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Quantitative weighted estimates for the Littlewood-Paley square function and Marcinkiewicz multipliers

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Quantitative weighted estimates are obtained for the Littlewood-Paley square function S associated with a lacunary decomposition of $\mathbb{R}$ and for the Marcinkiewicz multiplier operator. In particular, we find the sharp dependence on $[w]_{A_p}$ for the $L^p(w)$ operator norm of S for $1 < p \leq 2$.

1. Introduction

Given a weight w (i.e., a non-negative locally integrable function on $\mathbb{R}^n$), we say that $w \in A_p, 1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $\langle \cdot \rangle_Q$ is the integral mean over Q.

In the recent decade, it has been of great interest to obtain the $L^p(w)$ operator norm estimates (possibly optimal) in terms of $[w]_{A_p}$ for the different operators in harmonic analysis. In particular, it was established that the $L^p(w)$ operator norms of Calderón-Zygmund and a large class of Littlewood-Paley operators are bounded by a multiple of

$$[w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right)} \quad \text{and} \quad [w]_{A_p}^{\max\left(\frac{1}{2}, \frac{1}{p-1}\right)},$$

respectively, and these bounds are sharp for all $1 < p < \infty$ (see [6, 12, 17, 21]).

On the other hand, there are still a number of operators for which the sharp bounds in terms of $[w]_{A_p}$ are not known yet. For example, for rough

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homogeneous singular integrals T_Ω with angular part $\Omega \in L^\infty$ the currently best known result says that $\|T_\Omega\|_{L^2(w)\to L^2(w)}$ is at most a multiple of $[w]_{A_2}^2$, and it is an open question whether this bound is sharp (see [5, 15, 18]). Several other examples are the main objects of the present paper.

We consider the classical Littlewood-Paley square function associated with a lacunary decomposition of $\mathbb{R}$ and the Marcinkiewicz multiplier operator. Recall the definitions of these objects. For $k \in \mathbb{Z}$ set $\Delta_k = (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1})$. The Littlewood-Paley square function we shall deal with is defined by

$$Sf = \left(\sum_{k\in\mathbb{Z}} |S_{\Delta_k} f|^2\right)^{1/2},$$

where $\widehat{S_{\Delta_k} f} = \widehat{f}\chi_{\Delta_k}$. We say that T_m is the Marcinkiewicz multiplier operator if $\widehat{T_m f} = m\widehat{f}$, where $m \in L^\infty$ and

$$\sup_{k\in\mathbb{Z}} \int_{\Delta_k} |m'(t)|dt < \infty.$$

The fact that S and T_m are bounded on $L^p(w)$ for $w \in A_p$ is well known and due to D. Kurtz [16]. Tracking the dependence on $[w]_{A_p}$ in the known proofs yields, for example, that the $L^2(w)$ operator norms of S and T_m are bounded by a multiple of $[w]_{A_2}^2$ and $[w]_{A_2}^4$, respectively.

In this paper we give new proofs of the $L^p(w)$ boundedness of S and T_m providing better quantitative estimates it terms of $[w]_{A_p}$. Our main results are the following.

Theorem 1.1. *If α_p is the best possible exponent in*

$$\|S\|_{L^p(w)\to L^p(w)} \le C_p [w]_{A_p}^{\alpha_p},$$

then

$$\max\left(1, \frac{3}{2}\frac{1}{p-1}\right) \le \alpha_p \le \frac{1}{2}\frac{1}{p-1} + \max\left(1, \frac{1}{p-1}\right) \quad (1 < p < \infty);$$

in particular, $\alpha_p = \frac{3}{2}\frac{1}{p-1}$ for $1 < p \le 2$.

Theorem 1.2. *If β_p is the best possible exponent in*

$$\|T_m\|_{L^p(w)\to L^p(w)} \le C_{p,m}[w]_{A_p}^{\beta_p},$$

then

$$\frac{3}{2}\max\left(1, \frac{1}{p-1}\right) \le \beta_p \le \frac{p'}{2} + \max\left(1, \frac{1}{p-1}\right) \quad (1 < p < \infty).$$

Observe that the lower bounds for α_p and β_p are immediate consequences of several known results. By a general extrapolation argument due to T. Luque, C. Pérez and E. Rela [20], if an operator T is such that its unweighted L^p norms satisfy $\|T\|_{L^p\to L^p} \simeq \frac{1}{(p-1)^{\gamma_1}}$ as $p\to 1$ and $\|T\|_{L^p\to L^p} \simeq p^{\gamma_2}$ as $p\to\infty$, then the best possible exponent ξ_p in $\|T\|_{L^p(w)\to L^p(w)} \le C[w]_{A_p}^{\xi_p}$ satisfies $\xi_p \ge \max(\gamma_2, \frac{\gamma_1}{p-1})$. Therefore, the lower bounds for α_p and β_p follow from the sharp unweighted behavior of the L^p norms of S and T_m.

Such a behavior for S was found by J. Bourgain [3]:

$$\|S\|_{L^p\to L^p} \simeq \frac{1}{(p-1)^{3/2}} \text{ as } p\to 1 \quad \text{and} \quad \|S\|_{L^p\to L^p} \simeq p \text{ as } p\to\infty, \tag{1.1}$$

which implies the lower bound for α_p. These asymptotic relations were obtained in [3] for the circle version of the Littlewood-Paley square function but the arguments can be transferred to the real line version in a straightforward way. An alternative proof of the first asymptotic relation in (1.1) has been recently found by O. Bakas [1].

