

Positivity of Shimura operators

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In this paper we answer some questions of G. Shimura on the positivity of certain invariant differential operators that play a key role in the study of nearly holomorphic automorphic forms in higher rank. Our main result expresses the eigenvalues of Shimura operators in terms of certain polynomials introduced by A. Okounkov, which leads to an explicit answer to Shimura's questions.

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1. Introduction

In this paper we answer some questions of Shimura [24] concerning the positivity of certain differential operators on a Hermitian symmetric space $X = G/K$. These operators $\{\mathcal{L}_\lambda\}$, defined in (3.5) below, constitute a distinguished linear basis of the algebra $\mathbf{D}(X)$ of G -invariant differential operators on X , and have come to be known as Shimura operators in the literature. However despite several partial results [14, 19, 24, 29] the main questions of Shimura have remained unsolved for almost 30 years. The techniques

so far have involved representation theory, harmonic analysis, and explicit case-by-case computations; the novelty of our approach lies in incorporating ideas from algebraic combinatorics. Our main result relates the spectrum of Shimura operators to certain polynomials introduced by A. Okounkov [13] in Macdonald theory. This gives an explicit combinatorial formula for the spectrum and provides an answer to Shimura's questions.

Shimura's motivation for the study of these differential operators was to find a multivariate generalization of the theory of nearly holomorphic forms [21–24]. For a discussion of the classical (one variable) theory, which dates back to the work of Ramanujan on quasimodular forms, we refer the reader to the excellent survey by Zagier [28]. It would be extremely interesting to explore applications of our explicit formulas to multivariate nearly holomorphic forms, and while this aspect is not discussed in the present paper we hope to return to it in future work. We note that Okounkov polynomials also arise in the context of the “quadratic” Capelli operators recently studied by one of the authors together with H. Salmasian [17], and it would be interesting to understand their relation with Shimura operators.

The structure of the paper is as follows. In the next section we give a precise formulation of the main results, including the explicit description of the positivity domain, which we refer to as the Shimura set. The next three sections are devoted to the proofs of these results. In Section 3 we recall the necessary background on Hermitian symmetric spaces and invariant differential operators. In Section 4 we discuss symmetric functions and various properties on Okounkov polynomials, including some new results. In Section 5 we prove our main results relating Shimura operators and Okounkov polynomials, including the determination of the Shimura set. In Section 6 we study the precise description of the Shimura set for classical rank two domains, in particular for G/K with $G = SU(n, 2)$ in greater detail, and show that it can be described by *finitely* many non-polynomial inequalities.

2. Statement of the main results

In order to describe Shimura's question, and our answer, it is convenient to introduce the following notation for “partitions”:

$$(2.1) \quad \Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}, \quad |\lambda| = \lambda_1 + \cdots + \lambda_n.$$

We will write 1^j for the partition $(1, \dots, 1, 0, \dots, 0)$ with j “ones.”

Now suppose G/K is an irreducible Hermitian symmetric space of rank n . Let \mathfrak{g} and \mathfrak{k} denote the complexified Lie algebras of G and K , and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$$

be the corresponding Cartan decomposition. Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{k} , then \mathfrak{t} is also a Cartan subalgebra of \mathfrak{g} ; furthermore there is a distinguished family of strongly orthogonal roots for \mathfrak{t} in \mathfrak{p}^+

$$\{\gamma_1, \dots, \gamma_n\} \subseteq \Delta(\mathfrak{t}, \mathfrak{p}^+)$$

called the Harish-Chandra roots; see e.g. [4, pp. 66-68].

Now \mathfrak{p}^+ and \mathfrak{p}^- are abelian Lie algebras, which are contragredient as K -modules. Let W_λ be the K -module with highest weight $\sum_i \lambda_i \gamma_i$ and let W_λ^* be its contragredient, then by a result of Schmid we have K -module isomorphisms

$$U(\mathfrak{p}^+) \approx S(\mathfrak{p}^+) \approx \oplus_{\lambda \in \Lambda} W_\lambda, \quad U(\mathfrak{p}^-) \approx S(\mathfrak{p}^-) \approx \oplus_{\lambda \in \Lambda} W_\lambda^*.$$

Let u_λ denote the image of $1 \in \text{End}(W_\lambda)$ under the sequence of maps

$$\text{End}(W_\lambda) \approx W_\lambda^* \otimes W_\lambda \hookrightarrow U(\mathfrak{p}^-) \otimes U(\mathfrak{p}^+) \xrightarrow{\text{mult}} U(\mathfrak{g}).$$

Then u_λ belongs to $U(\mathfrak{g})^K$ and its right action on G descends to an operator $\mathcal{L}_\lambda \in \mathbf{D}(G/K)$. In fact $\{\mathcal{L}_\lambda : \lambda \in \Lambda\}$ is a linear basis and $\{\mathcal{L}_{1^j} : j = 1, \dots, n\}$ is an independent generating set for $\mathbf{D}(G/K)$. These are the Shimura operators.

The algebra $\mathbf{D}(G/K)$ is commutative and its eigenfunctions are the spherical functions Φ_x . These are parametrized by the set \mathfrak{a}^*/W_0 , where $\mathfrak{a} \subseteq \mathfrak{p}$ is a Cartan subspace and $W_0 = W(\mathfrak{a}, \mathfrak{g})$ is the restricted Weyl group. More precisely the parameter $x \in \mathfrak{a}^*$ defines an irreducible spherical subquotient $J(x)$ of a minimal principal series representation of G and Φ_x is its spherical matrix coefficient. We write \mathcal{U} for the set of spherical unitary parameters

$$\mathcal{U} = \{x \in \mathfrak{a}^* \mid J(x) \text{ is unitarizable}\}.$$

The determination of \mathcal{U} is an important problem, which is as yet unsolved in complete generality. In this connection, the Shimura operators have the following positivity property (see Proposition 5.1 below). Let $c_{\lambda,x}$ denote the

eigenvalue of the modified operator $\mathcal{L}'_\lambda = (-1)^{|\lambda|} \mathcal{L}_\lambda$ on Φ_x , then we have

$$(2.2) \quad c_{\lambda,x} \geq 0 \text{ for all } x \in \mathcal{U}.$$

Motivated in part by (2.2), Shimura asked for a determination of the sets

$$(2.3) \quad \mathcal{A} = \{x : c_{\lambda,x} \geq 0 \text{ for all } \lambda\},$$

$$(2.4) \quad \mathcal{G} = \{x : c_{1^j,x} \geq 0 \text{ for all } j\}.$$

Evidently we have $\mathcal{U} \subseteq \mathcal{A} \subseteq \mathcal{G}$ but these sets are quite different in general. In this paper we give an explicit formula for $c_{\lambda,x}$, thereby answering Shimura’s question. To describe our answer we need some further notation. First, by a classical result of Harish-Chandra we have

$$D\Phi_x = \eta_D(x) \Phi_x,$$

where $\eta_D = \eta(D)$ is the image of D under the Harish-Chandra homomorphism

$$(2.5) \quad \eta : \mathbf{D}(G/K) \rightarrow S(\mathfrak{a})^{W_0} \approx P(\mathfrak{a}^*)^{W_0}.$$

Thus we have $c_{\lambda,x} = c_\lambda(x)$, where

$$c_\lambda = \eta(\mathcal{L}'_\lambda) = (-1)^{|\lambda|} \eta(\mathcal{L}_\lambda),$$

and so the determination of the sets (2.3, 2.4) reduces to the determination of $\eta(\mathcal{L}_\lambda)$.

Our first result relates the Shimura operators to certain polynomials $P_\lambda(x; \tau, \alpha)$. These polynomials, denoted $P_\lambda^{ip}(x; \tau, \alpha)$ in [9], are the $q \rightarrow 1$ limits of a family of polynomials introduced by A. Okounkov [13, Definition 1.1], and they generalize an analogous family defined by one of the authors in connection with the Capelli identity [10, 11, 15] and studied together with F. Knop [8]. To describe the P_λ it is convenient to define $\delta = (n - 1, \dots, 1, 0)$ and to set

$$(2.6) \quad \rho = \rho_{\tau,\alpha} = (\rho_1, \dots, \rho_n), \quad \rho_i = \tau\delta_i + \alpha = \tau(n - i) + \alpha.$$

The polynomial $P_\lambda(x; \tau, \alpha)$ has total degree $2|\lambda|$ in x_1, \dots, x_n , its coefficients are rational functions in two parameters τ and α , it is even and symmetric, i.e. invariant under all permutations and sign changes of the x_i , and among

all such polynomials it is characterized up to scalar multiple by its vanishing at points of the form

$$\{\mu + \rho : \mu \in \Lambda, |\mu| \leq |\lambda|, \mu \neq \lambda\}.$$

For generic τ the set $\{P_\lambda; \lambda \in \Lambda\}$ is a linear basis of the space \mathcal{Q} of even symmetric polynomials.

Now suppose G/K is a Hermitian symmetric space as before. Then the restricted root system $\Sigma(\mathfrak{a}, \mathfrak{g})$ is of type BC_n , with (potentially) three root lengths, and we fix a choice of positive roots. The positive long roots have multiplicity 1 and constitute a basis of \mathfrak{a}^* , thus we may use them to identify \mathfrak{a}^* with \mathbb{C}^n . This identifies W_0 with the group of all permutation and sign changes, and $P(\mathfrak{a}^*)^{W_0}$ with the algebra \mathcal{Q} . Moreover if we denote the multiplicities of short and medium roots by $2b$ and d respectively, then the half-sum of positive roots $\Sigma(\mathfrak{a}, \mathfrak{g})$ is given by $\rho = \rho_{\tau, \alpha}$ as in (2.6) where

$$(2.7) \quad \tau = d/2, \quad \alpha = (b + 1)/2.$$

Theorem 2.1. *Let G/K be a Hermitian symmetric space with τ, α as in (2.7), then*

$$\eta(\mathcal{L}_\lambda) = k_\lambda P_\lambda(x; \tau, \alpha),$$

where $k_\lambda = k_\lambda(\tau, \alpha)$ is an explicit positive constant described in (4.2).

This is proved more generally for line bundles on G/K in Theorem 5.5 below. In view of this, we introduce the signed versions of the Okounkov polynomials

$$(2.8) \quad q_\lambda(x) := q_\lambda(x; \tau, \alpha) = (-1)^{|\lambda|} P_\lambda(x; \tau, \alpha).$$

Corollary 2.2. *The Shimura sets are given explicitly as follows:*

$$\begin{aligned} \mathcal{A} &= \{x : q_\lambda(x) \geq 0 \text{ for all } \lambda\}, \\ \mathcal{G} &= \{x : q_{1^j}(x) \geq 0 \text{ for all } j\}. \end{aligned}$$

This is Corollary 5.7 below.

Since one has explicit formula for P_λ , and hence q_λ , recalled in (4.1) below, this gives a complete characterization of the Shimura sets. In particular, we obtain the following explicit description of \mathcal{G} . If I is a subset of $\{1, \dots, n\}$ with j elements $i_1 < \dots < i_j$, then we define

$$\varphi_I(x) = \prod_{k=1}^j \left[(\rho_{i_k+j-k})^2 - x_{i_k}^2 \right], \quad \varphi_j(x) = \sum_{|I|=j} \varphi_I(x).$$

Theorem 2.3. *We have $q_{1j} = \varphi_j$ for all j , and hence*

$$\mathcal{G} = \{x : \varphi_j(x) \geq 0 \text{ for all } j\}.$$

This is Theorem 5.8 below.

The description of \mathcal{A} involves infinitely many polynomial inequalities $q_\lambda \geq 0$, and it is natural to ask whether \mathcal{A} can in fact be described by a *finite* set of inequalities. While we do not know the answer to this question in general, we show below that this is indeed the case for the real points in \mathcal{A} for the rank 2 groups $U(m, 2)$. However for $m > 2$ the characterization involves *non-polynomials* in an essential way. This is in sharp contrast with the unitary parameter set \mathcal{U} , whose description involves only *linear* functions (see Remark 5.9 below).

