text{ with } \gamma(Z_{C_{\widetilde{K}}}) = Z_{C_{\widetilde{\mathcal{Z}}}}\} \;\subset\; \mathrm{Aut}_Q(\widetilde{K}).$$

We conclude that

$$Q_{\mathcal{Z}} \;=\; \big\{x \in \widetilde{K} \colon \gamma(x) = x \text{ for all } \gamma \in \mathrm{Aut}_Q(\widetilde{K}) \text{ with } \gamma(Z_{C_{\widetilde{K}}}) = Z_{C_{\widetilde{K}}}\big\}.$$

As in the local situation we do not know whether in general $\mathcal{Z}$ has a representative $Z_{Q_{\mathcal{Z}}}$ over reflex curve $C_{\mathcal{Z}}$.

Lemma 3.1.6. *Consider the effective relative Cartier divisor $\Gamma_{\underline{s}}$ in C_S. Let $\mathcal{G}$ be a $\mathfrak{G}$-bundle over C_S and set $\widehat{\mathcal{G}} := \mathcal{G}|_{\mathbb{D}(\Gamma_{\underline{s}})}$. Then there is an fppf cover $T \to S$ such that the induced morphism $\mathbb{D}_T(\Gamma_{\underline{s}_T}) \to \mathbb{D}(\Gamma_{\underline{s}})$ is a trivializing cover for $\widehat{\mathcal{G}}$.*

Proof. Let $\mathcal{U} \to \mathbb{D}(\Gamma_{\underline{s}})$ be an étale covering that trivializes $\widehat{\mathcal{G}}$. Consider the closed immersion $\Gamma_{\underline{s}} \to \mathbb{D}_S(\Gamma_{\underline{s}})$ and set $T := \Gamma_{\underline{s}} \times_{\mathbb{D}(\Gamma_{\underline{s}})} \mathcal{U}$. To see that $T \to S$ is the desired covering notice that the closed immersion $\Gamma_{\underline{s}_T} \to \mathbb{D}_{T,n}(\Gamma_{\underline{s}_T})$ is defined by a nilpotent sheaf of ideal and moreover $\mathcal{U} \to \mathbb{D}_S(\Gamma_{\underline{S}})$ is étale, hence the natural morphism $\Gamma_{\underline{s}_T} \to \mathcal{U}$ lifts to a morphism $\mathbb{D}_{T,n}(\Gamma_{\underline{s}_T}) \to \mathcal{U}$ and consequently to $\mathbb{D}_T(\Gamma_{\underline{s}_T}) \to \mathcal{U}$. □

Theorem 3.1.7. *Let D be a proper closed subscheme of C and set $D_{\mathcal{Z}} := D \times_C C_{\mathcal{Z}}$. The stack $\nabla_n^{\mathcal{Z}} \mathscr{H}_D^1(C, \mathfrak{G})$ is a Deligne-Mumford stack locally of finite type and separated over $(C_{\mathcal{Z}} \smallsetminus D_{\mathcal{Z}})^n$. It is relatively representable over $\mathscr{H}^1(C, \mathfrak{G}) \times_{\mathbb{F}_q} (C_{\mathcal{Z}} \smallsetminus D_{\mathcal{Z}})^n$ by a separated morphism of finite type.*

Proof. The forgetful morphism $\nabla_n^{\mathcal{Z}}\mathscr{H}_D^1(C,\mathfrak{G}) \to \nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G}) \times_{C_{\mathcal{Z}}^n} (C_{\mathcal{Z}} \smallsetminus D_{\mathcal{Z}})^n$ is relatively representable by a finite étale surjective morphism, see Theorem 2.0.12. Thus it suffices to prove the statement for $\nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})$.

Let S be a scheme over $C_{\mathcal{Z}}^n$ and let Z_{C_K} be a representative of the bound $\mathcal{Z}$ for a finite extension $Q \subseteq K \subset Q^{\mathrm{alg}}$. As in Definition 3.1.3(d), we consider a trivialization of $\widehat{\mathcal{G}}'$ over an *fppf* covering $T \to S$. Recall that this induces a morphism $T \times_{C_{\mathcal{Z}}^n} C_K^n \to GR_n \times_{C^n} C_K^n$. By definition of boundedness condition 3.1.3(c), the closed subscheme $T \times_{GR_n \times_{C^n} C_K^n} \mathcal{Z}$ descends to a closed subscheme of S. Consequently $\nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})$ is a closed substack of $\nabla_n\mathscr{H}^1(C,\mathfrak{G})$ and we may now conclude by Theorem 2.0.12. □

3.2. The local model theorem for $\nabla_n^{\mathcal{Z}}\mathscr{H}_D^1(C,\mathfrak{G})$

The following theorem asserts that global affine Grassmannians (resp. global Schubert varieties inside global affine Grassmannians) may be regarded as local models for the moduli stacks of global $\mathfrak{G}$-shtukas (resp. bounded global $\mathfrak{G}$-shtukas). Here $\mathfrak{G}$ is a smooth affine group scheme over C. Notice that when $\mathfrak{G}$ is constant, i.e. $\mathfrak{G} = G_0 \times_{\mathbb{F}_q} C$ for a split reductive group G_0 over $\mathbb{F}_q$, this observation was first recorded by Varshavsky [Var04]. His proof relies on the well-known theorem of Drinfeld and Simpson [DS95] which assures that a G_0-bundle over a relative curve over S is Zariski-locally trivial after suitable étale base change $S' \to S$. Note however that this essentially follows from their argument about existence of B-structure on G_0-bundles, for a Borel subgroup B. Hence, one should not hope to implement this result to treat the general case, which we are mainly interested to study in this article (e.g. when $\mathfrak{G}$ ramifies at certain places of C). Accordingly, in the proof of the following theorem, we modify Varshavsky's method and provide a proof which is independent of the theorem of Drinfeld and Simpson. In the course of the proof we will see that it suffices to assume that $\mathfrak{G}$ is smooth affine group scheme over C. Additionally, we will evidently see that it is not possible to weaken this assumption any further.

Theorem 3.2.1. *Let $\mathfrak{G}$ be a smooth affine group scheme of finite type over C and let D be a proper closed subscheme of C. For any point y in $\nabla_n\mathscr{H}_D^1(C,\mathfrak{G})$ there exist an ind-étale neighborhood U_y of y and a roof*

$$\begin{tikzcd} & U_y \arrow[dl, "\acute{e}t"'] \arrow[dr, "\acute{e}t"] & \\ \nabla_n\mathscr{H}_D^1(C,\mathfrak{G}) & & GR_n, \end{tikzcd}$$

of ind-étale morphisms. In other words the global affine Grassmannian GR_n is a local model for the moduli stack $\nabla_n\mathscr{H}_D^1(C,\mathfrak{G})$ of global $\mathfrak{G}$-shtukas. Furthermore for a global bound $\mathcal{Z}$ as in Definition 3.1.3, with a representative $Z_K \subset GR_n \times_{C^n} C_K^n$ over finite extension $Q \subset K$, the pull back of the above diagram induces a roof of étale morphisms between $\nabla_n^{\mathcal{Z}}\mathscr{H}_D^1(C,\mathfrak{G}) \times_{C_{\mathcal{Z}}^n} C_K^n$ and Z_K.

Proof of Theorem 3.2.1. Regarding Theorem 2.0.12, we may ignore the D-level structure. Since the curve C, the group $\mathfrak{G}$ and the index n (which stands for the number of characteristic sections) are fixed, we drop them from the notation and simply write $\mathscr{H}^1 = \mathscr{H}^1(C,\mathfrak{G})$, $Hecke = Hecke_n(C,\mathfrak{G})$ and $\nabla\mathscr{H}^1 = \nabla_n\mathscr{H}^1(C,\mathfrak{G})$.

Let y' be the image of y in $C^n \times \mathscr{H}^1$ under the projection sending $(\mathcal{G}, \mathcal{G}', \underline{s}, \tau)$ to $(\underline{s}, \mathcal{G}')$.

According to Proposition 2.0.9, we may take an étale neighborhood $U \to C^n \times \mathscr{H}^1$ of y', such that the restriction U' of $Hecke$ to U and the restriction U'' of $GR_n \times \mathscr{H}^1$ to U become isomorphic. Now, set $\widetilde{U}_y := U' \times_{Hecke} \nabla\mathscr{H}^1$ and consider the following commutative diagram

$$\begin{tikzcd}
 & & U' \arrow[rr, "\acute{e}t"] \arrow[d] \arrow[dddl, "\cong"'] & & Hecke \arrow[dd] \\
\widetilde{U}_y \arrow[urr] \arrow[rrr, "\acute{e}t"] \arrow[drr] \arrow[dddrrr, bend right, "\varphi'"'] & & & \nabla\mathscr{H}^1 \arrow[ur] \arrow[dr] & \\
 & & U \arrow[rr, "\acute{e}t"] & & C^n \times \mathscr{H}^1 \\
 & U'' \arrow[ur] \arrow[rr, "\acute{e}t"] & & GR_n \times \mathscr{H}^1 \arrow[ur] \arrow[d, "pr_1"'] & \\
 & & & GR_n &
\end{tikzcd}$$

We pick an open substack of finite type $\mathscr{H}_\alpha^1$ that contains the image of y under projection to $\mathscr{H}^1$; see Remark 2.0.2. After restricting the above diagram to $\mathscr{H}_\alpha^1$ and imposing a D-level structure, we may obtain the following diagram

$$\begin{tikzcd}
& \widetilde{U}_{y,\alpha,D} \arrow[dl] \arrow[d] \arrow[drrr, "\varphi'"] & & & \\
\nabla\mathscr{H}^1_{\alpha,D} & U'_{\alpha,D} \arrow[r, "\cong"] \arrow[d, bend right, "f"'] & U''_{\alpha,D} \arrow[r, "\acute{e}t"] \arrow[d, bend left, "g"] & GR_n \times \mathscr{H}^1_{\alpha,D} \arrow[r, "pr_1"] & GR_n. \\
& \mathscr{H}^1_{\alpha,D} \arrow[r, "\sigma_{\mathscr{H}^1_\alpha}"] & \mathscr{H}^1_{\alpha,D} & &
\end{tikzcd}$$

Here f is the morphism induced by the projection $\pi\colon Hecke \to \mathscr{H}^1$ sending $(\mathcal{G}, \mathcal{G}', \underline{s}, \tau)$ to $\mathcal{G}$ and g is induced by $U'' \to GR_n \times \mathscr{H}^1$ followed by the projection. We may take D enough big such that $\mathscr{H}^1_{\alpha,D}$ admits an étale presentation $H^1_{\alpha,D} \to \mathscr{H}^1_{\alpha,D}$; see Remark 2.0.2 and [Beh91, Lemma 8.3.9, 8.3.10 and 8.3.11]. In addition, according to Theorem 2.0.12 the morphism $\nabla\mathscr{H}^1_{\alpha,D} \to \nabla\mathscr{H}^1_\alpha$ is étale. Consequently, we may replace $\mathscr{H}^1_{\alpha,D}$ by the scheme $H^1_{\alpha,D}$. Then define $U_y := \widetilde{U}_{y,\alpha,D} \times_{\mathscr{H}^1_{\alpha,D}} H^1_{\alpha,D}$, the theorem now follows from Remarks 2.0.2 and 2.0.8 and the lemma below. □

Lemma 3.2.2. *Let W, T, Y and Z be schemes locally of finite type over $\mathbb{F}_q$ and let Z be smooth. Assume that we have a morphism $f\colon W \to Z$, and étale morphisms $\iota\colon W \to T \times_{\mathbb{F}_q} Y$ and $\varphi\colon Y \to Z$. Let $g\colon W \to Z$ denote the morphism $\varphi \circ pr_2 \circ \iota$, where $pr_2\colon T \times_{\mathbb{F}_q} Y \to Y$ is the projection to the second factor. Consider the following diagram*

$$\begin{tikzcd}
V \arrow[d] \arrow[drr] & & \\
W \arrow[r, "\iota"'] \arrow[d, bend right, "\sigma_Z \circ f"'] \arrow[d, bend left, "g"] & T \times_{\mathbb{F}_q} Y \arrow[r, "pr_1"'] & T \\
Z & &
\end{tikzcd}$$

where $V := \mathrm{equi}(\sigma_Z \circ f, g\colon W \rightrightarrows Z)$; see Definition 1.1.1. Then the induced morphism $V \to T$ is étale.