The sharp unweighted L^p norm behavior of T_m is due to T. Tao and J. Wright [22]:

$$\|T_m\|_{L^p\to L^p} \simeq \max(p, p')^{3/2} \quad (1 < p < \infty),$$

which implies the lower bound for β_p.

Bourgain's proof [3] of the first relation in (1.1) was based on a dual restatement in terms of the vector-valued operator $\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k$ with its subsequent handling by means of the Chang-Wilson-Wolff inequality [4]. Our proof of the upper bound for α_p follows similar ideas but with some modifications. As the key tool we use Theorem 2.7, which is a discrete analogue of the sharp weighted continuous square function estimate proved by

M. Wilson [23]. Notice that the latter estimate is also based on the Chang-Wilson-Wolff inequality. We mention that the sharp $L^2(w)$ bound in Theorem 1.1,

$$\|S\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2}^{3/2},$$

by extrapolation yields yet another proof of the unweighted upper bound $\|S\|_{L^p\to L^p} \le \frac{C}{(p-1)^{3/2}}, 1 < p \le 2$ (see Remark 4.2 below).

Another important ingredient used both in the proofs of Theorems 1.1 and 1.2 is Lemma 3.2. This lemma establishes a two-weighted estimate for the multiplier operator $T_{m\chi_{[a,b]}}$. The need to consider two-weighted estimates comes naturally from the method of the proof of Theorem 1.2.

The paper is organized as follows. Section 2 contains some preliminaries and, in particular, the proof of Theorem 2.7. In Section 3 we prove two main technical lemmas. The proof of Theorems 1.1 and 1.2 is contained in Section 4. In Section 5 we make several conjectures related to the sharp upper bounds for α_p and β_p.

2. Preliminaries

Although the main objects we deal with are defined on $\mathbb{R}$, the results of subsections 2.1, 2.2 and 2.3 are valid on $\mathbb{R}^n$.

2.1. Dyadic lattices

The material of this subsection is taken from [19].

Given a cube $Q_0 \subset \mathbb{R}^n$, let $\mathcal{D}(Q_0)$ denote the set of all dyadic cubes with respect to Q_0, that is, the cubes obtained by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes.

Definition 2.1. A dyadic lattice $\mathscr{D}$ in $\mathbb{R}^n$ is any collection of cubes such that

(i) if $Q \in \mathscr{D}$, then each child of Q is in $\mathscr{D}$ as well;

(ii) every 2 cubes $Q', Q'' \in \mathscr{D}$ have a common ancestor, i.e., there exists $Q \in \mathscr{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$;

(iii) for every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathscr{D}$ containing K.

In order to construct a dyadic lattice $\mathscr{D}$, it suffices to fix an arbitrary cube Q_0 and to expand it dyadically (carefully enough in order to cover the whole space) by choosing one of 2^n possible parents for the top cube and including it into $\mathscr{D}$ together with all its dyadic subcubes during each step. Therefore, given $h > 0$, one can choose a dyadic lattice $\mathscr{D}$ such that for any $Q \in \mathscr{D}$ its sidelength ℓ_Q will be of the form $2^k h, k \in \mathbb{Z}$.

Theorem 2.2. (The Three Lattice Theorem) *For every dyadic lattice $\mathscr{D}$, there exist 3^n dyadic lattices $\mathscr{D}^{(1)}, \dots, \mathscr{D}^{(3^n)}$ such that*

$$\{3Q : Q \in \mathscr{D}\} = \bigcup_{j=1}^{3^n} \mathscr{D}^{(j)}$$

and for every cube $Q \in \mathscr{D}$ and $j = 1, \dots, 3^n$, there exists a unique cube $R \in \mathscr{D}^{(j)}$ of sidelength $\ell_R = 3\ell_Q$ containing Q.

2.2. Some Littlewood-Paley theory

Denote by $\mathscr{S}(\mathbb{R}^n)$ the class of Schwartz functions on $\mathbb{R}^n$. The following statement can be found in [11, Lemma 5.12] (see also [10, p. 783] for some details).

Lemma 2.3. *There exist $\varphi, \theta \in \mathscr{S}(\mathbb{R}^n)$ satisfying the following properties:*

(i) $\operatorname{supp}\theta \subset \{x : |x| \le 1\}$ *and* $\int \theta = 0$;

(ii) $\operatorname{supp}\widehat{\varphi} \subset \{\xi : 1/2 \le |\xi| \le 2\}$;

(iii) $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(2^{-k}\xi)\widehat{\theta}(2^{-k}\xi) \equiv 1$ *for all* $\xi \neq 0$.

Property (iii) implies, by taking the Fourier transform, the discrete version of the Calderón reproducing formula:

$$f = \sum_{k\in\mathbb{Z}} f * \varphi_{2^{-k}} * \theta_{2^{-k}}. \tag{2.1}$$

Remark 2.4. There are several interpretations of convergence in (2.1). In particular, we will use the following one. Let $1 < p < \infty$ and suppose $w \in A_p$. Given $f \in L^p(w)$ and $N \in \mathbb{N}$, set

$$f_N(x) = \sum_{k=-N}^{N} \int_{E_N} (f * \varphi_{2^{-k}})(y)\theta_{2^{-k}}(x - y)dy,$$

where $\{E_N\}$ is an increasing sequence of bounded measurable sets such that $E_N \to \mathbb{R}^n$. Then $f_N \to f$ in $L^p(w)$ as $N \to \infty$. For the continuous version of (2.1) this fact was proved by M. Wilson [24, Th. 7.1] (see also [25]), and in the discrete case the proof follows the same lines.