We give two independent derivations of this result. The first depends on a formula for $P_\lambda(x; \tau, \alpha)$ for $n = 2$ as a hypergeometric polynomial [9]. By symmetry it suffices to describe the sets

$$\mathcal{A}_0 = \mathcal{A} \cap \mathcal{C}, \quad \mathcal{G}_0 = \mathcal{G} \cap \mathcal{C}, \quad \mathcal{C} = \{x \in \mathbb{R}^n : x_1 \geq \dots \geq x_n \geq 0\}.$$

For $G = U(m, 2)$ we have

$$(\rho_1, \rho_2) = (\alpha + \tau, \alpha) = \left(\frac{m + 1}{2}, \frac{m - 1}{2} \right),$$

and we introduce two triangular regions as below

$$\begin{aligned} T_1 &= \{x \mid \rho_2 \geq x_1 \geq x_2 \geq 0\}, \\ T_2 &= \{x \mid x_1 \geq x_2 \geq \rho_2, \quad x_1 + x_2 \leq \rho_1 + \rho_2\}. \end{aligned}$$

We also write $(a)_k = a(a + 1) \cdots (a + k - 1)$ for the Pochhammer symbol of product of increasing factors and $(a)_k^- = a(a - 1) \cdots (a - k + 1)$ for the decreasing factors, and define

$$R(x_1, x_2) = \sum_{k=0}^{\infty} \frac{(\rho_2 + x_2)_k (\rho_2 - x_2)_k}{(\rho_1 + x_1)_k (\rho_1 - x_1)_k}.$$

Theorem 2.4. *For $G = U(m, 2)$ we have*

- 1) $\mathcal{G}_0 = T_1 \cup T_2$,
- 2) $\mathcal{A}_0 = T_1 \cup W$ where $W = \{x \in T_2 \mid R(x) \geq 0\}$.

This is proved in Theorems 6.2 and 6.4.

In Section 6.2 we give an alternative description of W in terms of the functions

$$s(t) = s_m(t) = \frac{\sin \pi t}{(t+1)_m}, \quad S(x_1, x_2) = \frac{s(x_1) - s(x_2)}{x_1 - x_2}.$$

Theorem 2.5. *The set W of Theorem 2.4 can also be described as follows:*

$$W = \{x \in T_2 \mid S(x_1 - \alpha, x_2 - \alpha) \geq 0\}.$$

This description of W is facilitated by a Weyl-type formula for the Okounkov polynomials $P_\lambda(x; \tau, \alpha)$ for $\tau = 1$, that we describe below. It turns out that for $\tau = 1$, the P_λ can be expressed in terms of rank 1 Okounkov polynomials, which are given explicitly as follows

$$p_l(z; \alpha) = \prod_{k=0}^{l-1} [z^2 - (k + \alpha)^2].$$

For $\mu \in \Lambda$ we define the *alternant* a_μ to be the determinant of the $n \times n$ matrix

$$[p_{\mu_j}(x_i; \alpha)]_{i,j=1}^n.$$

For $\delta = (n - 1, \dots, 1, 0)$, the alternant is in fact the Vandermonde determinant

$$a_\delta(x; \alpha) = a_\delta(x) = \prod_{i < j} (x_i^2 - x_j^2).$$

Theorem 2.6. *For $\tau = 1$ we have $P_\lambda(x; \tau, \alpha) = \frac{a_{\lambda+\delta}(x; \alpha)}{a_\delta(x)}$.*

This is proved in Theorem 4.6 below.

Corollary 2.7. *For $G = U(m, n)$ we have $(\tau, \alpha) = (1, \frac{m-n+1}{2})$ and*

$$\mathcal{A} = \left\{ x : (-1)^{|\lambda|} \frac{a_{\lambda+\delta}(x; \alpha)}{a_\delta(x; \alpha)} \geq 0 \text{ for all } \lambda \right\}.$$

This description of \mathcal{A} , although still infinite, is much more explicit. It plays a key role in the proof of Theorem 2.5, which involves a study of the limiting behavior of $a_{\lambda+\delta}(x; \alpha)$.

3. Hermitian symmetric spaces and Shimura operators

We shall introduce the Shimura operators following [24]. See also [29] for further details and references.

3.1. Lie algebras of Hermitian type

We will denote real Lie algebras by $\mathfrak{g}_0, \mathfrak{k}_0$ etc. and denote their complexifications by $\mathfrak{g}, \mathfrak{k}$ etc. Let $(\mathfrak{g}_0, \mathfrak{k}_0)$ be an irreducible Hermitian symmetric pair of real rank n , and let

$$\mathfrak{g} = \mathfrak{p}^- + \mathfrak{k} + \mathfrak{p}^+$$

be its Harish-Chandra decomposition into $(-1, 0, 1)$ eigenspaces with respect to a suitable central element $Z \in \mathfrak{k}$. Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{k} , then \mathfrak{t} is also a Cartan subalgebra of \mathfrak{g} , and we fix a compatible choice of positive root systems satisfying

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{k}, \mathfrak{t}) \cup \Delta(\mathfrak{p}^+, \mathfrak{t}).$$

Let $\gamma_1, \dots, \gamma_n \in \Delta(\mathfrak{p}^+, \mathfrak{t})$ be the Harish-Chandra strongly orthogonal roots, and let $h_j \in \mathfrak{t}$ be coroot corresponding to γ_j . Then we have a commuting family of sl_2 -triples

$$\{h_j, e_j^+, e_j^-\}, \quad e_j^\pm \in \mathfrak{p}^\pm.$$

We fix an invariant bilinear form on \mathfrak{g} such that

$$(3.1) \quad (e_1^+, e_1^-) = 1.$$

Let $\mathfrak{t}_- = \sum_{j=1}^n \mathbb{C}h_j$ be the span of the h_j , and let \mathfrak{t}_+ be the orthogonal complement of \mathfrak{t}_- in \mathfrak{t} ; then we have an orthogonal decomposition $\mathfrak{t} = \mathfrak{t}_- + \mathfrak{t}_+$. We also define

$$e_j = e_j^- + e_j^+, \quad \mathfrak{a} = \sum_{j=1}^n \mathbb{C}e_j, \quad \mathfrak{h} = \mathfrak{a} + \mathfrak{t}_+.$$

Then \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} and \mathfrak{h} is a maximally split Cartan subalgebra of \mathfrak{g} . The restricted root system $\Sigma(\mathfrak{a}, \mathfrak{g})$ is of type BC_n ; more precisely, if $\{\varepsilon_j\} \subset \mathfrak{a}^*$ is the basis dual to $\{e_j\} \subset \mathfrak{a}$, then we have

$$\Sigma(\mathfrak{a}, \mathfrak{g}) = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\}.$$

The long roots $\pm 2\varepsilon_i$ have multiplicity 1, and they are conjugate to $\pm\gamma_j$ via the Cayley transform that carries \mathfrak{a} to \mathfrak{t}_- and \mathfrak{h} to \mathfrak{t} [4]. We denote the

multiplicity of the medium roots $\pm\varepsilon_i \pm \varepsilon_j$ and the short roots $\pm\varepsilon_i$ by d and $2b$ respectively, and we fix the following choice of positive roots

$$\Sigma^+(\mathfrak{a}, \mathfrak{g}) = \{\varepsilon_i\} \cup \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{2\varepsilon_i\}.$$

Then the half sum of positive roots is given as follows

$$\rho = \rho(\mathfrak{a}, \mathfrak{g}) = \sum 2\rho_i\varepsilon_i, \quad \rho_i = \frac{1}{2}[d(n - i) + 1 + b].$$

Let G be the connected Lie group of the adjoint group of \mathfrak{g} with Lie algebras \mathfrak{g}_0 , and K the corresponding subgroup with Lie algebra \mathfrak{k}_0 . Then G/K is a non-compact Hermitian symmetric space.

3.2. Schmid decomposition

As before we define

$$(3.2) \quad \Lambda = \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

and for $\lambda \in \Lambda$ we let W_λ denote the irreducible K -module with highest weight $\sum_j \lambda_j \gamma_j$.

We write $S(V)$ for the symmetric algebra of a vector space and $U(\mathfrak{s})$ for the enveloping algebra of a Lie algebra. We note that $\mathfrak{p}^- \approx (\mathfrak{p}^+)^*$ as K -modules, and also that since \mathfrak{p}^\pm are abelian we have $U(\mathfrak{p}^\pm) \approx S(\mathfrak{p}^\pm)$. By a result of Schmid (see e.g. [4]) we have multiplicity free K -module decompositions

$$U(\mathfrak{p}^+) \approx \oplus_{\lambda \in \Lambda} W_\lambda, \quad U(\mathfrak{p}^-) \approx \oplus_{\lambda \in \Lambda} W_\lambda^*.$$

The space of all holomorphic polynomials on \mathfrak{p}^+ is naturally identified with $S(\mathfrak{p}^-)$. It is naturally equipped with the Fock space norm [4], with the inner product on $\mathfrak{p}^- \subset \mathfrak{g}$ being normalized above. We denote the corresponding reproducing kernel for the subspace W_λ^* by $K_\lambda(z, w)$, $z, w \in \mathfrak{p}^+$.

3.3. Line bundles over G/K

In the subsequent discussion we will need to study equivariant line bundles on G/K . Such bundles correspond to multiplicative characters of K , and one has the following standard result.

Lemma 3.1. *There exists a unique character ι of K whose differential restricts to $\frac{1}{2}(\gamma_1 + \dots + \gamma_n)$ on \mathfrak{t}_- .*

Proof. See [18]. □

Remark 3.2. In geometric terms, the character ι is a generator of the Picard group of holomorphic line bundles on G/K . We have

$$(3.3) \quad \iota^{p_0}(k) = \det(\text{ad}(k)|_{\mathfrak{p}^+}), \quad p_0 = 2 + (n-1)d + b,$$

and thus ι is the p_0 -th root of the canonical line bundle on G/K . The differential $d\iota$ vanishes on \mathfrak{t}_+ iff G/K is tube type, otherwise $d\iota|_{\mathfrak{t}_+}$ is as in [18, Section 5, p. 288] (with $l = 1$ there).

For any integer p , we consider the character $\iota^p(k) = \iota(k)^p$ and we write $C^\infty(G/K, p)$ for the space of smooth sections of the corresponding holomorphic line bundle over G/K . Explicitly we have

$$C^\infty(G/K, p) = \{f \in C^\infty(G); f(gk) = \iota(k)^p f(g), g \in G, k \in K\}.$$

The group G acts on $C^\infty(G/K, p)$ via the left regular action.

Remark 3.3. In [29] the eigenvalues of the Shimura operators on G/K were studied by explicit computations. The line bundle parameter p here corresponds to $-\nu$ there. When $-p = \nu > p_0 - 1$ there is a holomorphic discrete series representation in the space $L^2(G/K, p)$, and it is in the common kernels of the "adjoint" Shimura operators \mathcal{M}_μ in (3.4) below. Similar results hold for the compact dual U/K of G/K .

3.4. The Cartan-Helgason theorem

Finite dimensional representations of \mathfrak{k} and \mathfrak{g} are parametrized by their highest weights as linear functionals on \mathfrak{t} . The representations we shall treat in the present paper have their highest weights being determined by their restriction to \mathfrak{t}_- , so if a linear function on \mathfrak{t} is of the form $\sum_{j=1}^n \lambda_j \gamma_j$ on \mathfrak{t}_- and is dominant with respect to the roots $\Delta^+(\mathfrak{k}, \mathfrak{t})$ respectively $\Delta^+(\mathfrak{g}, \mathfrak{t})$, then the corresponding representations of \mathfrak{k} and \mathfrak{g} will be denoted by W_λ respectively V_λ .

