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Locating resonances on hyperbolic cones

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In this note we explicitly compute the resonances on hyperbolic cones. These are manifolds diffeomorphic to $\mathbb{R}_+ \times Y$ and equipped with the singular Riemannian metric $dr^2 + \sinh^2 r\ h$, where Y is a compact manifold without boundary and h is a Riemannian metric on Y. The calculation is based on separation of variables and Kummer's connection formulae for hypergeometric functions. To our knowledge this is the one of the few explicit resonance calculations that does not rely on the resolvent being a two-point function.

1. Introduction

In this note we explicitly calculate all of the resonances for a hyperbolic cone in terms of the eigenvalues of the cross-section. To fix notation, let X be a manifold of dimension $n+1$ diffeomorphic to $(\mathbb{R}_+)_r \times Y$, where Y is a compact n-manifold without boundary. Given a Riemannian metric h on Y, we equip X with the hyperbolic conic metric $dr^2 + \sinh^2 r\, h$. Except in the special case of hyperbolic space, X has an isolated conic singularity at $r = 0$.

Given a hyperbolic cone X and its associated metric g, we define the resolvent

$$R(\lambda) = \left(-\Delta_g - \lambda^2 - \frac{n^2}{4} \right)^{-1},$$

which is a bounded operator $L^2(X, g) \to L^2(X, g)$ for $\operatorname{Im} \lambda > 0$ (with the possible exception of finitely many poles). The resolvent $R(\lambda)$ admits a meromorphic continuation from $\{\operatorname{Im} \lambda > 0\}$ to the complex plane as an operator $L^2_c(X, g) \to L^2_{\text{loc}}(X, g)$, i.e., from compactly supported functions to locally L^2 functions. The poles of this meromorphic continuation (aside from potentially finitely many eigenvalues lying in the upper half plane) are called *resonances.*

This note establishes the following theorem:

Theorem 1. *Let $\{\mu_j^2\}_{j\in\mathbb{N}}$ be the eigenvalues of $-\Delta_h$. The resonances of $-\Delta_g$ are given by*

$$\lambda_{j,k} = -\imath\left(\frac{1}{2} + k + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}\right)$$

for $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, and j so that

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{Z}.$$

Here an eigenvalue μ_j^2 with multiplicity m for $-\Delta_h$ adds multiplicity m to $\lambda_{j,k}$.

Note that if

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} \in \frac{1}{2} + \mathbb{Z},$$

then μ_j contributes no resonances to $-\Delta_g$.

In fact, in the proof of Theorem 1 we find *all* poles of the meromorphic continuation of the resolvent. A simple calculation shows that $\lambda_{j,k}^2 + \frac{n^2}{4} \leq 0$ for all j and k, so that none of the poles corresponds to an eigenvalue. Although the hyperbolic cones we consider do not fit into the framework of Patterson–Sullivan theory [Pat76, Sul79], the lack of any eigenvalues is in line with their characterization of the bottom of the spectrum. Indeed, for convex co-compact quotients of hyperbolic space, the Laplacian has an eigenvalue in $(0, n^2/4)$ only when the dimension of the limit set (and hence the trapped set) is large enough. The hyperbolic cones considered here have no trapping, so the lack of eigenvalues in this range should not be surprising.

By appealing to the standard Weyl law on compact manifolds, Theorem 1 admits the following immediate consequence:

Corollary 2. *Suppose that (Y, h) is generic in the sense that*

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} \notin \frac{1}{2} + \mathbb{Z}$$

for all j. Then the resonances on the hyperbolic cone (X, g) obey the following Weyl law:

$$\#\{\lambda_{j,k} : |\lambda_{j,k}| \leq \lambda\} = \frac{|B_n|}{(2\pi)^n(n+1)} \operatorname{Vol}(Y, h)\lambda^{n+1} + O(\lambda^n).$$

Remark 3. Many examples of generic (Y, h) exist. Although the standard metrics on the sphere and flat torus are non-generic, there are arbitrarily small generic perturbations of them. Indeed, if h_0 is the round metric on the sphere and α is any transcendental number, then $\alpha^2 h_0$ satisfies the genericity condition. In particular, this shows that resonances can exist in principle in any dimension, even or odd.

