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Algebraization for zero-cycles and the p-adic cycle class map

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Using an idelic argument and assuming the Gersten conjecture for Milnor K-theory, we show that the restriction map from the Chow group of one-cycles on a smooth projective scheme over a henselian local ring to a pro-system of thickened zero-cycles is surjective. We relate this restriction map to the p-adic cycle class map.

1. Introduction

Let A be an excellent henselian discrete valuation ring with uniformising parameter π and residue field k. Let X be a smooth projective scheme over $\mathrm{Spec}(A)$ of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1.

For n invertible in k and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ the following commutative diagram has been studied extensively:

$$\begin{array}{ccc} \mathrm{CH}_1(X)/n & \xrightarrow{\rho} & \mathrm{CH}_0(X_1)/n \\ \downarrow{\scriptstyle cl_X} & & \downarrow{\scriptstyle cl_{X_s}} \\ H^{2d}_{\text{ét}}(X, \Lambda(d)) & \xrightarrow{\cong} & H^{2d}_{\text{ét}}(X_1, \Lambda(d)) \end{array} \tag{1}$$

The lower horizontal map is an isomorphism by proper base change. The map cl_{X_s} is an isomorphism assuming that k is finite or separably closed by unramified class field theory (see [6, Thm. 5, Rem. 3] and [20]). In [33], Saito and Sato show that cl_X is an isomorphism if k is finite or separably closed which implies that ρ is an isomorphism under these conditions. That ρ is in fact an isomorphism for arbitrary perfect residue fields is shown in [25] without using étale realizations by making use of a method introduced by Bloch in [7, App.]. In [28] these results are generalized to zero-cycles with coefficients in Milnor K-theory.

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Let $\mathcal{K}^M_{X,d}$ be the improved Milnor K-sheaf defined in [23] and $\mathcal{K}^M_{d,X_n}$ its restriction to X_n. In this article we study the restriction map

$$res_{X_n}: \mathrm{CH}^{d+j}(X,j) \xrightarrow{\cong} H^d(X, \mathcal{K}^M_{d+j,X}) \xrightarrow{res_{X_n}} H^d(X_1, \mathcal{K}^M_{d+j,X_n}).$$

If $j=0$, we assume the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{n,X}$ (see Definition 3.1) for the isomorphism on the left. By [22] and [23] it holds if X is equi-characteristic. If $j>0$, we additionally assume the Gersten conjecture for the sheaf $\mathcal{CH}^r(-q)$ associated to the presheaf $U \mapsto \mathrm{CH}^r(U,-q)$ for the isomorphism on the left (see [28]). This holds with finite coefficients or again if X is equi-characteristic (see e.g. [28]). For our applications we will need the following additional result which is well-known to the expert and easily follows from what is known about the Gersten conjecture for Quillen K-theory with finite coefficients (see Section 3):

Proposition 1.1. *Let* $\mathrm{ch}(k) = p > (n-1)$*. Then the Gersten conjecture holds for the sheaf* $\mathcal{K}^M_{n,X}/p^r$ *for all* $r \geq 1$*.*

The main theorem of this article is the following:

Theorem 1.2. *The restriction map* $res: H^d(X, \mathcal{K}^M_{d+j,X}) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n})$ *is surjective. In particular the map of pro-systems*

$$res: H^d(X, \mathcal{K}^M_{d+j,X}) \to \text{“}\lim_n\text{”} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in the category of pro-abelian groups pro-Ab for all $j \geq 0$*. Here we consider* $H^d(X, \mathcal{K}^M_{d+j,X})$ *as a constant pro-system.*

Theorem 1.2 is a partial response to a conjecture by Kerz, Esnault and Wittenberg saying that assuming the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{n,X}$ the restriction map

$$res: \mathrm{CH}^d(X) \otimes \mathbb{Z}/p^r\mathbb{Z} \to \text{“}\lim_n\text{”} H^d(X_1, \mathcal{K}^M_{d,X_n}/p^r)$$

is an isomorphism if $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and if k is perfect of characteristic $p>0$ (see [25, Sec. 10]). The proof of Theorem 1.2 relies on so-called idelic arguments. A different approach using differential forms is explained in [29].

Let again $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and k be perfect of characteristic $p>0$. In the final section of this article we relate the restriction map res to the p-adic cycle class map $\varrho^{d+j,j}_{p^r}: \mathrm{CH}^{d+j}(X,j)/p^r \to H^{2d+j}_{\text{ét}}(X, \mathcal{T}_r(j))$. The $\mathcal{T}_r(n)$ are the complexes defined in [36] and called p-adic étale Tate twists. $\mathcal{T}_r(n)$ is an object in the derived category $D^b(X, \mathbb{Z}/p^r\mathbb{Z})$ of bounded complexes of étale $\mathbb{Z}/p^r\mathbb{Z}$-sheaves on X. $\mathcal{T}_r(n)$ is expected to agree with $\mathbb{Z}(n)^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z}$, where $\mathbb{Z}(n)^{\text{ét}}$ denotes the conjectural étale motivic complex of Beilinson-Lichtenbaum (see [34, Sec. 1.3]). If $p > n+1$, then $i^*\mathcal{T}_r(n) \cong \mathcal{S}_r(n)$, where i is the inclusion $X_1 \to X$ and $\mathcal{S}_r(n)$ is the syntomic complex defined in [19] (see [36, Sec. 1.4]). In [34], Saito and Sato show the following result on the p-adic cycle class map:

Theorem 1.3. *([34, Thm. 1.3.1]) Let X be a regular scheme which is proper flat of finite type over the ring of integers A of a p-adic local field K. Assume that X has good or semistable reduction over A and let d be the fiber dimension of X over A. Then the cycle class map*

$$\varrho_{p^r}^{d,0} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

defined in [36, Cor. 6.1.4] is surjective.

We will show the following proposition:

Proposition 1.4. *Let $W(k)$ be the Witt ring of a finite field k of characteristic p and $p > d$. Let X be smooth and projective over $W(k)$. Then for all $j \geq d$ the map*

$$\text{“}\lim_n\text{”}\, H^d(X_1, \mathcal{K}^M_{j,X_n}/p^r) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$$

is an isomorphism of pro-abelian groups. Here we consider $H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$ as a constant pro-system.

In sum we establish, making use of the above result on the Gersten conjecture, the following commutative diagram analogous to diagram (1) for X smooth over $A = W(k)$ for a finite field k of characteristic p, $j \geq 0$ and $p > d+j+1$ (see Proposition/Definition 6.10):

(2)

$$\begin{array}{ccccc} \mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r\mathbb{Z}) & \xrightarrow{\cong} & H^d(X, \mathcal{K}^M_{d+j,X}/p^r) & \xrightarrow{res} & \text{“}\lim_n\text{”}\, H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r) \\ & & \downarrow & & \cong\downarrow \\ & & H^{2d+j}_{\text{ét}}(X, \mathcal{T}_r(d+j)) & \xrightarrow{\cong} & H^{2d+j}_{\text{ét}}(X_1, \mathcal{S}_r(d+j)) \end{array}$$

Notation. Unless otherwise specified, all cohomology groups are taken over the Zariski topology.

2. Parshin chains

Let X be an excellent scheme.

Definition 2.1. 1) A chain on X is a sequence of points $P = (p_0, \dots, p_s)$ on X such that

$$\overline{\{p_0\}} \subset \overline{\{p_1\}} \subset \cdots \subset \overline{\{p_s\}}.$$

2) A Parshin chain on X is a chain $P = (p_0, \dots, p_s)$ such that $\dim \overline{\{p_i\}} = i$ for all $0 \leq i \leq s$.

3) A Q-chain on X is a chain $Q = (p_0, \dots, p_{s-2}, p_s)$ such that $\dim \overline{\{p_i\}} = i$ for $i \in \{0, 1, \dots, s-2, s\}$. We denote by $B(Q)$ the set of all $x \in X$ such that $Q(x) = (p_0, \dots, p_{s-2}, x, p_s)$ is a Parshin chain.

4) Let Z be a closed subscheme of X and $U = X - Z$. A Parshin chain (resp. $\mathcal{Q}$-chain) on (X, Z) is a Parshin chain $P = (p_0, \dots, p_s)$ (resp. $\mathcal{Q}$-chain $Q = (p_0, \dots, p_{s-2}, p_s)$) such that $p_i \in Z$ for $i \leq s-1$ and $p_s \in U$ (resp. $p_i \in Z$ for $i \leq s-2$ and $p_s \in U$). A Parshin chain (resp. $\mathcal{Q}$-chain) on (X, X) is a Parshin chain in the sense of (2) (resp. $\mathcal{Q}$-chain in the sense of (3)).

5) We say that a Parshin chain $P = (p_0, \dots, p_s)$ on X is supported on a closed subscheme Z of X if $p_i \in Z$ for all $0 \leq i \leq s$.

6) The dimension $d(P)$ of a chain $P = (p_0, \dots, p_s)$ is defined to be $\dim \overline{\{p_s\}}$.

Definition 2.2. Let $P = (p_0, \dots, p_s)$ be a chain on X.

1) We define $\mathcal{O}_{X,P} = \mathcal{O}_{X,p_s}$ and $k(P) = k(p_s)$.

2) We define the finite product of henselian local rings $\mathcal{O}^h_{X,P}$ inductively as follows: If $s = 0$, then $\mathcal{O}^h_{X,P} = \mathcal{O}^h_{X,p_0}$. If $s > 0$, then assume that the ring $\mathcal{O}^h_{X,P'}$ over $\mathcal{O}_{X,p_0}$ has already been defined for $P' = (p_0, \dots, p_{s-1})$. Denote $\mathcal{O}^h_{X,P'}$ by R. Let T be the finite set of prime ideals of $\mathcal{O}^h_{X,P'}$ lying over p_s and

$$\mathcal{O}^h_{X,P} = \prod_{\mathfrak{p} \in T} R^h_{\mathfrak{p}}.$$

Let $k^h(P)$ denote the finite product of residue fields of $\mathcal{O}^h_{X,P}$.