By the Cartan-Helgason theorem the finite dimensional K -spherical representations of G can be parameterized in terms of their highest weights restricted to Cartan subspace \mathfrak{a} , they are precisely of the form $\sum 2\lambda_i \varepsilon_i$, where λ ranges over the same set (3.2).

We shall need a generalization by Schlichtkrull of the Cartan-Helgason theory for the line bundle case. We call a vector $v \in V$ in a representation

(π, V) of G a (K, ι^p) -spherical vector if

$$\pi(k)v = \iota(k)^p v.$$

Then one has the following result [18, Th 7.2].

Lemma 3.4. *Let p be an integer. For each $\lambda \in \Lambda$ there is a unique representation $V_{\lambda,p}$ of G in $C^\infty(G/K, p)$ whose highest weight restricts to*

$$\sum_{j=1}^n \left(\lambda_j + \frac{|p|}{2} \right) \gamma_j \quad \text{on } \mathfrak{t}_-.$$

In particular each space $V_{\lambda,p}$ contains a unique $(K, \iota^{\pm p})$ -spherical vector $v_{\pm p}$ up to non-zero scalars.

We denote $W_{\lambda,p} = W_\lambda \otimes \mathbb{C}_{\frac{p}{2}}$, which is an irreducible representation of \mathfrak{k} . Notice that the highest weight of $V_{\lambda,p}$ is the same as $W_{\lambda,|p|}$. Also, $V_{\lambda,p}$ contains both (K, ι^p) and (K, ι^{-p}) and spherical vectors. (This is not true for infinite dimensional highest weight representations.) It will be convenient to treat the space $V_{\lambda,p}$ as an abstract representation, more precisely if $V_{\lambda,p}$ is any irreducible representation space of G with highest weight $\sum_j \lambda_j \gamma_j$ and containing a (K, ι^{-p}) -spherical normalized vector v_{-p} , then the map

$$v \in V_{\lambda,p} \rightarrow f_v(g) = (\pi_\lambda(g^{-1})v, v_{-p})$$

is a realization of $V_{\lambda,p}$ in the space $C^\infty(G/K, \iota^p)$. Here (\cdot, \cdot) is the Hermitian inner product in $V_{\lambda,p}$.

3.5. Shimura operators

Shimura operators are parametrized by the set Λ . More precisely for each $\mu \in \Lambda$ the Shimura operator corresponds to the identity element $1 \in \text{Hom}(W_\mu, W_\mu)$ via the multiplication in the universal enveloping algebra $U(\mathfrak{g})$

$$1 \in \text{End}(W_\mu) \approx W_\mu^* \otimes W_\mu \hookrightarrow U(\mathfrak{p}^-) \otimes U(\mathfrak{p}^+) \xrightarrow{\text{mult}} U(\mathfrak{g}).$$

Explicitly let $\{\xi_\alpha\}$ be a basis of $W_\mu \subset S(\mathfrak{p}^+)$ and $\{\eta_\alpha\}$ be the dual basis $W_\mu^* \subset S(\mathfrak{p}^-)$ and define

$$(3.4) \quad \mathcal{L}_\mu = \sum_\alpha \eta_\alpha \xi_\alpha, \quad \mathcal{M}_\mu = \sum_\alpha \xi_\alpha \eta_\alpha,$$

viewed as elements of $U(\mathfrak{g})^K$ acting in $C^\infty(G)$ (or $C^\infty(U)$ for the compact dual U of G) as left invariant differential operators.

Alternatively we may take an orthonormal basis ξ_α of $W_\mu \subset S(\mathfrak{p}^+)$ and the dual basis is then given by $\eta_\alpha = \xi_\alpha^*$, where $v \rightarrow v^*$ is the conjugation in \mathfrak{p} with respect to the real form \mathfrak{p}_0 extended to $S(\mathfrak{p})$. Thus we have

$$(3.5) \quad \mathcal{L}_\mu = \sum_\alpha \xi_\alpha^* \xi_\alpha, \quad \mathcal{M}_\mu = \sum_\alpha \xi_\alpha \xi_\alpha^*.$$

Note that the operators $(-1)^\mu \mathcal{L}_\mu$ and $(-1)^\mu \mathcal{M}_\mu$ descend to G -invariant differential operator on $C^\infty(G/K, \mathfrak{p})$ and are formally non-negative on $C^\infty(G/K, \mathfrak{p})$; they also define U -invariant differential operators on $C^\infty(U/K, \mathfrak{p})$ for the compact symmetric space U/K with \mathcal{L}_μ and \mathcal{M}_μ being non-negative instead. See Section 5 below.

For $\mu \in \Lambda$, the Shimura operator \mathcal{L}_μ and \mathcal{M}_μ have order $2|\mu|$ where

$$|\mu| = \mu_1 + \dots + \mu_n.$$

The Harish-Chandra homomorphism for invariant differential operators on $C^\infty(G/K)$ can be generalized to line bundles over G/K ; see e.g. [20, 24] and references therein. More precisely there exists a Weyl group invariant polynomial $\eta_p(\mathcal{L}_\mu)$, the Harish-Chandra image of the Shimura operator \mathcal{L}_μ , such that \mathcal{L}_μ acts on the irreducible representations $V_{\lambda, p} \subset C^\infty(G/K, \mathfrak{p})$ above by

$$(3.6) \quad \mathcal{L}_\mu v = \eta_p(\mathcal{L}_\mu) \left(\lambda + \frac{p}{2} + \rho \right) v, \quad v \in V_{\lambda, p}.$$

Furthermore $\eta_p(\mathcal{L}_\mu) \in \mathcal{Q}_{2|\mu|}$.

4. Symmetric functions and Okounkov polynomials

In this section we discuss some key properties of a family of polynomials introduced by Okounkov [13], or rather their $q \rightarrow 1$ limit as discussed [9, (7.2)]. These polynomials play an important role in the theory of symmetric functions, and they generalize an earlier family of polynomials introduced by one of us in [15], and studied in [8, 16].

Let $\mathbb{F} = \mathbb{Q}(\tau, \alpha)$ be the field of rational functions in τ, α . Consider the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ equipped with the natural action of the group

$W = S_n \times (\mathbb{Z}/2)^n$ by permutations and sign changes, and let

$$\mathcal{Q} = \mathbb{F}[x_1, \dots, x_n]^W$$

be the subring of even symmetric polynomials. Okounkov polynomials $P_\lambda(x; \tau, \alpha)$ form a distinguished linear basis of \mathcal{Q} , indexed by the set Λ . We refer the reader to [13] and [9, (7.2)] for more background on these polynomials, noting that in the latter paper they are referred to as Okounkov’s BC_n type interpolation polynomials, and denoted $P_\lambda^{ip}(x; \tau, \alpha)$.

4.1. Combinatorial formula

We first recall [9, 12] some basic combinatorial terminology associated to partitions. The length of a partition $\lambda \in \Lambda$ is

$$l(\lambda) = \max \{i \mid \lambda_i > 0\}.$$

The Young diagram of λ is the collection of “boxes” $s = (i, j)$

$$\{(i, j) : 1 \leq i \leq l(\lambda), 1 \leq j \leq \lambda_i\}.$$

The *arm/leg/coarm/coleg* of $s = (i, j) \in \lambda$ are defined as follows:

$$a_\lambda(s) = \lambda_i - j, \quad l_\lambda(s) = \#\{k > i \mid \lambda_k \geq j\}, \quad a'_\lambda(s) = j - 1, \quad l'_\lambda(s) = i - 1.$$

We write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i . In this case the diagram of μ can be regarded as a subset of λ , and we define

$$(R \setminus C)_{\lambda \setminus \mu} = \{s \in \mu \mid a_\lambda(s) > a_\mu(s), l_\lambda(s) = l_\mu(s)\},$$

$$\psi_{\lambda \setminus \mu} = \prod_{s \in (R \setminus C)_{\lambda \setminus \mu}} \frac{b_\mu(s)}{b_\lambda(s)}, \quad b_\lambda(s) = \frac{\tau l_\lambda(s) + a_\lambda(s) + \tau}{\tau l_\lambda(s) + a_\lambda(s) + 1}.$$

A *reverse* tableau T of shape λ is an assignment of the boxes $s \in \lambda$ with numbers $T(s) \in \{1, \dots, n\}$ so that $T(i, j)$ is strongly decreasing in i and weakly decreasing in j . Such a tableau defines a sequence of partitions

$$0 = \lambda^{(n)} \subset \dots \subset \lambda^{(1)} \subset \lambda^{(0)} = \lambda, \quad \lambda^{(i)} = \{s \mid T(s) > i\},$$

and we set

$$\psi_T = \prod_{i=1}^n \psi_{\lambda^{(i-1)} \setminus \lambda^{(i)}}.$$

Definition 4.1. The Okounkov polynomial is

$$(4.1) \quad P_\lambda(x; \tau, \alpha) = \sum_T \psi_T \prod_{s \in \lambda} \left[x_{T(s)}^2 - (a'_\lambda(s) + \tau(n - T(s) - l'_\lambda(s)) + \alpha)^2 \right],$$

where the sum is over all reverse tableau T of shape λ .

The polynomial $P_\lambda(x; \tau, \alpha)$ is uniquely characterized by certain vanishing conditions. To state these we set $\delta = (n - 1, \dots, 1, 0)$ and we define

$$\rho = \rho_{\tau, \alpha} = (\rho_1, \dots, \rho_n), \quad \rho_i = \tau \delta_i + \alpha = \tau(n - i) + \alpha.$$

Theorem 4.2. ([9, 13]) *The polynomial $P_\lambda(x) = P_\lambda(x; \tau, \alpha)$ is in \mathcal{Q} and satisfies*

- 1) P_λ has degree $\leq 2|\lambda|$,
- 2) the coefficient of $x_1^{2\lambda_1} \cdots x_n^{2\lambda_n}$ in P_λ is 1.
- 3) $P_\lambda(\mu + \rho) = 0$ unless $\lambda \subseteq \mu$.

For future purposes we also define ([25], [2, (3.7)])

$$(4.2) \quad k_\mu = \prod_{s \in \mu} (\tau l_\mu(s) + a_\mu(s) + 1).$$

4.2. Uniqueness

We prove a slight strengthening of Theorem 4.2. For this we define

$$\mathcal{Q}_k = \{P \in \mathcal{Q} : \deg(P) \leq k\}, \quad \Lambda^d = \{\lambda \in \Lambda : |\lambda| \leq d\}.$$

Proposition 4.3. *Any polynomial in \mathcal{Q}_{2d} is characterized by its values on the set*

$$\Lambda^d + \rho = \left\{ \lambda + \rho : \lambda \in \Lambda^d \right\}.$$

Proof. Let \mathcal{V}_d be the vector space of functions on the set $\Lambda^d + \rho$, then we need to show that the restriction map

$$(4.3) \quad \text{res} : \mathcal{Q}_{2d} \rightarrow \mathcal{V}_d$$

is an isomorphism. Now \mathcal{Q}_{2d} has an explicit basis given by the set

$$\left\{ \tilde{m}_\lambda : \lambda \in \Lambda^d \right\}, \quad \tilde{m}_\lambda = \sum_{\sigma \in S_n} x_{\sigma(1)}^{2\lambda_1} \cdots x_{\sigma(n)}^{2\lambda_n};$$

thus both sides of (4.3) have the same dimension $|\Lambda^d|$. Since res is linear it suffices to prove that it is surjective. For this we consider the “ δ -basis” of \mathcal{V}_d given by

$$\delta_\mu(\lambda + \rho) = \delta_{\lambda\mu} \text{ for all } \lambda, \mu \in \Lambda^d.$$

Fix a total order on Λ^d compatible with $|\lambda| \geq |\mu|$. The restrictions of Okounkov polynomials $\{res(P_\mu) : \mu \in \Lambda^d\}$ belong to \mathcal{V}_d , and by Theorem 4.2 their expression in terms of the δ -basis is upper triangular with non-zero diagonal entries. Thus we can invert this to write δ_μ in terms of $res(P_\mu)$. This proves the Proposition. \square

Theorem 4.4. *The Okounkov polynomial $P_\lambda(x; \tau, \alpha)$ is the unique polynomial in \mathcal{Q} satisfying following conditions:*

- 1) P_λ has degree $\leq 2|\lambda|$;
- 2) the coefficient of $x_1^{2\lambda_1} \cdots x_n^{2\lambda_n}$ in P_λ is 1;
- 3) $P_\lambda(\mu + \rho) = 0$ if $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$.