A necessary ingredient for Theorem 1 is of course the meromorphic continuation of the resolvent in this setting. We provide only a sketch of that argument as it is nearly identical to the proof provided by Guillarmou–Mazzeo [GM12, Section 3.3], which is in turn based on the parametrix of Guillopé–Zworski [GZ95b, GZ95a] and previous analysis of Sjöstrand–Zworski [SZ91]. One defines the parametrix

$$Q(\lambda) = \tilde{\chi}_0 R_0(\lambda)\chi_0 + \tilde{\chi}_\infty R_\infty(\lambda)\chi_\infty,$$

where $R_i(\lambda)$ are model resolvents (R_0 is the resolvent on a compact manifold with a conic singularity containing a large compact region of X and R_∞ is the resolvent on a smooth manifold hyperbolic near infinity) and χ_i and $\tilde{\chi}_i$ are appropriately chosen cutoff functions. Applying $(-\Delta - \lambda^2 - n^2/4)$ yields a remainder of the form $I + \sum[-\Delta_X, \tilde{\chi}_i]R_i(\lambda)\chi_i$. Because the inclusion of the Friedrichs domain into L^2 is compact on the compact piece,[1] we can use the well-known mapping properties of the hyperbolic resolvent to conclude that the remainder is a meromorphic Fredholm operator that is invertible for large $\operatorname{Im}\lambda$. Applying $R(\lambda)$ to both sides and inverting the remainder shows that $R(\lambda)$ has a meromorphic continuation.

One can also compute the poles of the *scattering matrix* in this setting. Indeed, the scattering matrix can in general be written in terms of an extension operator and the resolvent [GZ95b, BP02, GZ03, Gui05, DZ16]. With possibly countably many exceptions, the poles of the scattering matrix are precisely those of the resolvent. These other poles (as in the case of odd-dimensional hyperbolic space, which has no resolvent poles but infinitely

[1]For an explicit characterization of the Friedrichs domain, we direct the reader to Melrose–Wunsch [MW04, Proposition 3.1].

many scattering poles) typically arise as poles of the extension operator and are localized on the boundary of the conformal compactification. Poles of the scattering matrix not arising as poles of the resolvent provide a deep connection between the scattering matrix and the conformal geometry of the boundary at infinity [Gui05].

Although many explicit calculations of resonances exist in the setting of potential scattering on Euclidean space, many fewer such calculations are known in geometric scattering theory. To our knowledge, Theorem 1 is one of the few explicit calculations of resonances where the resolvent is not necessarily a two-point function (i.e., where the resolvent depends on more than the distance between two points). The main such setting in which exact calculations are known is that of quotients of hyperbolic space. For hyperbolic cylinders, the exact resonance structure was worked out by Epstein [Eps] and Guillopé [Gui90]. For hyperbolic surfaces, Borthwick–Philipp [BP14] worked out the resonances for hyperbolic warped products and Datchev–Kang–Kessler [DKK15] studied resonances of surfaces of revolution obtained by removing a disk from a cone and attaching a hyperbolic cusp. Further examples for hyperbolic surfaces can be found in Appendix B of a paper of Patterson–Perry [PP01] and in the book by Borthwick [Bor07].

In other settings, it is sometimes possible to work out the asymptotic distribution of the resonances. For example, Sá Barreto–Zworski [SBZ97] worked out resonances on the Schwarzschild black hole asymptotically lie on a lattice, while Stefanov [Ste06] described the asymptotic distribution of resonances exterior to a disk in Euclidean space.