Proof. Let $v \in V$ be a point and let $w \in W$ and $z = \sigma_Z \circ f(w) = g(w) \in Z$, as well as $t \in T$ and $y \in Y$ be its images. Consider an affine open neighborhood Z' of z in Z which admits an étale morphism $\pi\colon Z' \to \widetilde{Z}$ to some affine space $\widetilde{Z} = \mathbb{A}^m_{\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q[z_1, \ldots, z_m]$, and consider an affine neighborhood T'

of t which we write as a closed subscheme of some $\widetilde{T} = \mathbb{A}^{\ell}_{\mathbb{F}_q}$. Replace Y by an affine neighborhood Y' of y contained in $\varphi^{-1}(Z')$ and W by an affine neighborhood W' of w contained in $(\sigma_Z \circ f)^{-1}(Z') \cap g^{-1}(Z') \cap \iota^{-1}(T' \times_{\mathbb{F}_q} Y')$. Then $V' := W' \times_W V = W' \times_{(\sigma_Z f, g), Z' \times Z', \Delta} Z'$ is an open neighborhood of v in V. We may extend the étale morphism $\iota \colon W' \to T' \times_{\mathbb{F}_q} Y'$ to an étale morphism $\tilde{\iota} \colon \widetilde{W} \to \widetilde{T} \times_{\mathbb{F}_q} Y'$ with $\widetilde{W} \times_{\widetilde{T}} T' = W'$. We also extend $\pi \circ f \colon W' \to \widetilde{Z}$ to a morphism $\tilde{f} \colon \widetilde{W} \to \widetilde{Z}$, let $\tilde{g} := \pi \, \varphi \, pr_2 \, \tilde{\iota} \colon \widetilde{W} \to \widetilde{Z}$, and set $\widetilde{V} := \mathrm{equi}(\sigma_{\widetilde{Z}} \circ \tilde{f}, \tilde{g} \colon \widetilde{W} \rightrightarrows \widetilde{Z}) = \widetilde{W} \times_{(\sigma_{\widetilde{Z}} \tilde{f}, \tilde{g}), \widetilde{Z} \times \widetilde{Z}, \Delta} \widetilde{Z}$. Since $\Delta \colon Z' \to Z' \times_{\widetilde{Z}} Z'$ is an open immersion, also the natural morphism

$$V' \longrightarrow \widetilde{V} \times_{\widetilde{T}} T' = W' \times_{(\sigma_{\widetilde{Z}} \tilde{f}, \tilde{g}), \widetilde{Z} \times \widetilde{Z}, \Delta} \widetilde{Z} = W' \times_{(\sigma_Z f, g), Z' \times Z', \Delta} (Z' \times_{\widetilde{Z}} Z')$$

is an open immersion. Since $\widetilde{W}$ is smooth over $\widetilde{T}$ of relative dimension m and $\widetilde{V}$ is given by m equations $\tilde{g}^*(z_j) - \tilde{f}^*(z_j)^q$ with linearly independent differentials $d\tilde{g}^*(z_j)$, the Jacobi-criterion [BLR90, §2.2, Proposition 7] implies that $\widetilde{V} \to \widetilde{T}$ is étale. The lemma follows from this. $\square$

Remark 3.2.3. Notice that the Varshavsky's arguments [Var04] about the intersection cohomology of the moduli stacks of (iterated) G-shtukas often rely on the following assumption that the Bott-Samelson-Demazure resolution for an affine Schubert variety is (semi-)small. This is the case for split constant reductive groups, but unfortunately for a parahoric group scheme $\mathfrak{G}$, the semi-smallness of the resolution may fail beyond the (co-)minuscule case.

4. Analogue of Rapoport-Zink local model

The PEL-Shimura varieties appear as moduli spaces for abelian varieties (together with additional structures, i.e. polarization, endomorphism and level structure). For such Shimura varieties, Rapoport and Zink [RZ96, Chapter 3] establish their theory of local models. They consider a moduli space whose S-valued points parametrizes abelian varieties X (with additional structures) together with a trivialization of the associated de Rham cohomology. Consequently, one may simultaneously view this space as a torsor on the Shimura variety by forgetting the trivialization, and on the other hand, it also maps to a flag variety which parametrizes filtrations on $Lie(X)^{\vee}$. The later map is formally smooth over its image by Grothendieck-Messing theory. This provides the local model roof, which they use to relate the local properties of the PEL-Shimura varieties and certain Schubert varieties inside flag varieties. In this chapter we investigate the analogous theory over function fields.

4.1. Preliminaries on local theory

In this section we provide some background material concerning the local theory of global $\mathfrak{G}$-shtukas.

Definition 4.1.1. Fix an n-tuple $\underline{\nu} := (\nu_i)_i$ of places on C with $\nu_i \neq \nu_j$ for $i \neq j$. Let $\widehat{A}_{\underline{\nu}}$ be the completion of the local ring $\mathcal{O}_{C^n,\underline{\nu}}$ of C^n at the closed point $\underline{\nu}$, and let $\mathbb{F}_{\underline{\nu}}$ be the residue field of the point $\underline{\nu}$. Then $\mathbb{F}_{\underline{\nu}}$ is the compositum of the fields $\mathbb{F}_{\nu_i}$ inside $\mathbb{F}_q^{\mathrm{alg}}$, and $\widehat{A}_{\underline{\nu}} \cong \mathbb{F}_{\underline{\nu}}[\![\zeta_1, \dots, \zeta_n]\!]$ where ζ_i is a uniformizing parameter of C at ν_i. Let the stack

$$Hecke_{n,D}(C, \mathfrak{G})^{\underline{\nu}} := Hecke_{n,D}(C, \mathfrak{G}) \,\widehat{\times}_{C^n} \operatorname{Spf} \widehat{A}_{\underline{\nu}}$$

(resp. $\nabla_n \mathscr{H}_D^1(C, \mathfrak{G})^{\underline{\nu}} := \nabla_n \mathscr{H}_D^1(C, \mathfrak{G}) \,\widehat{\times}_{C^n} \operatorname{Spf} \widehat{A}_{\underline{\nu}}$) be the formal completion of the ind-algebraic stack $Hecke_{n,D}(C, \mathfrak{G})$ (resp. $\nabla_n \mathscr{H}_D^1(C, \mathfrak{G})$) along $\underline{\nu} \in C^n$. It is an ind-algebraic stack over $\operatorname{Spf} \widehat{A}_{\underline{\nu}}$ which is ind-separated and locally of ind-finite type; see Remark 2.0.6 and Theorem 2.0.12. As before if $D = \emptyset$ we will drop it from our notation.

Here we recall the following notion of morphism between global $\mathfrak{G}$-shtukas.

Definition 4.1.2. Consider a scheme S together with characteristic sections $\underline{s} = (s_i)_i \in C^n(S)$ and let $\underline{\mathcal{G}} = (\mathcal{G}, \tau)$ and $\underline{\mathcal{G}}' = (\mathcal{G}', \tau')$ be two global $\mathfrak{G}$-shtukas over S with the same characteristics s_i. A *quasi-isogeny* from $\underline{\mathcal{G}}$ to $\underline{\mathcal{G}}'$ is an isomorphism $f\colon \mathcal{G}|_{C_S \smallsetminus D_S} \xrightarrow{\sim} \mathcal{G}'|_{C_S \smallsetminus D_S}$ satisfying $\tau' \sigma^*(f) = f\tau$, where D is some effective divisor on C.

Before introducing the category of local $\mathbb{P}$-shtukas, the global-local functor and the local boundedness condition, let us recall the following preparatory material.

Let $\mathbb{F}$ be a finite field and $\mathbb{F}[\![z]\!]$ be the power series ring over $\mathbb{F}$ in the variable z. We let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D} := \operatorname{Spec} \mathbb{F}[\![z]\!]$ with connected fibers, and we let $P := \mathbb{P} \times_{\mathbb{D}} \dot{\mathbb{D}}$ be the generic fiber of $\mathbb{P}$ over $\dot{\mathbb{D}} := \operatorname{Spec} \mathbb{F}(\!(z)\!)$.

Definition 4.1.3. The *group of positive loops associated with* $\mathbb{P}$ is the infinite dimensional affine group scheme $L^+\mathbb{P}$ over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$L^+\mathbb{P}(R) := \mathbb{P}(R[\![z]\!]) := \mathbb{P}(\mathbb{D}_R) := \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}_R, \mathbb{P})\,.$$

The *group of loops associated with* P is the *fpqc*-sheaf of groups LP over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$LP(R) := P(R(\!(z)\!)) := P(\dot{\mathbb{D}}_R) := \mathrm{Hom}_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, P)\,,$$

where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{\mathbb{D}}_R := \operatorname{Spec} R(\!(z)\!)$. It is representable by an ind-scheme of ind-finite type over $\mathbb{F}$; see [PR08, § 1.a], or [BD, §4.5], [NP01], [Fal03] when $\mathbb{P}$ is constant. Let $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P}) := [\operatorname{Spec}\mathbb{F}/L^+\mathbb{P}]$ (respectively $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, LP) := [\operatorname{Spec}\mathbb{F}/LP]$) denote the classifying space of $L^+\mathbb{P}$-torsors (respectively LP-torsors). It is a stack fibered in groupoids over the category of $\mathbb{F}$-schemes S whose category $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})(S)$ consists of all $L^+\mathbb{P}$-torsors (resp. LP-torsors) on S. The inclusion of sheaves $L^+\mathbb{P} \subset LP$ gives rise to the natural 1-morphism

$$\text{Ł}\colon \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P}) \longrightarrow \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, LP),\ \mathcal{L}_+ \mapsto \mathcal{L}\,. \tag{4.1}$$

Definition 4.1.4. The affine flag variety $\mathcal{F}\ell_{\mathbb{P}}$ is defined to be the ind-scheme representing the *fpqc*-sheaf associated with the presheaf

$$R \longmapsto LP(R)/L^+\mathbb{P}(R) \;=\; \mathbb{P}\left(R(\!(z)\!)\right)/\mathbb{P}\left(R[\![z]\!]\right).$$

on the category of $\mathbb{F}$-algebras; compare Definition 4.1.3.

Remark 4.1.5. Recall that Pappas and Rapoport [PR08, Theorem 1.4] show that $\mathcal{F}\ell_{\mathbb{P}}$ is ind-quasi-projective over $\mathbb{F}$, and hence ind-separated and of ind-finite type over $\mathbb{F}$. Additionally, they show that the quotient morphism $LP \to \mathcal{F}\ell_{\mathbb{P}}$ admits sections locally for the étale topology. They proceed as follows. When $\mathbb{P} = \mathrm{SL}_{r,\mathbb{D}}$, the *fpqc*-sheaf $\mathcal{F}\ell_{\mathbb{P}}$ is called the *affine Grassmanian.* It is an inductive limit of projective schemes over $\mathbb{F}$, that is, ind-projective over $\mathbb{F}$; see [BD, Theorem 4.5.1] or [Fal03, NP01]. By [PR08, Proposition 1.3] and [AH19, Proposition 2.1] there is a faithful representation $\mathbb{P} \to \mathrm{SL}_r$ with quasi-affine quotient. Pappas and Rapoport show in the proof of [PR08, Theorem 1.4] that $\mathcal{F}\ell_{\mathbb{P}} \to \mathcal{F}\ell_{\mathrm{SL}_r}$ is a locally closed embedding, and moreover, if $\mathrm{SL}_r/\mathbb{P}$ is affine, then $\mathcal{F}\ell_{\mathbb{P}} \to \mathcal{F}\ell_{\mathrm{SL}_r}$ is even a closed embedding and $\mathcal{F}\ell_{\mathbb{P}}$ is ind-projective. Moreover, if the fibers of $\mathbb{P}$ over $\mathbb{D}$

are geometrically connected, it was proved by Richarz [Ric13b, Theorem A] that $\mathcal{F}\ell_{\mathbb{P}}$ is ind-projective if and only if $\mathbb{P}$ is a parahoric group scheme in the sense of Bruhat and Tits [BT72, Définition 5.2.6]; see also [HR03].

Here we recall the definition of the category of local $\mathbb{P}$-shtukas.