Proof. This follows immediately from Theorem 4.2 and Proposition 4.3. \square

4.3. Explicit formulas for $\tau = 1$

In this section we give a determinantal formula for the Okounkov polynomials when $\tau = 1$. This involves the one variable polynomials discussed in the next result.

Lemma 4.5. *For $n = 1$ and $l \in \mathbb{Z}_+$ the Okounkov polynomial is given by*

$$(4.4) \quad p_l(x; \alpha) = \prod_{i=0}^{l-1} (x^2 - (i + \alpha)^2).$$

Proof. We verify that $p_l(x; \alpha)$ satisfies the three conditions of Theorem 4.4. The first two are immediate, while for the third we need to show

$$(4.5) \quad p_l(m + \alpha) = 0 \text{ for } m = 0, 1, \dots, l - 1,$$

which follows from the formula $p_l(m + \alpha) = \prod_{i=0}^{l-1} (m + 2\alpha + i) \prod_{i=0}^{l-1} (m - i)$. \square

For λ in Λ we define an $n \times n$ matrix A_λ and its determinant a_λ as follows

$$A_\lambda(x; \alpha) = (p_{\lambda_j}(x_i; \alpha))_{1 \leq i, j \leq n}, \quad a_\lambda = \det A_\lambda.$$

For $\delta = (n - 1, \dots, 1, 0)$ it is easy to see that a_δ is the Vandermonde determinant $\prod_{i < j} (x_i^2 - x_j^2)$, and is thus independent of α .

Theorem 4.6. *For $\tau = 1$ the Okounkov polynomials are given by*

$$(4.6) \quad P_\lambda(x; 1, \alpha) = \frac{a_{\lambda+\delta}(x; \alpha)}{a_\delta(x)}.$$

Proof. The proof is similar to [8]. Let us denote the right side of (4.6) by R_λ . We will show that R_λ satisfies the conditions of Theorem 4.4. The first condition is obvious. Also, the top degree component of R_λ is

$$\frac{\det(x_i^{2(\lambda_j + \delta_j)})}{\det(x_i^{2\delta_j})} = s_\lambda(x_1^2, \dots, x_n^2)$$

where s_λ is the Schur polynomial; this implies the second condition. To finish the proof it suffices to prove the third condition in the form

$$(4.7) \quad |\mu| \leq |\lambda| \text{ and } R_\lambda(\mu + \rho) \neq 0 \implies \mu = \lambda.$$

Suppose μ in Λ satisfies the assumptions of (4.7). Since $\mu + \rho$ has distinct components, the denominator in (4.6) is a nonzero Vandermonde determinant, and so the numerator must be non zero. Expanding the numerator we get

$$\sum_{\sigma \in S_n} (-1)^\sigma \prod_j p_{\lambda_j + \delta_j}(\mu_{\sigma(j)} + \delta_{\sigma(j)} + \alpha; \alpha) \neq 0,$$

and at least one term must be nonzero. Thus for some $\sigma \in S_n$ we must have

$$p_{\lambda_j + \delta_j}(\mu_{\sigma(j)} + \delta_{\sigma(j)} + \alpha; \alpha) \neq 0 \text{ for all } j.$$

By (4.5) we get

$$(4.8) \quad \mu_{\sigma(j)} + \delta_{\sigma(j)} \geq \lambda_j + \delta_j \text{ for all } j.$$

Summing this over j we obtain

$$(4.9) \quad |\mu| + |\delta| \geq |\lambda| + |\delta|.$$

If the inequality in (4.8) is strict for some j then strict inequality holds in (4.9), which contradicts the assumption that $|\mu| \leq |\lambda|$. Thus equality must hold in (4.8) for all j , which implies

$$\sigma(\mu + \delta) = \lambda + \delta.$$

Since $\lambda + \delta$ and $\mu + \delta$ are strictly decreasing sequences, this forces σ to be the identity permutation, and we get $\mu = \lambda$ as desired. \square

4.4. Explicit formulas for special partitions

In this section we give explicit formulas for Okounkov polynomials $P_\lambda(x; \tau, \alpha)$ for certain special partitions λ . For the reader's convenience we recall the definition of ρ

$$\rho_i = \tau\delta_i + \alpha = \tau(n - i) + \alpha.$$

Theorem 4.7. $P_{1^j}(x; \tau, \alpha)$ is the coefficient of t^j in the series expansion of

$$(4.10) \quad \frac{\prod_{i=1}^n (1 + tx_i^2)}{\prod_{i=j}^n (1 + t\rho_i^2)}.$$

Proof. Let R_j denote the coefficient of t^j in (4.10). We will prove that R_j satisfies the three conditions of Theorem 4.4 for $\lambda = 1^j$. The first two conditions are obvious. For the third condition it suffices to prove that in the expression

$$(4.11) \quad \frac{\prod_{i=1}^n (1 + t(\mu_i + \rho_i)^2)}{\prod_{i=j}^n (1 + t\rho_i^2)}$$

the coefficient of t^j is 0 if μ satisfies

$$(4.12) \quad |\mu| \leq j, \quad \mu \neq 1^j.$$

However under assumption (4.12) we have $\mu_j = \mu_{j+1} = \dots = \mu_n = 0$, and thus

$$\mu_i + \rho_i = \rho_i \text{ for } i = j, \dots, n.$$

It follows that in the expression (4.11) the denominator cancels completely, leaving behind a polynomial in t of degree $< j$. Hence the coefficient of t^j is 0. \square

Corollary 4.8. *We have*

$$(4.13) \quad P_{1^j}(x; \tau, \alpha) = \sum_{i_1 < \dots < i_j} \prod_{k=1}^j (x_{i_k}^2 - \rho_{i_k+j-k}^2).$$

Proof. This follows by a direct computation from Theorem 4.7. For an alternative argument, see [8, Proposition 3.1]. \square

We next give an explicit formula for $P_\lambda(x; \tau, \alpha)$ for $\lambda = l^n := 1^n = (l, l, \dots, l)$.

Proposition 4.9. *We have*

$$(4.14) \quad P_{l^n}(x; \tau, \alpha) = \prod_{i=0}^{l-1} \prod_{j=1}^n [x_j^2 - (i + \alpha)^2].$$

Proof. It suffices to show that right side of (4.14) satisfies the three conditions of Theorem 4.4 for $\lambda = l^n$. The first two conditions are obvious. For the third it suffices to show that if

$$(4.15) \quad |\mu| \leq nl, \quad \mu \neq l^n,$$

then we have

$$(4.16) \quad \prod_{i=0}^{l-1} \prod_{j=1}^n [(\mu_j + \rho_j)^2 - (i + \alpha)^2] = 0.$$

But if μ satisfies (4.15) then we have $\mu_n < l$, which implies

$$\mu_n + \rho_n = i + \rho_n = i + \alpha,$$

for some $i = 0, 1, \dots, l - 1$. Thus one of the factors of (4.16) is 0. \square

5. Eigenvalues of Shimura operators

We shall prove that the eigenvalues of the Shimura operators are given by the Okounkov polynomials.

5.1. Vanishing properties

Before turning to our main results we prove an elementary lemma. Let p be a non-negative integer. Recall the Schmid’s component W_ν of $S(\mathfrak{p}^+)$ and the \mathfrak{g} -representation $V_{\lambda,p}$ in Lemma 3.4. Recall the notation $W_{\nu,p} = W_\nu \otimes \mathbb{C}_2^p$.

Lemma 5.1. *If $\text{Hom}_K(W_{\nu,p}, V_{\lambda,p}) \neq 0$ then $\nu \subseteq \lambda$.*

Proof. The Lie algebra $\mathfrak{k} + \mathfrak{p}^-$ is the parabolic subalgebra opposite to $\mathfrak{k} + \mathfrak{p}^+$. Let $v_{\lambda,p}$ be a non-zero highest weight vector in the representation space $(V_{\lambda,p}, \pi)$ of \mathfrak{g} . Then by the Poincare-Birkhoff-Witt theorem we have

$$(5.1) \quad V_{\lambda,p} = \pi(U(\mathfrak{p}^-)U(\mathfrak{k}))v_{\lambda,p} = \pi(U(\mathfrak{p}^-))\pi(U(\mathfrak{k}))v_{\lambda,p}.$$

The space $\pi(U(\mathfrak{k}))v_{\lambda,p} = \pi(\mathfrak{k})v_{\lambda,p} = W_{\lambda,p}$ is a highest weight representation of \mathfrak{k} with highest weight $\sum_{j=1}^n(\lambda_j + \frac{p}{2})\gamma_j$ when restricted to \mathfrak{t}_- . If $\text{Hom}_K(W_{\nu,p}, V_{\lambda,p}) \neq 0$, equivalently $W_{\nu,p}$ occurs in $\pi(U(\mathfrak{p}^-))W_{\lambda,p}$ then $\sum_{j=1}^n(\nu_j + \frac{p}{2})\gamma_j$ must be of the form $\sum_j(\nu_j + \frac{p}{2})\gamma_j = \sum_j(\lambda_j + \frac{p}{2} + \mu_j)\gamma_j$ where $\sum_j\mu_j\gamma_j$ is a weight of $U(\mathfrak{p}^-)$, by [6, Theorem 20.2]. But then any such μ is of the form $-\sum_i\mu_i\gamma_i$ for some $\mu_i \geq 0$, proving our claim. \square

Theorem 5.2. *Let $\eta_p(\mathcal{L}_\mu)$ be the Harish-Chandra homomorphism of \mathcal{L}_μ defined in (3.6), then*

$$\eta_p(\mathcal{L}_\mu)\left(\lambda + \frac{p}{2} + \rho\right) = 0 \text{ unless } \mu \subseteq \lambda.$$

Proof. We equip $V_{\lambda,p}$ with a U -invariant unitary inner product. Now \mathcal{L}_μ is a sum of elements of the form $\bar{\xi}\xi$, where $\{\xi\}$ is a basis of $W_\mu \subset S(\mathfrak{p}^+)$. By Schur’s lemma the invariant differential operator \mathcal{L}_μ acts by the scalar $c = \eta_p(\mathcal{L}_\mu)(\lambda + \frac{p}{2} + \rho)$ on $V_{\lambda,p}$. Let v_p be the (K, ι^p) -spherical vector in $V_{\lambda,p}$, normalized to have $(v_p, v_p) = 1$. The (K, ι^{-p}) -spherical function in $V_{\lambda,p} \subset C^\infty(G/K, \iota^p)$ is of the form

$$\Phi_{\lambda,p}(g) = (\pi_\lambda(g^{-1})v_p, v_p).$$

Performing differentiation by \mathcal{L}_μ and evaluating at $g = 1 \in G$ we get

$$c = (\pi_\lambda(\mathcal{L}_\mu)v_p, v_p) = \sum (\pi_\lambda(\xi)v_p, \pi_\lambda(\bar{\xi})^*v_p) = \sum (\pi_\lambda(\xi)v_p, \pi_\lambda(\xi)v_p).$$

Here we have used the fact that $\pi_\lambda(x)^* = \pi_\lambda(\bar{x})$ since π_λ is a unitary representation of Lie algebra $\mathfrak{k} + i\mathfrak{p}$ of U . But the vectors $\pi_\lambda(\xi)v_p, \xi \in W_\mu$, are in the K -subspace of $V_{\lambda,p}$ of highest weight $\mu + \frac{p}{2}$, which is vanishing by Lemma 5.1. \square

Remark 5.3. By the same argument above there exists a polynomial $\eta_p(\mathcal{M}_\mu)$ such that \mathcal{M}_μ acts on $V_{\lambda,p}$ by the scalar $\eta_p(\mathcal{M}_\mu) (\lambda + \frac{p}{2} + \rho)$. Moreover the polynomial $\eta_p(\mathcal{M}_\mu)$ is related to $\eta_p(\mathcal{L}_\mu)$ by

$$\eta_{-p}(\mathcal{M}_\mu) = \eta_p(\mathcal{L}_\mu).$$

This relation is a simple consequence of the following observation: If $f \in C^\infty(G/K, p)$ then $\bar{f} \in C^\infty(G/K, -p)$ and

$$X\bar{f} = \overline{Xf},$$

where $X \rightarrow \bar{X}$ is the complex conjugation relative to the real form \mathfrak{g}_0 . The U -representations appearing in $C^\infty(G/K, -p)$ are the same as in $C^\infty(G/K, p)$ by Lemma 2.2 and are of the form $\lambda + \frac{p}{2}$. Thus $\eta_{-p}(\mathcal{M}_\mu)(\lambda + \frac{p}{2} + \rho) = \eta_p(\mathcal{L}_\mu)(\lambda + \frac{p}{2} + \rho)$ for $\mu \in \Lambda$, but Λ is Zariski dense in $\mathbb{C}^n = \mathfrak{a}^*$ so it holds also on \mathbb{C}^n . See further [29].