On perturbations of $\mathbb{R}^3$, solutions of the wave equation (or the wave equation with a potential) have a resonance expansion on compact sets. The resonances of $-\Delta$ (or $-\Delta + V$) provide the rates of decay and modes of oscillation seen in this expansion. On hyperbolic spaces, solutions of the corresponding wave equation also have a resonance expansion (see, for example, the work of Datchev [Dat16] and the references therein). Recent work of the first author and collaborators [BVW15, BVW16] shows that resonances on some asymptotically hyperbolic spaces also provide the decay rates for solutions of the wave equation on asymptotically Minkowski spaces. One interpretation of the difference in decay rates for the wave equation in even- and odd-dimensions is that even-dimensional hyperbolic space has resonances, while odd-dimensional hyperbolic space does not.

The proof of Theorem 1 has two main steps. First, we use the warped product structure to construct an explicit representation of the resolvent on hyperbolic cones (Proposition 4). This formula is found by using a coordinate representation (that is essentially the same as the one used in Patterson's computation of the hyperbolic resolvent [Pat75]) and then applying Kummer's connection formula for hypergeometric functions. We then compute poles of the resolvent by analyzing our resolvent formula; the poles arise as poles of the relevant Gamma functions. To deal with the other values of the parameters, we rely heavily on the formulae for hypergeometric functions coming from [DLMF, OLBC10]. In the final section of the paper, we give some explicit resonance expansions for a handful of relevant model problems.

2. An Explicit Expression For The Resolvent

We consider the resolvent $R(\lambda)$ given by

$$R(\lambda) = \left(-\Delta_g - \lambda^2 - \frac{n^2}{4} \right)^{-1}, \tag{1}$$

where we take the Friedrichs extension of the Laplacian. We adopt the convention that $R(\lambda)$ is a bounded operator on $L^2(X)$ when $\operatorname{Im}\lambda \gg 0$.

Because X is a warped product, we may analyze the resolvent by separating variables. Suppose that $\{\phi_j\}$ is an orthonormal basis of eigenfunctions of Δ_h with eigenvalues $-\mu_j^2$. For a function f on X, we then write

$$f(r,y) = \sum_{j=0}^{\infty} f_j(r)\phi_j(y).$$

As the ϕ_j are orthogonal, the resolvent decomposes into a family of one-dimensional resolvents:

$$u = R(\lambda)f = \sum_{j=0}^{\infty} \left(R_j(\lambda)f_j\right)(r)\phi_j(y).$$

In terms of the coordinate r and the eigenvalue $-\mu_j^2$, $R_j(\lambda)$ has the following expression acting on $L^2(\mathbb{R}_+, \sinh^n r\, dr)$:

$$R_j(\lambda) = \left(-\partial_r^2 - n\coth r\,\partial_r + \frac{\mu_j^2}{\sinh^2 r} - \lambda^2 - \frac{n^2}{4} \right)^{-1}.$$

The main result of this section is the following formula:

Proposition 4. *For $f_j \in C_c^\infty(\mathbb{R}_+ \times Y)$ and $\sigma = (\cosh^2(r/2))^{-1}$, the resolvent is given by*

$$\begin{aligned}(R_j(\lambda)f_j)(\sigma) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(1+s)}\Bigg[&- \int_1^\sigma f_j(\rho)F(a,b,c;\sigma)F(a,b,1+s;1-\rho) \\ &\times \left(\frac{\sigma}{\rho}\right)^{\frac{n}{2}-\imath\lambda}\left(\frac{1-\sigma}{1-\rho}\right)^{-\frac{n-1}{4}+\frac{s}{2}} \rho^{c-2}(1-\rho)^s\,d\rho \\ &+ \int_0^\sigma f_j(\rho)F(a,b,c;\rho)F(a,b,1+s;1-\sigma) \\ &\times \left(\frac{\sigma}{\rho}\right)^{\frac{n}{2}-\imath\lambda}\left(\frac{1-\sigma}{1-\rho}\right)^{-\frac{n-1}{4}+\frac{s}{2}} \rho^{c-2}(1-\rho)^s\,d\rho\Bigg],\end{aligned}$$

where a, b, c, and s are given by

$$\begin{aligned} a &= \frac{1}{2} - \imath\lambda, & b &= \frac{1}{2} - \imath\lambda + s, \\ c &= 1 - 2\imath\lambda, & s &= \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}. \end{aligned}$$