We note that T in Definition 2.2(2) is finite by [13, Thm. 18.6.9 (ii)] since X is excellent and therefore in particular noetherian. For a Parshin chain P on X we denote Spec$\mathcal{O}_{X,P}$ by X_P and Spec$\mathcal{O}^h_{X,P}$ by X^h_P. For more details on Parshin chains see [21, Sec. 1.6].

We will need the following facts (see e.g. [14, Ch. IV.]): Let X be a locally noetherian scheme, $\mathcal{F}$ be a sheaf of abelian groups on X and $\tau \in \{\text{Zar,Nis}\}$. Then there are coniveau spectral sequences

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X_\tau, \mathcal{F}) \Rightarrow H^{p+q}(X_\tau, \mathcal{F})$$

and isomorphisms

$$H^q_x(X_{\text{Zar}}, \mathcal{F}) \cong H^q_x(\mathcal{O}_{X,x}, \mathcal{F}) \text{ and } H^q_x(X_{\text{Nis}}, \mathcal{F}) \cong H^q_x(\mathcal{O}^h_{X,x}, \mathcal{F})$$

for every $x \in X$ and $q \geq 0$. From the coniveau spectral sequence we get complexes

$$\cdots \to \bigoplus_{x \in X^{(p-1)}} H_x^{p+q-1}(X_\tau, \mathcal{F}) \to \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X_\tau, \mathcal{F}) \to \bigoplus_{x \in X^{(p+1)}} H_x^{p+q+1}(X_\tau, \mathcal{F}) \to \cdots$$

We denote a morphism $H_y^{p+q}(X_\tau, \mathcal{F}) \to H_x^{p+q+1}(X_\tau, \mathcal{F})$ arising this way by ∂_{yx}. We explain this notation as follows: If $\mathcal{F} = \mathcal{K}^M_{n,X}, y \in X^{(p+q)}, x \in X^{(p+q+1)}$, and if the Gesten conjecture holds for $\mathcal{K}^M_{n,X}$ (see Section 3.2), then the diagram

$$\begin{array}{ccc} H_y^{p+q}(X_\tau, \mathcal{K}^M_{n,X}) & \longrightarrow & H_x^{p+q+1}(X_\tau, \mathcal{K}^M_{n,X}) \\ \downarrow \cong & & \downarrow \cong \\ K^M_{n-p-q}(k(y)) & \longrightarrow & K^M_{n-p-q-1}(k(x)) \end{array}$$

commutes, where the lower horizontal map is the tame symbol defined by passing to the normalisation and using the norm map for Milnor K-theory (see f.e. [10, 8.1.1]).

Finally recall that the cohomological dimension of X_{Zar} and X_{Nis} is at most equal to $\dim(X)$.

3. The Gersten conjecture for Milnor K-theory mod p

Let X be an excellent scheme and let $\mathcal{K}^M_{n,X}$ be the improved Milnor K-sheaf defined in [23].

Definition 3.1. We say that the Gersten conjecture holds for the (Milnor K-)sheaf $\mathcal{K}^M_{n,X}$ if the sequence of sheaves

$$0 \to \mathcal{K}^M_{n,X} \to \bigoplus_{x\in X^{(0)}} i_{x,*}K^M_n(k(x)) \to \bigoplus_{x\in X^{(1)}} i_{x,*}K^M_{n-1}(k(x)) \to \cdots$$

is exact.

This conjecture is known to hold for $\mathcal{K}^M_{n,X}$ if all local rings of X are regular and equi-characteristic (see [22] and [23, Prop. 10(8)]). If X is smooth over a henselian local discrete valuation ring of mixed characteristic $(0,p)$, then the Gersten conjecture is not known to hold for the sheaf $\mathcal{K}^M_{n,X}$. However, if $p > n-1$, then we have the following much weaker result which we will use in Section 6.

Proposition 3.2. *Let A be a discrete valuation ring with uniformising parameter π and residue field k of characteristic $p > 0$. Let B be a local ring, essentially smooth over A with field of fractions F and let $p > (n-1)$. Then the sequence*

$$0 \to K^M_n(B)/p^r \xrightarrow{i_n} K^M_n(F)/p^r \to \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r \to \cdots$$

is exact for all $r \geq 1$.

Proof. First note that the Gersten conjecture for Quillen K-theory with finite coefficients holds for B by [9, Thm. 8.2].

We consider the following commutative diagram:

$$\begin{array}{ccccc}
K^M_n(B)/p^r & \xrightarrow{i_n} & K^M_n(F)/p^r & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r \\
\downarrow & & \downarrow & & \downarrow \\
K^Q_n(B,\mathbb{Z}/p^r) & \xrightarrow{i^Q_n} & K^Q_n(F,\mathbb{Z}/p^r) & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^Q_{n-1}(x,\mathbb{Z}/p^r) \\
\downarrow & & \downarrow & & \\
K^M_n(B)/p^r & \longrightarrow & K^M_n(F)/p^r & \longrightarrow & \bigoplus_{x\in X^{(1)}} K^M_{n-1}(x)/p^r
\end{array}$$

The composition $\mu : K^M_n(B) \to K^Q_n(B) \to K^M_n(B)$ is multiplication by $(n-1)!$ by [30, Sec. 4] and [23, Prop. 10(6)]. Let us first show the injectivity of i_n: Let $\alpha \in K^M_n(B)/p^r$ and suppose that $i_n(\alpha) = 0$. Then $(n-1)! \cdot \alpha = 0$

since i_n^Q is injective. For $p > (n-1)$ we have that $(p, (n-1)!) = 1$. This implies that $\alpha = 0$. The exactness at $K_n^M(F)/p^r$ can be seen as follows: Let $\alpha \in \ker[K_n^M(F)/p^r \to \bigoplus_{x \in X^{(1)}} K_{n-1}^M(x)/p^r]$. Then $(n-1)!\alpha \in \mathrm{im}(i_n)$ since the square on the upper right commutes (see [38, p. 449f.]) and the middle row is exact at $K_n^Q(F, \mathbb{Z}/p^r)$. Again since $(p, (n-1)!) = 1$ it follows that $\alpha \in \mathrm{im}(i_n)$.

Exactness at the other places follows for example from [8, Cor. 4.3]. □

Remark 3.3. See [30, Cor. 4.4] for a similar result.

We will repeatedly use the following purity statement which follows from the Gersten conjecture:

Lemma 3.4. *Let D be an effective Cartier divisor on X. Let $i : D \to X$ be the inclusion and $U := X \setminus \mathrm{supp} D$. Let $\mathcal{K}_{n,X|D}^M$ be the sheaf defined in Definition 4.1(2). Let $x \in X$ be a point which is not be contained in D and assume the Gersten conjecture for the sheaf $\mathcal{K}_{n,U}^M$. Then for $t = \mathrm{codim}_X(x)$ there is a canonical isomorphism*

$$H_x^t(X, \mathcal{K}_{n,X|D}^M) \cong K_{n-t}^M(k(x)).$$

Proof. First note that

$$\begin{aligned} H_x^t(X, \mathcal{K}_{n,X|D}^M) &\cong H_x^t(\mathrm{Spec}\mathcal{O}_{X,x}, \mathcal{K}_{n,X|D}^M |_{\mathrm{Spec}\mathcal{O}_{X,x}}) \\ &\cong H_x^t(\mathrm{Spec}\mathcal{O}_{X,x}, \mathcal{K}_{n,\mathrm{Spec}\mathcal{O}_{X,x}}^M). \end{aligned}$$

The purity isomorphism now follows from the Gersten conjecture for $\mathcal{K}_{n,\mathrm{Spec}\mathcal{O}_{X,x}}^M$ which holds by assumption since $x \in U$. Indeed, applying Γ_x (see [14, p. 225]) to the exact sequence

$$\mathcal{K}_{n,\mathrm{Spec}\mathcal{O}_{X,x}}^M \to \cdots \to \bigoplus_{y \in \mathcal{O}_{X,x}^{(c-1)}} i_{y,*} K_{n-t+1}^M(k(y)) \to i_{x,*} K_{n-t}^M(k(x)) \to 0$$

gives the sequence

$$\cdots \to 0 \to K_{n-t}^M(k(x))$$

(the last term is in degree t) since $i_{y,*} K_{n-t+1}^M(k(y))$ is the constant sheaf on the integral scheme $\overline{\{y\}}$. □

4. Some topology on Milnor K-groups

In this section we define a topology on Milnor K-groups and state two lemmas which we will need in the proof of our main theorem.

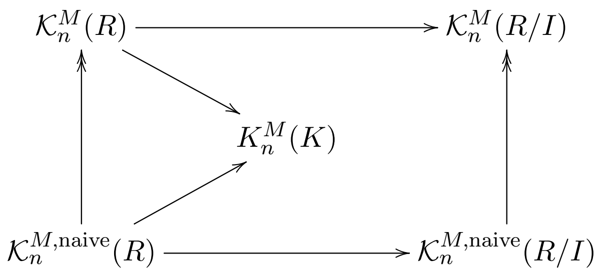
Recall that the naive Milnor K-sheaf $\mathcal{K}_n^{M,\text{naive}}$ is defined to be the sheafification of the functor

$$R \mapsto (R^\times)^{\otimes n} / < a_1 \otimes \cdots \otimes a_n | a_i + a_j = 1 \text{ for some } i \neq j >$$

from the category of commutative rings to abelian groups and that there is a natural homomorphism of sheaves

$$\mathcal{K}_n^{M,\text{naive}} \to \mathcal{K}_n^M$$

to the improved Milnor K-sheaf which is surjective (see [23]). In particular there is the following commutative diagram for a commutative local ring R, an ideal $I \subset R$ and $K = \text{Frac}(R)$:



This implies that that when defining a topology on $K_n^M(K)$ as in the following Definition 4.1(4) we may work with both $\mathcal{K}_n^{M,\text{naive}}$ or $\mathcal{K}_n^M$. We will use the improved Milnor K-sheaf and at some points implicitly use its generation by symbols.