The following result is also due to M. Wilson (see [23, Lemma 2.3] and [24, Th. 4.3]).

Theorem 2.5. *Let $\mathscr{D}$ be a dyadic lattice and let $\mathscr{G} \subset \mathscr{D}$ be a finite family of cubes. Assume that $f = \sum_{Q\in\mathscr{G}} \lambda_Q a_Q$, where* $\operatorname{supp} a_Q \subset Q$, $\|a_Q\|_{L^\infty} \le |Q|^{-1/2}$, $\|\nabla a_Q\|_{L^\infty} \le \ell_Q^{-1}|Q|^{-1/2}$ *and* $\int a_Q = 0$. *Then for all* $1 < p < \infty$ *and for every* $w \in A_p$,

$$\|f\|_{L^p(w)} \le C_{p,n}[w]_{A_p}^{1/2} \left\| \left(\sum_{Q\in\mathscr{G}} \frac{|\lambda_Q|^2}{|Q|}\chi_Q \right)^{1/2} \right\|_{L^p(w)}. \tag{2.2}$$

Remark 2.6. Notice that actually (2.2) was proved in [23] with a smaller $[w]_{A_\infty}$ constant defined by

$$[w]_{A_\infty} = \sup_Q \frac{1}{\int_Q w} \int_Q M(w\chi_Q),$$

where $Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f|$ is the Hardy-Littlewood maximal operator. See also [14] for various estimates in terms of $[w]_{A_\infty}$.

Theorem 2.5 along with the continuous version of (2.1) was applied in [23] in order to obtain the $L^p(w)$-norm relation between f and the continuous square function. In a similar way, using (2.1), we obtain the $L^p(w)$-norm relation between f and the discrete square function defined (for a given dyadic lattice $\mathscr{D}$) by

$$S_{\varphi,\mathscr{D}}(f)(x) = \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathscr{D}:\ell_Q=2^{-k}} \left(\frac{1}{|Q|}\int_Q |f * \varphi_{2^{-k}}|^2 \right) \chi_Q(x) \right)^{1/2}.$$

Theorem 2.7. *There exists a function* $\varphi \in \mathscr{S}(\mathbb{R}^n)$ *with* $\operatorname{supp}\widehat{\varphi} \subset \{\xi : 1/2 \le |\xi| \le 2\}$ *and there are* 3^n *dyadic lattices* $\mathscr{D}^{(j)}$ *such that for every* $w \in A_p$ *and for any* $f \in L^p(w), 1 < p < \infty$,

$$\|f\|_{L^p(w)} \le C_{p,n}[w]_{A_p}^{1/2} \sum_{j=1}^{3^n} \|S_{\varphi,\mathscr{D}^{(j)}}(f)\|_{L^p(w)}.$$

Proof. Let φ, θ be functions from Lemma 2.3. Let $\mathscr{D}$ be a dyadic lattice such that for every $Q \in \mathscr{D}$ its sidelength is of the form $\ell_Q = \frac{2^k}{3}, k \in \mathbb{Z}$. Let $\mathscr{D}^{(j)}, j = 1, \dots, 3^n$, be dyadic lattices obtained by applying Theorem 2.2 to $\mathscr{D}$. Then for every $Q \in \mathscr{D}^{(j)}$ its sidelength is of the form $\ell_Q = 2^k, k \in \mathbb{Z}$.

For $Q \in \mathscr{D}$ with $\ell_Q = 2^{-k}/3$ set

$$\gamma_Q(x) = \int_Q (f * \varphi_{2^{-k}})(y)\theta_{2^{-k}}(x-y)dy.$$

It is easy to check that $\operatorname{supp}\gamma_Q \subset 3Q$, $\int \gamma_Q = 0$ and

$$\max(\|\gamma_Q\|_{L^\infty}, \ell_Q\|\nabla\gamma_Q\|_{L^\infty}) \le c\left(\frac{1}{|Q|}\int_Q |f * \varphi_{2^{-k}}|^2\right)^{1/2}, \tag{2.3}$$

where c depends only on n and θ.

Take an increasing sequence of cubes $Q_N \in \mathscr{D}$ such that $\ell_{Q_N} = \frac{2^N}{3}, N \in \mathbb{N}$. Set

$$\mathscr{G}_N = \{Q \in \mathscr{D} : Q \subseteq Q_N, \ell_Q = 2^{-k}/3, k \in [-N, N]\}.$$

By Theorem 2.2, one can write

$$\{3Q : Q \in \mathscr{G}_N\} = \bigcup_{j=1}^{3^n} \mathscr{G}_N^{(j)},$$

where $\mathscr{G}_N^{(j)} \subset \mathscr{D}^{(j)}$. Then

$$\begin{aligned} f_N(x) &= \sum_{k=-N}^{N} \int_{Q_N} (f * \varphi_{2^{-k}})(y)\theta_{2^{-k}}(x-y)dy \\ &= \sum_{k=-N}^{N} \sum_{Q \in \mathscr{D}: Q \subseteq Q_N, \ell_Q = 2^{-k}/3} \gamma_Q(x) = \sum_{j=1}^{3^n} \sum_{P \in \mathscr{G}_N^{(j)}} \lambda_P^{(j)} a_P^{(j)}, \end{aligned}$$

where, for $P = 3Q, Q \in \mathscr{D}, \ell_Q = 2^{-k}/3$, we set

$$\lambda_P^{(j)} = c\left(\int_{3Q} |f * \varphi_{2^{-k}}|^2\right)^{1/2}$$

and $a_P^{(j)} = \frac{1}{\lambda_P^{(j)}}\gamma_Q$.