Definition 4.1.6. Let $\mathcal{X}$ be the fiber product

$$\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P}) \times_{\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, LP)} \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})$$

of groupoids. Let pr_i denote the projection onto the i-th factor. We define the groupoid of *local* $\mathbb{P}$*-shtukas* $Sht_{\mathbb{P}}^{\mathbb{D}}$ to be

$$Sht_{\mathbb{P}}^{\mathbb{D}} := \operatorname{equi}\big(\hat{\sigma} \circ pr_1, pr_2 \colon \mathcal{X} \rightrightarrows \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})\big) \,\widehat{\times}_{\operatorname{Spec}\mathbb{F}} \operatorname{Spf}\mathbb{F}[\![\zeta]\!].$$

(see Definition 1.1.1) where $\hat{\sigma} := \hat{\sigma}_{\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})}$ is the absolute $\mathbb{F}$-Frobenius of $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})$. The category $Sht_{\mathbb{P}}^{\mathbb{D}}$ is fibered in groupoids over the category $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ of $\mathbb{F}[\![\zeta]\!]$-schemes on which ζ is locally nilpotent. We call an object of the category $Sht_{\mathbb{P}}^{\mathbb{D}}(S)$ a *local* $\mathbb{P}$*-shtuka over* S.

More explicitly a local $\mathbb{P}$-shtuka over $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of an $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and an isomorphism of the associated loop group torsors $\hat{\tau} \colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.

Local $\mathbb{P}$-shtukas can be viewed as function field analogs of p-divisible groups. According to this analogy one may introduce the following notion.

Definition 4.1.7. A *quasi-isogeny* $f \colon \underline{\mathcal{L}} \to \underline{\mathcal{L}}'$ between two local $\mathbb{P}$-shtukas $\underline{\mathcal{L}} := (\mathcal{L}_+, \hat{\tau})$ and $\underline{\mathcal{L}}' := (\mathcal{L}'_+, \hat{\tau}')$ over S is an isomorphism of the associated LP-torsors $f \colon \mathcal{L} \to \mathcal{L}'$ satisfying $f \circ \hat{\tau} = \hat{\tau}' \circ \hat{\sigma}^* f$. We denote by $\operatorname{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}')$ the set of quasi-isogenies between $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}'$ over S.

Definition 4.1.8. Let $\hat{\mathbb{P}}$ be the formal group scheme over $\hat{\mathbb{D}} := \operatorname{Spf}\mathbb{F}[\![z]\!]$, obtained by the formal completion of $\mathbb{P}$ along $V(z)$. A *formal* $\hat{\mathbb{P}}$*-torsor* over an $\mathbb{F}$-scheme S is a z-adic formal scheme $\hat{\mathcal{P}}$ over $\hat{\mathbb{D}}_S := \hat{\mathbb{D}} \,\widehat{\times}_{\mathbb{F}} S$ together with an action $\hat{\mathbb{P}} \,\widehat{\times}_{\hat{\mathbb{D}}} \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ of $\hat{\mathbb{P}}$ on $\hat{\mathcal{P}}$ such that there is a covering $\hat{\mathbb{D}}_{S'} \to \hat{\mathbb{D}}_S$ where $S' \to S$ is an *fpqc*-covering and a $\hat{\mathbb{P}}$-equivariant isomorphism $\hat{\mathcal{P}} \,\widehat{\times}_{\hat{\mathbb{D}}_S} \hat{\mathbb{D}}_{S'} \xrightarrow{\sim} \hat{\mathbb{P}} \,\widehat{\times}_{\hat{\mathbb{D}}} \hat{\mathbb{D}}_{S'}$. Here $\hat{\mathbb{P}}$ acts on itself by right multiplication. Let $\mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{P}})$ be the category fibered in groupoids that assigns to each $\mathbb{F}$-scheme S the groupoid consisting of all formal $\hat{\mathbb{P}}$-torsors over $\hat{\mathbb{D}}_S$.

Remark 4.1.9. (a) There is a natural isomorphism

$$\mathscr{H}^1(\hat{\mathbb{D}},\hat{\mathbb{P}}) \xrightarrow{\sim} \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})$$

of groupoids. In particular all $L^+\mathbb{P}$-torsors for the *fpqc*-topology on S are already trivial étale locally on S; see [AH14, Proposition 2.4].

(b) Let ν be a place on C and let $\mathbb{D}_\nu := \operatorname{Spec}\widehat{A}_\nu$ and $\hat{\mathbb{D}}_\nu := \operatorname{Spf}\widehat{A}_\nu$. We set $\mathbb{P}_\nu := \mathfrak{G}\times_C \operatorname{Spec}\widehat{A}_\nu$ and $\hat{\mathbb{P}}_\nu := \mathfrak{G}\times_C \operatorname{Spf}\widehat{A}_\nu$. Let $\deg\nu := [\mathbb{F}_\nu : \mathbb{F}_q]$ and fix an inclusion $\mathbb{F}_\nu \subset \widehat{A}_\nu$. Assume that we have a section $s\colon S\to C$ which factors through $\operatorname{Spf}\widehat{A}_\nu$, that is, the image in $\mathcal{O}_S$ of a uniformizer of $\widehat{A}_\nu$ is locally nilpotent. In this case we have

$$\hat{\mathbb{D}}_\nu \widehat{\times}_{\mathbb{F}_q} S \cong \coprod_{\ell\in\mathbb{Z}/(\deg\nu)} \mathrm{V}(\mathfrak{a}_{\nu,\ell}) \cong \coprod_{\ell\in\mathbb{Z}/(\deg\nu)} \hat{\mathbb{D}}_{\nu,S},$$

where $\hat{\mathbb{D}}_{\nu,S} := \hat{\mathbb{D}}_\nu \widehat{\times}_{\mathbb{F}_\nu} S$ and $\mathrm{V}(\mathfrak{a}_{\nu,\ell})$ denotes the component identified by the ideal $\mathfrak{a}_{\nu,\ell} = \langle a\otimes 1 - 1\otimes s^*(a)^{q^\ell} : a\in\mathbb{F}_\nu\rangle$. Note that σ cyclically permutes these components and thus the $\mathbb{F}_\nu$-Frobenius $\sigma^{\deg\nu} =: \hat{\sigma}$ leaves each of the components $\mathrm{V}(\mathfrak{a}_{\nu,\ell})$ stable. Also note that there are canonical isomorphisms $\mathrm{V}(\mathfrak{a}_{\nu,\ell}) \cong \hat{\mathbb{D}}_{\nu,S}$ for all ℓ.

(c) Assume that we have a section $s\colon S\to C$ which factors through $\operatorname{Spf}\widehat{A}_\nu$ as above. By part (b) we may decompose

$$\mathcal{G}\widehat{\times}_{C_S}(\operatorname{Spf}\widehat{A}_{\nu_i}\widehat{\times}_{\mathbb{F}_q} S) \cong \coprod_{\ell\in\mathbb{Z}/(\deg\nu_i)} \mathcal{G}\widehat{\times}_{C_S}\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})$$

into a finite product with components $\mathcal{G}\widehat{\times}_{C_S}\mathrm{V}(\mathfrak{a}_{\nu_i,\ell}) \in \mathscr{H}^1(\hat{\mathbb{D}}_{\nu_i},\hat{\mathbb{P}}_{\nu_i})(S)$. According to (a), we view $\mathcal{G}\widehat{\times}_{C_S}\mathrm{V}(\mathfrak{a}_{\nu_i,0})$ as $L^+\mathbb{P}_\nu$-torsor over S, which we denote by $Ł_\nu^+\mathcal{G}$. Furthermore we denote by $Ł_\nu\mathcal{G}$ the LP_ν-torsor associated with $L^+\mathbb{P}_\nu$-torsor $Ł_\nu^+\mathcal{G}$ regarding the natural 1-morphism (4.1).

(d) Fix an n-tuple $\underline{\nu} := (\nu_i)$ of closed points of C. Let $\underline{s} := (s_i)$ be n-tuple of sections $s_i : S\to C$, such that s_i factors through $\operatorname{Spf}\widehat{A}_{\nu_i}$. We set $(\dot{C}_S)_{\underline{s}} := C_S \smallsetminus \Gamma_{\underline{s}}$. Let $[\mathbb{F}_q/\mathfrak{G}]_e((\dot{C}_S)_{\underline{s}})$ denote the category of $\mathfrak{G}$-bundles over $(\dot{C}_S)_{\underline{s}}$ that can be extended to a $\mathfrak{G}$-bundle over whole relative curve C_S. We denote by $(\dot{\ })_{\underline{s}}$ the restriction functor

$$(\dot{\ })_{\underline{s}}\colon \mathscr{H}^1(C,\mathfrak{G}) \longrightarrow [\mathbb{F}_q/\mathfrak{G}]_e((\dot{C}_S)_{\underline{s}})$$

which assigns to a $\mathfrak{G}$-torsor $\mathcal{G}$ over C_S, the $\mathfrak{G}$-torsor $(\dot{\mathcal{G}})_{\underline{s}} := \mathcal{G} \times_{C_S} (\dot{C}_S)_{\underline{s}}$ over $(\dot{C}_S)_{\underline{s}}$. Furthermore, for every i there is a functor

$$Ł_{\nu_i} \colon [\mathbb{F}_q/\mathfrak{G}]_e((\dot{C}_S)_{\underline{s}}) \longrightarrow \mathscr{H}^1(\operatorname{Spec}\mathbb{F}_\nu, LP_{\nu_i})(S)$$

which sends the $\mathfrak{G}$-torsor $\mathcal{G}'$ over $(\dot{C}_S)_{\underline{s}}$, having some extension $\mathcal{G}$ over C_S, to the LP_ν-torsor $Ł(Ł^+_{\nu_i}(\mathcal{G}))$ associated with $Ł^+_{\nu_i}\mathcal{G}$ under (4.1). One can verify that this assignment is well defined; see [AH14, Section 5.1].

(e) (Loop group version of Beauville-Laszlo gluing lemma) We define the groupoid $\mathcal{D}(\underline{s})$ whose objects consist of the tuples

$$(\mathcal{G}', \mathcal{L}_{+,\nu_i}, \alpha_i : Ł_{\nu_i}\mathcal{L}_{+,\nu_i} \to Ł_{\nu_i}\mathcal{G}'),$$

where $\mathcal{G}'$ lies in the category $[\mathbb{F}_q/\mathfrak{G}]_e((\dot{C}_S)_{\underline{s}})$, $\mathcal{L}_{+,\nu_i}$ is an $L^+\mathbb{P}_{\nu_i}$-torsor over S and finally an isomorphism $\alpha_i : Ł_{\nu_i}\mathcal{L}_{+,\nu_i} \to Ł_{\nu_i}\mathcal{G}'$ between associated loop torsors. One can prove that the functor $\mathscr{H}^1(C, \mathfrak{G})(S) \to \mathcal{D}(\underline{s})$, which sends $\mathcal{G}$ to $\left((\dot{\mathcal{G}})_{\underline{s}}, Ł^+_{\nu_i}\mathcal{G}, \alpha_i : Ł_{\nu_i}Ł^+_{\nu_i}\mathcal{G} \to Ł_{\nu_i}(\dot{\mathcal{G}})_{\underline{s}}\right)$ is an equivalence of categories; see [AH14, Lemma 5.1].

4.2. $\mathbb{P}$-Shtukas and the deformation theory of global $\mathfrak{G}$-shtukas

First of all, we recall an important feature of the morphisms of the category of local $\mathbb{P}$-shtukas. Namely, as in the theory of p-divisible groups, quasi-isogenies between local $\mathbb{P}$-shtukas enjoy the rigidity property. More explicitly, a quasi-isogeny between local $\mathbb{P}$-shtukas lifts over infinitesimal thickenings, thanks to the Frobenius connections.

Proposition 4.2.1 (Rigidity of quasi-isogenies for local $\mathbb{P}$-shtukas). *Let S be a scheme in $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ and let $j \colon \bar{S} \to S$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}$ which is locally nilpotent. Let $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}'$ be two local $\mathbb{P}$-shtukas over S. Then*

$$\operatorname{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}') \longrightarrow \operatorname{QIsog}_{\bar{S}}(j^*\underline{\mathcal{L}}, j^*\underline{\mathcal{L}}'), \quad f \mapsto j^*f$$

is a bijection of sets.

Proof. See [AH14, Proposition 2.11]. □

Notice that, like for abelian varieties, the corresponding statement for global $\mathfrak{G}$-shtukas only holds in fixed finite characteristics. For a detailed account see [AH19, Chapter 2 and Chapter 5].