Definition 4.1.

1) For a commutative ring R and an ideal $I \subset R$ we define $K_n^M(R, I)$ to be $\ker[\mathcal{K}_n^M(R) \to \mathcal{K}_n^M(R/I)]$ and similarly for $K_n^{M,\text{naive}}(R, I)$.

2) Let D be an effective Cartier divisor on X. We define $\mathcal{K}_{n,X|D}^M$ to be the kernel of the restriction map $\mathcal{K}_{n,X}^M \to i_*\mathcal{K}_{n,D}^M$ for $i : D \to X$ the inclusion. Again similarly for $K_{n,X|D}^{M,\text{naive}}$.

3) Let R be an excellent semi-local integral domain of dimension 1 with field of fractions K. We endow R with the J_R-adic topology, where J_R

is the Jacobsen radical of R. We endow $K_n^M(K)$ with the structure of a topological group by taking the subgroups generated by $\{U_1, \ldots, U_n\}$, where U_i ranges over all open subgroups of $R^\times$, as a fundamental system of neighbourhoods of 0 in $K_n^M(K)$.

4) For a Parshin chain $P = (p_0, \ldots, p_{s-1}, p_s)$, and $P' = (p_0, \ldots, p_{s-1})$, on an excellent scheme X and $Y = \overline{\{p_s\}}$ we define a topology on $K_n^M(k(P))$ (resp. $K_n^M(k^h(P))$) by taking the images of $K_n^M(\mathcal{O}_{Y,P'}, I)$ (resp. $K_n^M(\mathcal{O}^h_{Y,P'}, I)$) as a fundamental system of neighbourhoods of 0, where I ranges over all open ideals, with respect to the topology defined in (3), of the one dimensional local ring $\mathcal{O}_{Y,P'}$ (resp. semi-local ring $\mathcal{O}^h_{Y,P'}$).

Remark 4.2. If we set $R := \mathcal{O}_{Y,P'}$ (resp. $\mathcal{O}^h_{Y,P'}$), then the topologies on $K_n^M(K), K = \mathrm{Frac}(R)$, defined in (3) and (4) coincide.

Example 4.3. Let $m \geq 0$ be an integer. If R in (3) is a discrete valuation ring with quotient field K, maximal ideal $\mathfrak{p} \subset R$ and generic point η, then the subgroups generated by $K_n^M(K, m) := \{1 + \mathfrak{p}^m, R^\times, \ldots, R^\times\}$ of $K_n^M(K)$ generate the topology on $K_n^M(K)$ with respect to the Parshin chain $(\mathfrak{p}, \eta)$.

Lemma 4.4. *(Cf. [18, Prop. 2]) Let R be an excellent semi-local integral domain of dimension 1 with field of fractions K. Let $\tilde{R}$ be the integral closure of R in K. Then the topology of $K_n^M(K)$ defined by $\tilde{R}$ coincides with that defined by R.*

Proof. Since R is excellent, the normalisation is finite and there is some $f \in J_R \setminus \{0\}$ such that $f\tilde{R} \subset R$. Therefore for every $i \geq 1$

$$1 + f^{i+1}\tilde{A} \subset 1 + f^i A.$$

□

Lemma 4.5. *(cf. [21, Prop. 2.7], [24, Lem. 6.2]) Let X be an excellent integral scheme. Let U be a regular open subscheme of X and D an effective Weil divisor with support $X - U$. Let $y \in U$ and x be of codimension 1 on $\overline{\{y\}}$. Let $\dim \mathcal{O}_{X,y} = t$ and assume the Gersten conjecture for the sheaf $\mathcal{K}^M_{n,U}$. Then the map*

$$\partial_{yx} : K^M_{n-t}(k(y)) \cong H^t_y(X, \mathcal{K}^M_{n,X|D}) \to H^{t+1}_x(X, \mathcal{K}^M_{n,X|D})$$

annihilates the image of $K^M_{n-t}(\mathcal{O}_{Y,x}, J_x)$ for some non-zero ideal $J_x \subset \mathcal{O}_{Y,x}$. In particular the kernel of ∂_{yx} is open with respect to the topology defined in Definition 4.1(4) and the Parshin chain (x, y).

Proof. We proceed by induction on t. The case $t=0$ is clear since in that case $Y=X$ and $H^1_x(X,\mathcal{K}^M_{n,X|D})\cong K^M_n(k(y))/K^M_n(\mathcal{O}_{Y,x},J)$ for $x\in D^{(0)}$ and J corresponding to D.

If $t\geq 1$, then we take some point $z\in X^{t-1}$ such that y lies in the regular locus of $\overline{\{z\}}$. Consider the complex

$$(3)\qquad H^{t-1}_z(X,\mathcal{K}^M_{n,X|D})\to \bigoplus_{y'\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x}\setminus D} H^t_{y'}(X,\mathcal{K}^M_{n,X|D}) \oplus \bigoplus_{y''\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x}\cap D} H^t_{y''}(X,\mathcal{K}^M_{n,X|D}) \to H^{t+1}_x(X,\mathcal{K}^M_{n,X|D})$$

coming from the coniveau spectral sequence in Section 2. Applying the induction assumption to $H^{t-1}_z(X,\mathcal{K}^M_{n,X|D})\to H^t_{y''}(X,\mathcal{K}^M_{n,X|D})$ for all $y''\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z_x}\cap D$, we see that it suffices to show that the map

$$K^M_{n-t+1}(k(z))\xrightarrow{(\partial,\mathrm{Id})}\bigoplus_{y'\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x}\setminus D} K^M_{n-t}(k(y')) \oplus \bigoplus_{y''\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z,x}\cap D} K^M_{n-t+1}(K(z))/K^M_{n-t+1}(\mathcal{O}_{Z,y''},J_{y''})$$

annihilates the image of $K^M_{n-t}(\mathcal{O}_{Y,x},J_x)$ in $K^M_{n-t}(k(y))$ for some non-zero ideal $J_x\subset\mathcal{O}_{Y,x}$ given some non-zero ideals $J_{y''}\subset\mathcal{O}_{Z,y''}$. Indeed, in that case if $\alpha\in \mathrm{Im}(K^M_{n-t}(\mathcal{O}_{Y,x},J_x)\to K^M_{n-t}(k(y)))$, then there is some $\beta\in K^M_{n-t+1}(k(z))$ such that $\partial_{zy}(\beta)=\alpha$ and such that

$$\left(\bigoplus_{y'\neq y\in \mathrm{Spec}\mathcal{O}^{(1)}_{Z_x}} \partial_{zy''},\mathrm{Id}\right)(\beta)=0.$$

Since (3) is a complex, this implies that $\partial_{yx}(\alpha)=0$.

Now let $A:=\mathcal{O}_{Z,x}$. By Lemma 4.4 we may assume that the $\mathcal{O}_{Z,y''}$ are normal (semi-local) rings. By the definition of ∂ we may work with the normalisation $\tilde{A}$ of A. Let $\{y''_1,\dots,y''_r\}=\mathrm{Spec}\mathcal{O}^{(1)}_{Z,x}\cap D$ and let $J^{(y''_i)}$ be ideals in $\tilde{A}$ such that $J^{(y''_i)}\mathcal{O}_{Z,y''_i}=J_{y''_i}$. Let $\mathfrak{q}$ be the prime ideal corresponding to y. Let $\pi\in A$ such that $v_{\mathfrak{q}}(\pi)=1$. Let $\{\mathfrak{p}_{1+r},\dots,\mathfrak{p}_t\}$ be the finite set of prime ideals in $\tilde{A}$ such that $v_{\mathfrak{p}_i}(\pi)>0, 1+r\leq i\leq t$. By a standard approximation lemma (see e.g. [27, Lem. 9.1.9(b)]) we can choose an element π_i for all i

with $r+1 \leq i \leq t$ satisfying $v_{\mathfrak{p}_i}(\pi_i) = 1$ and $v_{\mathfrak{q}}(\pi_i) = 0$. Now we can choose a non-zero ideal

$$(4) \quad J^{(x)} \subset J^{(y_1'')} \cdots J^{(y_r'')}(\pi_{r+1}) \cdots (\pi_t)(\tilde{A}/\mathfrak{q}) \twoheadleftarrow J^{(y_1'')} \cdots J^{(y_r'')}(\pi_{r+1}) \cdots (\pi_t)\tilde{A}.$$

Let $J_x := J^{(x)}\mathcal{O}_{Y,x}$. Now given a symbol $\alpha := \{\bar{a}_1, \dots, \bar{a}_{n-t}\} \in K^M_{n-t}(\mathcal{O}_{Y,x}, J_x)$ with $\bar{a}_1 \in 1 + J_x$, lift α to $\beta := \{\pi, a_1, \dots, a_{n-t}\} \in K^M_{n-t+1}(k(z))$ lifting $\bar{a}_1$ via the surjection in (4) to a_1 and lifting the other $\bar{a}_i$ arbitrarily. Then β satisfies the required properties since

1) $\partial_{zy}(\beta) = \alpha$.
2) If $\tilde{y}' \notin \operatorname{div}(\pi)$, then $\partial_{z\tilde{y}'} = 0$ since $\pi, a_1, \dots, a_{n-t} \in \mathcal{O}^{\times}_{\tilde{A},\tilde{y}'}$.
3) If $\tilde{y}' \in \operatorname{div}(\pi)$, i.e. $\tilde{y}' \sim \mathfrak{p}_i$, then $\partial_{z\tilde{y}'} = 0$ since $a_1 = 1 \bmod (\pi_i)$.
4) $a_1 \in K^M_{n-t+1}(\mathcal{O}_{Z,y''}, J_{y''})$ for all $y'' \in \operatorname{Spec}\mathcal{O}^{(1)}_{Z,x} \cap D$.