Proof. Given some $g \in C_c^\infty(\mathbb{R}_+)$, we wish to find $u = R_j(\lambda)g$, depending meromorphically on λ so that $u_j \in L^2(\mathbb{R}_+, \sinh^n r\,dr)$ for $\operatorname{Im}\lambda \gg 0$ and

$$\left(-\partial_r^2 - n\coth r\,\partial_r + \frac{\mu_j^2}{\sinh^2 r} - \lambda^2 - \frac{n^2}{4}\right)u = g. \tag{2}$$

We start by reducing to a hypergeometric equation using the change of variable $\sigma = \big(\cosh^2(r/2)\big)^{-1}$. Under this change, $r \to 0$ corresponds to $\sigma \uparrow 1$, while $r \to \infty$ corresponds to $\sigma \downarrow 0$. In terms of σ, equation (2) becomes

$$\left(-\sigma^2(1-\sigma)\partial_\sigma^2 - \left[(1-n) - \frac{3-n}{2}\sigma\right]\sigma\partial_\sigma + \frac{\sigma^2\mu_j^2}{4(1-\sigma)} - \lambda^2 - \frac{n^2}{4}\right)u = g. \tag{3}$$

Both 0 and 1 are regular singular points for this equation, which has indicial roots given by

$$\begin{aligned} \alpha_\pm &= \frac{n}{2} \pm \imath\lambda & &\text{at } \sigma = 0, \\ \beta_\pm &= -\frac{n-1}{4} \pm \frac{1}{2}\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} & &\text{at } \sigma = 1. \end{aligned}$$

The requirement that $u \in L^2(\mathbb{R}_+, \sinh^n r\, dr)$ for $\operatorname{Im}\lambda \gg 0$ corresponds to requiring that

$$u(\sigma) \sim \sigma^{\frac{n}{2} - \imath\lambda}$$

as $\sigma \downarrow 0$ and that

$$u(\sigma) \sim \sigma^{-\frac{n-1}{4} + \frac{1}{2}\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}}$$

as $\sigma \uparrow 1$.

We now factor out the desired indicial behavior from u and define the function v by

$$u = \sigma^\alpha (1-\sigma)^\beta v,$$

where $\alpha = \frac{n}{2} - \imath\lambda$ and $\beta = -\frac{n-1}{4} + \frac{1}{2}\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}$ are the preferred indicial roots above. Making this substitution yields the following equation for v:

$$\begin{aligned} g = &-\sigma^{\alpha+2}(1-\sigma)^{\beta+1} v'' - \sigma^{\alpha+1}(1-\sigma)^\beta \\ &\quad\times \left((1-n+2\alpha) - \sigma \left[2\alpha + 2\beta - \frac{n-3}{2} \right] \right) v' \\ &- \sigma^\alpha (1-\sigma)^\beta \left(\alpha(\alpha-1) + (1-n)\alpha + \lambda^2 + \frac{n^2}{4} \right) v \\ &- \sigma^{\alpha+1}(1-\sigma)^{\beta-1} \left(\beta(\beta-1) + \frac{n+1}{2}\beta - \frac{\mu_j^2}{4} \right) \\ &+ \sigma^{\alpha+1}(1-\sigma)^\beta \left(2\alpha\beta + \alpha(\alpha-1) + \beta(\beta-1) - \frac{n-3}{2}(\alpha+\beta) - \frac{\mu_j^2}{4} \right) v. \end{aligned}$$

The exponents α and β were chosen precisely so that the new equation would have 0 as an indicial root at both 0 and 1 (i.e., so that the middle two terms would vanish). In other words, after dividing by $-\sigma^{\alpha+1}(1-\sigma)^\beta$, v must satisfy

$$\begin{aligned} -\frac{g}{\sigma^{\alpha+1}(1-\sigma)^\beta} = \sigma(1-\sigma)v'' + \left([1-n+2\alpha] - \sigma \left[2\alpha + 2\beta - \frac{n-3}{2} \right] \right) v' \\ - \left(2\alpha\beta + \alpha(\alpha-1) + \beta(\beta-1) - \frac{n-3}{2}(\alpha+\beta) - \frac{\mu_j^2}{4} \right) v. \end{aligned}$$