Analogously to the functor which assigns to an abelian variety over a $\mathbb{Z}_p$-scheme its p-divisible group, there is a global-local functor from the category of global $\mathfrak{G}$-shtukas to the category of local $\mathbb{P}_{\nu_i}$-shtukas. This functor has been introduced in [AH14, Section 5.2]. As an additional analogy, Hartl and the first author proved that the infinitesimal deformation of a global $\mathfrak{G}$-shtuka is completely ruled by the infinitesimal deformations of the associated local $\mathbb{P}$-shtukas at the characteristic places. Bellow we explain this phenomenon.

We set $\mathbb{P}_{\nu_i} := \mathfrak{G} \times_C \operatorname{Spec} \widehat{A}_{\nu_i}$ and $\hat{\mathbb{P}}_{\nu_i} := \mathfrak{G} \times_C \operatorname{Spf} \widehat{A}_{\nu_i}$. Let $(\mathcal{G}, \underline{s}, \tau) \in \nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$, that is, $s_i \colon S \to C$ factors through $\operatorname{Spf} \widehat{A}_{\nu_i}$. According to Remark 4.1.9(c) we define the *global-local functor* by

$$\begin{array}{rrcl} \widehat{\Gamma}_{\nu_i}(-)\colon & \nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S) & \longrightarrow & Sht_{\mathbb{P}_{\nu_i}}^{\operatorname{Spec} \widehat{A}_{\nu_i}}(S)\,, \\ & (\mathcal{G}, \tau) & \longmapsto & \big(L^+_{\nu_i}\mathcal{G}, L_{\nu_i}\tau^{\deg \nu_i}\big)\,, \end{array}$$

$$\underline{\widehat{\Gamma}} := \prod_i \widehat{\Gamma}_{\nu_i}\colon \ \nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S) \longrightarrow \prod_i Sht_{\mathbb{P}_{\nu_i}}^{\operatorname{Spec} \widehat{A}_{\nu_i}}(S)\,, \tag{4.2}$$

where $L_{\nu_i}\tau^{\deg \nu_i} \colon (\sigma^{\deg \nu_i})^* L_{\nu_i}\mathcal{G} \xrightarrow{\sim} L_{\nu_i}\mathcal{G}$ is the $\mathbb{F}_{\nu_i}$-Frobenius on the loop group torsor $L_{\nu_i}\mathcal{G}$ associated with $L^+_{\nu_i}\mathcal{G}$. These functors also transform quasi-isogenies into quasi-isogenies. For further explanation see [AH14, Sect. 5.2].

Let $S \in \mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}}$ and let $j : \overline{S} \to S$ be a closed subscheme defined by a locally nilpotent sheaf of ideals $\mathcal{I}$. Let $\underline{\overline{\mathcal{G}}}$ be a global $\mathfrak{G}$-shtuka $\nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(\bar{S})$. We let $Defo_S(\underline{\bar{\mathcal{G}}})$ denote the category of infinitesimal deformations of $\underline{\overline{\mathcal{G}}}$ over S. More explicitly $Defo_S(\underline{\bar{\mathcal{G}}})$ is the category of lifts of $\underline{\overline{\mathcal{G}}}$ to S, which consists of all pairs $(\underline{\mathcal{G}}, \alpha : j^*\underline{\mathcal{G}} \to \underline{\overline{\mathcal{G}}})$ where $\underline{\mathcal{G}}$ belongs to $\nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$, and α is an isomorphism of global $\mathfrak{G}$-shtukas over S.

Similarly for a local $\mathbb{P}$-shtuka $\bar{\mathcal{L}}$ in $Sht_{\mathbb{P}}^{\mathbb{D}}(S)$ we define the category of lifts $Defo_S(\underline{\bar{\mathcal{L}}})$ of $\underline{\bar{\mathcal{L}}}$ to S.

Theorem 4.2.2. *(The Analog of the Serre-Tate Theorem for $\mathfrak{G}$-shtukas) Keep the above notation. Let $\underline{\bar{\mathcal{G}}} := (\bar{\mathcal{G}}, \bar{\tau})$ be a global $\mathfrak{G}$-shtuka in*

$$\nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(\bar{S}).$$

Let $(\underline{\bar{\mathcal{L}}}_i)_i = \underline{\widehat{\Gamma}}(\underline{\bar{\mathcal{G}}})$. Then the functor

$$Defo_S(\underline{\bar{\mathcal{G}}}) \longrightarrow \prod_i Defo_S(\underline{\bar{\mathcal{L}}}_i)\,, \quad \big(\underline{\mathcal{G}}, \alpha\big) \longmapsto \big(\underline{\widehat{\Gamma}}(\underline{\mathcal{G}}), \underline{\widehat{\Gamma}}(\alpha)\big)$$

induced by the global-local functor (4.2), is an equivalence of categories.

Analyzing the proof of the above theorem, one can see that it essentially relies on the rigidity of quasdi-isogenies, Proposition 4.2.1, and the following significant observation. Namely, a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ can be pull-backed along a quasi-isogeny $\underline{\mathcal{L}}' \to \underline{\mathcal{L}}_{\nu_i}$ from a local $\mathbb{P}_{\nu_i}$-shtuka $\underline{\mathcal{L}}'$ to $\underline{\mathcal{L}}_{\nu_i}$. Where $\underline{\mathcal{L}}_{\nu_i}$ is the local $\mathbb{P}_{\nu_i}$-shtuka associated with $\underline{\mathcal{G}}$ at the characteristic place ν_i, via the global-local functor (4.2). Here we are not going to explain this phenomenon and we refer the interested reader to [AH14, Proposition 5.7]. However, regarding our goal in this chapter, we discuss the following simpler case. Namely, the corresponding fact for the moduli stack $\mathscr{H}^1(C, \mathfrak{G})$ of $\mathfrak{G}$-bundles. This will be needed later in the proof of Proposition 4.4.2 and the local model theorem 4.4.6.

Consider the formal scheme $\operatorname{Spf} \mathbb{F}[\![\zeta]\!]$ as the ind-scheme $\varinjlim \operatorname{Spec} \mathbb{F}[\![\zeta]\!]$ and let $\widehat{\mathcal{F}\ell}_{\mathbb{P}}$ be the fiber product $\mathcal{F}\ell_{\mathbb{P}} \times \operatorname{Spf} \mathbb{F}[\![\zeta]\!]$ in the category of ind-schemes ; see [BD, 7.11.1]. Thus $\widehat{\mathcal{F}\ell}_{\mathbb{P}}$ is the restriction of the sheaf $\mathcal{F}\ell_{\mathbb{P}}$ to the *fppf*-site of schemes in $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$.

The ind-scheme $\widehat{\mathcal{F}\ell}_{\mathbb{P}}$ pro-represents the functor

$$
(4.3)\quad \begin{aligned} (\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]})^o &\longrightarrow \mathcal{S}ets \\ S &\longmapsto \big\{\text{Isomorphism classes of } (\mathcal{L}_+, \delta)\colon \text{ where } \mathcal{L}_+ \text{ is an} \\ &\qquad\ L^+\mathbb{P}\text{-torsor over } S \text{ and a trivialization } \delta\colon \mathcal{L} \to LP_S \\ &\qquad\ \text{of the associated } LP\text{-torsor } \mathcal{L} \text{ over } S\big\}. \end{aligned}
$$

For details see [AH14, Remark 4.2, Theorem 4.4] and Proposition 4.2.1.

Note further that for an $L^+\mathbb{P}$-torsor $\mathcal{L}_{0,+}$ over S, one may further define a twisted variant $\mathcal{M}(\mathcal{L}_{0,+})$ of the above functor, which assigns to a scheme T over S, the set of isomorphism classes of tuples $(\mathcal{L}_+, \delta)$ consisting of an $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ over T and an isomorphism $\delta\colon \mathcal{L} \to \mathcal{L}_{0,T}$.

Now we construct the following uniformization map for $\mathscr{H}^1(C, \mathfrak{G})$.

Lemma 4.2.3. *Let S be a scheme in $\mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}}$. Fix a $\mathfrak{G}$-bundle $\mathcal{G}$ in $\mathscr{H}^1(C, G)(S)$. There is a uniformization map*

$$
\Psi_{\mathcal{G}} : \prod_{\nu_i} \mathcal{M}(L^+_{\nu_i}\mathcal{G}) \to \mathscr{H}^1(C, \mathfrak{G})|_S.
$$

Furthermore this map is formally smooth.

Proof. Suppose that the $\mathfrak{G}$-bundle $\mathcal{G}$ corresponds to the tuple

$$\Big((\dot{\mathcal{G}})_{\underline{s}}, (\L^+_{\nu_i}\mathcal{G}, \alpha_i : \L_{\nu_i}\L^+_{\nu_i}\mathcal{G} \to \L_{\nu_i}(\dot{\mathcal{G}})_{\underline{s}})_i\Big)\,;$$

see Remark 4.1.9(e). We define the map $\Psi_{\mathcal{G}}$, by sending the tuple

$$(\mathcal{L}_{+,\nu_i}, \delta_{\nu_i} : \L_{\nu_i}\mathcal{L}_{+,\nu_i} \tilde{\to} \L_{\nu_i}\L^+_{\nu_i}\mathcal{G})_i$$

to the $\mathfrak{G}$-bundle associated with the tuple

$$((\dot{\mathcal{G}_T})_{\underline{s}_T}, (\mathcal{L}_{+,\nu_i}, \alpha_i \circ \delta_{\nu_i} : \L_{\nu_i}\mathcal{L}_{+,\nu_i} \to \L_{\nu_i}(\dot{\mathcal{G}_T})_{\underline{s}_T})_i).$$

To see that the resulting map $\Psi_{\mathcal{G}}$ is formally smooth, first notice that we may reduce to the case where $\L_{\nu_i}\L^+_{\nu_i}\mathcal{G}$ is trivial. This is because being formally smooth is étale local on the source and target. Then using the assumption that $\mathfrak{G}$ is smooth and $\mathscr{H}^1(C, \mathfrak{G})$ is locally noetherian, one can easily argue by lifting criterion for smoothness. $\square$

4.3. Local boundedness conditions

Here we recall the notion of *local boundedness condition*, introduced in [AH14, Definition 4.8], and we further explain the relation to the global boundedness condition which we introduced in section 3.1.

Fix an algebraic closure $\mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}$ of $\mathbb{F}(\!(\zeta)\!)$. Since its ring of integers is not complete, we prefer to work with finite extensions of discrete valuation rings $R/\mathbb{F}[\![\zeta]\!]$ such that $R \subset \mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}$. For such a ring R we denote by κ_R its residue field, and we let $\mathcal{N}ilp_R$ be the category of R-schemes on which ζ is locally nilpotent. We also set $\widehat{\mathcal{F}\ell}_{\mathbb{P},R} := \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spf} R$ and $\widehat{\mathcal{F}\ell}_{\mathbb{P}} := \widehat{\mathcal{F}\ell}_{\mathbb{P},\mathbb{F}[\![\zeta]\!]}$. Before we can define (local) "bounds" let us make the following definition.

Definition 4.3.1. (a) For a finite extension of discrete valuation rings $\mathbb{F}[\![\zeta]\!] \subset R \subset \mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}$ we consider closed ind-subschemes $\widehat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$. We call two closed ind-subschemes $\widehat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ and $\widehat{Z}'_{R'} \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R'}$ *equivalent* if there is a finite extension of discrete valuation rings $\mathbb{F}[\![\zeta]\!] \subset \widetilde{R} \subset \mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}$ containing R and R' such that $\widehat{Z}_R \widehat{\times}_{\operatorname{Spf} R} \operatorname{Spf} \widetilde{R} = \widehat{Z}'_{R'} \widehat{\times}_{\operatorname{Spf} R'} \operatorname{Spf} \widetilde{R}$ as closed ind-subschemes of $\widehat{\mathcal{F}\ell}_{\mathbb{P},\widetilde{R}}$.

(b) Let $\widehat{Z} = [\widehat{Z}_R]$ be an equivalence class of closed ind-subschemes $\widehat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ and let $G_{\widehat{Z}} := \{\gamma \in \operatorname{Aut}_{\mathbb{F}[\![\zeta]\!]}(\mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}) : \gamma(\widehat{Z}) = \widehat{Z}\,\}$. We define the *ring of definition $R_{\widehat{Z}}$ of $\widehat{Z}$* as the intersection of the fixed field of $G_{\widehat{Z}}$ in $\mathbb{F}(\!(\zeta)\!)^{\mathrm{alg}}$

with all the finite extensions $R \subset \mathbb{F}((\zeta))^{\mathrm{alg}}$ of $\mathbb{F}[\![\zeta]\!]$ over which a representative $\widehat{Z}_R$ of $\widehat{Z}$ exists.