By (2.3), we have that the functions $a_P^{(j)}$ satisfy all conditions from Theorem 2.5. Therefore, by (2.2),

$$\|f_N\|_{L^p(w)} \le C_{p,n}[w]_{A_p}^{1/2}\sum_{j=1}^{3^n}\left\|\left(\sum_{P\in\mathscr{G}_N^{(j)}}\frac{|\lambda_P^{(j)}|^2}{|P|}\chi_P\right)^{1/2}\right\|_{L^p(w)}$$
$$\le C_{p,n}[w]_{A_p}^{1/2}\sum_{j=1}^{3^n}\|S_{\varphi,\mathscr{D}^{(j)}}(f)\|_{L^p(w)}.$$

Applying the convergence argument as described in Remark 2.4 completes the proof. □

2.3. The sharp extrapolation

The following result was proved in [9].

Theorem 2.8. *Assume that for some f, g and for all weights $w \in A_{p_0}$,*

$$\|f\|_{L^{p_0}(w)} \le CN([w]_{A_{p_0}})\|g\|_{L^{p_0}(w)},$$

where N is an increasing function and the constant C does not depend on w. Then for all $1 < p < \infty$ and all $w \in A_p$,

$$\|f\|_{L^p(w)} \le CK(w)\|g\|_{L^p(w)},$$

where

$$K(w) = \begin{cases} N\big([w]_{A_p}(2\|M\|_{L^p(w)\to L^p(w)})^{p_0-p}\big), & \text{if } p < p_0;\\ N\left([w]_{A_p}^{\frac{p_0-1}{p-1}}(2\|M\|_{L^{p'}(w^{1-p'})\to L^{p'}(w^{1-p'})})^{\frac{p-p_0}{p-1}}\right), & \text{if } p > p_0.\end{cases}$$

In particular, $K(w) \le C_1 N\left(C_2[w]_{A_p}^{\max\left(1,\frac{p_0-1}{p-1}\right)}\right)$ for $w \in A_p$.

2.4. Some two-weighted estimates

Let

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(y)}{x-y}dy \quad \text{and} \quad H^\star f(x) = \sup_{\varepsilon>0}\frac{1}{\pi}\left|\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy\right|$$

be the Hilbert and the maximal Hilbert transforms, respectively.

Given two weights u and v, set

$$[u,v]_{A_2} = \sup_Q \langle u\rangle_Q \langle v^{-1}\rangle_Q.$$

Then the following two-weighted estimates hold:

$$\max(\|M\|_{L^2(v)\to L^2(u)}, \|H^\star\|_{L^2(v)\to L^2(u)}) \le C[u,v]_{A_2}^{1/2}\big([u]_{A_2}^{1/2} + [v]_{A_2}^{1/2}\big). \tag{2.4}$$

The proofs of these estimates can be found in [13, 14] (notice that stronger versions of (2.4) in terms of the $[w]_{A_\infty}$ constants are proved there).

2.5. The partial sum operator

Given an interval $[a,b]$, the partial sum operator $S_{[a,b]}$ is defined by $\widehat{S_{[a,b]}f} = \widehat{f}\chi_{[a,b]}$. We will use two standard facts about $S_{[a,b]}$ (see, e.g., [8]). First,

$$S_{[a,b]} = \frac{i}{2}(\mathcal{M}_a H \mathcal{M}_{-a} - \mathcal{M}_b H \mathcal{M}_{-b}), \tag{2.5}$$

where $\mathcal{M}_a f(x) = e^{2\pi i a x} f(x)$. Second, if $(T_{m\chi_{[a,b]}} f)\widehat{\ } = m\chi_{[a,b]}\widehat{f}$, then

$$T_{m\chi_{[a,b]}} f = m(a) S_{[a,b]} f + \int_a^b (S_{[t,b]} f) m'(t)\,dt. \tag{2.6}$$

3. Two key lemmas

Given a dyadic lattice $\mathscr{D}$ in $\mathbb{R}$, a weight w and $k \in \mathbb{Z}$, denote

$$w_{k,\mathscr{D}} = \sum_{I\in\mathscr{D}:|I|=2^{-k}} \langle w\rangle_I \chi_I.$$

Lemma 3.1. *Let $w \in A_2$. Then $w_{k,\mathscr{D}} \in A_2$ and*

$$[w_{k,\mathscr{D}}]_{A_2} \le 9[w]_{A_2}. \tag{3.1}$$

Also, for two arbitrary dyadic lattices $\mathscr{D}$ and $\mathscr{D}'$,

$$[w_{k,\mathscr{D}}, ((w^{-1})_{k,\mathscr{D}'})^{-1}]_{A_2} \le 9[w]_{A_2}. \tag{3.2}$$

Proof. Denote $u = w_{k,\mathscr{D}}$ and $\mathcal{P}_k = \{I \in \mathscr{D} : |I| = 2^{-k}\}$. Take an arbitrary interval $J \subset \mathbb{R}$. Notice that

$$\langle u \rangle_J = \frac{1}{|J|} \sum_{I \in \mathcal{P}_k : I \cap J \neq \emptyset} \frac{|I \cap J|}{|I|} \int_I w. \tag{3.3}$$