Plugging in the values of α and β yields the following equation for v:

$$\begin{aligned}
(4) \qquad & -g\sigma^{-\alpha-1}(1-\sigma)^{-\beta} \\
&= \sigma(1-\sigma)v'' - \left(\frac{1}{2} - \imath\lambda\right)\left(\frac{1}{2} - \imath\lambda + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}\right) v \\
&+ \left((1-2\imath\lambda) - \sigma\left[2 - 2\imath\lambda + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}\right]\right) v'.
\end{aligned}$$

Equation (4) is an inhomogeneous hypergeometric equation with parameters a, b, and c given by

$$\begin{aligned}
c &= 1 - 2\imath\lambda, \\
a + b &= 1 - 2\imath\lambda + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}, \\
ab &= \left(\frac{1}{2} - \imath\lambda\right)\left(\frac{1}{2} - \imath\lambda + \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}\right).
\end{aligned}$$

In particular, we have that

$$\begin{aligned}
a &= \frac{1}{2} - \imath\lambda, \qquad & b &= a + s, \\
c &= 2a, \qquad & s &= \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}.
\end{aligned}$$

For generic a, b, and c, the solution u_1 of the homogeneous equation (i.e., equation (4) with $g = 0$) that is regular at $\sigma = 0$ is given by

$$u_1(\sigma) = F(a, b, c; \sigma),$$

while the solution u_2 that is regular at $\sigma = 1$ is given by

$$u_2(\sigma) = F(a, b, a + b + 1 - c; 1 - \sigma),$$

where $F(a, b, c; z)$ denotes the standard hypergeometric function with parameters a, b, and c. In general, these two solutions are linearly independent and one can compute their Wronskian using standard facts about hypergeometric functions and Kummer's connection formula [DLMF, 15.10.17]:

$$W[u_1, u_2](\sigma) = \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}(1-c)\sigma^{-c}(1-\sigma)^{c-a-b-1}.$$

This yields the following formula for the solution of equation (4) that is regular at both 0 and 1:

$$\begin{aligned} v(\sigma) = {}& \frac{\Gamma(\frac{1}{2}-\imath\lambda)\Gamma(\frac{1}{2}-\imath\lambda+s)}{\Gamma(1-2\imath\lambda)\Gamma(1+s)} \\ & \times \Bigg[-u_1(\sigma)\int_1^\sigma g(\rho)u_2(\rho)\rho^{c-\alpha-2}(1-\rho)^{1+a+b-c-\beta-1}\,d\rho \\ & \quad + u_2(\sigma)\int_0^\sigma g(\rho)u_1(\rho)\rho^{c-\alpha-2}(1-\rho)^{1+a+b-c-\beta-1}\,d\rho \Bigg]. \end{aligned}$$

Here the exponents are given by

$$\begin{aligned} c-\alpha-2 &= -1-\frac{n}{2}-\imath\lambda, \\ 1+a+b-c-\beta-1 &= \frac{n-1}{4}+\frac{1}{2}\sqrt{\left(\frac{n-1}{2}\right)^2+\mu_j^2}. \end{aligned}$$

Multiplying v by $\sigma^\alpha(1-\sigma)^\beta$ finishes the proof. □

2.1. Connection to Past Resolvent Computations

The resolvent formula in Proposition 4 allows us to quickly recover a formula for the resolvent on hyperbolic space. The resolvent on $\mathbb{H}^{n+1}$ is a two-point function and so its integral kernel $K_\lambda(z, z')$ depends only on the distance between z and z'. It thus suffices to consider the case where the pole is at the origin (and so only the first integral is relevant). As the delta function is spherically symmetric, only the zero mode on the cross-section ($\mathbb{S}^n$) contributes to the formula and so we should multiply our formula by $1/(2\pi)$ to account for the eigenfunction. Accounting for the natural scaling for the delta function on hyperbolic space then yields an expression matching those in the literature (see, e.g., the work of Guillarmou–Mazzeo [GM12, Section 3.1]).