Remark 5.4. Let $p = 0$. The Harish-Chandra spherical function ϕ_x on G/K in [5, Ch. IV, Theorem 4.3] and [20] is our $\Phi_{ix,0}$. Thus there is a change of variable $x \rightarrow ix$ from the parameterization in [5] to ours here.

5.2. Eigenvalue polynomials $\eta_p(\mathcal{L}_\mu)$ in terms of P_μ

We can now find the precise relation between $\eta_p(\mathcal{L}_\mu)$ and Okounkov’s BC-interpolation polynomials. So let $P_\lambda(x, \tau, \alpha)$ be as above the BC-type interpolation polynomials with two parameters (τ, α) with the normalization that the coefficient of m_λ in the expression is 1. We prove now one of our main results, stated as Theorem 1.1 in Section 1. Recall the reproducing kernel $K_\lambda(z, w)$ of the space W_λ in Section 3.2 equipped with the Fock norm.

We define

$$(5.2) \quad \tau = \tau(d) := \frac{d}{2}, \quad \alpha := \alpha(b, p) = \frac{b + 1 + p}{2},$$

so that $\rho + \frac{p}{2} = \rho(\tau, \alpha)$. Also recall the constant k_μ defined in (4.2).

Theorem 5.5. *The Harish-Chandra image of \mathcal{L}_μ is*

$$\eta_p(\mathcal{L}_\mu)(x) = k_\mu P_\mu(x; \tau, \alpha),$$

where (τ, α) are as in (5.2).

Proof. It follows from Theorem 5.2 and Theorem 4.4 that $\eta_p(\mathcal{L}_\mu)(x)$ is a scalar multiple of $P_\mu(x; \tau, \alpha)$, $\eta_p(\mathcal{L}_\mu)(x) = kP_\mu(x; \tau, \alpha)$. To find the scalar constant k we compare their leading terms. Recall the Cartan subspace $\mathfrak{a} = \sum_j \mathbb{C}e_j$ of \mathfrak{p} . Now each element in \mathfrak{p} can be written as $u = u^+ + u^-$, and $u^\pm = \frac{1}{2}(u \pm i\tilde{u})$ for $\tilde{u} = [Z_0, u] \in \mathfrak{p}$ with Z_0 defining the complex structure on \mathfrak{p}_0 . In particular we have $e_j^+ = \frac{1}{2}(e_j + i\tilde{e}_j)$. Any $x = \sum_j x_j(2\epsilon_j) \in \mathfrak{a}^*$ can be extended to an element in \mathfrak{p}^* , and thus $x(e_j) = 2x_j$, $x(e_j^+) = x_j$. It follows then from the definition of \mathcal{L}_μ that the Harish-Chandra homomorphism $\eta_p(\mathcal{L}_\mu)$ of \mathcal{L}_μ has its leading term the polynomial

$$x = \sum_j x_j e_j \in \mathfrak{a}_0^* \mapsto K_\mu \left(\sum_j x_j e_j^+, \sum_j x_j e_j^+ \right).$$

Now the sum of $\sum_{|\mu|=m} K_\mu \left(\sum_j x_j e_j^+, \sum_j x_j e_j^+ \right)$ is

$$\sum_{|\mu|=m} K_\mu \left(\sum_j x_j e_j^+, \sum_j x_j e_j^+ \right) = \frac{1}{m!} (x_1^2 + \dots + x_n^2)^m$$

by the definition of the reproducing kernel K_μ . On the other hand the top homogeneous term of $P_\mu(x; \tau, \alpha)$ is precisely the monic Jack polynomial $P_\mu^{Jac}(x_1^2, \dots, x_n^2)$ with parameter τ , [9], thus the constant k is precisely the coefficient k'_μ in the expansion

$$\frac{1}{m!} (x_1^2 + \dots + x_n^2)^m = \sum_{|\mu|=m} k'_\mu P_\mu^{Jac}(x_1^2, x_2^2, \dots, x_n^2).$$

But it is well-known ([25], [27, (iii)-(iv)-(vii), p. 1319]) that the constant k'_μ is given by (4.2). □

Remark 5.6. We can also give a different proof of the evaluation formula for the constant k_μ above. Let $e^+ = e_1^+ + \dots + e_n^+$ be the sum of the strongly orthogonal positive root vectors. Recall [4, Lemma 3.1] that

$$(5.3) \quad K_\mu(z, e^+) = \frac{\beta_\mu}{\left(\frac{d}{2}(n-1) + 1\right)_\mu} \psi_\mu(z),$$

where ψ_μ is the spherical polynomial of the K -homogeneous space Ke^+ (i.e., the Shilov boundary of G/K) normalized by $\psi_\mu(e^+) = 1$,

$$(5.4) \quad \beta_\mu := \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j + \frac{d}{2}(j - i)}{\frac{d}{2}(j - i)} \frac{\left(\frac{d}{2}(j - i + 1)\right)_{\mu_i - \mu_j}}{\left(\frac{d}{2}(j - i - 1) + 1\right)_{\mu_i - \mu_j}},$$

and

$$(5.5) \quad (a)_\mu := \prod_{i=1}^n \left(a - \frac{d}{2}(i - 1)\right)_{\mu_i} = \prod_{i=1}^n \frac{\Gamma\left(a - \frac{d}{2}(i - 1) + \mu_i\right)}{\Gamma\left(a - \frac{d}{2}(i - 1)\right)}$$

is the generalized Pochhammer symbol. See e.g. [3, (2.6)-(2.7)] where our β_μ is denoted by π_μ . (The coefficient $\frac{\beta_\mu}{\left(\frac{d}{2}(n-1)+1\right)_\mu}$ is now independent of the root multiplicity $2b$.) Now the top homogeneous term of $\eta_p(\mathcal{L}_\mu)$ is given by $K_\mu\left(\sum_j x_j e_j^+, \sum_j x_j e_j^+\right)$, which in turn is ([4])

$$\begin{aligned} K_\mu\left(\sum_j x_j e_j^+, \sum_j x_j e_j^+\right) &= K_\mu\left(\sum_j x_j^2 e_j^+, e^+\right) \\ &= \frac{\beta_\mu}{\left(\frac{d}{2}(n - 1) + 1\right)_\mu} \psi_\mu\left(\sum_j x_j^2 e_j^+\right), \end{aligned}$$

where the last equation is just (5.3). Now $\psi_\mu\left(\sum_j x_j^2 e_j^+\right) = \psi_\mu(x_1^2, \dots, x_n^2)$ is the Jack symmetric polynomial

$$\psi_\mu(x_1^2, \dots, x_n^2) = \frac{1}{P_\mu^{Jac}(1^n)} P_\mu^{Jac}(x_1^2, \dots, x_n^2),$$

whereas the Okounkov polynomial P_μ has the same leading term as $P_\mu^{Jac}(x_1^2, \dots, x_n^2)$; see [9]. Thus the constant $k = k_\mu$ is $k = \frac{\beta_\mu}{\left(\frac{d}{2}(n-1)+1\right)_\mu} \frac{1}{P_\mu^{Jac}(1^n)}$. Now by the known evaluation formula (see e.g. [9, (4.8)])

$$(5.6) \quad P_\mu^{Jac}(1^n) = \prod_{1 \leq i < j \leq n} \frac{\left((j - i + 1)\frac{d}{2}\right)_{\mu_i - \mu_j}}{\left((j - i)\frac{d}{2}\right)_{\mu_i - \mu_j}},$$

we can write k

$$k = \frac{1}{\left(\frac{d}{2}(n - 1) + 1\right)_\mu} \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j + \frac{d}{2}(j - i)}{\frac{d}{2}(j - i)} \frac{\left(\frac{d}{2}(j - i)\right)_{\mu_i - \mu_j}}{\left(\frac{d}{2}(j - i - 1) + 1\right)_{\mu_i - \mu_j}}.$$

This can be simplified using the Gamma function

$$k = \frac{1}{\left(\frac{d}{2}(n-1) + 1\right)_\mu} \prod_{1 \leq i < j \leq n} \frac{\gamma(\mu_i - \mu_j, j - i)}{\gamma(0, j - i)}$$

where

$$\gamma(x, j - i) := \frac{\Gamma\left(x + 1 + \frac{d}{2}(j - i)\right)}{\Gamma\left(x + 1 + \frac{d}{2}(j - i - 1)\right)}.$$

By a straightforward computation using (5.5) we find

$$k = \prod_{i=1}^n \frac{1}{\Gamma\left(1 + \frac{d}{2}(n - i) + \mu_i\right)} \prod_{1 \leq i < j \leq n} \gamma(\mu_i - \mu_j, j - i),$$

which is precisely (4.2), by [2, Proposition 3.5].

We can now describe the Shimura sets in terms of the Okounkov polynomials. Using Theorem 5.5 and the definition of the sets \mathcal{A} and \mathcal{G} we have

Corollary 5.7. *The Shimura sets are given explicitly as follows:*

$$\begin{aligned} \mathcal{A} &= \{x : q_\lambda(x) \geq 0 \text{ for all } \lambda\}, \\ \mathcal{G} &= \{x : q_{1^j}(x) \geq 0 \text{ for all } j\}. \end{aligned}$$

The following is a restatement of Theorem 1.2, the notation being the same, and it is an immediate consequence of Corollaries 5.7 and 4.8.

Theorem 5.8. *The Shimura set \mathcal{G} is also given by*

$$\mathcal{G} = \{\xi : \varphi_j(\xi) \geq 0 \text{ for all } j\}.$$

In the rest of the paper we let $p = 0$ and write $\eta_p(\mathcal{L}_\mu) = \eta(\mathcal{L}_\mu)$, namely we consider the trivial line bundle over G/K .

5.3. The Shimura sets and unitary spherical representations of G

We introduce now the set

$$\mathcal{U} = \{x \in \mathbb{C}^n : \text{the spherical function } \Phi_x \text{ is positive definite}\}.$$

In other words \mathcal{U} is the set of unitary spherical representations. This set has been studied intensively, and in [7] it is determined for the group $G = U(N, 2)$.