Next, by Hölder's inequality,

$$|I|^2 \leq \left(\int_I w\right)\left(\int_I w^{-1}\right),$$

which implies

$$\begin{aligned} \langle u^{-1} \rangle_J &= \frac{1}{|J|} \sum_{I \in \mathcal{P}_k : I \cap J \neq \emptyset} |I \cap J| \frac{|I|}{\int_I w} \\ &\leq \frac{1}{|J|} \sum_{I \in \mathcal{P}_k : I \cap J \neq \emptyset} \frac{|I \cap J|}{|I|} \int_I w^{-1}. \end{aligned} \tag{3.4}$$

Denote

$$J^* = \bigcup_{I \in \mathcal{P}_k : I \cap J \neq \emptyset} I.$$

If $|J| > 2^{-k}$, then $|J^*| \leq 3|J|$, and hence, by (3.3) and (3.4),

$$\langle u \rangle_J \leq \frac{1}{|J|} \int_{J^*} w \leq 3 \langle w \rangle_{J^*} \quad \text{and} \quad \langle u^{-1} \rangle_J \leq 3 \langle w^{-1} \rangle_{J^*}. \tag{3.5}$$

Assume that $|J| \leq 2^{-k}$. Then $|J^*| \leq 2^{-k+1}$. Hence in this case,

$$\langle u \rangle_J \leq \frac{1}{|I|} \int_{J^*} w \leq 2 \langle w \rangle_{J^*} \quad \text{and} \quad \langle u^{-1} \rangle_J \leq 2 \langle w^{-1} \rangle_{J^*},$$

which along with (3.5) implies (3.1).

The proof of (3.2) is identically the same. Denote $v = ((w^{-1})_{k,\mathscr{D}'})^{-1}$. If $|J| > 2^{-k}$, then by (3.5),

$$\langle u \rangle_J \leq 3 \langle w \rangle_{J^*} \quad \text{and} \quad \langle v^{-1} \rangle_J \leq 3 \langle w^{-1} \rangle_{J^*}.$$

Similarly, if $|J| \leq 2^{-k}$, then

$$\langle u \rangle_J \leq 2 \langle w \rangle_{J^*} \quad \text{and} \quad \langle v^{-1} \rangle_J \leq 2 \langle w^{-1} \rangle_{J^*},$$

which along with the previous estimate proves (3.2). □

Define the operator $T_{m\chi_{[a,b]}}$ by $(T_{m\chi_{[a,b]}}f)\widehat{\ }= m\chi_{[a,b]}\widehat{f}$. In the lemma below we use the same notation $u_{k,\mathscr{D}}$ as in Lemma 3.1.

Lemma 3.2. *Assume that m is a bounded and differentiable function on $[a,b]$. Then for all $u, v \in A_2$,*

$$\|T_{m\chi_{[a,b]}}f\|_{L^2(u_{k,\mathscr{D}})} \le cK(m)N(u,v)(2^{-k}(b-a)+1)\|f\|_{L^2(v)},$$

where $K(m) = \|m\|_{L^\infty} + \int_a^b |m'(t)|dt$,

$$N(u,v) = \min\big([u,v]_{A_2}, [u_{k,\mathscr{D}}, v]_{A_2}\big)^{1/2}\big([u]_{A_2}^{1/2} + [v]_{A_2}^{1/2}\big)$$

and $c > 0$ is an absolute constant.

Proof. Let $t \in [a,b)$. Take an arbitrary $I \in \mathscr{D}$ with $|I| = 2^{-k}$. Notice that

$$\|S_{[t,b]}f\|_{L^\infty} \le (b-a)\|f\|_{L^1}.$$

Therefore, for all $x, y \in I$,

$$\begin{aligned} |S_{[t,b]}f(y)| &\le (b-a)\int_{3I}|f| + |S_{[t,b]}(f\chi_{\mathbb{R}\setminus 3I})(y)| \\ &\le 3(b-a)2^{-k}Mf(x) + |S_{[t,b]}(f\chi_{\mathbb{R}\setminus 3I})(y)|. \end{aligned} \tag{3.6}$$

Applying (2.5) yields

$$\begin{aligned} |S_{[t,b]}(f\chi_{\mathbb{R}\setminus 3I})(y)| \le{}& |H\mathcal{M}_{-t}(f\chi_{\mathbb{R}\setminus 3I})(y)| \\ &+ |H\mathcal{M}_{-b}(f\chi_{\mathbb{R}\setminus 3I})(y)|. \end{aligned} \tag{3.7}$$

For every $t \in [a,b]$,

$$\begin{aligned} &|H\mathcal{M}_{-t}(f\chi_{\mathbb{R}\setminus 3I})(y) - H\mathcal{M}_{-t}(f\chi_{\mathbb{R}\setminus 3I})(x)| \\ &\le c|I|\int_{\mathbb{R}\setminus 3I}|f(\xi)|\frac{1}{|x-\xi|^2}d\xi \le cMf(x). \end{aligned} \tag{3.8}$$

Further,

$$\begin{aligned} |H\mathcal{M}_{-t}(f\chi_{\mathbb{R}\setminus 3I})(x)| &\le |H\mathcal{M}_{-t}(f\chi_{\mathbb{R}\setminus[x-|I|/2,x+|I|/2]})(x)| \\ &\quad + |H\mathcal{M}_{-t}(f\chi_{3I\setminus[x-|I|/2,x+|I|/2]})(x)| \\ &\le H^\star\mathcal{M}_{-t}f(x) + cMf(x), \end{aligned}$$

which, combined with (3.6), (3.7) and (3.8), implies

$$|S_{[t,b]}f(y)| \le H^\star\mathcal{M}_{-b}f(x) + H^\star\mathcal{M}_{-t}f(x) + (3(b-a)2^{-k}+c)Mf(x).$$