3. Locating the resonances

We handle resonances on a case by case basis using the structure of the hypergeometric and Gamma functions in the expression for the resolvent. Since the Gamma functions will play a dominant role in our discussion, we note that Gamma functions have simple poles at the non-positive integers, which we denote here by $\mathbb{Z}_-$. This motivates the following case by case analysis. The main observation (noted in standard special functions texts [DLMF, 15.2.2]) is that the function

$$\frac{1}{\Gamma(c)} F(a,b,c;z)$$

is entire in a, b, and c.

3.1. $c \notin \mathbb{Z}_-$

When $c \notin \mathbb{Z}_-$, the hypergeometric functions occurring in the resolvent formula in Proposition 4 are regular. Hence, the only possible poles occur due to the Γ prefactors. Since it is impossible for a to be a negative integer since $c = 2a$, we have two remaining scenarios.

1) $c \notin \mathbb{Z}_-$, $b \notin \mathbb{Z}_-$, $a \notin \mathbb{Z}_-$:
   No resolvent poles: In this case, the Γ function pre-factor for the resolvent has no poles, and since none arise from the hypergeometric functions (see Formula (15.2.1) from [DLMF]), there are no poles.

2) $c \notin \mathbb{Z}_-$, $b \in \mathbb{Z}_-$, $a \notin \mathbb{Z}_-$:
   Resolvent poles: In this case b is a pole of the Gamma function. We see that this pole is non-removable as we can easily see that the hypergeometric functions in the resolvent formula are non-zero.

3.2. $c \in \mathbb{Z}_-$

When $c \in \mathbb{Z}_-$, we must work a bit harder and examine the interplay between the parameters a, b, and c. Because the function $\frac{1}{\Gamma(c)} F(a,b,c;z)$ is entire in a, b, and c, the poles must arise as poles of $\Gamma(a)\Gamma(b)$. This leads naturally to four scenarios.

1) $c \in \mathbb{Z}_-$, $b \notin \mathbb{Z}_-$, $a \notin \mathbb{Z}_-$:
   No resolvent poles: The numerator $\Gamma(a)\Gamma(b)$ does not have poles here, so there is no resonance in this case.

2) $c \in \mathbb{Z}_-$, $b \notin \mathbb{Z}_-$, $a \in \mathbb{Z}_-$:
No resolvent poles: The apparent pole arising from $\Gamma(a)$ is in fact removable. Indeed, the power series expansion [DLMF, 15.2.1]

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k$$

and the fact that $c = 2a \leq a$ show that the resolvent is regular here.

3) $c \in \mathbb{Z}_-$, $b \in \mathbb{Z}_-$, $a \notin \mathbb{Z}_-$:
No resolvent poles: As in the previous case, the resolvent has a removable singularity; the same power series expansion (and the fact that $c \leq b$) shows that the pole is removable.

4) $c \in \mathbb{Z}_-$, $b \in \mathbb{Z}_-$, $a \in \mathbb{Z}_-$:
Resolvent poles: In this case, we again see that

$$(\Gamma(a)/\Gamma(c))F(a,b,c;z)$$

has a removable singularity at a. However, as can be seen once again from Formula (15.2.1) of [DLMF], the pole of $\Gamma(b)$ is a pole of the resolvent. That it is a true pole (and not a removable singularity) follows from the observation that the power series of

$$\frac{\Gamma(a)}{\Gamma(c)}F(a,b,c;z)$$

is non-zero.

Having understood all cases, we can now finish the proof of the main theorem.