□

Remark 4.6. In [21, Prop. 2.7] the above lemma was proved in the Nisnevich topology. The proof in the Zariski topology follows the argument in *loc. cit.* We recall the proof for the convenience of the reader and to convince them of this claim. In [24, Lem. 6.2] the last step of the proof is given under the assumption that A is a two-dimensional excellent henselian local ring.

Lemma 4.7. *Given a family of inequivalent discrete valuations $v_1, \dots, v_s$ on a valued field F, the diagonal map*

$$K^M_n(F) \to \bigoplus_{v_i} K^M_n(F_{v_i}),$$

has dense image, were we write F_{v_i} instead of F in order to indicate which valuation defines the topology on F.

Proof. This follows from standard approximation theorems for F. See e.g. [31, II.3.4]. □

5. Main theorem

In this section we prove Theorem 1.2.

We return to the situation of the introduction. Let A be an excellent henselian discrete valuation ring with uniformising parameter π and residue field k and let X be a smooth projective scheme over $\operatorname{Spec}(A)$ of relative dimension d. Let $X_n := X \times_A A/(\pi^n)$, i.e. X_1 is the special fiber and the X_n are the respective thickenings of X_1.

Proposition 5.1. *For all $j \geq 0$ the group*

$$H^{d+1}_{\mathrm{Zar}}(X, \mathcal{K}^M_{j+d,X|X_n}) = 0.$$

Proof. By the coniveau spectral sequence and cohomological vanishing we have to show that the map

$$\bigoplus_{y\in X^{(d)}} H^d_y(X, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{x\in X^{(d+1)}} H^{d+1}_x(X, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective. In order to show this, we show that the map

$$\bigoplus_{y\in(\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d} H^d_y(X, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d+1}_x(X, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective for any $x \in X^{(d+1)}$. This suffices since $(\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d \subset X^{(d)}$ and since, as A is henselian, any $y \in (\mathrm{Spec}\mathcal{O}_{X,x}[\frac{1}{\pi}])^d$ restricts to just one closed point $x \in X^{(d+1)}$. Let us start with the case $d = 0$: Let $X'_x := X_x - x$. Then

$$H^1_x(X, \mathcal{K}^M_{j,X|X_n}) \cong H^0(X'_x, \mathcal{K}^M_{j,X|X_n})/H^0(X_x, \mathcal{K}^M_{j,X|X_n}),$$

and $H^0_\mu(X, \mathcal{K}^M_{j,X|X_n})$, μ being the generic point of X, surjects onto $H^1_x(X, \mathcal{K}^M_{j,X|X_n})$ since $H^0_\mu(X, \mathcal{K}^M_{j,X|X_n})$ is isomorphic to $H^0(X'_x, \mathcal{K}^M_{j,X|X_n})$.

Let $d \geq 1$ and $x \in X^{(d+1)}$. We have that

$$H^{d+1}_x(X, \mathcal{K}^M_{d+j,X|X_n}) \cong H^d(X'_x, \mathcal{K}^M_{d+j,X|X_n})$$

and again it follows from the coniveau spectral sequence and cohomological vanishing that $H^d(X'_x, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to

$$\mathrm{coker}\left(\bigoplus_{z\in(X_x)^{d-1}} H^{d-1}_z(X_x, \mathcal{K}^M_{d+j,X|X_n}) \to \bigoplus_{y\in(X_x)^d} H^d_y(X_x, \mathcal{K}^M_{d+j,X|X_n})\right).$$

By Lemma 3.4 we have that

$$H^d_y(X_x, \mathcal{K}^M_{d+j,X|X_n}) \cong \mathcal{K}^M_{j,X}(k(x,y))$$

for $y \in (X_x[\frac{1}{\pi}])^d$. It therefore suffices to move elements of $H^d_y(X_x, \mathcal{K}^M_{d+j,X|X_n})$ for $y \in X_x^{d-r} \setminus (X_x[\frac{1}{\pi}])^d$ to the horizontal components, i.e. with $y \in (X_x[\frac{1}{\pi}])^d$, using the 'Q-chains' $H^{d-1}_z(X_x, \mathcal{K}^M_{d+j,X|X_n})$.

We write P_r for a Parshin chain $(x, \dots)$ of dimension r and let x_{P_r} denote the closed point of X_{P_r} and X'_{P_r} the open subscheme $X_{P_r} \setminus \{x_{P_r}\}$. We proceed by descending in induction in $r \geq 0$, starting with $r = d$, to show that the map

$$\bigoplus_{y \in (X_{P_r}[\frac{1}{\pi}])^{d-r}} H^{d-r}_y(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d-r+1}_{x_{P_r}}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective for all Parshin chains P_r supported on X_1.

The group $H^{d-r+1}_{x_{P_r}}(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to

$$\begin{aligned} &\operatorname{coker}\Bigg(\bigoplus_{z \in (X_{P_r})^{d-1-r}} H^{d-1-r}_z(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \\ &\to \bigoplus_{y \in (X_{P_r})^{d-r}} H^{d-r}_y(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \Bigg) \\ \cong\ & H^{d-r}(X'_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \end{aligned}$$

for $r < d$ and to

$$H^0(X'_{P_d}, \mathcal{K}^M_{d+j,X|X_n}) / H^0(X_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$$

for $r = d$. If $r = d$, then $H^0_y(X_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$ is isomorphic to

$$H^0(X'_{P_d}, \mathcal{K}^M_{d+j,X|X_n})$$

which implies the induction beginning.

We now do the induction step. Let $\alpha \in H^{d-r}_y(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$ for $y \in X_{P_r}^{d-r} \setminus (X_{P_r}[\frac{1}{\pi}])^{d-r}$. Then the map

$$\bigoplus_{z \in (X_{(P_r,y)})^{d-r-1}} H^{d-r-1}_z(X_{(P_r,y)}, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d-r}_y(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n})$$

is surjective. This follows again from the coniveau spectral sequence and the isomorphism $H^{d-r}_y(X_{P_r}, \mathcal{K}^M_{d+j,X|X_n}) \cong H^{d-r-1}(X'_{(P_r,y)}, \mathcal{K}^M_{d+j,X|X_n})$.

By assumption we have that

$$\operatorname{coker}\Bigg(\bigoplus_{t\in (X_{(P_r,y)})^{d-r-2}} H_t^{d-r-2}(X_{(P_r,y)},\mathcal{K}^M_{d+j,X|X_n}) \\ \to \bigoplus_{z\in (X_{(P_r,y)})^{d-r-1}} H_z^{d-r-1}(X_{(P_r,y)},\mathcal{K}^M_{d+j,X|X_n})\Bigg)$$

is generated by $K^M_{r+j+1}(k(P))$ for all Parshin chains $P=(P_r,y,z)$ of dimension $r+2$ on (X,X_1). We may assume that α is in the image of $K^M_{r+j+1}(k(P))$ for some such P. Then by Lemma 4.5 the kernel of the map

$$\partial_{zy}-\alpha : K^M_{r+j+1}(k(P)) \to H_y^{d-r}(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

is open in $K^M_{r+j+1}(k(P))$ and the kernel of the map

$$\partial_{zy'} : K^M_{r+j+1}(k(P)) \to H_{y'}^{d-r}(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

is open in $K^M_{r+j+1}(k(P_r,y',z))$ for all $y'\neq y\in X^{d-r}_{P_r}\cap X_1$ with $\overline{\{x_{P_r}\}}\subset \overline{\{y'\}}\subset\overline{\{z\}}$. By Lemma 4.7 and Lemma 4.4 the diagonal image of

$$K^M_{r+j+1}(k(Q))\cong H_z^{d-r-1}(X_{P_r},\mathcal{K}^M_{d+j,X|X_n})$$

for a Q-chain $Q=(P_r,z)$ is dense in the direct sum (with finitely many summands) $K^M_{r+j+1}(k(P))\oplus_{y'\neq y\in X^{d-r}_{P_r}\cap\overline{\{z\}}_1} K^M_{r+j+1}(k(P_r,y',z))$ which implies that α is in the image of some $\beta\in H_z^{d-r-1}(X_x,\mathcal{K}^M_{d+j,X|X_n})$ with β mapping to zero in $H_{y'}^{d-r}(X_x,\mathcal{K}^M_{d+j,X|X_n})$ for all $y'\neq y\in((X_x)_1)^d$. □

Remark 5.2. The proof of Proposition 5.1 is inspired by the proof of Theorem 2.5 in [21] and the proof of Theorem 8.2 in [24]. In both of these articles the authors work in the Nisnevich topology. We note that the proof of Proposition 5.1 also works in the Nisnevich topology if for every Parshin chain $P=(p_0,\dots,p_s)$ we replace $\mathcal{O}_{X,P}=\mathcal{O}_{X,p_s}$ by $\mathcal{O}^h_{X,P}$ according to Definition 2.2. We therefore get that

$$H^{d+1}_{\mathrm{Nis}}(X,\mathcal{K}^M_{j+d,X|X_n})=0$$

for all $j\geq 0$.