Definition 4.3.2. (a) We define a (local) *bound* to be an equivalence class $\widehat{Z} := [\widehat{Z}_R]$ of closed ind-subschemes $\widehat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$, such that all the ind-subschemes $\widehat{Z}_R$ are stable under the left $L^+\mathbb{P}$-action on $\mathcal{F}\ell_{\mathbb{P}}$, and the special fibers $Z_R := \widehat{Z}_R \widehat{\times}_{\mathrm{Spf}\, R} \operatorname{Spec} \kappa_R$ are quasi-compact subschemes of $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spec} \kappa_R$. The ring of definition $R_{\widehat{Z}}$ of $\widehat{Z}$ is called the *reflex ring* of $\widehat{Z}$. Since the Galois descent for closed ind-subschemes of $\mathcal{F}\ell_{\mathbb{P}}$ is effective, the Z_R arise by base change from a unique closed subscheme $Z \subset \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \kappa_{R_{\widehat{Z}}}$. We call Z the *special fiber* of the bound $\widehat{Z}$. It is a quasi-projective scheme over $\kappa_{R_{\widehat{Z}}}$, and even projective when $\mathbb{P}$ is parahoric.

(b) Let $\widehat{Z}$ be a bound with reflex ring $R_{\widehat{Z}}$. Let $\mathcal{L}_+$ and $\mathcal{L}'_+$ be $L^+\mathbb{P}$-torsors over a scheme S in $\mathcal{N}ilp_{R_{\widehat{Z}}}$ and let $\delta\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ be an isomorphism of the associated LP-torsors. We consider an étale covering $S' \to S$ over which trivializations $\alpha\colon \mathcal{L}_+ \xrightarrow{\sim} (L^+\mathbb{P})_{S'}$ and $\alpha'\colon \mathcal{L}'_+ \xrightarrow{\sim} (L^+\mathbb{P})_{S'}$ exist. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1}$ of $(LP)_{S'}$ corresponds to a morphism $S' \to LP \widehat{\times}_{\mathbb{F}} \operatorname{Spf} R_{\widehat{Z}}$. We say that δ satisfies *local boundedness condition* (**LBC**) by $\widehat{Z}$ if for any such trivialization and for all finite extensions R of $\mathbb{F}[\![\zeta]\!]$ over which a representative $\widehat{Z}_R$ of $\widehat{Z}$ exists, the induced morphism $S' \widehat{\times}_{R_{\widehat{Z}}} \operatorname{Spf} R \to LP \widehat{\times}_{\mathbb{F}} \operatorname{Spf} R \to \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ factors through $\widehat{Z}_R$. Furthermore we say that a local $\mathbb{P}$-shtuka $(\mathcal{L}_+, \hat{\tau})$ is *bounded by* $\widehat{Z}$ if the isomorphism $\hat{\tau}^{-1}$ satisfies LBC by $\widehat{Z}$. Assume that $\widehat{Z} = \mathcal{S}(\omega) \widehat{\times}_{\mathbb{F}} \operatorname{Spf} \mathbb{F}[\![\zeta]\!]$ for a *Schubert variety* $\mathcal{S}(\omega) \subseteq \mathcal{F}\ell_{\mathbb{P}}$, with $\omega \in \widetilde{W}$; see [PR08]. Then we say that δ satisfies LBC by ω.

(c) Fix an n-tuple $\underline{\nu} = (\nu_i)$ of places on the curve C with $\nu_i \neq \nu_j$ for $i \neq j$. Let $\widehat{Z}_{\underline{\nu}} := (\widehat{Z}_{\nu_i})_i$ be an n-tuple of bounds in the above sense and set $R_{\widehat{Z}_{\underline{\nu}}} := R_{\widehat{Z}_{\nu_1}} \hat{\otimes}_{\mathbb{F}_q} \cdots \hat{\otimes}_{\mathbb{F}_q} R_{\widehat{Z}_{\nu_n}}$. We say that a tuple $(\mathcal{G}, \mathcal{G}', \underline{s}, \varphi)$ in $Hecke_n(C, \mathfrak{G})^{\underline{\nu}} \times_{\widehat{A}_{\underline{\nu}}} \operatorname{Spf} R_{\widehat{Z}_{\underline{\nu}}}$ is bounded by $\widehat{Z}_{\underline{\nu}}$ if for each i the inverse $\widehat{\varphi}_{\nu_i}^{-1}$ of the associated isomorphism $\widehat{\varphi}_{\nu_i} := \L_{\nu_i}(\varphi_i) : \L_{\nu_i}\mathcal{G}' \to \L_{\nu_i}\mathcal{G}$ satisfies LBC by $\widehat{Z}_{\nu_i}$ in the above sense. We denote the resulting formal stack by $Hecke_n^{\widehat{Z}_{\underline{\nu}}}(C, \mathfrak{G})^{\underline{\nu}}$, and sometimes we abbreviate this notation by $Hecke_n^{\widehat{Z}_{\underline{\nu}}}$. This accordingly defines boundedness condition on global $\mathfrak{G}$-shtukas. Equivalently, one says that a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$ is bounded by $\widehat{Z}_{\underline{\nu}}$ if for each i the associated local $\mathbb{P}_{\nu_i}$-shtuka $\widehat{\Gamma}_{\nu_i}(\underline{\mathcal{G}})$ under the global-local functor $\widehat{\Gamma}_{\nu_i}(-)$ is bounded by

$\widehat{Z}_{\nu_i}$. We denote by $\nabla_n^{\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ the formal substack obtained by imposing the boundedness condition $\widehat{Z}_{\underline{\nu}}$.

(d) Assume that the bound $\widehat{Z}_{\underline{\nu}}$ comes from a tuple of affine Schubert varieties $\mathcal{S}(\underline{\omega}) := (\mathcal{S}(\omega_i))_i$, where $\underline{\omega} := (\omega_i)_i \in \prod_{i=1}^n \widetilde{W}_i$. Here $\widetilde{W}_i$ denotes the Iwahori-Weyl group corresponding to P_{ν_i}; see Definition 1.1.2. Then we use the notation $Hecke_n^{\underline{\omega}}(C,\mathfrak{G})^{\underline{\nu}}$ (resp. $\nabla_n^{\underline{\omega}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$) for the corresponding moduli stack obtained by imposing the bound $\widehat{Z}_{\underline{\nu}}$.

Proposition 4.3.3. *Fix an n-tuple $\underline{\nu} = (\nu_i)_i$ of pairwise distinct places on the curve C. Let $\mathcal{Z}$ be a global bound in the sense of Definition 3.1.3. Furthermore, let $\underline{\nu}' = (\nu_i')_i$ be an n-tuple of places on the reflex field $Q_{\mathcal{Z}}$, such that ν_i' lies over ν_i. Then one can associate an n-tuple $(\widehat{Z}_{\nu_i})$ of local bounds to the global bound $\mathcal{Z}$.*

Proof. Let $Z := Z_K$ be a representative of $\mathcal{Z}$ over a field K which has minimal inseparable degree over Q. Let Z_i denote the image of Z under the morphism $GR_n \times_{C^n} \widetilde{C}_K^n \to GR_1 \times_C \widetilde{C}_K$ given by $(\underline{s} := (s_i)_i, \widehat{\mathcal{G}}, \dot{\varepsilon}) \mapsto (s_i, \widehat{\mathcal{G}}|_{\mathbb{D}(\Gamma_{s_i})}, \dot{\varepsilon}|_{\dot{\mathbb{D}}(\Gamma_{s_i})})$; see Remark 3.1.2. This is a closed subscheme of $GR_1 \times_C \widetilde{C}$ which is stable under the action of the global loop group $\mathfrak{L}_1^+\mathfrak{G}$. Let $\widehat{Z}_i^j$ denote the scheme defined by the following Cartesian diagram

$$\begin{tikzcd}
 & Z_i \arrow[rr] & & GR_1 \times_C \widetilde{C}_K \arrow[rr] \arrow[dd] & & GR_1 \arrow[dd] \\
\coprod \widehat{Z}_i^j \arrow[ur] \arrow[rr] & & \coprod_j \mathcal{F}\ell_{\mathbb{P}_{\nu_i},\widetilde{A}_{\nu_i^j}} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{F}\ell_{\mathbb{P}_{\nu_i},\widehat{A}_{\nu_i}} \arrow[ur] \arrow[dd] & \\
 & & & \widetilde{C}_K \arrow[rr] & & C \\
 & & \coprod_j \operatorname{Spf} \widetilde{A}_{\nu_i^j} \arrow[ur] \arrow[rr] & & \operatorname{Spf} \widehat{A}_{\nu_i} \arrow[ur] &
\end{tikzcd}$$

Now we choose an element $\widehat{Z}_{K,i} := \widehat{Z}_i^j$ from the set $\{\widehat{Z}_i^j\}_j$ such that the corresponding place ν_i^j lies above ν_i'. We assign to a global bound $\mathcal{Z}$, the tuple $([\widehat{Z}_{K,i}])_i$ of local bounds. Note that since the extension $Q_{\mathcal{Z}} \subset K$ is separable, see Remark 3.1.5, this assignment is independent of the choice of $\widehat{Z}_{K,i}$. The fact that the assignment is independent of the choice of the representative Z_K of $\mathcal{Z}$ is obvious. □

4.4. The local model theorem for $\nabla_n^{H,\widehat{Z}_\nu}\mathscr{H}^1(C,\mathfrak{G})^\nu$

Before proving the local model theorem for the moduli stack of global $\mathfrak{G}$-shtukas, let us first establish it for the stack Hecke.

Definition 4.4.1. Let $D \subset C$ be a finite subscheme, disjoint from $\underline{\nu}$. We denote by $\widetilde{Hecke}_{n,D}(C, \mathfrak{G})^{\underline{\nu}}$ the formal stack whose S-points parametrize tuples

$$\big(((\mathcal{G},\psi),(\mathcal{G}',\psi'),\underline{s},\tau),(\varepsilon_i)_{i=1,\dots,n}\big)\,,$$

consisting of

i) $((\mathcal{G},\psi),(\mathcal{G}',\psi'),\underline{s},\tau)$ in $Hecke_{n,D}(C,\mathfrak{G})^{\underline{\nu}}(S)$ and
ii) trivializations $\varepsilon_i : \text{Ł}^+_{\nu_i}(\mathcal{G}') \tilde{\to} L^+\mathbb{P}_{\nu_i,S}$ of the associated $L^+\mathbb{P}_{\nu_i}$-torsors $\text{Ł}^+_{\nu_i}(\mathcal{G}')$; see Remark 4.1.9(c).

Moreover, for a bound $\widehat{Z}_{\underline{\nu}} := (\widehat{Z}_{\nu_i})_i$, we denote by $\widetilde{Hecke}_{n,D}^{\widehat{Z}_{\underline{\nu}}}(C,\mathfrak{G})^{\underline{\nu}}$ the corresponding formal substack obtained by imposing the boundedness condition $\widehat{Z}_{\underline{\nu}}$. Since we fixed the curve C and the group $\mathfrak{G}$, and moreover, the number of characteristic places is implicit in $\underline{\nu}$, we sometimes drop them from our notation and we write $\widetilde{Hecke}_D^{\underline{\nu}}$ (resp. $\widetilde{Hecke}_D^{\widehat{Z}_{\underline{\nu}}}$) instead of $\widetilde{Hecke}_{n,D}(C,\mathfrak{G})^{\underline{\nu}}$ (resp. $\widetilde{Hecke}_{n,D}^{\widehat{Z}_{\underline{\nu}}}(C,\mathfrak{G})^{\underline{\nu}}$). We further drop the subscript D when it is empty.

Let $\widehat{Z}_{\nu_i,R_{\nu_i}}$ be a representative of $\widehat{Z}_{\nu_i}$ over R_{ν_i}. Set

$$\begin{aligned} R_{\widehat{Z}_{\underline{\nu}}} &:= R_{\widehat{Z}_{\nu_1}} \hat{\otimes}_{\mathbb{F}_q} \cdots \hat{\otimes}_{\mathbb{F}_q} R_{\widehat{Z}_{\nu_n}} \\ \text{and} \quad R_{\underline{\nu}} &:= R_{\nu_1} \hat{\otimes}_{\mathbb{F}_q} \cdots \hat{\otimes}_{\mathbb{F}_q} R_{\nu_n} \end{aligned}$$

and let $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ (resp. $\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$) denote the base change $Hecke_D^{\widehat{Z}_{\underline{\nu}}} \times_{R_{\widehat{Z}_{\underline{\nu}}}} R_{\underline{\nu}}$ (resp. $\widetilde{Hecke}_D^{\widehat{Z}_{\underline{\nu}}} \times_{R_{\widehat{Z}_{\underline{\nu}}}} R_{\underline{\nu}}$).