Proposition 5.9. *We have*

$$\mathcal{U} \subseteq \mathcal{A} \subseteq \mathcal{G}.$$

Proof. Let $x \in \mathcal{U}$. The spherical function Φ_x defines a unitary irreducible representation (H, π) of G with a K -fixed vector v so that Φ_x is the matrix coefficient

$$\Phi_x(g) = (v, \pi(g)v),$$

where (\cdot, \cdot) is the Hilbert Hermitian product in H ; see e.g. [5, Ch. IV]. For any element $X \in \mathfrak{p}$ we have

$$X\Phi_x(e) = (v, \pi(X)v) = (\pi(X)^*v, v) = -(\pi(\bar{X})v, v),$$

where \bar{X} is the complex conjugation with respect to the real form \mathfrak{g}_0 in \mathfrak{g} . Now let \mathcal{L}_μ act on Φ_x and evaluate at $g = e$. We have

$$\begin{aligned} (-1)^{|\mu|} \mathcal{L}_\mu \Phi_x(e) &= (-1)^{|\mu|} \sum_{\alpha} (v, \pi(\xi_{\alpha}^*)\pi(\xi_{\alpha})v) \\ &= (-1)^{|\mu|} \sum_{\alpha} (\pi(\xi_{\alpha}^*)^*v, \pi(\xi_{\alpha})v) \\ &= (-1)^{|\mu|} \sum_{\alpha} (\pi(\bar{\xi}_{\alpha}^*)v, \pi(\xi_{\alpha})v) \\ &= \sum_{\alpha} (\pi(\xi_{\alpha})v, \pi(\xi_{\alpha})v) \geq 0, \end{aligned}$$

proving $\mathcal{U} \subseteq \mathcal{A}$. □

5.4. Positivity for real parameters

We shall study a real version of the sets $\mathcal{A}, \mathcal{G}, \mathcal{U}$. Denote $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{C}$, $\mathcal{G}_0 = \mathcal{G} \cap \mathcal{C}$, $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{C}$ where

$$\mathcal{C} = \{x : x_1 \geq \dots \geq x_n \geq 0\}$$

is a Weyl chamber.

Theorem 5.10. *Suppose the rank $n > 1$ then we have*

$$[0, \rho_n]^n \cap \mathcal{C} \subsetneq \mathcal{A}_0.$$

Proof. We shall need an explicit formula for $P_\lambda(x)$ by Koornwinder [9].

By Theorem 5.5 we have, using the notation q_λ in (2.8), that

$$\begin{aligned} q_\lambda(x) &= (-1)^{|\lambda|} P_\lambda(x) \\ &= k_\mu \sum_T \psi_T \prod_{s \in \lambda} \left((a'_\lambda(s) + \frac{d}{2}(n - T(s) - l'_\lambda(s)) + \frac{b+1}{2})^2 - x_{T(s)}^2 \right) \end{aligned}$$

with k_μ being positive. Now if $x \in [0, \rho_n]^n$, i.e., if $0 \leq x_j \leq \rho_n$ for all j , we have for any fixed T in the sum and $s = (i, j) \in \lambda$ in the product, writing $T(s) = k$, that $a'_\lambda(s) = j - 1$, $l'_\lambda(s) = i - 1$ and

$$\begin{aligned} a'_\lambda(s) + \frac{d}{2}(n - k - l'_\lambda(s)) + \frac{b+1}{2} &\geq \frac{d}{2}(n - k - l'_\lambda(s)) + \frac{b+1}{2} \\ &= \frac{d}{2}(n - k - i + 1) + \frac{db+1}{2} \\ &= \rho_{k+i-1} \geq \rho_n \geq x_k. \end{aligned}$$

Here we have used the fact that $T(i, j)$ is strongly decreasing in i , implying $T(s) = T(i, j) = k \leq n - i + 1$ and ρ_{k+i-1} makes sense. Thus each factor in the product is nonnegative and $(-1)^{|\lambda|} \eta(\mathcal{L}_\lambda)(x) \geq 0$, proving $x \in \mathcal{A}_0$. The element ρ is in \mathcal{A}_0 since it is a zero point of all $\eta(\mathcal{L}_\lambda)$, but $\rho \notin [0, \rho_n]^n$. This finishes the proof. \square

Note that if the rank $n = 1$ then the three sets are the same

$$\mathcal{A} = \mathcal{G} = \mathcal{U} = [-\rho, \rho] \cup i\mathbb{R},$$

and

$$\mathcal{A}_0 = \mathcal{G}_0 = \mathcal{U}_0 = [0, \rho].$$

In other words, the set of unitary spherical representations are characterized by one relation, namely $x^2 - \rho^2 \leq 0$, with the complementary series parameters corresponding to the real points.

6. Shimura sets for rank two domains

6.1. The case of classical rank two domains

$(\mathfrak{g}_0, \mathfrak{k}_0) = (\mathfrak{u}(b + 2, 2), \mathfrak{u}(b + 2) + \mathfrak{u}(2))$ and $(\mathfrak{sp}(2, \mathbb{R}), \mathfrak{u}(2))$

We shall determine the set \mathcal{A}_0 for the domains G/K of rank $n = 2$ and with $d = 2$, namely $(\mathfrak{g}_0, \mathfrak{k}_0)$ being the pair $(\mathfrak{u}(b + 2, 2), \mathfrak{u}(b + 2) + \mathfrak{u}(2))$ and prove an inclusion for the pair $(\mathfrak{sp}(2, \mathbb{R}), \mathfrak{u}(2))$; we refer the two pairs as $I_{2,2+b}$ and II_2 . The variable x will be in the Weyl Chamber $\mathcal{C} \subset \mathbb{R}_{\geq 0}^2$ throughout the discussions below. Recall the Pochhammer symbol $(a)_m = (a)(a + 1) \cdots (a + m - 1)$, and introduce its multiparameter version

$$(a_1, \dots, a_p)_k = (a_1)_k \cdots (a_p)_k.$$

To simplify notation still further we will write $(a \pm x)_k$ for $(a + x)_k (a - x)_k$.

In [9, (10.13)] Koornwinder found explicit formulas for the interpolation polynomials $P_{(m_1, m_2)}(x_1, x_2)$ of rank two in terms of hypergeometric series ${}_pF_q\left(\begin{smallmatrix} a \\ b \end{smallmatrix}; t\right)$, $a = (a_1, \dots, a_p)$, $b = (b_1, \dots, b_p)$. We shall be only dealing with the series evaluated at $t = 1$. To simplify notation we write

$$F\left(\begin{smallmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{smallmatrix}\right) = {}_pF_q\left(\begin{smallmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{smallmatrix}; 1\right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_p)_k}{(b_1, \dots, b_q)_k} \frac{1}{k!}$$

and

$$F^{[m]}\left(\begin{smallmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{smallmatrix}\right) = \sum_{k=0}^m \frac{(a_1, \dots, a_p)_k}{(b_1, \dots, b_q)_k} \frac{1}{k!}.$$

In the formulas below we adapt also the short-hand notation $\alpha \pm \beta$ to indicate that the both terms appear in parallel positions.

Lemma 6.1. ([9]) *The Okounkov polynomial*

$$q_{(m_1, m_2)}(x) = (-1)^{m_1 + m_2} P_{(m_1, m_2)}(x)$$

of two variables $x = (x_1, x_2)$ with the parameter (τ, α) is given in terms of ${}_4F_3$ -series by

$$(6.1) \quad q_{(m_1, m_2)}(x) = (\rho_2 \pm x_1, \rho_2 \pm x_2)_{m_2} (m_2 + \rho_1 \pm x_1)_{m_1 - m_2} \\ \times F\left(\begin{smallmatrix} -m_1 + m_2, m_2 + \rho_2 \pm x_2, \frac{d}{2} \\ 1 - m_1 + m_2 - \frac{d}{2}, m_2 + \rho_1 \pm x_1 \end{smallmatrix}\right).$$

In particular if $d = 2$ the polynomial $q_{(m_1, m_2)}(x_1, x_2)$ can be written in terms of the partial sum of an ${}_3F_2$ -series

$$(6.2) \quad q_{(m_1, m_2)}(x_1, x_2) = (\rho_2 \pm x_1, \rho_2 \pm x_2)_{m_2} (m_2 + \rho_1 \pm x_1)_{m_1 - m_2} \\ \times F^{[m_1 - m_2]} \left(\begin{matrix} m_2 + \rho_2 \pm x_2, 1 \\ m_2 + \rho_1 \pm x_1 \end{matrix} \right).$$

Denote

$$(6.3) \quad R(x_1, x_2) := F \left(\begin{matrix} \rho_2 \pm x_2, \frac{d}{2} \\ \rho_1 \pm x_1 \end{matrix} \right).$$

Theorem 6.2. *Let*

$$\mathcal{B} = \{x \in \mathcal{C} \mid q_{1,0}(x) \geq 0, \quad q_{1,1}(x) \geq 0, R(x) \geq 0\}$$

Then the set \mathcal{A}_0 of real points λ for the positivity of all $q_\mu(\lambda)$ is $\mathcal{A}_0 = \mathcal{B}$ if $(\mathfrak{g}_0, \mathfrak{k}_0)$ is of type $I_{2,2+b}$, and $\mathcal{A}_0 \subseteq \mathcal{B}$ for type II_2 .

Proof. To simplify the discussion we take all x below to be in the first quadrant $x_1, x_2 \geq 0$ instead of the Weyl chamber \mathcal{C} . It follows immediately from the formulas in Lemma 6.1 that $q_{(1,0)}(x) \geq 0, q_{(1,1)}(x) \geq 0$ if and only if $x \in [0, \rho_2]^2$ or $x_1, x_2 \geq \rho_2, \|x\| \leq \|\rho\|$, namely, x is in the square $[0, \rho_2]^2$ or in disc $\{\|x\| \leq \|\rho\|\}$ cut by the square $[\rho_2, \rho_1]^2$, i.e. $\{\|x\| \leq \|\rho\|\} \cap [\rho_2, \rho_1]^2$. However the triangle $[0, \rho_2]^2 \cap \mathcal{C}$ is in \mathcal{A}_0 by Theorem 5.10 above so we need only consider x in the square $[\rho_2, \rho_1]^2$ and we restrict x to this square.

We prove first the inclusion $\mathcal{A}_0 \subseteq \mathcal{B}$ for $d = 1, 2$. Note first that $\rho_1 = \rho_2 + \frac{d}{2}$ and observe that $\rho_2 - x_2 \leq 0$ and $\rho_2 - x_2 + l \geq 0$ if $l \geq 1$ for all x in the square $[\rho_2, \rho_1]^2$. Suppose $q_{(m,0)}(x_1, x_2) = (-1)^m P_{(m,0)}(x_1, x_2) \geq 0$ for all m . We fix $N > 0$ and let $m \geq N$. Denote the partial sum in $a_{(m,0)}(x_1, x_2)$ by

$$f_{m,N}(x) := \sum_{j=0}^N \frac{(m)_j^- (\rho_2 \pm x_2,)_j (\frac{d}{2})_j}{(m + \frac{d}{2} - 1)_j^- (\rho_1 \pm x_1)_j j!}.$$

Now by the above observation $q_{(m,0)}(x_1, x_2)$ has leading term 1 with the rest being nonpositive, we have

$$f_{m,N}(x) \geq q_{(m,0)}(x_1, x_2) \geq 0.$$

Letting $m \rightarrow \infty$ we find

$$\sum_{j=0}^N \frac{(\rho_2 \pm x_2)_j (\frac{d}{2})_j}{(\rho_1 \pm x_1)_j j!} = \lim_{m \rightarrow \infty} f_{m,N}(x) \geq 0.$$

Now take the limit $N \rightarrow \infty$:

$$R(x) = \lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{(\rho_2 \pm x_2)_j (\frac{d}{2})_j}{(\rho_1 \pm x_1)_j j!} \geq 0,$$

proving $\mathcal{A}_0 \subseteq \mathcal{B}$.