From this and from (2.6), for all $x, y \in I$ we have

$$|T_{m\chi_{[a,b]}}f(y)| \le cK(m)\mathcal{T}(f)(x) + \int_a^b H^\star\mathcal{M}_{-t}f(x)|m'(t)|dt,$$

where

$$\mathcal{T}(f)(x) = H^\star\mathcal{M}_{-b}f(x) + H^\star\mathcal{M}_{-a}f(x) + (2^{-k}(b-a)+1)Mf(x).$$

Therefore,

$$\frac{1}{|I|}\int_I |T_{m\chi_{[a,b]}}f|^2 \le \inf_I \left(cK(m)\mathcal{T}(f) + \int_a^b H^\star\mathcal{M}_{-t}f|m'(t)|dt\right)^2. \tag{3.9}$$

Hence, applying Minkowski's inequality and using (2.4), we obtain

$$\begin{aligned}\|T_{m\chi_{[a,b]}}f\|_{L^2(u_{k,\mathscr{D}})} &\le \left\|cK(m)\mathcal{T}(f) + \int_a^b H^\star\mathcal{M}_{-t}f|m'(t)|dt\right\|_{L^2(u)}\\ &\le cK(m)\|\mathcal{T}(f)\|_{L^2(u)} + \int_a^b \|H^\star\mathcal{M}_{-t}f\|_{L^2(u)}|m'(t)|dt\\ &\le cK(m)(2^{-k}(b-a)+1)[u,v]_{A_2}^{1/2}([u]_{A_2}^{1/2}+[v]_{A_2}^{1/2})\|f\|_{L^2(v)}.\end{aligned}$$

On the other hand, (3.9) also implies

$$\|T_{m\chi_{[a,b]}}f\|_{L^2(u_{k,\mathscr{D}})} \le \left\|cK(m)\mathcal{T}(f) + \int_a^b H^\star\mathcal{M}_{-t}f|m'(t)|dt\right\|_{L^2(u_{k,\mathscr{D}})}.$$

Therefore, by the previous arguments and Lemma 3.1,

$$\begin{aligned}&\|T_{m\chi_{[a,b]}}f\|_{L^2(u_{k,\mathscr{D}})}\\ &\le cK(m)(2^{-k}(b-a)+1)[u_{k,\mathscr{D}},v]_{A_2}^{1/2}([u]_{A_2}^{1/2}+[v]_{A_2}^{1/2})\|f\|_{L^2(v)},\end{aligned}$$

which completes the proof. □

4. Proof of Theorems 1.1 and 1.2

The lower bounds for α_p and β_p are explained in the Introduction. Therefore, we are left with establishing the upper bounds.

Proof of Theorem 1.1. By duality, the estimate

$$\|Sf\|_{L^p(w)} \leq C[w]_{A_p}^{\frac{1}{2}\frac{1}{p-1}+\max\left(1,\frac{1}{p-1}\right)}\|f\|_{L^p(w)} \tag{4.1}$$

is equivalent to

$$\left\|\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k\right\|_{L^{p'}(\sigma)} \leq C[\sigma]_{A_{p'}}^{\frac{1}{2}+\max(1,p-1)}\left\|\left(\sum_{k\in\mathbb{Z}}|\psi_k|^2\right)^{1/2}\right\|_{L^{p'}(\sigma)},$$

where $\sigma = w^{1-p'}$. Changing here p' by p and σ by w, we see that it suffices to prove that

$$\left\|\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k\right\|_{L^p(w)} \leq C[w]_{A_p}^{\frac{1}{2}+\max\left(1,\frac{1}{p-1}\right)}\left\|\left(\sum_{k\in\mathbb{Z}}|\psi_k|^2\right)^{1/2}\right\|_{L^p(w)}. \tag{4.2}$$

Applying Theorem 2.7 yields

$$\left\|\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k\right\|_{L^p(w)} \leq C[w]_{A_p}^{\frac{1}{2}}\sum_{j=1}^{3}\left\|S_{\varphi,\mathscr{D}^{(j)}}\left(\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k\right)\right\|_{L^p(w)}.$$

Therefore, by Theorem 2.8, (4.2) will follow from

$$\left\|S_{\varphi,\mathscr{D}}\left(\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k\right)\right\|_{L^2(w)} \leq C[w]_{A_2}\left\|\left(\sum_{k\in\mathbb{Z}}|\psi_k|^2\right)^{1/2}\right\|_{L^2(w)}. \tag{4.3}$$

Using that $\operatorname{supp}\widehat{\varphi_{2^{-k}}} \subset \{\xi : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$, we have

$$\left(\sum_{j\in\mathbb{Z}} S_{\Delta_j}\psi_j\right) * \varphi_{2^{-k}} = (S_{\Delta_{k-1}}\psi_{k-1} + S_{\Delta_k}\psi_k) * \varphi_{2^{-k}},$$

which implies

$$S_{\varphi,\mathscr{D}}\left(\sum_{j\in\mathbb{Z}} S_{\Delta_j}\psi_j\right)(x)^2 = \sum_{k\in\mathbb{Z}}\sum_{I\in\mathscr{D}:\ell_I=2^{-k}}\left(\frac{1}{|I|}\int_I |(S_{\Delta_{k-1}}\psi_{k-1} + S_{\Delta_k}\psi_k) * \varphi_{2^{-k}}|^2\right)\chi_I(x).$$