Proposition 4.4.2. *There is a roof of morphisms*

$$\begin{tikzcd}[column sep=small] & \widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i,R_{\nu_i}}. \end{tikzcd} \tag{4.4}$$

Furthermore, in the above roof, the formal stack $\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ is an $\prod_i L^+\mathbb{P}_{\nu_i}$-torsor over $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ under the projection π. Moreover for a geometric *point y of $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$, the $\prod_i L^+\mathbb{P}_{\nu_i}$-torsor $\pi : \widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \to Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ admits a section s, over an étale neighborhood of y, such that the composition $f \circ s$ is formally smooth.*

Proof. Forgetting the trivialization ε_i, it is clear that the formal stack $\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ is an $\prod_i L^+\mathbb{P}_{\nu_i}$-torsor over $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$.

Recall that $\widehat{\mathcal{F}\ell}_{\mathbb{P}}$ represents the functor (4.3). Consider a tuple

$$((\mathcal{G}, \mathcal{G}', \underline{s}, \tau), (\varepsilon_i)_i)$$

in $\widetilde{Hecke}^{\underline{\nu}}$ and let α_i (resp. α_i') denote the canonical isomorphism

$$Ł_{\nu_i} Ł_{\nu_i}^+ \mathcal{G} \tilde{\to} Ł_{\nu_i}(\dot{\mathcal{G}})_{\underline{s}} \text{ (resp. } Ł_{\nu_i} Ł_{\nu_i}^+ \mathcal{G}' \tilde{\to} Ł_{\nu_i}(\dot{\mathcal{G}}')_{\underline{s}}).$$

We define the morphism $\widetilde{Hecke}^{\underline{\nu}} \to \prod_{\nu_i} \widehat{\mathcal{F}\ell}_{\mathbb{P}_{\nu_i}}$, which assigns $(Ł_{\nu_i}^+\mathcal{G}, \varphi_i := Ł_{\nu_i}(\varepsilon_i) \circ \alpha_i'^{-1} \circ Ł_{\nu_i}(\tau)^{-1} \circ \alpha_i)$ to the tuple $((\mathcal{G}, \mathcal{G}', \underline{s}, \tau), (\varepsilon_i)_i)$. According to the Definition 4.3.2(c) this morphism induces the following map

$$f : \widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \longrightarrow \prod_i \widehat{Z}_{\nu_i, R_{\nu_i}}. \tag{4.5}$$

Let $\mathcal{A} := \mathcal{A}_y$ be the stalk of the structure sheaf of $Hecke^{\widehat{Z}_{\underline{\nu}}}$ at the geometric point y. As $\mathcal{A}$ is strictly henselian, we may fix a trivialization $\varepsilon_i^{\mathcal{A}}$ of the restriction of the universal $L^+\mathbb{P}_{\nu_i}$-torsor over $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}_\nu, L^+\mathbb{P}_{\nu_i})$ to $\operatorname{Spec}\mathcal{A}$; see Remark 4.1.9(a). This induces the section s.

Notice that as $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ is a torsor over $Hecke_{R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$ for the smooth group scheme $\mathfrak{G}_D = \operatorname{Res}_{D/k}\mathfrak{G}$, we may ignore the D-level structure. The morphism $f : \widetilde{Hecke}_{R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \longrightarrow \prod_i \widehat{Z}_{\nu_i, R_{\nu_i}}$ factors through $f_1 : \widetilde{Hecke}_{R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \longrightarrow \mathscr{H}^1(C, \mathfrak{G}) \times_{\mathbb{F}_q} \prod_i \widehat{Z}_{\nu_i, R_{\nu_i}}$ followed by the projection to the second factor. Consider a closed immersion $\overline{S} \hookrightarrow S$, defined by a nilpotent sheaf of ideal $\mathcal{I}$. Let $(\overline{\mathcal{G}}, \overline{\mathcal{G}}', (\overline{s}_i), \overline{\tau})$ be an object of $Hecke^{\widehat{Z}_{\underline{\nu}}}(\overline{S})$ and assume that it maps to the tuple

$$\left(\overline{\mathcal{G}}, (Ł_{\nu_i}^+\overline{\mathcal{G}}, \overline{\varphi}_i : Ł_{\nu_i}\overline{\mathcal{G}} \tilde{\to} LP_{\nu_i, \overline{S}})_i\right)$$

(resp. $(Ł_{\nu_i}^+\overline{\mathcal{G}}, \overline{\varphi}_i)_i$) via $f_1 \circ s$ (resp. $f \circ s$). Let $(\mathcal{L}_{+,i}, \varphi_i : \mathcal{L}_i \tilde{\to} LP_{\nu_i, S})_i$ be a lift of $(Ł_{\nu_i}^+\overline{\mathcal{G}}, \overline{\varphi}_i)_i$ over S. As any formal $\hat{\mathbb{P}}_\nu$-torsor over $\mathbb{D}(\Gamma_{\underline{s}})$ extends to a $\mathfrak{G}$-bundle over C_S, and $\mathscr{H}^1(C, \mathfrak{G})$ is smooth [AH19, Theorem 2.5], one may take a lift $\big(\mathcal{G}, (\mathcal{L}_{+,i} = Ł^+_{\nu_i}\mathcal{G}, \varphi_i : \mathcal{L}_i \tilde{\to} LP_{\nu_i,S})_i\big)$ of $\big(\overline{\mathcal{G}}, (Ł^+_{\nu_i}\overline{\mathcal{G}}, \overline{\varphi}_i)_i\big)$ over S, which maps to $(\mathcal{L}_{+,i}, \varphi_i : \mathcal{L}_i \tilde{\to} LP_{\nu_i,S})_i$. Again by smoothness of the algebraic stack $\mathscr{H}^1(C, \mathfrak{G})$, we may choose a $\mathfrak{G}$-bundle $\mathcal{G}'$ which lifts $\overline{\mathcal{G}}'$ over S. We let $\varepsilon'_{i,S}$ denote the trivialization of $Ł^+_{\nu_i}\mathcal{G}'$ induced by $\varepsilon_i^{\mathcal{A}}$.

Consider the isomorphism $\delta_{\nu_i} := \varphi_i^{-1} \circ Ł(\varepsilon'_{i,S}) : Ł_{\nu_i}Ł^+_{\nu_i}\mathcal{G}' \tilde{\to} Ł_{\nu_i}Ł^+_{\nu_i}\mathcal{G}$ and let $\mathcal{G}''$ denote the image of S-point $(Ł^+_{\nu_i}\mathcal{G}', \delta_{\nu_i})_i$ under the uniformization map

$$\Psi_{\mathcal{G}} : \prod_{\nu_i} \mathcal{M}_{\nu_i}(Ł_{\nu_i}\mathcal{G}) \to \mathscr{H}^1(C, \mathfrak{G})|_S,$$

see Lemma 4.2.3. Note that by construction there is an isomorphism $\tau : (\dot{\mathcal{G}''})_{\underline{s}} \to (\dot{\mathcal{G}})_{\underline{s}}$. The S-point $(\mathcal{G}, \mathcal{G}'', (s_i), \tau)$ of $Hecke_n(C, \mathfrak{G})^{\underline{\nu}}$ provides the desired lift of the $\overline{S}$-point $(\overline{\mathcal{G}}, \overline{\mathcal{G}}', (\overline{s}_i), \overline{\tau})$ in the following sense. Namely, there is an isomorphism between $\overline{\mathcal{G}}' = ((\dot{\overline{\mathcal{G}}'})_{\underline{s}}, (Ł^+_{\nu_i}\overline{\mathcal{G}}', \overline{\alpha}'_i : Ł_{\nu_i}Ł^+_{\nu_i}\overline{\mathcal{G}}' \to Ł_{\nu_i}(\dot{\overline{\mathcal{G}}'})_{\underline{s}})_i)$ and the pull back $\overline{\mathcal{G}}'' = \Big((\dot{\overline{\mathcal{G}}})_{\underline{s}}, (Ł^+_{\nu_i}\overline{\mathcal{G}}', \alpha''_i : Ł_{\nu_i}Ł^+_{\nu_i}\overline{\mathcal{G}}' \to Ł_{\nu_i}(\dot{\overline{\mathcal{G}}})_{\underline{s}})_i\Big)$ of $\mathcal{G}''$ over $\overline{S}$, which is given by $\overline{\tau}$ on the first factor and by identity and the following commutative diagram

$$\begin{tikzcd}
Ł_{\nu_i}Ł^+_{\nu_i}\overline{\mathcal{G}}' \arrow[r, "\overline{\alpha}'_i"] \arrow[d, equal] & Ł_{\nu_i}(\dot{\overline{\mathcal{G}}'})_{\underline{s}} \arrow[d, "Ł_{\nu_i}(\overline{\tau})"] \\
Ł_{\nu_i}Ł^+_{\nu_i}\overline{\mathcal{G}}' \arrow[r, "\overline{\alpha}''_i"] & Ł_{\nu_i}(\dot{\overline{\mathcal{G}}})_{\underline{s}} .
\end{tikzcd}$$

on the second factor. To justify the commutativity of the above diagram notice that

$$\begin{aligned}
\overline{\alpha}''_i &= \overline{\alpha}_i \circ \overline{\varphi}_i^{-1} \circ Ł(\varepsilon'_{i,\overline{S}}) \\
&= \overline{\alpha}_i \circ \Big(Ł(\varepsilon'_{i,\overline{S}}) \circ \overline{\alpha}'^{-1}_i \circ Ł_{\nu_i}(\overline{\tau})^{-1} \circ \overline{\alpha}_i\Big)^{-1} \circ Ł(\varepsilon'_{i,\overline{S}}) \\
&= Ł_{\nu_i}(\overline{\tau}) \circ \overline{\alpha}'_i
\end{aligned}$$

Regarding the definition of the uniformization map, it can be easily seen that $(\mathcal{G}, \mathcal{G}'', \underline{s}, \tau)$ maps to $(\mathcal{L}_{+,i}, \varphi_i : \mathcal{L}_i \tilde{\to} LP_{\nu_i,S})_i$ under $f \circ s$. $\square$

Using tannakian formalism, we can equip the moduli stack $\nabla\mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}$ of global $\mathfrak{G}$-shtukas with H-level structure, for a compact open subgroup $H \subseteq \mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$; details are explained in [AH19, Chapter 6]. Here we briefly recall the definition.

Definition 4.4.3 (H-level structure). Assume that $S \in \mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}}$ is connected and fix a geometric point $\bar{s}$ of S. Let $\pi_1(S, \bar{s})$ denote the algebraic fundamental group of S.

(a) The rational Tate functor constructed in [AH19, Chapter 6]

$$\begin{array}{rcl} \check{\mathcal{V}}_{-} : \nabla_n \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S) & \longrightarrow & Funct^{\otimes}(\mathrm{Rep}_{\mathbb{A}^{\underline{\nu}}} \mathfrak{G}\,,\, \mathfrak{Mod}_{\mathbb{A}_Q^{\underline{\nu}}[\pi_1(S,\bar{s})]}) \\ \underline{\mathcal{G}} & \mapsto & \check{\mathcal{V}}_{\underline{\mathcal{G}}} : \rho \to \varprojlim_{D \subset C'} \rho_*(\underline{\mathcal{G}}|_{D_{\bar{s}}})^{\tau} \otimes_{\mathbb{A}^{\underline{\nu}}} \mathbb{A}_Q^{\underline{\nu}} \end{array}$$

assigns to a global $\mathfrak{G}$-shtuka over S, a tensor functor from the category $\mathrm{Rep}_{\mathbb{A}^{\underline{\nu}}} \mathfrak{G}$ of adelic representations of $\mathfrak{G}$, to the category $\mathfrak{Mod}_{\mathbb{A}_Q^{\underline{\nu}}[\pi_1(S,\bar{s})]}$, of $\mathbb{A}_Q^{\underline{\nu}}[\pi_1(S, \bar{s})]$-modules. The limit is taken over all finite subschemes D of $C' := C \smallsetminus \{\nu_1, \ldots, \nu_n\}$.