Remark 5.3. If $\mathrm{ch}(k) = 0$ and $A = k[[t]]$ or if A is the Witt ring $W(k)$ of a perfect field k of $\mathrm{ch}(k) > 2$, then there are exact sequences of sheaves

$$0 \to \Omega^{r-1}_{X_1} \to \mathcal{K}^M_{r,X_n} \to \mathcal{K}^M_{r,X_{n-1}} \to 0$$

and

$$0 \to \Omega^{r-1}_{X_1}/B_{n-2}\Omega^{r-1}_{X_1} \to \mathcal{K}^M_{r,X_n} \to \mathcal{K}^M_{r,X_{n-1}} \to 0$$

respectively, by [3, Sec. 2] and [4, Sec. 12]. Under the above assumptions this implies that the canonical map

$$H^d_{\mathrm{Zar}}(X_1, \mathcal{K}^M_{d+j,X_n}) \to H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an isomorphism for all $n \in \mathbb{N}_{>0}$. Indeed, it follows from the Gersten conjecture for the Milnor K-sheaf $\mathcal{K}^M_{*,X_1}$ that the maps $H^i_{\mathrm{Zar}}(X_1, \mathcal{K}^M_{d+j,X_1}) \to H^i_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{d+j,X_1})$ are isomorphisms for all i and the sheaves $\Omega^{r-1}_{X_1}$ and $\Omega^{r-1}_{X_1}/B_{n-2}\Omega^{r-1}_{X_1}$ are coherent. The claim now follows by induction on n.

Corollary 5.4.

1) *The restriction map* $res : H^d(X, \mathcal{K}^M_{d+j,X}) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n})$ *is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}) \to \text{“}\lim_n\text{”}\, H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in pro-Ab for all $j \geq 0$.

2) *The restriction map* $res : H^d(X, \mathcal{K}^M_{d+j,X}/p^r) \to H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$ *is surjective. In particular the map of pro-systems*

$$res : H^d(X, \mathcal{K}^M_{d+j,X}/p^r) \to \text{“}\lim_n\text{”}\, H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$$

is an epimorphism in pro-Ab for all $j \geq 0$.

Here and in the following we always consider

$$H^d(X, \mathcal{K}^M_{d+j,X}) \quad (\textit{resp. } H^d(X, \mathcal{K}^M_{d+j,X}/p^r))$$

as a constant pro-system in in pro-Ab.

Proof. For (1) consider the short exact sequence

$$0 \to \mathcal{K}^M_{d+j,X|X_n} \to \mathcal{K}^M_{d+j,X} \to \mathcal{K}^M_{d+j,X_n} \to 0$$

and the induced long exact sequence

$$\begin{aligned} \cdots \to H^d(X, \mathcal{K}^M_{d+j,X|X_n}) &\xrightarrow{i} H^d(X, \mathcal{K}^M_{d+j,X}) \\ &\xrightarrow{res} H^d(X_1, \mathcal{K}^M_{d+j,X_n}) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) \to \cdots \end{aligned}$$

The statement now follows from Proposition 5.1 and the fact that "$\lim_n$" is exact when considered as a functor $\mathrm{Hom}(I^{\mathrm{op}}, \mathrm{Ab}) \to$ pro-Ab, where I is a small filtering category (see [1, App. Prop. 4.1]).

(2) can be seen as follows: Since $\otimes\mathbb{Z}/p^r\mathbb{Z}$ is right exact, there is a short exact sequence

$$0 \to \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I} \to \mathcal{K}^M_{d+j,X}/p^r \to \mathcal{K}^M_{d+j,X_n}/p^r \to 0$$

for some sheaf of abelian groups $\mathcal{I}$. This induces an exact sequence

$$H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I}) \to H^{d+2}(X, \mathcal{I}).$$

By [12, Thm. 3.6.5] the group $H^{d+2}(X, \mathcal{I})$ vanishes for dimensional reasons. The group $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r)$ vanishes by the same arguments as in the integral case or in fact from the surjectivity of the map

$$H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}) \to H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r)$$

which also holds for dimensional reasons. Together this implies the vanishing of $H^{d+1}(X, \mathcal{K}^M_{d+j,X|X_n}/p^r/\mathcal{I})$. This implies the statement by the same argument as in the proof of (1). □

Corollary 5.5. *If A is equi-characteristic, then the map*

$$res : \mathrm{CH}^{d+j}(X, j) \to \text{“}\lim_n\text{”} H^d(X_1, \mathcal{K}^M_{d+j,X_n})$$

is an epimorphism in pro-Ab for all $j \geq 0$. If A is of mixed characteristic $(0, p)$ with $p > d + j - 1$, then the map

$$res : \mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r) \to \text{“}\lim_n\text{”} H^d(X_1, \mathcal{K}^M_{d+j,X_n}/p^r)$$

is an epimorphism in pro-Ab for all $j \geq 0$.

Proof. For $j = 0$, Corollary 5.4 implies the first assertion since the Gersten conjecture for the sheaf $\mathcal{K}^M_{n,X}$ holds for regular schemes of equal characteristic and the second assertion since the Gersten conjecture holds for $\mathcal{K}^M_{n,X}/p^r$ if $p > n - 1$ by Proposition 3.2.

If $j > 0$, then the identifications of $\mathrm{CH}^{d+j}(X, j)$ with $H^d(X, \mathcal{K}^M_{d+j,X})$ in the equi-characteristic case and $\mathrm{CH}^{d+j}(X, j, \mathbb{Z}/p^r\mathbb{Z})$ with $H^d(X, \mathcal{K}^M_{d+j,X}/p^r)$ in the mixed characteristic case require in addition to the above mentioned results on the Gersten conjecture the Gersten conjecture for higher Chow groups (see [28, Sec. 2]). This holds if A is equi-characteristic by [2, Sec. 10] and the method developed by Panin in [32] to extend the Gersten conjecture to the equi-dimensional setting. In mixed characteristic the Gersten conjecture for higher Chow groups with $\mathbb{Z}/p^r\mathbb{Z}$-coefficients holds by [8, Cor. 4.3]. □

Remark 5.6. Consider again the short exact sequence

$$0 \to \mathcal{K}^M_{d,X|X_n} \to \mathcal{K}^M_{d,X} \to \mathcal{K}^M_{d,X_n} \to 0$$

and the induced long exact sequence

$$\begin{aligned} \cdots \to H^d(X, \mathcal{K}^M_{d,X|X_n}) &\xrightarrow{i} H^d(X, \mathcal{K}^M_{d,X}) \\ &\xrightarrow{res} H^d(X_1, \mathcal{K}^M_{d,X_n}) \xrightarrow{0} H^{d+1}(X, \mathcal{K}^M_{d,X|X_n}) \to \cdots \end{aligned}$$

We denote the image of $H^d(X, \mathcal{K}^M_{d,X|X_n})$ under i by F^X_n.

As mentioned in the introduction, Kerz, Esnault and Wittenberg conjecture in [25, Sec. 10] that if $\mathrm{ch}(\mathrm{Quot}(A)) = 0$ and k is perfect of characteristic $p > 0$ and if we assume that the Gersten conjecture for $\mathcal{K}^M_X$ holds, then the map

$$res : \mathrm{CH}_1(X)/p^r \to \text{“}\lim_n\text{”}\, H^d(X_n, \mathcal{K}^M_{X_n,d}/p^r)$$

is an isomorphism in pro-Ab. We note that this conjecture would be implied by the following conjecture:

Conjecture 5.7.

$$\begin{aligned} F^X_n = <\, g_*F^Y_n | g : Y \to X \text{ projective}, \\ Y/A \text{ smooth projective relative curve} > . \end{aligned}$$

This can be seen as follows: By definition, $F^Y_n \subset H^1(Y, \mathcal{K}^M_{1,Y})$ is the image of $H^1(Y, \mathcal{K}^M_{1,Y|Y_n}) = H^1(Y, \mathcal{O}^\times_{Y|Y_n})$ under i. By the theorem on formal functions and the p-adic logarithm isomorphism, assuming that p is large enough, $H^1(Y, \mathcal{O}^\times_{Y|Y_n}) \cong H^1(Y, p^n\mathcal{O}_Y)$ and therefore the composition

$$H^1(Y, \mathcal{O}^\times_{Y|Y_{n+1}}) \to H^1(Y, \mathcal{O}^\times_{Y|Y_n})$$

is multiplication by p. This implies that "$\lim_n$"$F_n^Y \otimes \mathbb{Z}/p^r = 0$ and therefore that "$\lim_n$"$F_n^X \otimes \mathbb{Z}/p^r = 0$.

Corollary 5.8. *Let k be a finite field of characteristic $p > 2$ and $A = W(k)$ the Witt ring of k. Let X be a smooth projective scheme of relative dimension 1 over A. Then the map*

$$res : \mathrm{CH}^1(X)/p^r \to \text{“lim”}\, H^1(X_1, \mathcal{K}^M_{1,X_n}/p^r)$$

is an isomorphism in the category of pro-systems of abelian groups.

Proof. The injectivity follows from the arguments in Remark 5.6 assuming $p > 2$ for the p-adic logarithm isomorphism. The surjectivity follows from Corollary 5.4. □

6. Relation with the p-adic cycle class map

In this section we prove Proposition 1.4.

Let k be a finite field of $\mathrm{ch}(k) = p > 0$, $A = W(k)$ and X be a smooth projective scheme over A of fiber dimension d. We let X_1/k denote the reduced special fiber. Let $\tau \in \{\text{Nis, ét}\}$ and $X_{1,\tau}$ be the respective small site. Let $\epsilon : X_{1,\text{ét}} \to X_{1,\text{Nis}}$ be the canonical map of sites.