Suppose now $d = 2$, $x \in \mathcal{B}$ and is in the square $[\rho_2, \rho_1]^2$. Thus $R(x) \geq 0$. If $m_1 = m_2 \geq 1$ then $q_{(m_1, m_2)}$ is a product of m_1 pairs of nonpositive numbers and is nonnegative. Let $m_1 = m \geq m_2 = 0$. By Lemma 6.1 the polynomial $q_{(m,0)}(x_1, x_2)$ is a partial sum of a ${}_3F_2$ series,

$$q_{(m,0)}(x_1, x_2) = (\rho_1 \pm x_1)_m F^{[m]} \left(\begin{matrix} \rho_2 \pm x_2, 1 \\ \rho_1 \pm x_1 \end{matrix} \right)$$

with the factor $(\rho_1 \pm x_1)_m \geq 0$. The second factor is

$$F^{[m]} \left(\begin{matrix} -m, \rho_2 \pm x_2, \frac{d}{2} \\ 1 - m - \frac{d}{2}, \rho_1 \pm x_1 \end{matrix} \right) = \sum_{j=0}^m \frac{(\rho_2 \pm x_2)_j (1)_j}{(\rho_1 \pm x_1)_j j!}.$$

All terms in the sum are nonpositive except the leading term 1. Thus adding infinitely many negative terms we find

$$\begin{aligned} F^{[m]} \left(\begin{matrix} \rho_2 \pm x_2, 1 \\ \rho_1 \pm x_1 \end{matrix} \right) &= \sum_{j=0}^m \frac{(\rho_2 \pm x_2)_j (1)_j}{(\rho_1 \pm x_1)_j j!} \\ &\geq \sum_{j=0}^{\infty} \frac{(\rho_2 \pm x_2)_j (1)_j}{(\rho_1 \pm x_1)_j j!} \\ &= F \left(\begin{matrix} \rho_2 \pm x_2, 1 \\ \rho_1 \pm x_1 \end{matrix} \right) \\ &= R(x) \geq 0. \end{aligned}$$

Now if $m_1 > m_2 > 0$ the positivity of $q_{(m_1, m_2)}(x)$ for $x \in [\rho_2, \rho_1]^2$, $\rho_2 \leq x_2 \leq \rho_2 + 1 = \rho_1$ follows immediately using Lemma 6.1 as all terms in the summation of $F^{[m_1 - m_2]}$ are positive.

The proof is now completed. □

When $b = 0$, namely when $\rho_2 = \frac{1}{2}$ the above ${}_3F_2$ -series can be evaluated. We have [1, Theorem 3.5.5(ii)]

Lemma 6.3. *Suppose $a_1 + a_2 = 1, b_1 + b_2 = 2a_3 + 1$. Then*

$$F\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right) = \frac{\pi\Gamma(b_1)\Gamma(b_2)}{2^{2a_3-1}\Gamma\left(\frac{a_1+b_1}{2}\right)\Gamma\left(\frac{a_1+b_2}{2}\right)\Gamma\left(\frac{a_2+b_1}{2}\right)\Gamma\left(\frac{a_2+b_2}{2}\right)}.$$

Theorem 6.4. *Let $(\mathfrak{g}_0, \mathfrak{k}_0)$ be the symmetric pair $(\mathfrak{su}(2, 2), \mathfrak{u}(2) + \mathfrak{u}(2))$. Then $\mathcal{A}_0 = T_1 \cup T_2$ is a union of two triangles, $T_1 = [0, \rho_2]^2 \cap \mathcal{C}$, and*

$$T_2 = \{(x_1, x_2), x_1 \geq x_2 \geq \rho_2, \quad x_1 + x_2 \leq \rho_1 + \rho_2 = 2\}.$$

Proof. The polynomial $q_{1,0}(x) = -x_1^2 - x_2^2 + \rho_1^2 + \rho_2^2$ and $q_{1,1}(x)$ is by Lemma 6.1 the polynomial

$$(\rho_2^2 - x_1^2)(\rho_2^2 - x_2^2).$$

The nonnegativity of $q_{1,0}(x)$ is equivalent to $x_1^2 + x_2^2 \leq \rho_1^2 + \rho_2^2$ whereas that of $q_{1,1}(x)$ is $x_1, x_2 \leq \rho_2$ or $x_1, x_2 \geq \rho_2$.

The function $R(x)$ can now be evaluated by Lemma 6.3, viz.,

$$\begin{aligned} R(x) &= F\left(\begin{matrix} \rho_2 \pm x_2, \frac{d}{2} \\ \rho_1 \pm x_1 \end{matrix}\right) \\ &= \frac{\pi\Gamma(\rho_1 + x_1)\Gamma(\rho_1 - x_1)}{2^{d-1}\Gamma\left(\frac{\rho_1+\rho_2+x_1+x_2}{2}\right)\Gamma\left(\frac{\rho_1+\rho_2+x_1-x_2}{2}\right)\Gamma\left(\frac{\rho_1+\rho_2-x_1+x_2}{2}\right)\Gamma\left(\frac{\rho_1+\rho_2-x_1-x_2}{2}\right)}. \end{aligned}$$

From which we see that $R(x) \geq 0$ for $0 \leq x_1, x_2 \leq \rho_2$, and $R(x) \geq 0$ for $\rho_2 \leq x_1, x_2 \leq \rho_1$ if and only if

$$x_1 + x_2 \leq \rho_1 + \rho_2.$$

Our claim then follows from Theorem 6.2. □

Remark 6.5. If $b > 0$ the triangle $T_2 = \{(x_1, x_2), x_1 \geq x_2 \geq \rho_2, x_1 + x_2 \leq \rho_1 + \rho_2 = 2 + b\}$ is not in the positivity domain \mathcal{A}_0 . Indeed if we put $x_1 =$

$x_2 = \frac{\rho_1 + \rho_2}{2}$, then $(x_1, x_2) \in T$ and the function $R(x)$ is

$$\begin{aligned} R(x) &= \sum_{k=0}^{\infty} \frac{(2\rho_2 + \frac{1}{2})_k (-\frac{1}{2})_k}{(2\rho_2 + \frac{3}{2})_k (\frac{1}{2})_k} \\ &= 1 + \sum_{j=0}^{\infty} \frac{(2\rho_2 + \frac{1}{2})_{j+1} (-\frac{1}{2})_{j+1}}{(2\rho_2 + \frac{3}{2})_{j+1} (\frac{1}{2})_{j+1}} \\ &= 1 + \sum_{j=0}^{\infty} \frac{(2\rho_2 + \frac{1}{2})(-\frac{1}{2})}{(2\rho_2 + \frac{3}{2} + j)(\frac{1}{2} + j)}, \end{aligned}$$

by cancelling the common factors in the Pochhammer symbols. This sum then can be explicitly evaluated, viz.,

$$\begin{aligned} R(x) &= 1 + \left(2\rho_2 + \frac{1}{2}\right) \left(-\frac{1}{2}\right) \sum_{j=0}^{\infty} \frac{1}{(2\rho_2 + \frac{3}{2} + j)(\frac{1}{2} + j)} \\ &= 1 + \left(2\rho_2 + \frac{1}{2}\right) \left(-\frac{1}{2}\right) \frac{1}{2\rho_2 + 1} \sum_{j=0}^{\infty} \left(\frac{1}{\frac{1}{2} + j} - \frac{1}{2\rho_2 + \frac{3}{2} + j}\right) \\ &= 1 - \frac{1}{2} \left(2\rho_2 + \frac{1}{2}\right) \frac{1}{2\rho_2 + 1} \sum_{k=0}^{2\rho_2} \frac{1}{\frac{1}{2} + k} \end{aligned}$$

since it is a telescopic series. Now as a function of $2\rho_2 = 1, 2, \dots$,

$$\frac{1}{2} \left(2\rho_2 + \frac{1}{2}\right) \frac{1}{2\rho_2 + 1} \sum_{k=0}^{2\rho_2} \frac{1}{\frac{1}{2} + k}$$

attains its minimum 1 when $2\rho_2 = 1$ namely when $b = 0$. Thus for $b > 0$,

$$R(x) = 1 - \frac{1}{2} \left(2\rho_2 + \frac{1}{2}\right) \frac{1}{2\rho_2 + 1} \sum_{k=0}^{2\rho_2} \frac{1}{\frac{1}{2} + k} < 1 - 1 = 0.$$

In the next section we shall give a different description of \mathcal{A}_0 and a different proof that the triangle T_2 is not in \mathcal{A}_0 .

Remark 6.6. We note that the unitarity set $\mathcal{U} \cap \mathcal{C}$ is the parameter set for the spherical complementary series of G and it has been determined for $U(2, N)$ by Knapp and Spohn [7]. Let k be the largest positive integer such that $k \leq \frac{b-1}{2}$. Then $\mathcal{U} \cap \mathcal{C}$ is the union of the following sets

- 1) the triangle $\{x \in \mathbb{R}_{\geq 0}^2; 0 \leq x_1 + x_2 \leq 1\}$;
- 2) the triangles bordered by $x_1 - x_2 \geq j$ and $x_1 + x_2 \leq j + 1$ in the triangle $[0, \rho_2]^2 \cap \mathcal{C}$, $j = 1, \dots, k$;
- 3) line segments $x_1 - x_2 = j$ in the triangle $[0, \rho_2]^2 \cap \mathcal{C}$, $j = 1, \dots, k$.

Thus in this case \mathcal{U}_0 is a proper subset of \mathcal{A}_0 .

6.2. Alternative approach to $U(m + 2, 2)$

We will need the following beautiful and simple identity for the Γ -function.

Lemma 6.7. *Suppose $a, b, c, d \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ satisfy $a + b = c + d$ then*

$$(6.4) \quad \prod_{n=0}^{\infty} \frac{(n + a)(n + b)}{(n + c)(n + d)} = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)}.$$

Proof. We recall the Weierstrass formula for the Γ -function

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left\{ (1 + z/n) e^{-z/n} \right\}, \quad z \neq 0, -1, -2, \dots$$

(see e.g. [26, P. 236]). Using this the right side of (6.4) becomes

$$\frac{ab}{cd} \prod_{n=1}^{\infty} \left\{ \frac{(1 + a/n)(1 + b/n) e^{-(a+b)/n}}{(1 + c/n)(1 + d/n) e^{-(c+d)/n}} \right\}.$$

After canceling $e^{-(a+b)/n} = e^{-(c+d)/n}$ we get the left side of (6.4). □

Let $\psi_l(t) = (-1)^k p_l(t) = \prod_{n=0}^{l-1} [(n + \alpha)^2 - t^2]$ be the rank 1 Okounkov polynomial. We will show how to compute the limit of the rescaled polynomial

$$(6.5) \quad r_l(t) = \frac{\psi_l(t)}{\psi_l(0)} = \prod_{n=0}^{l-1} \frac{(n + \alpha)^2 - t^2}{(n + \alpha)^2}, \quad r(t) = \lim_{l \rightarrow \infty} r_l(t).$$

We are mainly interested in the case $\alpha = \frac{m+1}{2}$ where m is a non-negative integer. In this case the limit can be expressed in terms of the function

$$(6.6) \quad s(t) = \frac{\sin \pi t}{(t + 1) \cdots (t + m)}$$

with $s(t) = \sin \pi t$ for $m = 0$.