(b) For a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over S let us consider the set of isomorphisms of tensor functors $\mathrm{Isom}^{\otimes}(\check{\mathcal{V}}_{\underline{\mathcal{G}}}, \omega^{\circ})$, where $\omega^{\circ} \colon \mathrm{Rep}_{\mathbb{A}^{\underline{\nu}}} \mathfrak{G} \to \mathfrak{Mod}_{\mathbb{A}_Q^{\underline{\nu}}}$ denote the neutral fiber functor. The set $\mathrm{Isom}^{\otimes}(\check{\mathcal{V}}_{\underline{\mathcal{G}}}, \omega^{\circ})$ admits an action of $\mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}}) \times \pi_1(S, \bar{s})$ where $\mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$ acts through ω° by tannakian formalism and $\pi_1(S, \bar{s})$ acts through $\check{\mathcal{V}}_{\underline{\mathcal{G}}}$. For a compact open subgroup $H \subseteq \mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$ we define a *rational H-level structure* $\bar{\gamma}$ on a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over $S \in \mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}}$ to be a $\pi_1(S, \bar{s})$-invariant H-orbit $\bar{\gamma} = H\gamma$ in $\mathrm{Isom}^{\otimes}(\check{\mathcal{V}}_{\underline{\mathcal{G}}}, \omega^{\circ})$.

(c) We denote by $\nabla_n^{H, \widehat{Z}_{\underline{\nu}}} \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}$ the category fibered in groupoids, whose category of S-valued points $\nabla_n^{H, \widehat{Z}_{\underline{\nu}}} \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$ has tuples $(\underline{\mathcal{G}}, \bar{\gamma})$, consisting of a global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n^{\widehat{Z}_{\underline{\nu}}} \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}(S)$ together with a rational H-level structure $\bar{\gamma}$, as its objects. The morphisms are quasi-isogenies of global $\mathfrak{G}$-shtukas that are isomorphisms at the characteristic places ν_i and are compatible with the H-level structures.

The above definition of level structure generalizes the initial Definition 2.0.11, according to the following proposition.

Proposition 4.4.4. *We have the following statements*

(a) *For a finite subscheme $D \subset C$, disjoint from $\underline{\nu}$, there is a canonical isomorphism $\nabla_n \mathscr{H}_D^1(C, \mathfrak{G})^{\underline{\nu}} \xrightarrow{\sim} \nabla_n^{H_D} \mathscr{H}^1(C, \mathfrak{G})^{\underline{\nu}}$ of formal stacks. Here H_D denotes the compact open subgroup* $\ker\big(\mathfrak{G}(\mathbb{A}^{\underline{\nu}}) \to \mathfrak{G}(\mathcal{O}_D)\big)$ *of* $\mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$.

(b) *For any compact open subgroup* $H \subseteq \mathfrak{G}(\mathbb{A}_Q^{\underline{\nu}})$ *the stack* $\nabla_n^H \mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ *is an ind-algebraic stack, ind-separated and locally of ind-finite type over* $\operatorname{Spf} A_{\underline{\nu}}$*. The forgetful morphism*

$$\nabla_n^H \mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}} \to \nabla_n \mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$$

is finite étale.

Proof. For part (a) see [AH19, Theorem 6.5]. Part (b) follows from (a) and Theorem 2.0.12. □

Definition 4.4.5. We define the formal stack $\nabla_n^{\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1(C,\mathfrak{G})}^{\underline{\nu}}$ by the following pull back diagram

$$\begin{array}{ccc} \nabla_n^{\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1(C,\mathfrak{G})}^{\underline{\nu}} & \longrightarrow & \widetilde{Hecke}_n^{\widehat{Z}_{\underline{\nu}}} \\ \downarrow & & \downarrow \\ \nabla_n^{\widehat{Z}_{\underline{\nu}}} \mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}} & \longrightarrow & Hecke_n^{\widehat{Z}_{\underline{\nu}}} . \end{array}$$

More explicitly the S-points of the formal stack $\nabla_n^{\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1(C,\mathfrak{G})}^{\underline{\nu}}$ parametrizes the tuples $(\underline{\mathcal{G}}, (\varepsilon_i)_i)$ consisting of

(a) a $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n^{\widehat{Z}_{\underline{\nu}}} \mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S)$ and

(b) trivializations $\varepsilon_i : \hat{\sigma}_S^* \mathcal{L}_{+,i} \xrightarrow{\sim} L^+\mathbb{P}_{i,S}$, where $(\mathcal{L}_{+,i}, \hat{\tau}_i) := \widehat{\Gamma}_{\nu_i}(\underline{\mathcal{G}})$.

Furthermore we use the notation $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1(C,\mathfrak{G})}^{\underline{\nu}}$ when the $\mathfrak{G}$-shtukas in (a) are additionally equipped with H-level structure. Since we fix the curve C, the group $\mathfrak{G}$ and the characteristic places $\underline{\nu}$, we sometimes drop them from our notation and write $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}} \mathscr{H}^1$ and $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1}$ to denote the corresponding formal stacks.

Theorem 4.4.6. *Keep the above notation. Consider the following roof*

$$\begin{tikzcd}[column sep=small] & \nabla_n^{H,\widehat{Z}_{\underline{\nu}}} \widetilde{\mathscr{H}^1_{R_{\underline{\nu}}}} \arrow[dl, "\pi'"'] \arrow[dr, "f'"] & \\ \nabla_n^{H,\widehat{Z}_{\underline{\nu}}} \mathscr{H}^1_{R_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i, R_{\nu_i}}, \end{tikzcd} \tag{4.6}$$

induced from (4.4). Let y be a geometric point of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}}$. The $\prod_i L^+\mathbb{P}_{\nu_i}$-torsor $\pi' : \nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}^1_{R_{\underline{\nu}}}} \to \nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}}$ admits a section s', locally over an étale neighborhood of y, such that the composition $f' \circ s'$ is formally étale.

Proof. According to Proposition 4.4.4 we may forget the H-level structure. Let $\mathcal{A}' := \mathcal{A}'_y$ be the henselian local ring at the geometric point $y \in \nabla_n^{\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}}$. The restriction of the section s obtained in the course of the proof of Proposition 4.4.2 to $\mathcal{U}' := \operatorname{Spec}\mathcal{A}'$ provides a section s'.

Now we check that the morphism $f' \circ s'$ is formally étale. Let $\overline{S}$ be a closed subscheme of S, defined by a nilpotent sheaf of ideal $\mathcal{I}$. Take an $\overline{S}$-valued point $\overline{\underline{\mathcal{G}}} = (\overline{\mathcal{G}}, (s_i), \overline{\tau})$ of $\mathcal{A}'$ and let $(\overline{\underline{\mathcal{L}}_i})_i = \big((\overline{\mathcal{L}}_{+,i}, \hat{\overline{\tau}}_i)\big)_i$ denote the associated tuple of local $\mathbb{P}_{\nu_i}$-shtukas under the global-local functor. Let $(\overline{\mathcal{L}}_{+,i}, \overline{\varphi}_i : \overline{\mathcal{L}}_i \xrightarrow{\sim} L\mathbb{P}_{\nu_i,\overline{S}})$ denote the image of $\overline{\underline{\mathcal{G}}} = (\overline{\mathcal{G}}, (s_i), \overline{\tau})$ in $\prod_i \widehat{Z}_{\nu_i}(\overline{S})$ under the morphism $f' \circ s'$, and let $(\mathcal{L}_{+,i}, \varphi_i : \mathcal{L}_i \xrightarrow{\sim} L\mathbb{P}_{\nu_i,S})$ be an infinitesimal deformation of $(\overline{\mathcal{L}}_{+,i}, \varphi_i : \overline{\mathcal{L}}_i \xrightarrow{\sim} L\mathbb{P}_{\nu_i,\overline{S}}) \in \prod_i \widehat{Z}_{\nu_i}(\overline{S})$ over S. We consider the infinitesimal deformation $(\underline{\mathcal{L}}_i)_i := \big((\mathcal{L}_{+,i}, \hat{\tau}_i := \varphi_i^{-1} \circ \mathrm{Ł}_{\nu_i}(\varepsilon_{i,S}))_i\big)_i$ of the n-tuple $(\overline{\underline{\mathcal{L}}_i})_i$ of local $\mathbb{P}_{\nu_i}$-shtukas over S. Here $\varepsilon_{i,S}$ denotes the trivialization of $\sigma_S^*\mathcal{L}_+$ induced by $\varepsilon_i^{\mathcal{A}}$; see proof of the Proposition 4.4.2. By the analog of Serre-Tate, theorem 4.2.2, we have

$$Defo_S(\bar{\underline{\mathcal{G}}}) \cong \prod_i Defo_S(\bar{\underline{\mathcal{L}}}_i).$$

Consequently, the n-tuple $(\underline{\mathcal{L}}_i)_i$ provides a unique infinitesimal deformation $\underline{\mathcal{G}} = (\mathcal{G}, \underline{s}, \tau)$ of $\overline{\underline{\mathcal{G}}} = (\overline{\mathcal{G}}, \underline{s}, \overline{\tau})$ over S, which by construction maps to $(\mathcal{L}_{+,i}, \varphi_i)$ under $f' \circ s'$. The fact that $\underline{\mathcal{G}}$ is bounded by $\widehat{Z}_{\underline{\nu}}$ is tautological. $\square$

Remark 4.4.7. Analyzing the proof of Theorem 3.2.1, Proposition 2.0.9 and also Proposition 4.4.2, one can immediately see that the assumption that the group $\mathfrak{G}$ is smooth over C can not be weakened.

Remark 4.4.8. (The generalized Lang Correspondence) We have the following roof from the local model diagram

$$\begin{tikzcd}[column sep=small] & \nabla_n^{\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}^1_{R_{\underline{\nu}}}} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \nabla_n^{\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}} & & \prod_i \widehat{Z}_{\nu_i, R_{\nu_i}}, \end{tikzcd}$$

see Theorem 4.4.6. Let $y := \underline{\mathcal{G}}$ be a global $\mathfrak{G}$-shtuka over S. Then, the above diagram together with the uniformization morphism $\Theta := \Theta(\underline{\mathcal{G}})$ from [AH19, Theorem 7.11] induces the following roof

$$\begin{tikzcd}[column sep=small] & \prod_i \underline{\mathcal{M}}_{\mathbb{L}_0}^{\widehat{Z}_{\nu_i}} \times_{\nabla_n^{\hat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}}} \nabla_n^{\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}^1_{R_{\underline{\nu}}}} \arrow[dl] \arrow[dr] & \\ \prod_i \underline{\mathcal{M}}_{\mathbb{L}_0}^{\widehat{Z}_{\nu_i}} & & \prod_i \widehat{Z}_{i,R_{\nu_i}} \end{tikzcd}$$

Thus up to a choice of a section for π we obtain a local morphism from the product of Rapoport-Zink spaces to $\prod_i \hat{Z}_i$. Note that $\prod_i \hat{Z}_i$ can be also viewed as a parameter space for Hodge-Pink structures; see [Har11].

4.5. Some applications

In this section we assume that $\mathfrak{G}$ is a parahoric Bruhat-Tits group scheme (i.e. a smooth affine group scheme with connected fibers and reductive generic fiber) over C.