Definition 6.1. ([4, Def. A.3])

(a) By $\mathrm{Sh}(X_{1,\tau})$ we denote the category of sheaves of abelian groups on $X_{1,\tau}$. By $\mathrm{C}(X_{1,\tau})$ we denote the category of unbounded complexes in $\mathrm{Sh}(X_{1,\tau})$.

(b) By $\mathrm{Sh}_{\text{pro}}(X_{1,\tau})$ we denote the category of pro-sytems in $\mathrm{Sh}(X_{1,\tau})$.

(c) By $\mathrm{C}_{\text{pro}}(X_{1,\tau})$ we denote the category of pro-systems in $\mathrm{C}(X_{1,\tau})$.

(d) By $\mathrm{D}_{\text{pro}}(X_{1,\tau})$ we denote the Verdier localization of the homotopy category of $\mathrm{C}_{\text{pro}}(X_{1,\tau})$, where we kill objects which are represented by systems of complexes which have level-wise vanishing cohomology sheaves.

Definition 6.2. We define

$$W_{\cdot}\Omega^{\bullet}_{X_1} \in \mathrm{C}_{\mathrm{pro}}(X_1)_{\tau}$$

to be the pro-system of de Rham-Witt complexes in the étale or Nisnevich topology (see [15]). We define

$$W_{\cdot}\Omega^{r}_{X_1,\log} \in \mathrm{Sh}_{\mathrm{pro}}(X_1)_{\tau}$$

to be the pro-system of étale or Nisnevich subsheaves in $W_r\Omega^j_{X_1}$ which are locally generated by symbols

$$d\log\{[a_1]\} \cdots d\log\{[a_j]\}$$

with $a_1, \ldots, a_j \in \mathcal{O}^{\times}_{X_1}$ local sections and where $[-]$ is the Teichmüller lift (see [15, p. 505, (1.1.7)]).

Definition 6.3. Assuming $j < p$, we define $\mathcal{S}_r(j)_{\text{ét}}$ to be the syntomic complex defined in [19, Def. 1.6]. We denote the corresponding object in $\mathrm{D}_{\mathrm{pro}}(X_1)_{\text{ét}}$ by $\mathcal{S}_{X_{\cdot}}(j)_{\text{ét}}$.

Definition 6.4. ([4, Sec. 4]) We define

$$\mathcal{S}_r(j)_{\mathrm{Nis}} := \tau_{\leq j} R\epsilon_* \mathcal{S}_r(j)_{\text{ét}} \quad \text{and} \quad \mathcal{S}_{X_{\cdot}}(j)_{\mathrm{Nis}} := \tau_{\leq j} R\epsilon_* \mathcal{S}_{X_{\cdot}}(j)_{\text{ét}},$$

where $\tau_{\leq j}$ is the good truncation.

Let $j < p$. In [4, Sec. 7], Bloch, Esnault and Kerz define a motivic pro-complex

$$\mathbb{Z}_{X_{\cdot}}(j) := \mathrm{cone}(\mathcal{S}_{X_{\cdot}}(j) \oplus \mathbb{Z}_{X_1}(j) \to W_{\cdot}\Omega^j_{X_1,\log}[-j])[-1]$$

in the Nisnevich topology. $\mathbb{Z}_{X_{\cdot}}(j)$ is an object in $\mathrm{D}_{\mathrm{pro}}(X_{1,\mathrm{Nis}})$ with the following properties:

Proposition 6.5. *([4, Prop. 7.2])*

(0) $\mathbb{Z}_{X_{\cdot}}(0) = \mathbb{Z}$, *the constant sheaf in degree* 0.

(1) $\mathbb{Z}_{X_{\cdot}}(j) = \mathbb{G}_{m,X_{\cdot}}[-1]$.

(2) $\mathbb{Z}_{X_{\cdot}}(j)$ *is supported in degrees* $\leq j$ *and in* $[1, j]$ *if* $j \geq 1$ *and if the Beilinson-Soulé conjecture holds.*

(3) $\mathbb{Z}_{X_\cdot}(j) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\cdot = \mathcal{S}_{X_\cdot}(j)_{\mathrm{Nis}}$ *in* $\mathrm{D}_{\mathrm{pro}}(X_{1,\mathrm{Nis}})$.

(4) $\mathcal{H}^j(\mathbb{Z}_{X_\cdot}(j)) = \mathcal{K}^M_{j,X_\cdot}$ *in* $\mathrm{Sh}_{\mathrm{pro}}(X_{1,\mathrm{Nis}})$.

(5) *There is a canonical product structure* $\mathbb{Z}_{X_\cdot}(j) \otimes^L_{\mathbb{Z}} \mathbb{Z}_{X_\cdot}(j') \to \mathbb{Z}_{X_\cdot}(j + j')$.

We now start the proof of Proposition 1.4 proving the following lemmas:

Lemma 6.6. *Let $j < p$. Then the map*

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \to H^{d+j}_{\mathrm{Nis}}(X_1, \mathcal{S}_{X_\cdot}(j))$$

is an isomorphism in the category of pro-abelian groups.

Proof. By properties (2)–(4) of Proposition 6.5 we have that $\mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot \cong \mathcal{H}^j(\mathcal{S}_\cdot(j)_{\mathrm{Nis}})$. Let us be more precise:

$$\mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot \overset{(4)}{\cong} \mathcal{H}^j(\mathbb{Z}_{X_\cdot}(j)) \otimes \mathbb{Z}/p^\cdot \overset{(2)}{\cong} \mathcal{H}^j(\mathbb{Z}_{X_\cdot}(j) \otimes^{\mathbb{L}} \mathbb{Z}/p^\cdot) \overset{(3)}{\cong} \mathcal{H}^j(\mathcal{S}_\cdot(j)_{\mathrm{Nis}}).$$

For the isomorphism in the middle consider the short exact sequence

$$0 \to \mathcal{H}^j(\mathbb{Z}_{X_\cdot}(j)) \otimes \mathbb{Z}/p^\cdot \to \mathcal{H}^j(\mathbb{Z}_{X_\cdot}(j) \otimes^{\mathbb{L}} \mathbb{Z}/p^\cdot) \to \mathcal{H}^{j+1}(\mathbb{Z}_{X_\cdot}(j))[p^\cdot] \to 0$$

in which the term on the right vanishes by (2). This implies that

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \cong H^d_{\mathrm{Nis}}(X_1, \mathcal{H}^j(\mathcal{S}_\cdot(j))).$$

The hypercohomology spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Nis}}(X_1, \mathcal{H}^q(\mathcal{S}_r(j))) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Nis}}(X_1, \mathcal{S}_r(j))$$

together with the Nisnevich cohomological dimension of X_1 and the concentration of $\mathcal{S}_r(j)_{\mathrm{Nis}}$ in degrees $\leq j$ implies that $H^d_{\mathrm{Nis}}(X_1, \mathcal{H}^j(\mathcal{S}_r(j))) \cong H^{d+j}_{\mathrm{Nis}}(X_1, \mathcal{S}_r(j))$ and therefore that

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \to H^{d+j}_{\mathrm{Nis}}(X_1, \mathcal{S}_\cdot(j)).$$ □

Lemma 6.7. *The natural map*

$$\rho^{d,q}_{X_{\mathrm{Nis}}} : H^{2d-q}_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log}[-d]) \to H^{2d-q}_{\text{ét}}(X_1, W_r\Omega^d_{X_1,\log}[-d])$$

is an isomorphism for $q \in \{0, 1\}$.

Proof. Let $KH^0_a(X_1, \mathbb{Z}/p^r\mathbb{Z})$ denote the so called Kato homology groups, i.e. the homology in degree a of the complex $C^0_{p^r}$ defined in [17]. By [16, Lem. 6.2] (see also [26, Sec. 9]) there is a long exact sequence

$$\cdots \to KH^0_{q+2}(X_1, \mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{CH}^d(X_1, q; \mathbb{Z}/p^r\mathbb{Z}) \xrightarrow{\rho^{d,q}_{X_{\mathrm{Zar}}}} H^{2d-q}_{\text{ét}}(X_1, \mathbb{Z}/p^r\mathbb{Z}(d)) \to KH^0_{q+1}(X_1, \mathbb{Z}/p^r\mathbb{Z}) \to \mathrm{CH}^d(X_1, q-1; \mathbb{Z}/p^r\mathbb{Z}) \to H^{2d-q+1}_{\text{ét}}(X_1, \mathbb{Z}/p^r\mathbb{Z}(d)) \to \cdots$$

where $\mathbb{Z}/p^r\mathbb{Z}(d) = W_r\Omega^d_{X_1,\log}[-d]$. We first identify the group

$$\mathrm{CH}^d(X_1, q; \mathbb{Z}/p^r\mathbb{Z}) \quad \text{with} \quad H^{2d-q}_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log}[-d]) \quad \text{for} \quad q = 0, 1.$$