Proposition 6.8. *If $\alpha = \frac{m+1}{2}$ where m is a non-negative integer then*

$$(6.7) \quad r(t + \alpha) = -\frac{\Gamma(\alpha)^2}{\pi} s(t).$$

Proof. Applying (6.4) to (6.5) we get

$$r(t) = \prod_{n=0}^{\infty} \frac{(n + \alpha + t)(n + \alpha - t)}{(n + \alpha)(n + \alpha)} = \frac{\Gamma(\alpha)^2}{\Gamma(\alpha + t)\Gamma(\alpha - t)},$$

for $\alpha \notin \{0, -1, -2, \dots\}$. For $\alpha = (m + 1)/2$ this gives

$$r(t + \alpha) = \frac{\Gamma(\alpha)^2}{\Gamma(m + 1 + t)\Gamma(-t)} = \left[\frac{\Gamma(\alpha)^2}{(t + 1) \cdots (t + m)} \right] \frac{1}{\Gamma(1 + t)\Gamma(-t)},$$

and (6.7) now follows from the elementary identity $\Gamma(t)\Gamma(1-t) = -\pi/\sin \pi t$. □

We consider now the real points of the Shimura sets for the rank 2 groups $U(m + 2, 2)$. (So the root multiplicity $2b$ is now $2m$.) For this we fix as before

$$\alpha = \frac{m + 1}{2},$$

and write $q_\lambda(x)$ for $q_\lambda(x; 1, \alpha)$. As above we restrict attention to the Weyl chamber \mathcal{C} in the first quadrant \mathbb{R}_+^2 and we define

$$\begin{aligned} \mathcal{G}_0 &= \{x \in \mathcal{C} \mid q_{(1,0)}(x), q_{(1,1)}(x) \geq 0\}, \\ \mathcal{A}_0 &= \{x \in \mathcal{C} \mid q_\lambda(x) \geq 0 \text{ for all } \lambda\}. \end{aligned}$$

Our description of these set will involve the triangles

$$T_1 = [0, \alpha] \times [0, \alpha] \cap \mathcal{C}, \quad T_2 = [\alpha, \alpha + 1] \times [\alpha, \alpha + 1] \cap \mathcal{C}.$$

For \mathcal{G}_0 we consider the following subset of T_2

$$V = \{x \in T_2 : q_{1,0}(x) \geq 0\}.$$

Theorem 6.9. *We have $\mathcal{G}_0 = T_1 \cup V$.*

Proof. For $x \in \mathbb{R}_+^2$ the inequalities $q_{(1,1)}(x) \geq 0$ and $q_{(1,0)}(x) \geq 0$ are respectively

$$(\alpha^2 - x_1^2)(\alpha^2 - x_2^2) \geq 0, \quad x_1^2 + x_2^2 \leq \alpha^2 + (\alpha + 1)^2.$$

The $q_{(1,1)}$ inequality holds iff either (a) $x \in T$ or (b) $x_1, x_2 \geq \alpha$. In case (a) the $q_{(1,0)}$ inequality is automatic, in case (b) it forces $x \in T_2$. The result follows. \square

For $\lambda = (l + k, k)$ we have

$$P_\lambda(x) = \frac{1}{x_1^2 - x_2^2} \det \begin{bmatrix} p_{l+k+1}(x_1) & p_{l+k+1}(x_2) \\ p_k(x_1) & p_k(x_2) \end{bmatrix},$$

which gives

$$(6.8) \quad q_\lambda(x) = (-1)^{|\lambda|} P_\lambda(x) = \frac{\psi_{l+1}^k(x_2) - \psi_{l+1}^k(x_1)}{x_1^2 - x_2^2} \psi_k(x_1) \psi_k(x_2)$$

where

$$(6.9) \quad \psi_{l+1}^k(t) = \frac{\psi_{l+k+1}(t)}{\psi_k(t)} = \prod_{i=0}^l [(i + k + \alpha)^2 - t^2].$$

Lemma 6.10. *The inequality $q_\lambda(x) \geq 0$ holds in the following cases.*

- 1) *If $x \in T$ and λ is arbitrary.*
- 2) *If $x \in T_2$ and $\lambda_2 = k > 0$.*

Proof. By continuity and symmetry it suffices to prove $q_\lambda(x) \geq 0$ for x satisfying the additional conditions

$$(6.10) \quad x_1 > x_2, \quad x_1, x_2 \notin \{\alpha, \alpha + 1\}.$$

In this case we have

$$x_1^2 - x_2^2 > 0, \quad 0 < x_2 < x_1 < \alpha.$$

Now $0 < t < \alpha$, $\psi_k(t)$ is positive and $\psi_{l+1}^k(t)$ is positive and decreasing. It follows that

$$(6.11) \quad \psi_k(x_1) \psi_k(x_2) > 0 \quad \text{and} \quad \psi_{l+1}^k(x_2) - \psi_{l+1}^k(x_1) > 0.$$

Thus by (6.8) we have $q_\lambda(x) \geq 0$.

Let $\lambda = (l + k, k)$ with $k \geq 1$, and suppose $x \in T_2$ satisfies the assumptions (6.10). Then we have

$$x_1^2 - x_2^2 > 0, \quad \alpha < x_2 < x_1 < \alpha + 1.$$

For $\alpha < t < \alpha + 1$ and $k \geq 1$, $\psi_k(t)$ is negative and $\psi_{l+1}^k(t)$ is positive and decreasing. Once again (6.11) holds and so $q_\lambda(x) \geq 0$. □

We now describe \mathcal{A}_0 and for this we recall the function $s(t) = \frac{\sin \pi t}{(t+1)\cdots(t+m)}$ as in the previous section, and we let $S(x, y)$ denote its symmetrized divided difference

$$(6.12) \quad S(x_1, x_2) = \frac{s(x_1) - s(x_2)}{x_1 - x_2} \text{ for } x_1 \neq x_2, \quad S(x, x) = s'(x),$$

and we put

$$W = \{x \in T_2 : S(x - \alpha) \geq 0\}.$$

Here and elsewhere $x - \alpha$ denotes the pair $(x_1 - \alpha, x_2 - \alpha)$.

Theorem 6.11. *We have $\mathcal{A}_0 = T_1 \cup W$.*

Proof. By Theorem 6.9 we know that

$$\mathcal{A}_0 \subseteq \mathcal{G}_0 = T \cup V \subseteq T \cup T_2.$$

By Lemma 6.10 it remains only to prove that for $x \in T_2$

$$(6.13) \quad q_{l,0}(x) \geq 0 \text{ for all } l \iff S(x - \alpha) \geq 0.$$

Let $x_1 \geq x_2$. We divide the proof of (6.13) into three cases.

Case 1: We first consider $x \in T_2$ satisfying

$$(6.14) \quad \alpha + 1 > x_1 \geq x_2 > \alpha.$$

This implies that $-\psi_{l+1}(x_2)$, $s(x_2 - \alpha)$, and $x_1 + x_2$ are all > 0 , and we define

$$c_l(x) = \frac{q_{l,0}(x)}{-\psi_{l+1}(x_2)}, \quad c(x) = \frac{S(x - \alpha)}{(x_1 + x_2)(s(x_2 - \alpha))}.$$

By positivity (6.13) is equivalent to the assertion

$$(6.15) \quad c_l(x) \geq 0 \text{ for all } l \iff c(x) \geq 0.$$

We will prove a stronger statement, namely

$$(6.16) \quad c_l(x) \text{ is a decreasing sequence with limit } c(x).$$

By continuity it suffices to prove (6.16) under the additional assumption $x_1 > x_2$, and we may consider then the simpler expressions

$$\begin{aligned} b_l &= (x_1^2 - x_2^2) c_l(x) + 1 = \frac{\psi_{l+1}(x_1)}{\psi_{l+1}(x_2)}, \\ b &= (x_1^2 - x_2^2) c(x) + 1 = \frac{s(x_1 - \alpha)}{s(x_2 - \alpha)} \end{aligned}$$

Then b_l and b are strictly positive and we have

$$\frac{b_{l+1}}{b_l} = \frac{\alpha + l + 1 - x_1}{\alpha + l + 1 - x_2} \leq 1.$$

Moreover by Proposition 6.8 we have $b_l \rightarrow b$. Thus b_l is a decreasing sequence with limit b . This implies (6.16) and hence (6.15) and (6.13).

Case 2: We now suppose that $x_2 = \alpha$, so that x is of the form (x_1, α) . We claim that we have

$$q_{l,0}(x) \geq 0 \text{ for all } l, \quad S(x - \alpha) \geq 0.$$

By continuity it suffices to prove this for $x_1 \neq \alpha$ in which case it follows from the explicit formula

$$q_{l,0}(x) = \frac{-q_l(x_1)}{x_1^2 - \alpha^2}, \quad S(x - \alpha) = \frac{s(x_1 - \alpha)}{x_1 - \alpha}.$$

Thus both sides of (6.13) are true and hence equivalent.

Case 3: Finally suppose that $x_1 = \alpha + 1$, so that x is of the form $(\alpha + 1, x_2)$. By Case 2 we may further suppose that $x_2 > \alpha$. With these assumptions we have $q_{l,0}(x) < 0$. So the left side of (6.13) is false and we need only prove that

$$(6.17) \quad S(x - \alpha) < 0.$$

If $x_2 \neq \alpha + 1$ this follows from the explicit formula

$$S(x - \alpha) = \frac{-s(x_2 - \alpha)}{(\alpha + 1) - x_2}.$$

If $x_2 = \alpha + 1$ then x is the point $(\alpha + 1, \alpha + 1)$ and we have

$$S(x - \alpha) = S(1, 1) = s'(1).$$

To compute this derivative we recall the formula

$$s(t) = \frac{\sin(\pi t)}{g(t)}, \quad g(t) = (t+1) \cdots (t+m).$$

Thus we have $g(1) = (m+1)!$ and

$$s'(t) = \frac{(\pi \cos \pi t) g(t) - (\sin \pi t) g'(t)}{g(t)^2}, \quad s'(1) = -\frac{\pi}{(m+1)!}.$$

This proves (6.17) and hence (6.13).

Cases 1, 2, 3 establish then (6.13) for $x \in T_2$. □

Now if $m = 0$ then it is clear that the set W is the triangle T_2 bordered by $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, x_1 + x_2 = 2$ so this agrees with Theorem 6.2.

7. Appendix

In this appendix we write (x, y) instead of (x_1, x_2) . The set W of Theorem 6.11 is the (α, α) translate of the region in the positive quadrant bounded by the coordinate axes and the curve defined implicitly by the equation $S(x, y) = 0$.

We write $S_m(x, y)$ for $S(x, y)$ to indicate its dependence on m , and we give the graph of $S_m(x, y) = 0$ for $m = 0, 1, 2, 3$.

This graph is symmetric about the line $x = y$, and it is of some interest to determine the point c_m where the graph crosses the line $x = y$.

Lemma 7.1. *The point $c = c_m$ satisfies the equation*

$$(7.1) \quad \pi \cot \pi c = \sum_{i=1}^m \frac{1}{(c+i)}.$$

Proof. It is easy to see that c_m is a critical point of $s(t)$. Since $s(x)$ is positive in the open interval $(0, 1)$, its critical points are the same as those

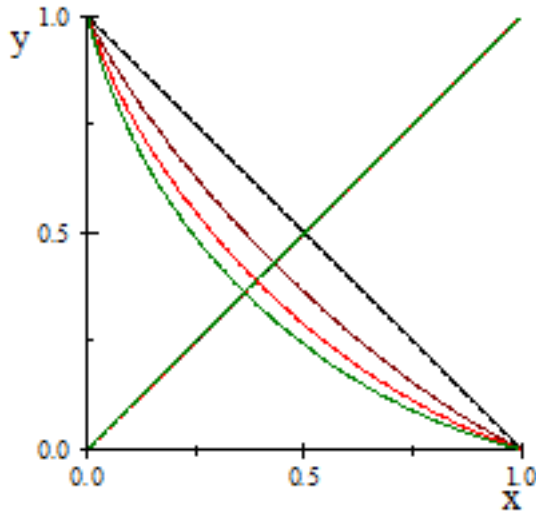


Figure 1: $S_m(x, y) = 0$, $m = 0, 1, 2, 3$.

of the function

$$\ln(s(x)) = \ln(\sin \pi x) - \sum_{i=1}^m \ln(x+i).$$

This gives

$$\frac{d}{dx} \ln(s(x)) = \pi \cot \pi x - \sum_{i=1}^m \frac{1}{(x+i)}.$$

The result follows by setting the derivative equal to 0. \square

Corollary 7.2. *We have $c_m \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. As $m \rightarrow \infty$ the right side of (7.1) approaches ∞ for all c in the interval $(0, 1)$. Thus we must have $\pi \cot(\pi c_m) \rightarrow \infty$ as well, which implies $c_m \rightarrow 0$. \square

It seems likely that as $m \rightarrow \infty$ the region collapses to the union of the unit intervals on the coordinate axes. However this requires an extra convexity argument for the graph.

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