Kottwitz-Rapoport stratification. Fix an n-tuple $\underline{\nu}$ of pairwise distinct characteristic places. Let $\widehat{Z}_{\underline{\nu}}$ be a bound in the sense of Definition 4.3.2 and $\widehat{Z}_{\nu_i,R_{\nu_i}}$ be a representative of the bound $\widehat{Z}_{\nu_i}$ over R_{ν_i}. Let $K_{\underline{\nu}} = \prod_i K_{\nu_i} \subset \prod_i L^+\mathbb{P}_{\nu_i}$ denote the normal subgroup which acts trivially on $\widehat{Z}_{\underline{\nu}}$. The diagram 4.6 (resp. 4.4) induces the following Cartesian diagram

$$\begin{tikzcd} & \left(\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}^1_{R_{\underline{\nu}}}}\right)^{K_{\underline{\nu}}} \arrow[dl, "\pi^{K_{\underline{\nu}}}"'] \arrow[dr, "f^{K_{\underline{\nu}}}"] & \\ \nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1_{R_{\underline{\nu}}} \arrow[dr] & & \prod_i [K_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}] \arrow[dl] \\ & \prod_i [L^+\mathbb{P}_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}] & \end{tikzcd}$$

(resp.

$$\begin{tikzcd}
& \left(\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}\right)^{K_{\underline{\nu}}} \arrow[dl, "\pi^{K_{\underline{\nu}}}"'] \arrow[dr, "f^{K_{\underline{\nu}}}"] & \\
Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}} \arrow[dr] & & \prod_i [K_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}] \arrow[dl] \\
& \prod_i [L^+\mathbb{P}_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}] &
\end{tikzcd}$$

)

of algebraic stacks. Here $[L^+\mathbb{P}_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}]$ and $[K_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}]$ denote the quotient stacks and $\left(\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}_{R_{\underline{\nu}}}^1}\right)^{K_{\underline{\nu}}}$ (resp. $\left(\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}\right)^{K_{\underline{\nu}}}$), denote the formal algebraic stack whose S-points parametrizes $K_{\underline{\nu}}$-orbits of the S-points $(\underline{\mathcal{G}},(\varepsilon_i)_i)$ of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\widetilde{\mathscr{H}_{R_{\underline{\nu}}}^1}(S)$ (resp. $((\mathcal{G},\mathcal{G}',\underline{s},\tau),(\varepsilon_i)_i)$ of $\widetilde{Hecke}_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}(S)$). This stack is a $\prod_i K_{\nu_i}\backslash L^+\mathbb{P}_{\nu_i}$-fiber bundle over $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1$ (resp. $Hecke_{D,R_{\underline{\nu}}}^{\widehat{Z}_{\underline{\nu}}}$) via the morphism $\pi^{K_{\underline{\nu}}}$, and therefore locally of finite type. Note that since $\widehat{Z}_{\nu_i,R_{\nu_i}}$ is projective (see Remark 4.1.5), the scheme of morphisms

$$Mor(\widehat{Z}_{\nu_i,R_{\nu_i}},\widehat{Z}_{\nu_i,R_{\nu_i}})$$

is of finite type, see [FGA, IV.4.c], and thus $K_{\nu_i}\backslash L^+\mathbb{P}_{\nu_i}$ is of finite type. Let $\widetilde{W}_i$ denote the Iwahori-Weyl group corresponding to P_{ν_i}. Recall that by definition of the boundedness condition, the special fiber of $\widehat{Z}_{\nu_i}$ is a finite union of closed affine Schubert varieties. Therefore the special fiber of the formal stack $[L^+\mathbb{P}_{\nu_i}\backslash \widehat{Z}_{\nu_i,R_{\nu_i}}]$ is a discrete set indexed by a finite subset of $\widetilde{W}_i$, corresponding to the $L^+\mathbb{P}_{\nu_i}$-orbits, see [Ric13a, Proposition 0.1]. Accordingly, we obtain a natural stratification $\{(\nabla_n^{H,Z_{\underline{\nu}}}\mathscr{H}_s^1)^{\underline{\lambda}}\}_{\underline{\lambda}}$ (resp. $\{\left(Hecke_{D,s}^{Z_{\underline{\nu}}}\right)^{\underline{\lambda}}\}_{\underline{\lambda}}$) of the special fiber $\nabla_n^{H,Z_{\underline{\nu}}}\mathscr{H}_s^1$ (resp. $Hecke_{D,s}^{Z_{\underline{\nu}}}$) of $\nabla_n^{H,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1$ (resp. $Hecke_D^{Z_{\underline{\nu}}}$), indexed by $\underline{\lambda}\in\prod_i\widetilde{W}_i$, such that the incidence relation between strata is given by the obvious partial order on the product $\prod_i\widetilde{W}_i$, induced by the natural Bruhat order on each factor $\widetilde{W}_i$. The fiber over $\underline{\lambda}\in\prod_i\widetilde{W}_i$ appearing in this stratification is called *Kottwitz-Rapoport stratum* corresponding to $\underline{\lambda}$.

Assume that the special fiber $Z_{\underline{\nu}}$ of $\widehat{Z}_{\underline{\nu}}$ equals the fiber product $\mathcal{S}(\underline{\omega}) = \mathcal{S}(\omega_1) \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} \mathcal{S}(\omega_n)$ for $\underline{\omega} := (\omega_i) \in \prod_i \widetilde{W}_i$. In particular, we have

$$\nabla_n^{H,Z_{\underline{\nu}}} \mathscr{H}_s^1 = \left(\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1\right)^{\preceq \underline{\omega}} = \bigcup_{\underline{\lambda} \preceq \underline{\omega}} \left(\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1\right)^{\underline{\lambda}}$$

(resp. $Hecke_{D,s}^{Z_{\underline{\nu}}} = \left(Hecke_{D,s}^{\underline{\omega}}\right)^{\preceq \underline{\omega}} := \bigcup_{\underline{\lambda} \preceq \underline{\omega}} \left(Hecke_{D,s}^{\underline{\omega}}\right)^{\underline{\lambda}}$).

For each $\underline{\omega}$, let $\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^{1\circ}$ (resp. $Hecke_{D,s}^{\underline{\omega}^\circ}$) be the complement in $\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1$ (resp. $Hecke_{D,s}^{\underline{\omega}}$) of the union of all $\left(\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1\right)^{\underline{\lambda}}$ (resp. $\left(Hecke_{D,s}^{\underline{\omega}}\right)^{\underline{\lambda}}$) with $\underline{\lambda} \prec \underline{\omega}$. We have the following

Proposition 4.5.1. *Keep the above assumption about the bound $\widehat{Z}_{\underline{\nu}}$. The stack $\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^{1\circ}$ (resp. $Hecke_{D,s}^{\underline{\omega}^\circ}$) is an smooth open and dense substack of $\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1$ (resp. $Hecke_{D,s}^{\underline{\omega}\circ}$) of dimension $\sum_{i=1}^n \ell_i(\omega_i)$ (resp. $d + \sum_{i=1}^n \ell_i(\omega_i)$). Here ℓ_i denotes the Bruhat-Chevally length function on $\widetilde{W}_i$ and $d = \dim \mathscr{H}_D^1(C, \mathfrak{G})$.*

Proof. This follows from [Ric13a, Propposition 0.1], Theorem 4.4.6 (resp. proposition 4.4.2) and the above discussion. □

Intersection cohomology. Fix an n-tuple $\underline{\nu}$ of pairwise distinct characteristic places ν_i on C. Following Varshavsky's argument [Var04, Corollary 2.21 c)], together with the analogue of Rapoport-Zink local model, Theorem 4.4.6 and the above discussion, we can now prove the following.

Proposition 4.5.2. *Keep the assumption of Proposition 4.5.1. The IC-sheaves $IC(\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1)$ and the restriction of $IC(Hecke_s^{\underline{\omega}})$ coincide up to some shift and Tate twists. In particular when $\mathfrak{G}$ is constant split reductive group, the restriction of $IC(\nabla_n^{H,\underline{\omega}} \mathscr{H}_s^1)$ to each stratum is a direct sum of complexes of the form $\overline{\mathbb{Q}}_\ell(k)[2k]$.*

Proof. According to Proposition 4.4.4 we may forget the H-level structure. The stratification on $\nabla_n^{\underline{\omega}} \mathscr{H}_s^1$ is induced by that of $Hecke_s^{\underline{\omega}}$. Moreover by Proposition 4.4.2 and Theorem 4.4.6 and [Ric13a, Remark 2.6] the smooth open stratum $\nabla_n^{\underline{\omega}} \mathscr{H}_s^{1\circ}$ lies inside the pull back of the open smooth stratum $Hecke_s^{\underline{\omega}^\circ}$ of $Hecke_s^{\underline{\omega}}$, thus the first argument is obvious over the open stratum.

Regarding Proposition 4.4.2 and Theorem 4.4.6, we have the following diagram

$$\begin{tikzcd}
U_y \arrow[r] \arrow[rr, bend left, "s"] \arrow[rrr, bend left=30, "Smooth"] & Hecke^{\omega}_{n,s} & \widetilde{Hecke}^{\omega}_{n,s} \arrow[l, "\pi"'] \arrow[r, "f"] & \prod_i \mathcal{S}(\omega_i) \arrow[d, equal] \\
U'_y \arrow[u, "i_{U_y}"] \arrow[r] \arrow[rr, bend right, "s'"'] \arrow[rrr, bend right=30, "Etale"'] & \nabla^{\omega}_{n}\mathscr{H}^1_s \arrow[u, "i"] & \widetilde{\nabla^{\omega}_{n}\mathscr{H}_s}^{1} \arrow[l, "\pi'"'] \arrow[u] \arrow[r, "f'"] & \prod_i \mathcal{S}(\omega_i)\ .
\end{tikzcd}$$

Now it suffices to show that the statement holds for the restriction of the IC-sheaves to the étale neighborhoods U_y and U'_y

$$\begin{aligned}
IC(U'_y) &= (f' \circ s')^* IC(\prod_i \mathcal{S}(\omega_i)) \\
&= (f \circ s \circ i_{U_y})^* IC(\prod_i \mathcal{S}(\omega_i)) \\
&= i^*_{U_y} IC(U_y)(-\dim \mathscr{H}^1/2)[-\dim \mathscr{H}^1] \\
&= i^*_{U_y} IC(Hecke^{\omega}_{n,s})|_{U_y}(-\dim \mathscr{H}^1/2)[-\dim \mathscr{H}^1] \\
&= i^* IC(Hecke^{\omega}_{n,s})|_{U'_y}(-\dim \mathscr{H}^1/2)[-\dim \mathscr{H}^1],
\end{aligned}$$

where the first equality follows from Theorem 4.4.6, the third equality follows from Proposition 4.4.2 and the rest follows from commutativity of the above diagram.

Consider the Bott-Samelson resolution of singularities $\Sigma(\omega_i) \to \mathcal{S}(\omega_i)$ of the affine Schubert variety $\mathcal{S}(\omega_i)$. Note that it is a convolution morphism with cellular fibers and the variety $\Sigma(\omega_i)$ may be viewed as a tower of iterated projective homogeneous fiber bundles. Recall that projective homogeneous varieties admit cellular decomposition (i.e. can be stratified by affine spaces). Now the last assertion follows from the first statement of the proposition, Proposition 2.0.9 and the decomposition theorem [BBD]. See also proof of [Var04, Proposition A. 13]. $\square$

Flatness. Let $\{\mu_i\}_{i=1,\dots,n}$ be a set of geometric conjugacy classes of cocharacters in G which are defined over a finite separable extension E/Q. We say that a global boundedness condition $\mathcal{Z}$ is *generically defined by* $\{\mu_i\}$ if it arises in the following way. Namely, observe that each μ_i defines a closed subscheme of $GR_1 \times_C \widetilde{C}_E$ which we denote by $GR_{\preceq\mu_i}$. Note that $GR_{\preceq\mu_i}$ is proper flat with geometrically connected equi-dimensional fibers over $\widetilde{C}_E$. Now consider the bound $\mathcal{Z}$ which is given by the class of the Zariski closure in $GR_n \times_{C^n} \widetilde{C}_E^n$ of the restriction of the fiber product $GR_{\preceq\mu_1} \times \cdots \times GR_{\preceq\mu_n}$ of global affine Schubert varieties $GR_{\preceq\mu_i}$ to the complement of the big diagonal in $\widetilde{C}_E^n$.

Proposition 4.5.3. *Assume that $\mathcal{Z}$ is a global boundedness condition generically defined by $\{\mu_i\}$ in the above sense. Furthermore assume that G is tamely ramified and that p does not divide the order of the fundamental group of the derived group G_{der}. Then the moduli stack $\nabla_n^{H,\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ is flat over $\widehat{\mathcal{O}}_{C^n_{\mathcal{Z}},\underline{\nu}}$ and has Cohen-Macaulay singularities.*

Proof. We may ignore the H-level structure by Proposition 4.4.4(b). Since the stack $\nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})$ is locally of finite type, the ring homomorphism from the stalk at a point $y \in \nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ to the completion is faithfully flat. As the natural morphism $C_{\mathcal{Z}} \to C$ is faithfully flat, we may assume that $\mathcal{Z}$ is defined over C. Note that according to Theorem 3.2.1, [PR08, Theorem 8.4] and [B-K] $\nabla_n^{\mathcal{Z}}\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}$ has Cohen-Macaulay singularities. Since $\mathcal{O}_{C^n,\underline{\nu}}$ is regular, the assertion follows from the construction of $\mathcal{Z}$ and [EGA, IV_2, Proposition 6.1.5]. □

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