Proof of Theorem 1. We start by fixing an eigenvalue μ_j^2 on the cross-section and setting

$$s = \sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2}.$$

The only possible resonances for this eigenvalue occur when

$$b = \frac{1}{2} - \imath\lambda + s = -k \in \mathbb{Z}_-,$$

i.e., for

$$\lambda = -\imath\left(\frac{1}{2} + k + s\right).$$

If $s \in \frac{1}{2} + \mathbb{N}$ and λ as above, then we must have

$$\imath\lambda = s + \frac{1}{2} - b \in \mathbb{N},$$

and so $c \in \mathbb{Z}_-$ but $a \notin \mathbb{Z}_-$. This is a case above in which the resolvent has a removable pole and so there is no resonance arising from this μ_j^2.

Suppose now that $s \notin \frac{1}{2} + \mathbb{N}$ and λ is as above. Note that we must then have

$$\imath\lambda = s + \frac{1}{2} - b \notin \mathbb{N},$$

and so $c \in \mathbb{Z}_-$ if and only if $a \in \mathbb{Z}_-$. In both of these cases we end up with a resonance. □

4. Examples

4.1. Resonances for $\mathbb{H}^{n+1}$

In this section we recover the calculation of the location of resonances on hyperbolic space. In this case we have that the cross-section Y is $\mathbb{S}^n$ with its standard round metric. The associated eigenvalues are given by

$$\mu_j^2 = j(j+n-1),$$

with multiplicities

$$m_j = \binom{n+j-2}{n} \frac{2j+n-1}{j},$$

where $\binom{n+j-2}{n}$ is the binomial coefficient.

We then have that

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \mu_j^2} = j + \frac{n-1}{2}.$$

If n is even, then $\frac{n-1}{2}$ is a half integer and the conclusion of Theorem 1 tells us that odd-dimensional hyperbolic spaces have no resonances. On the other hand, if n is odd, then $\frac{n-1}{2}$ is an integer so that even-dimensional hyperbolic spaces have resonances precisely at

$$\lambda_j = -\imath \left(\frac{1}{2} + \frac{n-1}{2} + j \right).$$

This recovers the well-known calculation of the resonances of hyperbolic space.

4.2. Resonances in the presence of a conic singularity

Let us now take $n = 1$ so that X is a surface with (potentially) an isolated conic singularity. This means that

$$\begin{aligned} a &= \frac{1}{2} - i\lambda, \\ b &= \frac{1}{2} - i\lambda + \mu_j. \end{aligned}$$

Let $HC(\mathbb{S}^1_\rho)$ denote the hyperbolic cone of radius $\rho > 0$, defined as the product manifold $HC(\mathbb{S}^1_\rho) = \mathbb{R}_+ \times \big(\mathbb{R}/2\pi\rho\mathbb{Z}\big)$, equipped with the metric $\mathrm{g}(r,\theta) = \mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\theta^2$. This is an incomplete manifold which is locally isometric to $\mathbb{H}^2$ away from the conic singularity and hence hyperbolic there. (In the case of the Euclidean cone ($\mathrm{g}(r,\theta) = \mathrm{d}r^2 + r^2 \, \mathrm{d}\theta^2$), see, for example, works of the second author and collaborators [BFHM12, BFM11].) Recall from above, that our methods for computing resonances suggest that we should see resonances at

$$\lambda = -i \left(\mu_j + \frac{1}{2} \right) - ik, \quad k \in \mathbb{N}.$$

In this case, the spectrum of $-\Delta_\rho$ is easily described and we have

$$\mu_j^2 \in \sigma(-\Delta_\rho) = \left\{ 0, \frac{1}{\rho^2}, \frac{4}{\rho^2}, \frac{9}{\rho^2}, \dots \right\},$$

the spectrum of the Laplacian on a circle with circumference $2\pi\rho$. Thus, we observe that in such a case we have resonances for

$$\lambda = -i \left(\frac{1}{2} + \frac{j}{\rho} - k \right)$$

for $j \in \mathbb{N}$ and $k = 1, 2, 3, \dots$. In particular, we observe that for cone angles much larger than 2π ($\rho \gg 1$), there are resonances much closer to $\frac{1}{2}$ than in the setting without a conic singularity, whereas for small cone angles $\rho < 1$, the resonances introduced here occur at much larger values.

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