Consider the spectral sequence

$$^{\mathrm{CH}}E^{p,q}_1(X_1) = \oplus_{x \in X_1^{(p)}} \mathrm{CH}^{d-p}(\mathrm{Spec} k(x), -p-q, \mathbb{Z}/p^r\mathbb{Z}) \Rightarrow \mathrm{CH}^d(X_1, -p-q, \mathbb{Z}/p^r\mathbb{Z})$$

from [2, Sec. 10] and note that

$$\mathrm{CH}^a(\mathrm{Spec} k(x), a, \mathbb{Z}/p^r\mathbb{Z}) \cong K^M_a(k(x))/p^r \cong W_r\Omega^a_{k(x),\log}$$

for all $a \geq 0$. The first isomorphism follows from [30, Thm. 4.9] (see also [37]) and the fact that $\mathrm{CH}^a(\mathrm{Spec} k(x), a, \mathbb{Z}/p^r\mathbb{Z}) \cong \mathrm{CH}^a(\mathrm{Spec} k(x), a) \otimes \mathbb{Z}/p^r\mathbb{Z}$. The second isomorphism follows from the Bloch-Gabber-Kato theorem (see [5]). This implies the identification since $\mathrm{CH}^0(k(x), 1) = 0$ and since

$$\bigoplus_{x \in X^0} i_{x*} W_r\Omega^d_{k(x),\log} \to \bigoplus_{x \in X^1} i_{x*} W_r\Omega^{d-1}_{k(x),\log} \to \cdots \to \bigoplus_{x \in X^d} i_{x*} W_r\Omega^0_{k(x),\log}$$

is a (Gersten-)resolution for the sheaf $W_r\Omega^d_{X_1,\log}$ considered in the Zariski topology (see [11]) and therefore also in the Nisnevich topology. In particular $H^i_{\mathrm{Zar}}(X_1, W_r\Omega^d_{X_1,\log}) \cong H^i_{\mathrm{Nis}}(X_1, W_r\Omega^d_{X_1,\log})$ for all $i \geq 0$. Note furthermore that $\rho^{d,q}_{X_{\mathrm{Zar}}}$ factors through $\rho^{d,q}_{X_{\mathrm{Nis}}}$ since it comes from the change of sites $\epsilon : X_{\text{ét}} \to X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$. In fact, Nisnevich and Zariski motivic cohomology coincide.

Now the Kato homology groups $KH^0_i(X_1, \mathbb{Z}/p^r\mathbb{Z})$ vanishes for $1 \leq i \leq 4$ by [16, Thm. 0.3] (see also [26, Thm. 8.1]) which implies the lemma. □

Lemma 6.8. *Let $j < p$. Then*

$$H^{j+d}_{\mathrm{Nis}}(X_1, \mathcal{S}_{X.}(j)) \to H^{j+d}_{\text{ét}}(X_1, \mathcal{S}_{X.}(j))$$

is an isomorphism for all $j \geq d$.

Proof. By [4, Thm 5.4] we have an exact triangle

$$p(j)\Omega^{\leq j}_{X.}[-1] \to S_{X.}(j)_{\mathrm{Nis}} \to W.\Omega^j_{X_1,\log}[-j] \xrightarrow{[1]} \cdots$$

in $\mathrm{D}_{\mathrm{pro}}(X_1)_{\mathrm{Nis}}$ which comes from the exact triangle

$$p(j)\Omega^{\leq j}_{X.}[-1] \to S_{X.}(j)_{\text{ét}} \to W.\Omega^j_{X_1,\log}[-j] \xrightarrow{[1]} \cdots$$

in $\mathrm{D}_{\mathrm{pro}}(X_1)_{\text{ét}}$ by applying the functor $\tau_{\leq j} \circ R\epsilon_*$. This induces the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^{d+j-1}_{\mathrm{Nis}}(X_1, W.\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, p(j)\Omega^{\leq j}_{X.}[-1]) & \longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, S_{X.}(j)_{\mathrm{Nis}}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow \\
H^{d+j-1}_{\text{ét}}(X_1, W.\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j}_{\text{ét}}(X_1, p(j)\Omega^{\leq j}_{X.}[-1]) & \longrightarrow & H^{d+j}_{\text{ét}}(X_1, S_{X.}(j)_{\text{ét}})
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{d+j}_{\mathrm{Nis}}(X_1, W.\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j+1}_{\mathrm{Nis}}(X_1, p(j)\Omega^{\leq j}_{X.}[-1]) \\
& \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
\longrightarrow & H^{d+j}_{\text{ét}}(X_1, W.\Omega^j_{X_1,\log}[-j]) & \longrightarrow & H^{d+j+1}_{\text{ét}}(X_1, p(j)\Omega^{\leq j}_{X.}[-1])
\end{array}$$

Now α and γ are isomorphisms by Lemma 6.7 and the fact that

$$W.\Omega^j_{X_1,\log}[-j] = 0 \quad \text{for} \quad j > d.$$

The maps β and δ are isomorphisms since $p(j)\Omega^{\leq j}_{X.}[-1]$ is a complex of coherent sheaves. The result follows by the five-lemma. $\square$

Proposition 6.9. *Let $p > j$ and $j \geq d$. Then the map*

$$\text{“}\lim_n\text{''} H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j))$$

is an isomorphism of pro-abelian groups.

Proof. It follows from Lemma 6.6 and Lemma 6.8 that

$$H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_\cdot(j)).$$

Tensoring with $\mathbb{Z}/p^r$ gives the desired result: Since $\otimes\mathbb{Z}/p^r$ is right exact and the cohomology group on the left is taken in the top degree, it follows that

$$\text{“}\lim_n\text{”}\, H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \cong H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_\cdot} \otimes \mathbb{Z}/p^\cdot) \otimes \mathbb{Z}/p^r.$$

For the right side note first that $\mathcal{S}_\cdot(j)_{\text{ét}} \otimes \mathbb{Z}/p^r \cong \mathcal{S}_r(j)_{\text{ét}}$. This follows for example from [8, Thm. 1.3]. Now consider the short exact sequence

$$\begin{aligned} 0 &\to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(j))/p^r \\ &\to H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(j) \otimes^{\mathbb{L}} \mathbb{Z}/p^r) \to H^{d+j+1}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(j))[p^r] \to 0 \end{aligned}$$

and note that $H^{d+j+1}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(j))[p^r] = 0$. For $j \geq d+1$ this is clear since in that case $H^{d+j+1}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(j)) = 0$. For $j = d$ we have that

$$\begin{aligned} H^{2d+1}_{\text{ét}}(X_1, \mathcal{S}_{X_\cdot}(d)) &\cong H^{2d+1}_{\text{ét}}(X_1, W_\cdot\Omega^d_{X_1,\log}[-d]) \\ &\cong KH^0_0(X_1, \mathbb{Z}/p^\cdot\mathbb{Z}) \cong \mathbb{Z}/p^\cdot\mathbb{Z}. \end{aligned}$$

Here the first isomorphism follows from the diagram in the proof of Lemma 6.8, the second from the long exact sequence in the proof of Lemma 6.7 extended to the right and the third from the Kato conjectures (see [26, Thm. 8.1]). But $\mathbb{Z}/p^\cdot\mathbb{Z}[p^r] = 0$ since for every $\mathbb{Z}/p^m\mathbb{Z}[p^r] = 0$ we can find an m' (f.e. $m' = m + r$) such that $\mathbb{Z}/p^{m'}\mathbb{Z}[p^r] \to \mathbb{Z}/p^m\mathbb{Z}[p^r]$ is the zero map. □

Proposition/Definition 6.10. *Diagram (2) of the introduction commutes.*

Proof. Let X_K be the generic fiber of X. Let $i : X_1 \hookrightarrow X$ and $j : X_K \hookrightarrow X$ be the canonical inclusions.

The exact Kummer sequence

$$0 \to \mu_{p^n} \to \mathcal{O}^\times_{X_K} \xrightarrow{p^n} \mathcal{O}^\times_{X_K} \to 0$$

on $X_{K,\text{ét}}$ induces an exact sequence

$$j_*\mathcal{O}^\times_{X_K} \xrightarrow{p^n} j_*\mathcal{O}^\times_{X_K} \to R^1 j_*\mu_{p^n} \to 0$$

on $X_{\text{ét}}$ which induces a Galois symbol map

$$j_*\mathcal{K}^M_{q,X_K} \to R^q j_*(\mathbb{Z}/p^r\mathbb{Z}(q))$$

(see [5, (1.2)]). This map induces a map

$$\mathcal{K}^M_{q,X} \to i_*\mathrm{ker}(\sigma^q_{X,r} : i^*R^q j_*(\mathbb{Z}/p^r\mathbb{Z}(q)) \to W_r\Omega^{q-1}_{X_1,\log}) \cong \mathcal{H}^q(\mathcal{T}_r(q))$$

in the étale topology. For the definition of $\sigma^q_{X,r}$ see [36, Sec. 3.2] and for the isomorphism on the right see [36, Def. 4.2.4]. Furthermore, $\ker(\sigma^q_{X,r} : i^*R^q j_*(\mathbb{Z}/p^r\mathbb{Z}(q)) \to W_r\Omega^{q-1}_{X_1,\log}) \cong \mathcal{H}^q(\mathcal{S}_r(q))$ by [19]. We have the following commutative diagram in the étale topology:

$$\begin{array}{ccc} \mathcal{K}^M_{q,X}/p^r & \longrightarrow & i_*\mathcal{K}^M_{q,X_{r+1}}/p^r \\ \downarrow & & \downarrow (*) \\ \mathcal{H}^q(\mathcal{T}_r(q)) & \xrightarrow{\cong} & i_*\mathcal{H}^q(\mathcal{S}_r(q)) \end{array}$$

Here $(*)$ is induced by Kato's syntomic regulator map ([19, Sec. 3]) and the commutativity follows from [19, Lem. 4.2].