Hence, in order to prove (4.3), it suffices to establish that for every $k \in \mathbb{Z}$,

$$\|(S_{\Delta_{k-1}} f) * \varphi_{2^{-k}}\|_{L^2(w_{k,\mathscr{D}})} \le C[w]_{A_2}\|f\|_{L^2(w)} \tag{4.4}$$

and

$$\|(S_{\Delta_k} f) * \varphi_{2^{-k}}\|_{L^2(w_{k,\mathscr{D}})} \le C[w]_{A_2}\|f\|_{L^2(w)}. \tag{4.5}$$

Since

$$((S_{\Delta_{k-1}} f) * \varphi_{2^{-k}})\widehat{\ }(\xi) = \widehat{\varphi}(2^{-k}\xi)\chi_{\{2^{k-1}\le|\xi|\le 2^k\}}\widehat{f}(\xi),$$

(4.4) is an immediate corollary of Lemma 3.2 (applied in the case of equal weights). Estimate (4.5) follows in the same way. Notice that the constants C in (4.4) and (4.5) can be taken as

$$C = c\left(\|\widehat{\varphi}\|_{L^\infty} + \int_{1/2\le|\xi|\le 2} |(\widehat{\varphi})'(\xi)|d\xi\right)$$

with some absolute $c > 0$. □

Remark 4.1. There is a minor inaccuracy in the proof, namely, applying Theorem 2.7, we have used that $\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k \in L^p(w)$ as an *a priori* assumption. This point can be fixed in several ways. First, by [16], $f \in L^p(w)$ implies $Sf \in L^p(w)$ for $w \in A_p$ for all $1 < p < \infty$. By duality, this means that $\left(\sum_{k\in\mathbb{Z}} |\psi_k|^2\right)^{1/2} \in L^p(w)$ implies $\sum_{k\in\mathbb{Z}} S_{\Delta_k}\psi_k \in L^p(w)$.

However, one can avoid the use of [16] as follows. Defining

$$S_N f = \left(\sum_{k=-N}^{N} |S_{\Delta_k} f|^2\right)^{1/2},$$

we have that (4.1) with $S_N f$ instead of Sf is equivalent to (4.2) with $\sum_{k=-N}^{N} S_{\Delta_k}\psi_k$ on the left-hand side. But the fact that $\sum_{k=-N}^{N} S_{\Delta_k}\psi_k \in L^p(w)$ follows immediately from (2.5). The rest of the proof is exactly the same, and we obtain (4.1) with $S_N f$ instead of Sf with the corresponding constant independent of N. Letting $N \to \infty$ yields the desired bound for S.

Remark 4.2. Theorem 1.1 in the case $p = 2$ says that

$$\|S\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2}^{3/2}.$$

From this, by Theorem 2.8,

$$\|S\|_{L^p\to L^p} \le C\|M\|_{L^p\to L^p}^{3/2} \quad (1 < p \le 2).$$

Since $\|M\|_{L^p\to L^p} \simeq \frac{1}{p-1}$ for $1 < p \le 2$, we obtain the sharp upper bound

$$\|S\|_{L^p\to L^p} \le \frac{C}{(p-1)^{3/2}} \quad (1 < p \le 2)$$

found by J. Bourgain [3].

Proof of Theorem 1.2. Using the fact that

$$\|T_m\|_{L^p(w)\to L^p(w)} = \|T_m\|_{L^{p'}(\sigma)\to L^{p'}(\sigma)}$$

and $[\sigma]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$, it suffices to prove that

$$\|T_m\|_{L^p(w)\to L^p(w)} \le C_{p,m}[w]_{A_p}^{\frac{1}{2}+\frac{3}{2}\frac{1}{p-1}} \quad (1 < p \le 2). \tag{4.6}$$

By Theorems 2.7 and 2.8, (4.6) will follow from

$$\|S_{\varphi,\mathscr{D}}(T_m f)\|_{L^2(w)} \le C_m[w]_{A_2}^{3/2}\|f\|_{L^2(w)}. \tag{4.7}$$

Notice that

$$\|S_{\varphi,\mathscr{D}}(T_m f)\|_{L^2(w)} = \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |(T_m f) * \varphi_{2^{-k}}|^2 w_{k,\mathscr{D}}dx\right)^{1/2}.$$

Therefore, by duality, (4.7) is equivalent to

$$\left\|\sum_{k\in\mathbb{Z}}(T_m\psi_k) * \varphi_{2^{-k}}\right\|_{L^2(w^{-1})} \le C_m[w]_{A_2}^{3/2}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |\psi_k|^2(w_{k,\mathscr{D}})^{-1}dx\right)^{1/2}.$$

Applying Theorem 2.7 again, we see that the question is reduced to the estimate

$$\left(\sum_{k\in\mathbb{Z}}\left\|\left(\sum_{j\in\mathbb{Z}}(T_m\psi_j)*\varphi_{2^{-j}}\right)*\varphi_{2^{-k}}\right\|^2_{L^2((w^{-1})_{k,\mathscr{D}'})}\right)^{1/2} \le C_m[w]_{A_2}\left(\sum_{k\in\mathbb{Z}}\|\psi_k\|^2_{L^2((w_{k,\mathscr{D}})^{-1})}\right)^{1/2} \tag{4.8}$$

for some dyadic lattices $\mathscr{D}$ and $\mathscr{D}'$.