Taking cohomology groups we get the following commutative diagram:

$$\begin{array}{ccc} H^d_{\mathrm{Nis}}(X, \mathcal{K}^M_{q,X}/p^r) & \longrightarrow & H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{q,X_{r+1}}/p^r) \\ \downarrow & & \downarrow (*) \\ H^d_{\mathrm{Nis}}(X, \mathcal{H}^q(\mathcal{T}_r(q)_{\mathrm{Nis}})) & \xrightarrow{\cong} & H^d_{\mathrm{Nis}}(X_1, \mathcal{H}^q(\mathcal{S}_r(q)_{\mathrm{Nis}})) \\ \downarrow \cong & & \downarrow \cong \\ H^{d+q}_{\mathrm{Nis}}(X, \mathcal{T}_r(q)_{\mathrm{Nis}}) & \xrightarrow{\cong} & H^{d+q}_{\mathrm{Nis}}(X_1, \mathcal{S}_r(q)_{\mathrm{Nis}}) \\ \downarrow & & \downarrow \\ H^{d+q}_{\mathrm{\acute{e}t}}(X, \mathcal{T}_r(q)) & \xrightarrow{\cong} & H^{d+q}_{\mathrm{\acute{e}t}}(X_1, \mathcal{S}_r(q)), \end{array}$$

where $\mathcal{T}_r(q)_{\mathrm{Nis}} := \tau_{\leq q} R\epsilon_* \mathcal{T}_r(q)_{\mathrm{\acute{e}t}}$. The lower horizontal isomorphism follows from proper base change and the fact that $i^*\mathcal{T}_r(n) \cong \mathcal{S}_r(n)$ if $p > n+1$. For the isomorphism on the right see the proof of Proposition 6.6. The change of sites $\epsilon : X_{\mathrm{Nis}} \to X_{\mathrm{Zar}}$ now gives the result. □

As a corollary we get the following result:

Corollary 6.11. *Let $j+1 < p$. Then the cycle class map*

$$\varrho^{j,j-d}_{p^r} : \mathrm{CH}^j(X, j-d, \mathbb{Z}/p^r\mathbb{Z}) \to H^{d+j}_{\mathrm{\acute{e}t}}(X, \mathcal{T}_r(j))$$

is surjective for all $j \geq d$.

Proof. By Corollary 5.5 and Remark 5.3 the map

$$\begin{aligned} res : \mathrm{CH}^j(X, j-d, \mathbb{Z}/p^r\mathbb{Z}) &\to \text{“}\mathrm{lim}_n\text{”} H^d_{\mathrm{Zar}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \\ &\cong H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \end{aligned}$$

is surjective for $j-1 < p$. By Proposition 6.9 we have that

$$\text{“}\mathrm{lim}_n\text{”} H^d_{\mathrm{Nis}}(X_1, \mathcal{K}^M_{j,X_n}/p^r) \cong H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j)) \quad \text{for} \quad j < p.$$

Furthermore, for $j+1 < p$ we have that $H^{d+j}_{\text{ét}}(X_1, \mathcal{S}_r(j)) \cong H^{d+j}_{\text{ét}}(X, \mathcal{T}_r(j))$ (see [36, Sec. 1.4]). The result now follows from the commutativity of (2). □

Remark 6.12. As we noted in the introduction, Saito and Sato show in [34] that the the cycle class map

$$\varrho^{d,0}_{p^r} : \mathrm{CH}^d(X)/p^r \to H^{2d}_{\text{ét}}(X, \mathcal{T}_r(d))$$

defined in [36, Cor. 6.1.4] is surjectiv for X a regular scheme which is proper, flat, of finite type and which has semistable reduction over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a p-adic local field K. We expect that this map coincides with the map defined in Proposition 6.10.

Finally we note the following injectivity result for curves:

Proposition 6.13. *Let X be smooth projective of relative dimension* 1 *over a p-adic local ring A. Then the cycle class map*

$$\varrho^{1,0}_{p^r} : \mathrm{CH}_1(X)/p^r \to H^2_{\text{ét}}(X, \mathcal{T}_r(1))$$

is injective.

Proof. This follows immediately from the spectral sequence

$$E_1^{u,v} = \oplus_{x \in X^u} H^{v+u}_{\text{ét},x}(X, \mathcal{T}_r(d)) \Rightarrow H^{v+u}_{\text{ét}}(X, \mathcal{T}_r(d))$$

since by absolute cohomological purity and the purity property of p-adic étale Tate twists $E_2^{1,1} \cong \mathrm{CH}_1(X)/p^r$ (see [35]). □

Under the assumptions of Corollary 5.8 we therefore get a sequence of isomorphisms

$$H^2_{\text{ét}}(X, \mathcal{T}_r(1)) \xleftarrow{\cong} \mathrm{CH}_1(X)/p^r \xrightarrow{\cong} \text{“}\lim_n\text{”} H^1(X_1, \mathcal{K}^M_{X_n,1}/p^r).$$

The isomorphism on the left, induced by $\varrho^{1,0}_{p^r}$, follows from Proposition 6.13 and the theorem of Saito and Sato mentioned in the introduction and the isomorphism on the right follows from Corollary 5.8. It would be interesting to have a similar result for $\mathrm{CH}^2(X, 1, \mathbb{Z}/p^r\mathbb{Z})$.

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References

[1] M. Artin and B. Mazur, Etale Homotopy, Lecture Notes in Mathematics, No. 100, Springer-Verlag, Berlin-New York (1969).

[2] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304.

[3] S. Bloch, H. Esnault, and M. Kerz, *Deformation of algebraic cycle classes in characteristic zero*, Algebr. Geom. **1** (2014), no. 3, 290–310.

[4] S. Bloch, H. Esnault, and M. Kerz, *p-adic deformation of algebraic cycle classes*, Invent. Math. **195** (2014), no. 3, 673–722.

[5] S. Bloch and K. Kato, *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. (1986), no. 63, 107–152.

[6] J.-L. Colliot-Thélène, J.-J. Sansuc, and C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. **50** (1983), no. 3, 763–801.

[7] H. Esnault and O. Wittenberg, *On the cycle class map for zero-cycles over local fields*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 2, 483–520. With an appendix by Spencer Bloch.

[8] T. Geisser, *Motivic cohomology over Dedekind rings*, Math. Z. **248** (2004), no. 4, 773–794.

[9] T. Geisser and M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), no. 3, 459–493.

[10] P. Gille and T. Szamuely, Central Simple Algebras and Galois Cohomology, Vol. 101 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2006). ISBN 978-0-521-86103-8; 0-521-86103-9.

[11] M. Gros and N. Suwa, *La conjecture de Gersten pour les faisceaux de Hodge-Witt logarithmique*, Duke Math. J. **57** (1988), no. 2, 615–628.

[12] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

[13] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.

[14] R. Hartshorne, Residues and Duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard (1963/64). With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York (1966).

[15] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 501–661.

[16] U. Jannsen and S. Saito, *Kato conjecture and motivic cohomoogy over finite fields*, preprint.

[17] K. Kato, *A Hasse principle for two-dimensional global fields*, J. Reine Angew. Math. **366** (1986), 142–183. With an appendix by Jean-Louis Colliot-Thélène.

[18] K. Kato, *Milnor K-theory and the Chow group of zero cycles*, in: Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colo., 1983), Vol. 55 of Contemp. Math., 241–253, Amer. Math. Soc., Providence, RI (1986).

[19] K. Kato, *On p-adic vanishing cycles (application of ideas of Fontaine-Messing)*, in: Algebraic Geometry, Sendai (1985), Vol. 10 of Adv. Stud. Pure Math., 207–251, North-Holland, Amsterdam (1987).

[20] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275.

[21] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, in: Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, II (Boulder, Colo., 1983), Vol. 55 of Contemp. Math., 255–331, Amer. Math. Soc., Providence, RI (1986).

[22] M. Kerz, *The Gersten conjecture for Milnor K-theory*, Invent. Math. **175** (2009), no. 1, 1–33.

[23] M. Kerz, *Milnor K-theory of local rings with finite residue fields*, J. Algebraic Geom. **19** (2010), no. 1, 173–191.

[24] M. Kerz, *Ideles in higher dimension*, Math. Res. Lett. **18** (2011), no. 4, 699–713.

[25] M. Kerz, H. Esnault, and O. Wittenberg, *A restriction isomorphism for cycles of relative dimension zero*, Camb. J. Math. **4** (2016), no. 2, 163–196.

[26] M. Kerz and S. Saito, *Cohomological Hasse principle and motivic cohomology for arithmetic schemes*, Publ. Math. Inst. Hautes Études Sci. **115** (2012), 123–183.

[27] Q. Liu, Algebraic Geometry and Arithmetic Curves, Vol. 6 of Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford (2002). ISBN 0-19-850284-2. Translated from the French by Reinie Erné, Oxford Science Publications.

[28] M. Lüders, *A restriction isomorphism for zero-cycles with coefficients in Milnor K-theory*, Camb. J. Math. **7** (2019), no. 1-2, 1–31.

[29] M. Lüders, *Deformation theory of the Chow group of zero-cycles*, preprint, (2018).

[30] Y. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.

[31] J. Neukirch, Algebraische Zahlentheorie, Springer-Verlag, Berlin (1992). ISBN 3-540-54273-6.

[32] I. A. Panin, *The equicharacteristic case of the Gersten conjecture*, Tr. Mat. Inst. Steklova **241** (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 169–178.

[33] S. Saito and K. Sato, *A finiteness theorem for zero-cycles over p-adic fields*, Ann. of Math. (2) **172** (2010), no. 3, 1593–1639. With an appendix by Uwe Jannsen.

[34] S. Saito and K. Sato, *Zero-cycles on varieties over p-adic fields and Brauer groups*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 3, 505–537.

[35] K. Sato, *Higher-dimensional arithmetic using p-adic étale Tate twists*, Homology Homotopy Appl. **7** (2005), no. 3, 173–187.

[36] K. Sato, *p-adic étale Tate twists and arithmetic duality*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 519–588. With an appendix by Kei Hagihara.

[37] B. Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189.

[38] C. A. Weibel, The K-Book, Vol. 145 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI (2013). ISBN 978-0-8218-9132-2. An introduction to algebraic K-theory.

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