Since

$$\left(\sum_{j\in\mathbb{Z}}(T_m\psi_j)*\varphi_{2^{-j}}\right)*\varphi_{2^{-k}} = \sum_{j=k-1}^{k+1}(T_m\psi_j)*\varphi_{2^{-j}}*\varphi_{2^{-k}},$$

in order to prove (4.8), it suffices to show that for every $k\in\mathbb{Z}$ and every $j=k-1,k,k+1$,

$$\|(T_mf)*\varphi_{2^{-j}}*\varphi_{2^{-k}}\|_{L^2((w^{-1})_{k,\mathscr{D}'})} \le C_m[w]_{A_2}\|f\|_{L^2((w_{k,\mathscr{D}})^{-1})}. \tag{4.9}$$

By Lemma 3.1,

$$[(w^{-1})_{k,\mathscr{D}'},(w_{k,\mathscr{D}})^{-1})]^{1/2}_{A_2}\big([(w^{-1})_{k,\mathscr{D}'}]^{1/2}_{A_2}+[(w_{k,\mathscr{D}})^{-1})]^{1/2}_{A_2}\big)\le c[w]_{A_2}.$$

From this and from Lemma 3.2 we obtain (4.9) with

$$C_m = cC_\varphi\left(\|m\|_{L^\infty}+\sup_{k\in\mathbb{Z}}\int_{\Delta_k}|m'(t)|dt\right),$$

which completes the proof. □

Remark 4.3. As in Remark 4.1, it is not difficult to justify the use of Theorem 2.7. We omit the details.

5. Concluding remarks

5.1. On the sharpness of α_p and β_p

The extrapolation principle explained in the Introduction says that if ξ_p is the best possible exponent in $\|T\|_{L^p(w)\to L^p(w)} \le C[w]_{A_p}^{\xi_p}$, then

$$\xi_p \ge \max\left(\gamma_2, \frac{\gamma_1}{p-1}\right),$$

where γ_1 and γ_2 are the constants appearing in the endpoint asymptotic relations for $\|T\|_{L^p\to L^p}$. In fact, for many particular operators we have that $\xi_p = \max(\gamma_2, \frac{\gamma_1}{p-1})$.

Therefore, it is plausible that the upper bounds for α_p and β_p from Theorems 1.1 and 1.2 are not sharp for $p > 2$ and $1 < p < \infty$, respectively, and it is natural to make the following.

Conjecture 5.1. The best possible exponent α_p in

$$\|S\|_{L^p(w)\to L^p(w)} \le C_p[w]_{A_p}^{\alpha_p}$$

is

$$\alpha_p = \max\left(1, \frac{3}{2}\frac{1}{p-1}\right) \quad (1 < p < \infty).$$

Conjecture 5.2. The best possible exponent β_p in

$$\|T_m\|_{L^p(w)\to L^p(w)} \le C_{p,m}[w]_{A_p}^{\beta_p}$$

is

$$\beta_p = \frac{3}{2}\max\left(1, \frac{1}{p-1}\right) \quad (1 < p < \infty).$$

Observe that by Theorem 2.8, in order to establish Conjectures 5.1 and 5.2, it suffices to show that

$$\|S\|_{L^{5/2}(w)\to L^{5/2}(w)} \le C[w]_{A_{5/2}} \quad \text{and} \quad \|T_m\|_{L^2(w)\to L^2(w)} \le C_m[w]_{A_2}^{3/2},$$

respectively.

5.2. Sparse bounds for S and T_m?

A family of cubes $\mathcal{S}$ is called sparse if there exist $0<\eta<1$ and a family of pairwise disjoint sets $\{E_Q\}_{Q\in\mathcal{S}}$ such that $E_Q\subset Q$ and $|E_Q|\ge\eta|Q|$ for all $Q\in\mathcal{S}$. By a sparse bound for a given operator T we mean an estimate of the form

$$|\langle Tf,g\rangle|\le C\sum_{Q\in\mathcal{S}}\langle f\rangle_{r,Q}\langle g\rangle_{s,Q}|Q|,$$

with suitable $1\le r,s<\infty$, where $\langle f\rangle_{p,Q}=\langle|f|^p\rangle_Q^{1/p}$, and $\mathcal{S}$ is a sparse family.

Sparse bounds have become a powerful tool for obtaining sharp quantitative weighted estimates in recent years (see, e.g., [2, 5, 18]). Therefore it would be natural to try to attack Conjectures 5.1 and 5.2 by means of the corresponding sparse bounds for S and T_m.

At this point, we mention that it is not clear to us what is the sparse bound for S leading to Conjecture 5.1. For example, it is plausible that S satisfies

$$|\langle Sf,g\rangle|\le\frac{C}{(r-1)^{1/2}}\sum_{Q\in\mathcal{S}}\langle f\rangle_{r,Q}\langle g\rangle_{1,Q}|Q|\quad(1<r\le 2)$$

but one can show that this estimate leads to the same upper bound for α_p as obtained in Theorem 1.1.

Contrary to this, the sparse bound

$$|\langle T_mf,g\rangle|\le\frac{C}{(r-1)^{1/2}}\sum_{Q\in\mathcal{S}}\langle f\rangle_{r,Q}\langle g\rangle_{r,Q}|Q|\quad(1<r\le 2)\tag{5.1}$$

would imply Conjecture 5.2. The technique developed in [22] probably may play an important role in establishing (5.1).

Added in proof. In a recent paper [7], the authors consider similar questions in the Walsh-Fourier setting. In particular they establish Conjecture 5.1 for all $1<p<\infty$ and Conjecture 5.2 for $\max\{p,p'\}\ge 5/2$ in this setting.

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